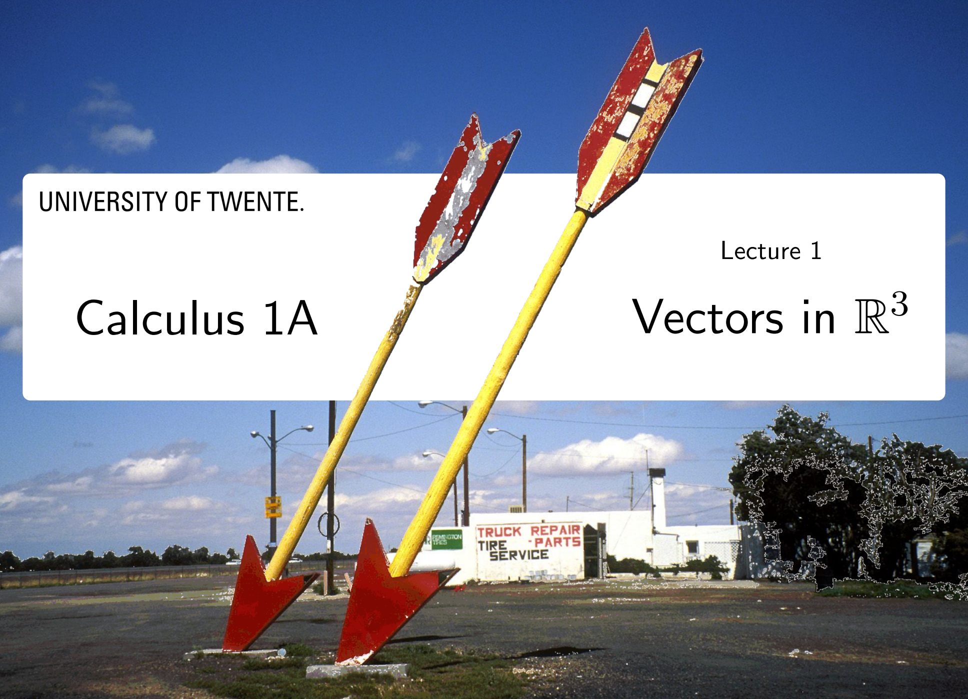
UNIVERSITY OF TWENTE.

Calculus 1A

Lecture 1

Vectors in $\mathbb{R}^3$



Twin Arrows, Arizona (http://wallpaperweb.org/wallpaper/miscellaneous/twin-arrows-gas-station-twin-arrows-arizona_21200.htm)

The organization of Calculus 1A

- Every week:
  - A plenary session, highlighting important topics.
  - A self-study in which you study the material and make exercises.
  - A tutorial with exercises and (sometimes) explanation.
- In week 8, Friday: 2-hour written test.
- In week 10, Friday: 3-hour written test, combined with *Intro To Math*.
- For dates, times and locations, consult MyTimeTable.
- All other information is published on Canvas.

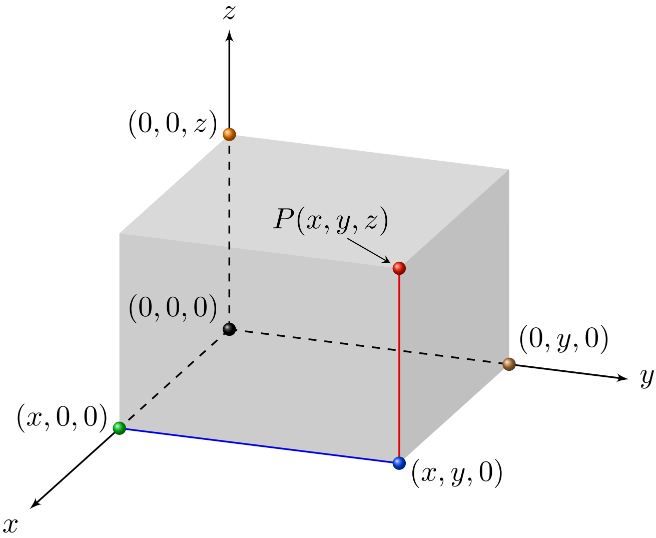
Overview

This lecture is based on the material of chapter 12 of Thomas' Calculus.

