Calculus 1A

UNIVERSITY OF TWENTE.

Lecture 2

Limits and continuity

Overview

This lecture is based on the material of chapter 2 of Thomas' Calculus.

Topics:

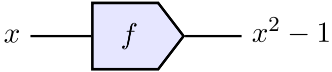
- Functions (recap).
- Informal introduction to limits.
- A formal way to describe limits.
- Limit laws.
- Special types of limits: one-sided limits, limits to infinity, limits that are infinity.
- The sandwich theorem.
- Continuity.

How to perceive functions

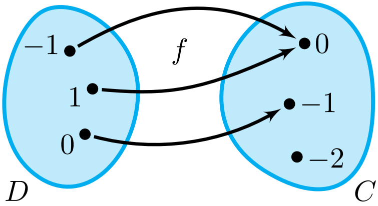
formula

$$f(x) = x^2 - 1$$

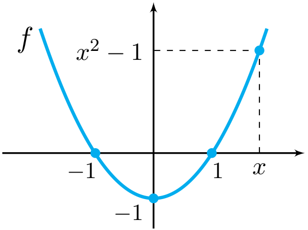
machine model



arrow diagram



graph



Functions, formal definition

Definition

Let D and C be sets. A **function from D to C** is a subset $f \subseteq D \times C$ such that

$$\forall\, x \in D\; \exists!\, y \in C\; \big[\, (x, y) \in f \,\big].$$

- The set $D \times C$ is the **Cartesian product of D and C**:

$$D \times C = \{(x, y) \,|\, x \in D \wedge y \in C\}.$$

- The quantification $\exists!\, y \in C\, \big[\, p(y) \,\big]$ means: "there is *exactly one* y that satisfies $p(y)$".
- The set D is the **domain of f**, while C is the **codomain of f**.
- We denote $y = f(x)$ rather than $(x, y) \in f$.

The limit of a function

Limits, the informal way

Let $f\colon D \to \mathbb{R}$ be a function with domain $D \subseteq \mathbb{R}$. We say

$$\lim_{x\to c} f(x) = L$$

if $f(x)$ tends to L whenever x tends to c.

- The limit point c does not need to be an element of D.
- While x approaches c (but is not equal to c), it must be an element of D.

The limit of a function

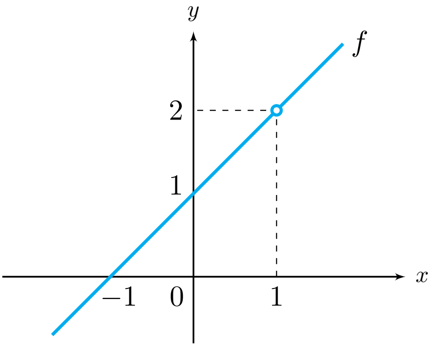
Example

Try to find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

x	$f(x)$
1	undefined
2	
1.1	
1.01	
1.001	

x	$f(x)$
1	undefined
0.9	
0.99	
0.999	
0.9999	

The limit of a function



- For all $x \neq 1$ we have

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x - 1} = x + 1.$$

A more mathematical approach

What does $\lim_{x \to c} f(x) = L$ mean?

- ★ If x approaches c, then $f(x)$ approaches L.
- ★★ Make $f(x)$ approach L by moving x to c.
- ★★★ Make the distance between $f(x)$ and L small by making the distance between x and c small.
- ★★★★ Make the distance between $f(x)$ and L smaller than a given positive number, say 0.001, by making the distance between x and c small.
- ★★★★★ Make $|f(x) - L|$ smaller than an *arbitrary* positive number, say ε, by making $|x - c|$ small.
- ★★★★★★ For all $\varepsilon > 0$, make $|f(x) - L| < \varepsilon$, by making $|x - c|$ sufficiently small.

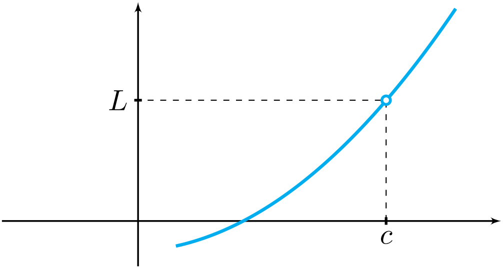
The formal definition of the limit

Quantify the statements:

★ $\forall\, \varepsilon > 0$[if $|x - c|$ is sufficiently small (but not 0), then $|f(x) - L| < \varepsilon$.]

