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Lecture 3

Differentiation

Calculus 1A

UNIVERSITY OF TWENTE.

Overview

This lecture is based on the material of chapters 3 and 4 of Thomas' Calculus.

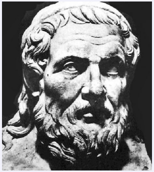
Topics:

- The definition of the derivative.
- Calculating derivatives.
- Extreme values.
- L'Hôpital's rule.

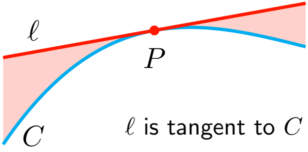
A bit of history

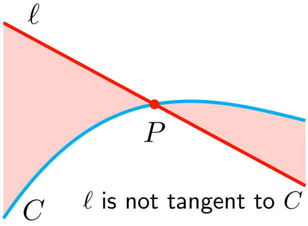
Apollonius of Perga

A **tangent to a curve** C is a line ℓ that intersects C, such that no other line could fall between ℓ and C.

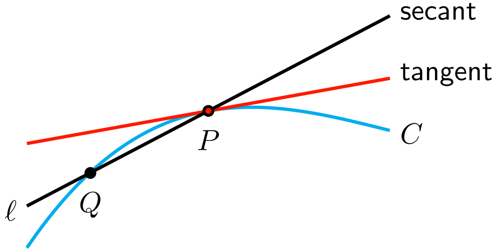


Appolonius of Perga, ca. 225 BC (source: `https://cmap.ihmc.us/`)



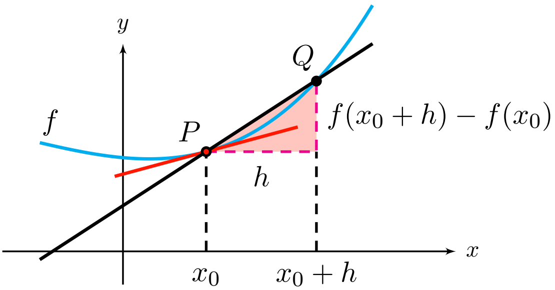


Finding the tangent using a limit



- The line through P and Q is called a **secant**.
- The tangent to C (at P) is the limit of ℓ for Q approaching P.

Finding the tangent to the graph of a function



- The slope of the secant through P and Q is

$$\frac{f(x_0+h)-f(x_0)}{h}$$

- The slope of the tangent to C at P is

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

Differentiability and continuity

Definition

Let f be a function defined on an open interval I, and let $x_0 \in I$. The **derivative of f at x_0** is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

- If the limits exists we say: f **is differentiable at x_0**.
- Alternative notations:

$$\frac{df}{dx}(x_0) \qquad \text{and} \qquad f_x(x_0).$$

Differentiability is a local property

Example

Let $f(x) = |x|$.

- If $c \neq 0$, then f is differentiable at c.
- The function f is not differentiable at 0.

- Let $c > 0$, then $|x| = x$ in an open interval containing c, so

$$f'(c) = \lim_{h \to 0} \frac{|c+h| - |c|}{h} = \lim_{h \to 0} \frac{(c+h) - c}{h} = 1.$$

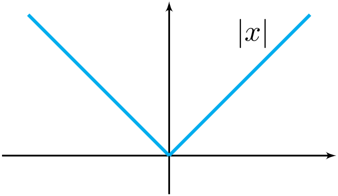
- For $c < 0$ we have

$$f'(c) = \lim_{h \to 0} \frac{|c+h| - |c|}{h} = \lim_{h \to 0} \frac{-(c+h) - (-c)}{h} = -1.$$

Example (continued)

The function $f(x) = |x|$ is not differentiable at $c = 0$:

- $$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h| - |0|}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$$
- $$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} \lim_{h\to 0^-} \frac{|h| - |0|}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1.$$
- The graph of $|x|$ has no tangent line at 0.



Differentiability and continuity

Theorem

If f is differentiable at c, then it is continuous at c.