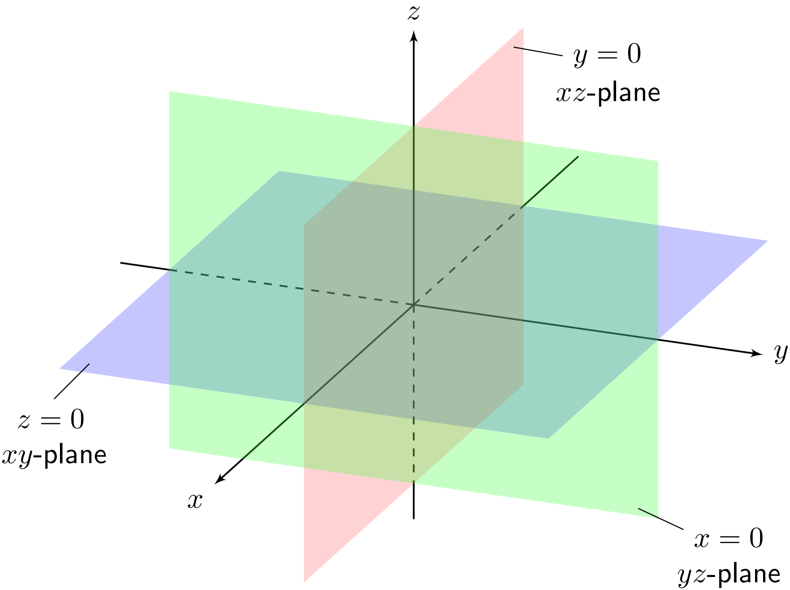
Topics:

- Vectors in three-dimensional space.
- The dot product and orthogonality.
- The cross product.
- Lines and planes in three-dimensional space.
- Applications.

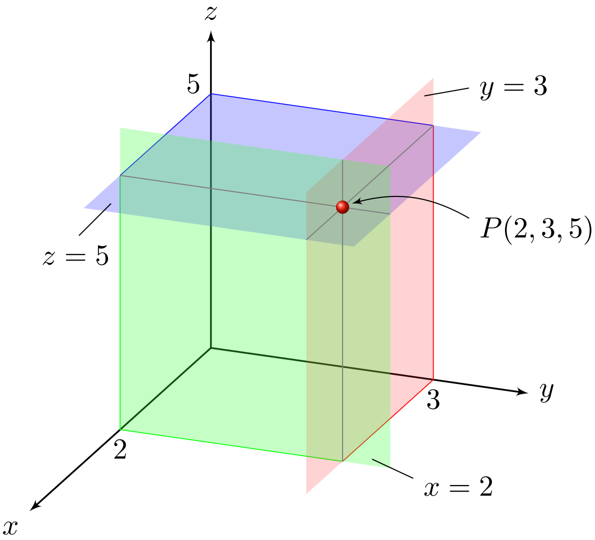
Coordinates in $\mathbb{R}^3$



Coordinate planes



Coordinates and planes

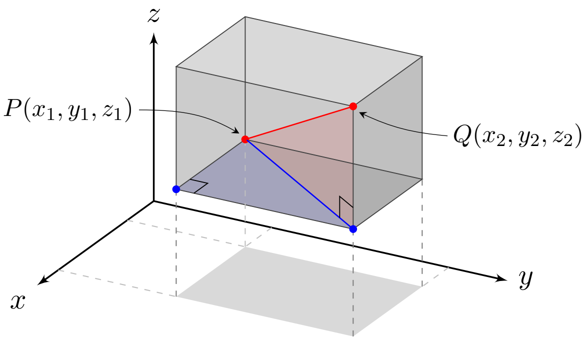


Distance

Definition

The **distance** between the points $P(x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is defined as

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Spheres

Definition

The **sphere with center** P **and radius** $r \geq 0$ is defined as the set of all points $X \in \mathbb{R}^3$ with distance r to P.

- If $P = (a, b, c)$ then (x, y, z) is on the sphere whenever

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

- This is equivalent to

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

- The formal definition of the sphere S is

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2\}.$$

Sphere

Example

Find its center and radius of the sphere described by

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Note that

$$\begin{aligned}(x-a)^2 + (y-b)^2 + (z-c)^2 \\ &= x^2 + y^2 + z^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2.\end{aligned}$$

Vectors

Vectors

Definition

The vector $\overrightarrow{PQ}$ is the directed line segment with initial point P and terminal point Q.