★★ $\forall\, \varepsilon > 0\ \exists\, \delta > 0 \left[\text{if } 0 < |x - c| < \delta, \text{ then } |f(x) - L| < \varepsilon. \right]$

★★★ $$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in \mathbb{R} \left[0 < |x - c| < \delta \ \Rightarrow\ |f(x) - L| < \varepsilon \right].$$



Simplifying limits

Theorem (replacement theorem)

If $f(x) = g(x)$ *for all* $x \neq c$*, then* $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$*.*

- Calculate $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$:

Existence of the limit

- The limit $\lim_{x \to c} f(x)$ exists:

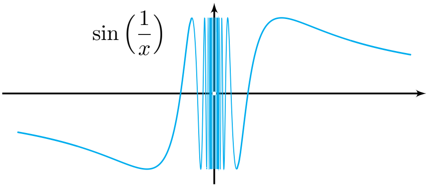
$$\exists\, L \in \mathbb{R}\; \forall\, \varepsilon > 0\; \exists\, \delta > 0\; \forall\, x \in \mathbb{R} \left[\, 0 < |x - c| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon \,\right]$$

- The limit $\lim_{x \to c} f(x)$ does not exist:

$$\forall\, L \in \mathbb{R}\; \exists\, \varepsilon > 0\; \forall\, \delta > 0\; \exists\, x \in \mathbb{R} \left[\, 0 < |x - c| < \delta \;\wedge\; |f(x) - L| \geq \varepsilon \,\right]$$

When doesn't the limit exist?

$$\forall L \in \mathbb{R}\, \exists \varepsilon > 0\, \forall \delta > 0\, \exists x \in \mathbb{R} \left[0 < |x - c| < \delta \ \wedge \ |f(x) - L| \geq \varepsilon \right]$$



- The limit $\lim\limits_{x \to 0} \sin(1/x)$ does not exist.
- Proof for $L > 0$:

One-sided limits

Definition

We say

$$\lim_{x \to c^-} f(x) = L$$

if $f(x)$ tends to L whenever x approaches c from the left, and

$$\lim_{x \to c^+} f(x) = L$$

if $f(x)$ tends to L whenever x approaches c from the right.

- Formal definition of the left limit:

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in \mathbb{R} \left[c - \delta < x < c \;\Rightarrow\; |f(x) - L| < \varepsilon \right]$$

- Formal definition of the right limit:

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in \mathbb{R} \left[c < x < c + \delta \;\Rightarrow\; |f(x) - L| < \varepsilon \right]$$

One-sided limits

Theorem

- *If* $\lim_{x\to c} f(x) = L$ *exists, then* $\lim_{x\to c^-} f(x) = L$ *and* $\lim_{x\to c^+} f(x) = L$ *exist, and they are both equal to* L*.*
- *If* $\lim_{x\to c^-} f(x) = L$ *and* $\lim_{x\to c^+} f(x) = R$ *exist, and if* $L = R$*, then* $\lim_{x\to c} f(x) = L$ *exists, and it is equal to* $L = R$*.*

- The limit $\lim_{x\to c}$ does not exist, if one of the following is true:
  1. the left limit $\lim_{x\to c^-} f(x)$ does not exist,
  2. the right limit $\lim_{x\to c^+} f(x)$ does not exist,
  3. both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist, but they are not equal.

One-sided limits

Example

Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Limits to infinity

Definition

We say

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ tends to L whenever x tends to ∞, and

$$\lim_{x\to-\infty} f(x) = L$$

if $f(x)$ tends to L whenever x tends to $-\infty$.