- Suppose f is differentiable at c, so this limit exists:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

- Then

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c} \big(f(x) - f(c) + f(c)\big) \\
&= \lim_{x \to c} \big(f(x) - f(c)\big) + \lim_{x \to c} f(c) \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) + f(c) \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) + f(c) \\
&= f'(c) \cdot 0 + f(c) = f(c).
\end{aligned}$$

Quiz

Define

$$f(x) = \begin{cases} x & \text{if } x < 1, \\ x + 1 & \text{if } x \geq 1. \end{cases}$$

Is f differentiable in 1?

Assume that f and g are differentiable at x.

1. *Sum rule* $$\frac{d}{dx}(f+g)(x) = f'(x) + g'(x)$$

2. *Difference rule* $$\frac{d}{dx}(f-g)(x) = f'(x) - g'(x)$$

3. *Constant multiple rule* $$\frac{d}{dx}(k \cdot f)(x) = k \cdot f'(x)$$

4. *Product rule* $$\frac{d}{dx}(f \cdot g)(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

5. *Quotient rule*[1] $$\frac{d}{dx}\left(\frac{f}{g}\right)(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

6. *Chain rule* $$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x)$$

[1] Assuming $g(x) \neq 0$.

The derivatives of elementary functions

	$f(x)$	$f'(x)$	condition		$f(x)$	$f'(x)$	condition
1.	c	0	$c \in \mathbb{R}$	**7.**	$\sin x$	$\cos x$	
2.	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$x > 0$	**8.**	$\cos x$	$-\sin x$	
3.	$\frac{1}{x}$	$-\frac{1}{x^2}$	$x \neq 0$	**9.**	$\arcsin x$	$-\frac{1}{\sqrt{1-x^2}}$	$\|x\| < 1$
4.	x^α	$\alpha\, x^{\alpha-1}$	$\alpha \in \mathbb{R}$	**10.**	$\arccos x$	$\frac{1}{\sqrt{1-x^2}}$	$\|x\| < 1$
5.	e^x	e^x		**11.**	$\arctan x$	$\frac{1}{x^2+1}$	
6.	$\ln x$	$\frac{1}{x}$	$x > 0$				

Example

Problem

Calculate $\dfrac{d}{dx}(x^x)$.

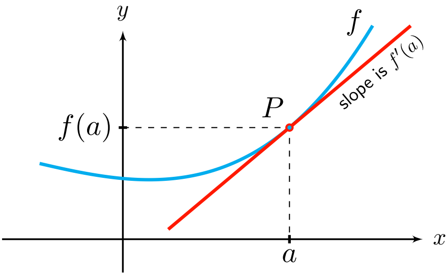
Linearization

The tangent line

Theorem

Let f be differentiable at a. The tangent line to the graph of f at $P\big(a, f(a)\big)$ is given by the equation