Conventions and notations

- Two vectors with the same length and the same direction are considered to be the same.
- The name of a vector is bold in printed text: $\mathbf{v}$, or underlined in hand written texts: $\underline{v}$.

Standard position and component form

Definition

- A vector is in **standard position** if the origin is the initial point of the vector.
- If the vector is equal to $\overrightarrow{OP}$, where O is the origin, and P has coordinates (x, y, z), then we write $\mathbf{v}$ in **component form**:

$$\mathbf{v} = \langle x, y, z \rangle.$$

Theorem

If vector $\mathbf{v}$ *has initial point* $P(x_1, y_1, z_1)$ *and terminal point* $Q(x_2, y_2, z_2)$, *then*

$$\boxed{\mathbf{v} = \overrightarrow{PQ} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle}$$

The length of a vector

Definition

The **length** of the vector $\mathbf{v}$ with initial point P and terminal point Q is denoted as $|\mathbf{v}|$, and is defined as the distance between P and Q.

$$|\mathbf{v}| = |PQ|.$$

- If $\mathbf{v} = \langle x, y, z \rangle$ is in standard position, then

$$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}.$$

- If $\mathbf{v} = \overrightarrow{PQ}$ with initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ then

$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Special vectors

Notation

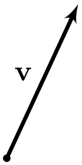
- The **zero vector** is the vector where all components are 0, and is denoted as $\mathbf{0}$.
- The **standard base vectors in** $\mathbb{R}^2$ are the vectors

$$\mathbf{i} = \langle 1, 0 \rangle, \quad \mathbf{j} = \langle 0, 1 \rangle.$$

- The **standard base vectors in** $\mathbb{R}^3$ are the vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

- **Scalar multiplication** of a vector $\mathbf{v}$ with a real number α results in a shorter or longer vector with the same or opposite direction as $\mathbf{v}$.



The sum of two vectors

- For the **head-to-tail construction** of the sum of two vectors $\mathbf{u}$ and $\mathbf{v}$: shift $\mathbf{v}$ such that the initial point of $\mathbf{v}$ is the same as the terminal point of $\mathbf{u}$.
- For the **parallelogram construction** of the sum of $\mathbf{u}$ and $\mathbf{v}$: shift $\mathbf{v}$ such that the initial points of $\mathbf{u}$ and $\mathbf{v}$ coincide.

Addition and scalar multiplication, algebraically

Theorem

If $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ *and* $\mathbf{v} = \langle u_1, \ldots, u_n \rangle$*, then*

- $\alpha\mathbf{v} = \langle \alpha v_1, \ldots, \alpha v_n \rangle$ *for all* $\alpha \in \mathbb{R}$*, and*
- $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, \ldots, u_n + v_n \rangle$*.*

Properties

For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and real numbers a and b we have

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

5. $0\mathbf{u} = \mathbf{0}$

6. $1\mathbf{u} = \mathbf{u}$

7. $a(b\mathbf{u}) = (ab)\mathbf{u}$

8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

9. $(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$

The difference of two vectors

For the **head-to-head construction** of the difference of two vectors $\mathbf{u}$ and $\mathbf{v}$: shift the vectors such that the initial points of coincide.

- $\mathbf{v} - \mathbf{u}$ is the vector from the head of $\mathbf{u}$ to the head of $\mathbf{v}$.
- $\mathbf{u} - \mathbf{v}$ is the vector from the head of $\mathbf{v}$ to the head of $\mathbf{u}$.

Unit vectors

Definition

A **unit vector** is a vector with length 1.

- If $\mathbf{v} \neq \mathbf{0}$, then $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector.

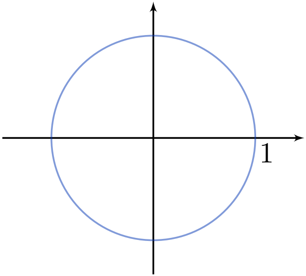
Example: normalize $\langle 2, -4, 4 \rangle$.