- Formal definition of the limit to ∞:

$$\forall\, \varepsilon > 0\ \exists\, M \in \mathbb{R}\ \forall\, x \in \mathbb{R}\ \Big[\, x > M \ \Rightarrow\ |f(x) - L| < \varepsilon \,\Big]$$

- Formal definition of the limit to $-\infty$:

$$\forall\, \varepsilon > 0\ \exists\, M \in \mathbb{R}\ \forall\, x \in \mathbb{R}\ \Big[\, x < M \ \Rightarrow\ |f(x) - L| < \varepsilon \,\Big]$$

Infinite limits

Definition

We say

$$\lim_{x\to c} f(x) = \infty$$

if $f(x)$ tends to ∞ whenever x tends to a, and

$$\lim_{x\to c} f(x) = -\infty$$

if $f(x)$ tends to $-\infty$ whenever x tends to a.

- Formal definition of a limit that is ∞:

$$\forall\, K \in \mathbb{R}\, \exists\, \delta > 0\, \forall\, x \in \mathbb{R} \left[\, 0 < |x - c| < \delta \;\Rightarrow\; f(x) > K \,\right]$$

- Formal definition of a limit that is $-\infty$:

$$\forall\, K \in \mathbb{R}\, \exists\, \delta > 0\, \forall\, x \in \mathbb{R} \left[\, 0 < |x - c| < \delta \;\Rightarrow\; f(x) < K \,\right]$$

Infinity is not a number

Warning!

Even if $\lim_{x \to c} f(x) = \pm\infty$, then the limit $\lim_{x \to c} f(x)$ does not exist!

Both statements are true:

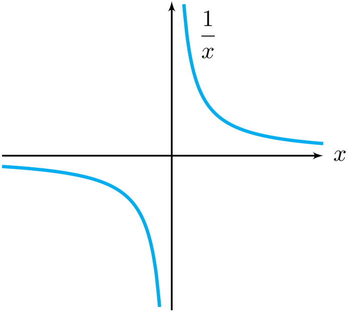
- $\lim_{x \to 0} \frac{1}{x^2} = \infty$.
- $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.

Standard limits with infinity

1. $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$ "$\frac{1}{\infty} = \frac{1}{-\infty} = 0$"

2. $\lim_{x\to 0} \frac{1}{x}$ does not exist.

3. $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\frac{1}{0}$ is undefined



Well-behaved functions

- Functions f like polynomials, exponential functions, logarithms, sine and cosine are **well-behaved** in the sense that a limit to a point a in the domain of f can be calculated by direct substitution:

$$\lim_{x \to c} f(x) = f(c).$$

Example

Calculate the limit $\lim_{x \to 2} x^3 - 7x + 10$.

Replacement rule

Theorem (weak version)

If $f(x) = g(x)$ *for all* $x \neq c$, *then* $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$.

Theorem (strong version)

Let I *be an open interval containing* c. *If* $f(x) = g(x)$ *for all* $x \in I$, *then* $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$.

- These theorems can be extended to all types of limits:
  - one-sided limits;
  - limits to plus or minus infinity;
  - limits that are plus or minus infinity.

Limit laws

Assume that both $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ exist.[1]

1. *Sum rule* $\lim_{x\to c} \big(f(x) + g(x)\big) = L + M$
2. *Difference rule* $\lim_{x\to c} \big(f(x) - g(x)\big) = L - M$
3. *Constant multiple rule* $\lim_{x\to c} \big(k \cdot f(x)\big) = k \cdot L$
4. *Product rule* $\lim_{x\to c} \big(f(x) \cdot g(x)\big) = L \cdot M$
5. *Quotient rule* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power rule* $\lim_{x\to c} \Big(f(x)\Big)^n = L^n$, with $n \in \mathbb{N}$
7. *Root rule* [2] $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, with $n \in \mathbb{N}$

[1] In particular: L and M are not ∞ or $-\infty$.

[2] If n is even, assume that $L > 0$.