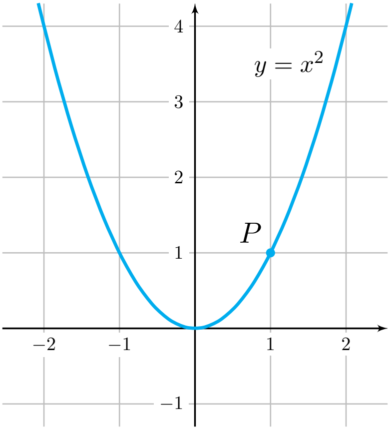
$$y = f(a) + f'(a)(x - a).$$



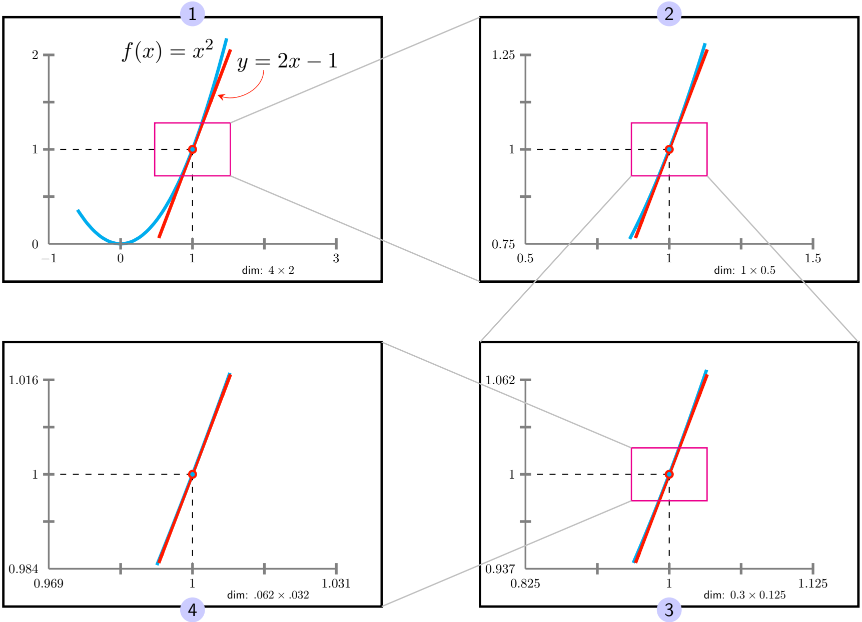
The tangent line

Example

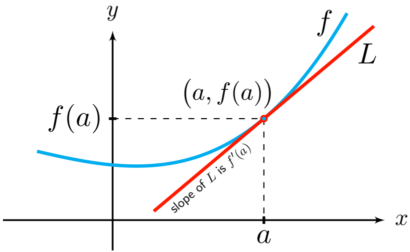
Calculate an equation of the tangent line to $y = x^2$ at $P(1, 1)$.



The tangent line as linear approximation



Linearization



- The tangent to the graph of f at a is the graph of the function

$$L(x) = f(a) + f'(a)(x - a),$$

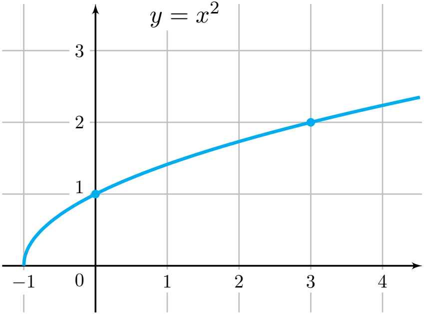
Definition

The funtion L is called the **linearization of f at a**.

Linearization

Example

Calculate the linerizations of $f(x) = \sqrt{x+1}$ at 0 and at 3.



Linearization as approximation

	approximation		true value	error
x	$f(x) \approx \frac{1}{2}x + 1$	$= L(x)$	$f(x)$	$\lvert f(x) - L(x) \rvert$
0.2	$\sqrt{1.2} \approx \dfrac{0.2}{2} + 1$	$= 1.1$	1.095445	$< 10^{-2}$
0.05	$\sqrt{1.05} \approx \dfrac{0.05}{2} + 1$	$= 1.025$	1.024695	$< 10^{-3}$
0.005	$\sqrt{1.005} \approx \dfrac{0.005}{2} + 1$	$= 1.00250$	1.002497	$< 10^{-5}$

- The table shows approximations of $f(x) = \sqrt{x+1}$ using the linearization $L(x) = \frac{1}{2}x + 1$ at 0.
- The closer x is to 0, the better the approximation is.

Differentials

- If $y = f(x)$, define $\Delta y = f(x + \Delta x) - f(x)$. You cannot use the quotient rule here:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\text{``}\underset{???}{=}\text{''} \; \frac{\displaystyle\lim_{\Delta x \to 0} \Delta y}{\displaystyle\lim_{\Delta x \to 0} \Delta x} = \frac{0}{0}.$$

- Early mathematicians used "infinitisimal small numbers" to overcome this problem.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx},$$

where dx indicates: "make Δx infinitisimal small", and where dy indicates: "the value of Δy obtained by making Δx infinitisimal small".

Differentials

Axiom of differentials

If $y = f(x)$ then $dy = f'(x)dx$.

- **Example**: let $y = x^3 + 1$, then

$$dy = 3x^2 dx.$$

- While the true nature of differentials remain obscure, you can still do calculations with them!
- Differentials play an important role in the calculation of integrals.