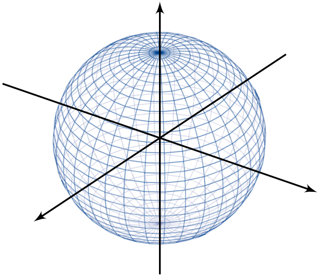
Direction

Definition

The **direction** of a non-zero vector $\mathbf{v}$ is the vector $\dfrac{\mathbf{v}}{|\mathbf{v}|}$.

- Directions are unit vectors
- If a direction is in standard position, its end points lies on the unit circle (in $\mathbb{R}^2$), or the unit sphere (in $\mathbb{R}^3$).





The dot product

By Daderot [Public domain or CC0], from Wikimedia Commons

https://commons.wikimedia.org/wiki/File:Protractor_used_for_drawing_maps,_German,_c._1580_-_Mathematisch-Physikalischer_Salon,_Dresden_-_DSC08011.JPG

The dot product

Definition

The **dot product** or **inner product** of the vectors $\mathbf{u} = \langle u_1, \ldots, u_n \rangle \in \mathbb{R}^n$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle \in \mathbb{R}^n$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

- The dot product is a *number*.
- The dot product can be used to describe the length of a vector:

$$\boxed{|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}}$$

Properties

Theorem

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} =\in \mathbb{R}^n$ *and* $c \in \mathbb{R}$ *the following properties holds*

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$.
5. $\mathbf{u} \cdot \mathbf{0} = 0$.

- Properties 2 and 3 can be combined for any number of vectors:

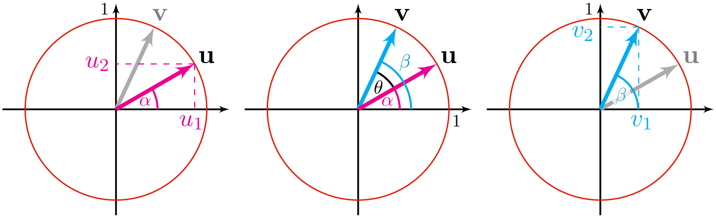
$$(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1\mathbf{u}_1 \cdot \mathbf{w} + \cdots + c_p\mathbf{u}_p \cdot \mathbf{w}.$$

The angle between unit vectors

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are unit vectors, and* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*, then*

$$\mathbf{u} \cdot \mathbf{v} = \cos\theta.$$

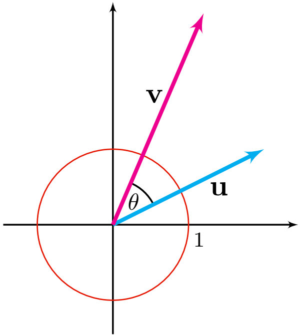


The angle between vectors

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are non-zero vectors, and* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*, then*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta.$$

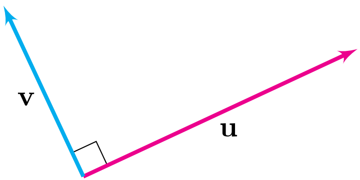


Orthogonality

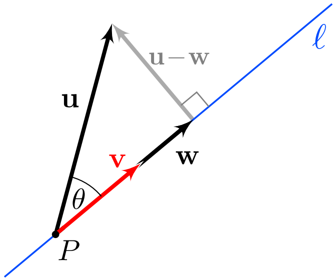
Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** or **perpendicular** if $\mathbf{u} \cdot \mathbf{v} = 0$.

- If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, then the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/2$.
- If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then there is no angle between $\mathbf{u}$ and $\mathbf{v}$, but we still say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
- We denote orthogonal vectors with the "$\perp$"-symbol: $\mathbf{u} \perp \mathbf{v}$.