Warning

The condition "*both limits must exist*" is essential:

- Define
$$f(x) = x - 1 \quad \text{and} \quad g(x) = \frac{2}{x-1}.$$

- This is wrong: $\lim_{x\to 1} f(x) = \lim_{x\to 1} x - 1 = 0$ so
$$\lim_{x\to 1} f(x)\cdot g(x) = \lim_{x\to 1} f(x) \cdot \lim_{x\to 1} g(x) = 0 \cdot \lim_{x\to 1} g(x) = 0.$$

- In stead, use the replacement rule:
$$\lim_{x\to 1} f(x)\cdot g(x) = \lim_{x\to 1} \cancel{x-1} \cdot \frac{2}{\cancel{x-1}} = \lim_{x\to 1} 2 = 2.$$

∞ **is not a number!**

- A limit is called called an **indeterminate form** if applying the limit laws leads to an indecisive result.
- Example: in $\lim_{x\to 1^+} \frac{x-1}{\sqrt{x}-1}$ we can not apply the quotient rule, because $\lim_{x\to 1} \sqrt{x}-1 = 0$. But also $\lim_{x\to 1} x-1 = 0$, so we cannot decide.
- We denote this situation with the pseudo-formula "$\frac{0}{0}$".

limit law	limit	notation
difference rule	$\lim_{x\to c} f(x) - g(x)$	$\infty - \infty$
product rule	$\lim_{x\to c} f(x) \cdot g(x)$	$0 \cdot \infty$
quotient rule	$\lim_{x\to c} \frac{f(x)}{g(x)}$	$\frac{\infty}{\infty}$ or $\frac{0}{0}$
power rule	$\lim_{x\to c} f(x)^{g(x)}$	1^∞, 0^0 or ∞^0

Not really indeterminate forms

- Some limits look like indeterminate forms, but they are not.

limit law	limit	notation	value
sum rule	$\lim_{x \to c} f(x) + g(x)$	$\infty + \infty$	∞
product rule	$\lim_{x \to c} f(x) \cdot g(x)$	$\infty \cdot \infty$	∞
quotient rule	$\lim_{x \to c} \frac{f(x)}{g(x)}$	$\frac{c}{\infty}$	0 for any $c \in \mathbb{R}$
power rule	$\lim_{x \to c} f(x)^{g(x)}$	∞^α	∞ if $\alpha > 0$
		∞^α	0 if $\alpha < 0$

Expelling indeterminate-ness

- If $x \to 0$, factor out the lowest power of x:

$$\lim_{x\to 0} \frac{2x^2 - 6x}{3x^2 + 2x} =$$

- If $x \to \infty$, factor out the highest power of x:

$$\lim_{x\to \infty} \frac{2x^2 - 6x}{3x^2 + 2x} =$$

The Conjugate Trick

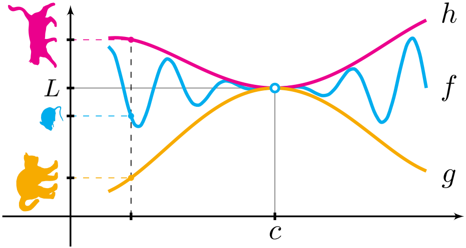
- The **Conjugate Trick** is based on the following identity:

$$(a+b)(a-b) = a^2 - b^2$$

- The factor $a+b$ is called the **conjugate** of $a-b$ (and vice versa).
- Example: evaluate the following limit:

$$\lim_{x \to 7} \frac{\sqrt{x+2}-3}{x-7} =$$

The sandwich theorem



- Let f, g and h be functions for which the following holds:

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } x \neq c$$

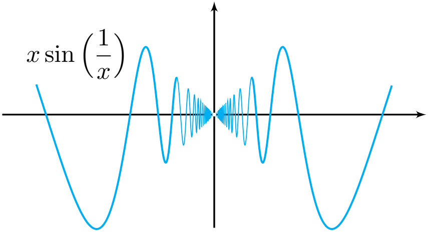
- The graph of f is "sandwiched" between the graphs of g and h.

Theorem

If $\lim_{x \to c} g(x)$ and $\lim_{x \to c} h(x)$ exist, and moreover, they are equal (say L), then $\lim_{x \to c} f(x)$ exists, and it is equal to L.

Example

Show that $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.



Continuity

Moab Fault splays (Moab Canyon, eastern Utah, USA). James St. John/Flickr/CC BY 2.0, from Wikimedia Commons

Continuity at a point

Definition

Let $f\colon [a,b] \to \mathbb{R}$ be a function. Let $c \in [a,b]$.