Extreme values

Photo by Jeremy Thompson (Busch Gardens Tampa Bay Themepark)

Absolute extreme values

Definition

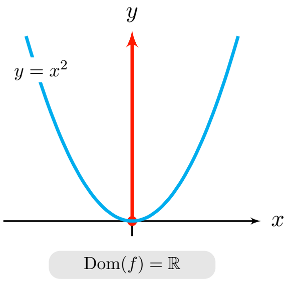
Let f be a real function with domain D. Let $c \in D$. Then f has an **absolute maximum value on** D **at** c if

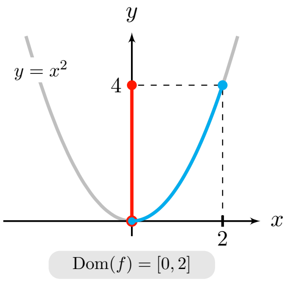
$$\forall\, x \in D \left[\, f(x) \leq f(c) \,\right],$$

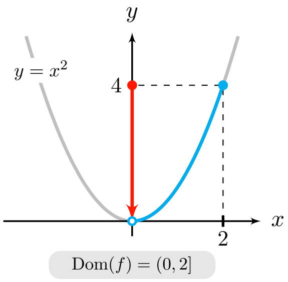
and f has an **absolute minimum value on** D **at** c if

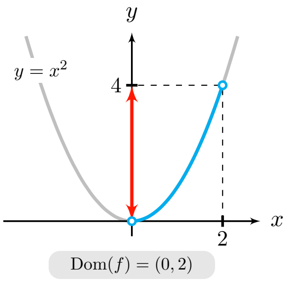
$$\forall\, x \in D \left[\, f(x) \geq f(c) \,\right].$$

Absolute extreme values





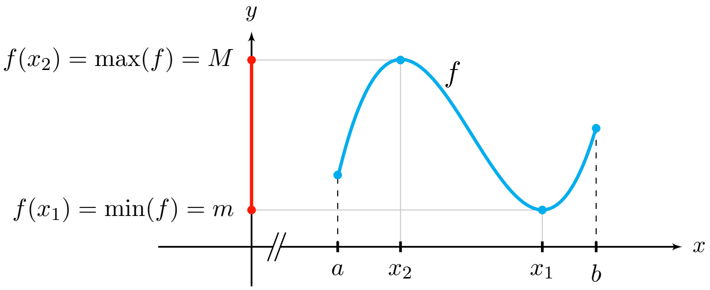




The Extreme Value Theorem

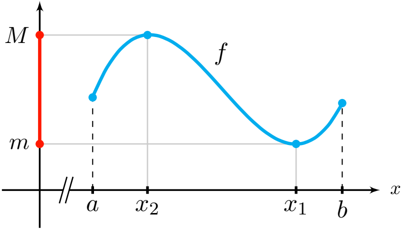
Theorem (the Extreme Value Theorem)

If a real function f is **continuous** *on a* **closed** *and* **bounded** *interval $[a, b]$, then f attains both an absolute maximum value and an absolute minimum value on $[a, b]$.*

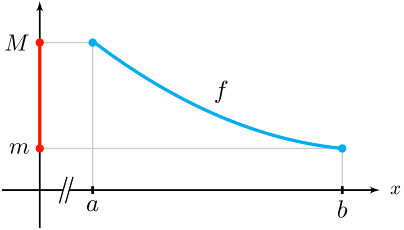


- Due to the Intermediate Value Theorem, the range of f is $[m, M]$.