Projection



- The blue line has direction $\mathbf{v}$, hence $\mathbf{w} = \alpha\mathbf{v}$ for some $\alpha \in \mathbb{R}$.
- We require that $\mathbf{u} - \mathbf{w} \perp \mathbf{v}$, in other words:

$$(\mathbf{u} - \alpha\mathbf{v}) \cdot \mathbf{v} = 0$$

Projection

- From the equation $(\mathbf{u} - \alpha\mathbf{v}) \cdot \mathbf{v} = 0$ we derive

$$\mathbf{u} \cdot \mathbf{v} - \alpha \mathbf{v} \cdot \mathbf{v} = 0$$
$$\mathbf{u} \cdot \mathbf{v} = \alpha |\mathbf{v}|^2$$
$$\alpha = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}$$

- Hence

$$\mathbf{w} = \alpha \mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$$

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors, $\mathbf{v} \neq \mathbf{0}$. The **projection of $\mathbf{u}$ onto $\mathbf{v}$** is

$$\boxed{\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}}$$

The cross product

Antique corkscrews, by Gixie (http://www.antiquariatoerestauro.it)

The cross product

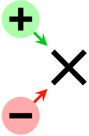
Definition

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be two vectors in $\mathbb{R}^3$. The **cross product van $\mathbf{u}$ and $\mathbf{v}$** is defined as

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \rangle.$$

The cross product can be computed using this template:

$$\begin{array}{llllll} \mathbf{u} = & \langle u_1 & u_2 & u_3 \rangle & \dots & \dots \\ \mathbf{v} = & \langle v_1 & v_2 & v_3 \rangle & \dots & \dots \end{array}$$



1. Copy the first two leftmost entries of $\mathbf{u}$ and $\mathbf{v}$.
2. Calculate the first entry of $\mathbf{u} \times \mathbf{v}$.
3. Calculate the second entry of $\mathbf{u} \times \mathbf{v}$.
4. Calculate the third entry of $\mathbf{u} \times \mathbf{v}$.

The cross product

Example

Calculate the cross product of $\langle 2, -1, 2 \rangle$ and $\langle -1, 2, 2 \rangle$.

Algebraic properties of the cross product

Theorem

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} =\in \mathbb{R}^n$ *and* $r, s \in \mathbb{R}$ *we have*

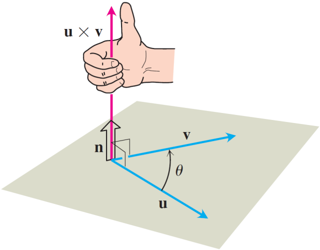
1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{u} \times \mathbf{0} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$!!!

Geometric properties of the cross product

Theorem

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. The cross product of $\mathbf{u} \times \mathbf{v}$ is has the following properties

1. $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ *and* $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$.
2. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$, *with* θ *the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.
3. *The direction is determined by the* **right-hand rule**.



The cross product

Example

For the vectors $\mathbf{u} = \langle 2, -1, 2 \rangle$ and $\mathbf{v}\langle -1, 2, 2 \rangle$, check the first two geometric properties of the cross product.

Lines in $\mathbb{R}^3$

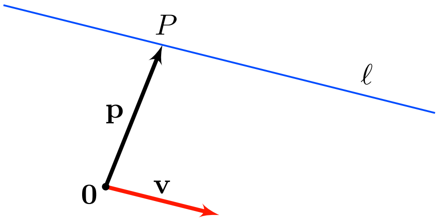
Definition

The **line through** P **parallel to** $\mathbf{v}$ is given by

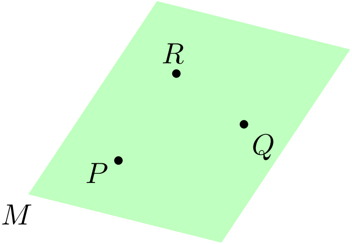
$$\{\mathbf{p} + t\mathbf{v} \in \mathbb{R}^3 \,|\, t \in \mathbb{R}\},$$

where $\mathbf{p} = \overrightarrow{OP}$.

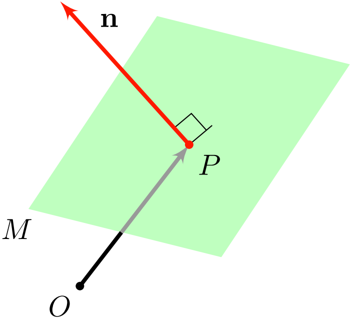
- The vector $\mathbf{p}$ is called a **support vector** of ℓ.
- The vector $\mathbf{v}$ is called a **direction vector** of ℓ.