- *Interior points:*
  If $a < c < b$, then f is **continuous at** c if

$$\lim_{x \to c} f(x) = f(c).$$

- *Endpoints:*
  If $c = a$ or $c = b$, then f is **continuous at** a if

$$\lim_{x \to a^+} f(x) = f(a),$$

  and f is **continuous at** b if

$$\lim_{x \to b^-} f(x) = f(b),$$

Practical approach

Continuity test

A function $f(x)$ is continuous at an interior point c of its domain if and only if meets the following three conditions

1. $f(c)$ exists $\rightarrow$ c lies in the domain of f;
2. $\lim_{x \to c} f(x)$ exists $\rightarrow$ f has a limit as x approaches c;
3. $\lim_{x \to c} f(x) = f(c)$ $\rightarrow$ the limit equals the function value.

- If one (or more) of the conditions is not satisfied, the function f is not continuous at c.

Discontinuities

In all following cases f is not continuous at c:

Function	c	*Violation*
$f(x) = \dfrac{x^2 - 1}{x - 1}$	1	f is not defined at c
$f(x) = \begin{cases} \dfrac{\lvert x \rvert}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$	0	$\lim_{x \to c} f(x)$ does not exist
$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{if } x \neq 1, \\ 5 & \text{if } x = 1 \end{cases}$	1	$\lim_{x \to c} f(x) = 2 \neq f(c)$

Continuity is a local property

Example

Define $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

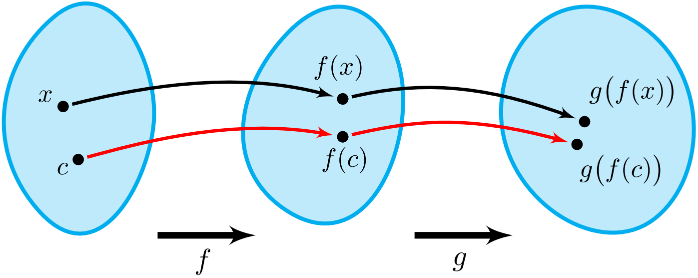
The function f is continuous at 0, but nowhere else.

Laws of continuity

Assume that f and g are continuous at c, then the following combinations are continuous at c:

1. *Sums* $f + g$
2. *Differences* $f - g$
3. *Constant multiples* $k \cdot f$, with $k \in \mathbb{R}$
4. *Products* $f \cdot g$
5. *Quotients* $\dfrac{f}{g}$, $\quad g(c) \neq 0$
6. *Powers* f^n, with $n \in \mathbb{N}$
7. *Roots* $\sqrt[n]{f}$, with $n \in \mathbb{N}$

Composition of continuous functions



Theorem

If f is continuous at c, and g is continuous at $f(c)$, then $g \circ f$ is continous at c.

- The **composition** $g \circ f$ is the function that maps x to $g\big(f(x)\big)$.

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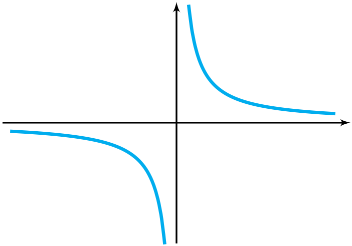
Global continuity

Definition

- Let I be an interval in $\mathbb{R}$. A function f is **continuous on** I if for all $c \in I$ the function f is continuous in c.
- A function f is **continuous** if f is continuous on its domain.

- Definition may differ from book to book.
- According to above definition $f(x) = \frac{1}{x}$ is continuous!



Formula functions

Definition (sort of)

A **formula function** is a function constructed from elementary functions, like polynomials, power functions, trigonometric and exponential functions, and logarithms, using algebraic operations like addition, subtraction, multiplication, division and composition.

All formula functions are continuous

For all formula functions f and for all $c \in \mathrm{Dom}(f)$:

$$\lim_{x \to c} f(x) = f(c)$$

Self study and preview

Self study

1. The *Intermediate Value Theorem*, study the last part of Thomas section 2.5.

2. Asymptotes, study Thomas section 2.6.

Next lecture

1. L'Hôpitals rule.