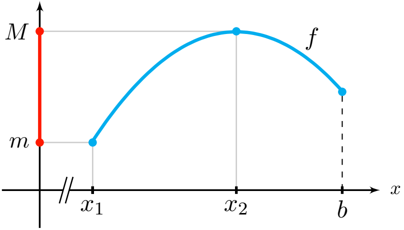
The Extreme Value Theorem



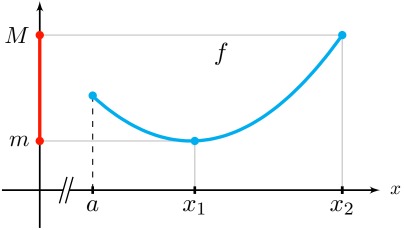
Minimum and maximum
at interior points



Minimum and maximum
at endpoints



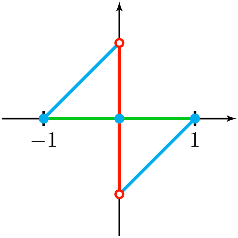
Maximum at interior point,
minimum at enpoint



Minimum at interior point,
maximum at enpoint

The Extreme Value Theorem

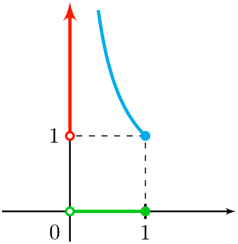
- The conditions ‘closed’, ‘bounded’ and ‘continuous’ must all be satisfied.



$f\colon [-1,1] \to \mathbb{R}$

$$f(x) = \begin{cases} x+1 & \text{if } -1 \le x < 0, \\ 0 & \text{if } x = 0, \\ x-1 & \text{if } 0 < x \le 1. \end{cases}$$

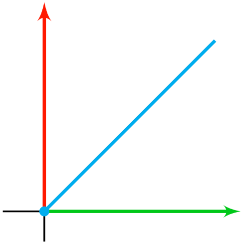
f is not coninuous on I



$f\colon (0,1] \to \mathbb{R}$

$$f(x) = \frac{1}{x}$$

I is not closed



$f\colon [0,\infty) \to \mathbb{R}$

$$f(x) = x$$

I is not bounded

Local Extrema

Definition

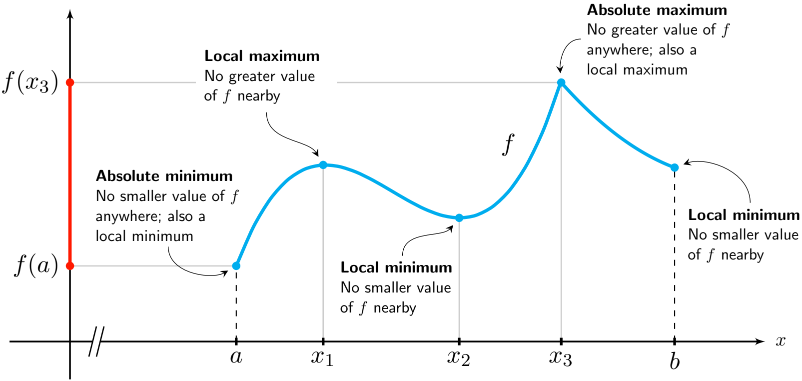
Let f be a real function with domain D. Let $c \in D$. Then f has a **local maximum value at** c if there exists an open interval I containing c such that

$$\forall\, x \in I \cap D \left[\, f(x) \leq f(c) \,\right],$$

and f has a **local minimum value at** c if there exists an open interval $I \subseteq D$ containing c such that

$$\forall\, x \in I \cap D \left[\, f(x) \geq f(c) \,\right].$$

Local extrema



- The function f has an absolute minimum at a.
- The function f has an absolute maximum at x_3.
- The function f has local minima at a, x_2 and b.
- The function f has local maxima at x_1 and x_3.

Local extrema and derivatives

Theorem (First Derivative Theorem)

If f has a local maximum or minimum value at an interior point c of its domain, and f' is defined at c, then $f'(c) = 0$.

If $f: I \to \mathbb{R}$ has a local extreme value at c, at least one of the following statements is true:

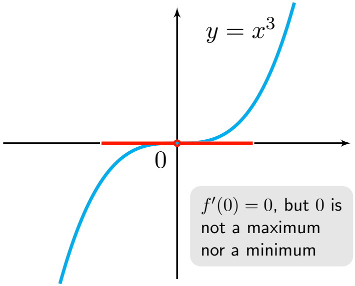
- c is not an interior point of I,
- f is not differentiable at c,
- $f'(c) = 0$.

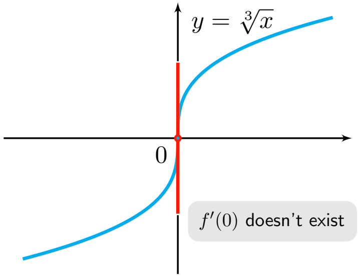
Critical points

Definition

An interior point of the domain of f where f' is undefined or where $f'(c) = 0$ is called a **critical point of** f.

- A critical point need not to be a local extreme.





Finding absolute extrema

Step-by-step procedure

Let f be a continuous function on a closed and bounded interval $[a, b]$.

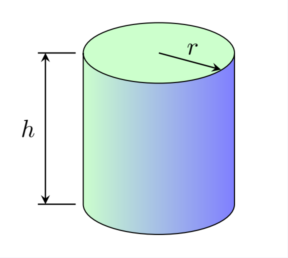
1. Find all points in (a, b) where f' does not exist.
2. Find all $c \in (a, b)$ where $f'(c) = 0$.
3. Calculate the function values in the previous points, and in a and b.
4. The largest value is the absolute maximum value.
5. The smallest value is the absolute minimum value.

An optimization problem

Example

A cylindrical can has a volume of 1 liter. Minimize the surface of the can.

The can has a bottom and a lid. The radius is r and the height is h.



An optimization problem (continued)

L'Hôpital's rule

Photo Pille Kirsi, from www.pexels.com

https://www.pexels.com/photo/rock-balancing-1093199/

Indeterminate forms

Recap

A limit is called called an **indeterminate form** if applying the limit laws leads to an indecisive result.

limit law	limit	notation
difference rule	$\lim_{x \to c} f(x) - g(x)$	$\infty - \infty$
product rule	$\lim_{x \to c} f(x) \cdot g(x)$	$0 \cdot \infty$
quotient rule	$\lim_{x \to c} \frac{f(x)}{g(x)}$	$\frac{\infty}{\infty}$ or $\frac{0}{0}$
power rule	$\lim_{x \to c} f(x)^{g(x)}$	1^{∞}, 0^0 or ∞^0

Limits of type $\frac{0}{0}$

Theorem (L'Hôpital's rule)

Suppose that $f(c) = g(c) = 0$, that f and g are differentiable on an open interval I containing c, and that $g'(x) \neq 0$ on I if $x \neq c$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

- This rule is also correct when $c = \infty$ or $c = -\infty$.
- If the limit on the right side is of type $\frac{0}{0}$, you can apply L'Hôpital's again.

L'Hôpital's rule

Example

Calculate $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.

Limits of type $\frac{\infty}{\infty}$

Theorem

Suppose that f and g are differentiable on an open interval I containing c, that $g'(x) \neq 0$ on I if $x \neq c$. Moreover, assume that

$$\lim_{x \to c} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x \to c} g(x) = \pm\infty,$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

- This rule is also correct when $c = \infty$ or $c = -\infty$.

L'Hôpital's rule

Example

Calculate $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$.

Limits of type $0 \cdot \infty$

- Limits of type $\lim_{x \to c} f(x) \cdot g(x)$ that on substitution give $0 \cdot \infty$, should be written as a fraction, so that L'Hôpital's rule can be applied.
- Either write

$$\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} \frac{f(x)}{\frac{1}{g(x)}},$$

or write

$$\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} \frac{g(x)}{\frac{1}{f(x)}}.$$

L'Hôpital's rule

Example

Calculate $\lim_{x \to \infty} x\, e^{-x}$.

Limits with indeterminate powers

- Limits of type $\lim_{x \to c} f(x)^{g(x)}$ that after substitution gives a limit of type 0^0, ∞^0 or 1^∞, should be written as follows:

$$\lim_{x \to c} f(x)^{g(x)} = \lim_{x \to c} e^{g(x)\, \ln f(x)}.$$

- First calculate $L = \lim_{x \to c} g(x)\, \ln f(x)$.

- Then

$$\lim_{x \to c} f(x)^{g(x)} = e^L.$$

Example

Calculate $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$.