Planes in $\mathbb{R}^3$



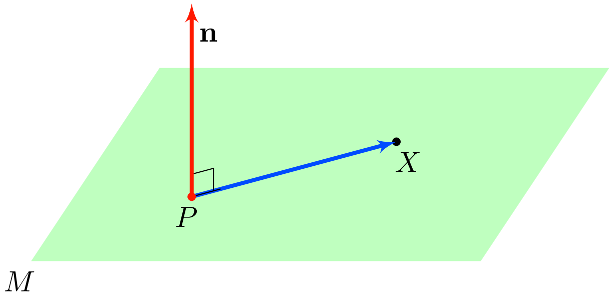
(a)



(b)

- A **normal vector of** M is a non-zero vector orthogonal to M.
- A plane is determined by:
  (a) three points, not on one line,
  (b) or by a support vector and a normal vector.

Planes determined by a normal vector



- If $\mathbf{n}$ is a normal vector of M, then for every $X \in M$:

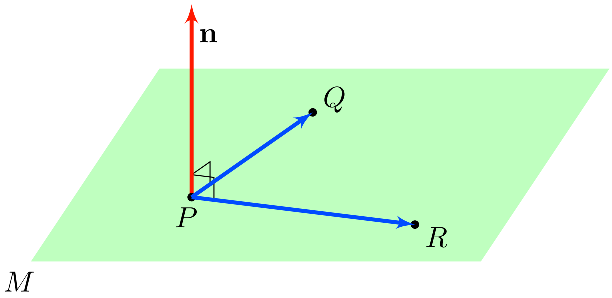
$$\mathbf{n} \perp \overrightarrow{PX}.$$

- If $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{x} = \overrightarrow{OX}$, then

$$\boxed{\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0}$$

This equation is called the **normal equation of** M.

Planes determined by three points



- If P, Q and R are three points (not on a line), then a normal vector of the plane can be calculated by taking a cross product, for example:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}.$$

Planes determined by three points

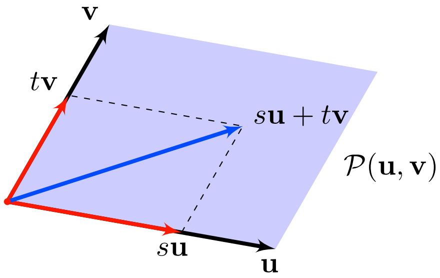
Example Sec. 12.5, Example 7

Find an equation for the plane through the points $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.

British Museum - Queen Elizabeth II Great Court - Roof Tessellations, London

https://commons.wikimedia.org/wiki/File:British_Museum_1120002_1120026.jpg

Parallelograms



Definition

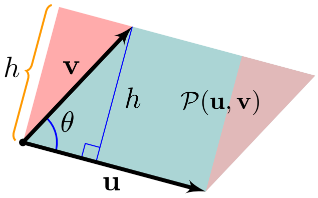
The **parallelogram spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$** is the set $\mathcal{P}(\mathbf{u}, \mathbf{v})$ defined by

$$\mathcal{P}(\mathbf{u}, \mathbf{v}) = \{s\mathbf{u} + t\mathbf{v} \,|\, 0 \le s \le 1, 0 \le t \le 1\}$$

Parallelograms

Theorem

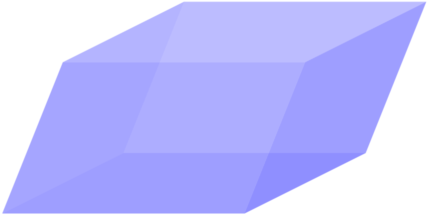
Let $\mathbf{u}$ and $\mathbf{v}$ vectors in $\mathbb{R}^3$. The area of $\mathcal{P}(\mathbf{u}, \mathbf{v})$ is equal to $|\mathbf{u} \times \mathbf{v}|$.



- Observe that $\sin\theta = \dfrac{h}{|\mathbf{v}|}$, so $h = |\mathbf{v}| \sin\theta$.
- The area of $\mathcal{P}(\mathbf{u}, \mathbf{v})$ is

$$|\mathbf{u}|\, h = |\mathbf{u}|\, |\mathbf{v}| \sin\theta = |\mathbf{u} \times \mathbf{v}|\,.$$

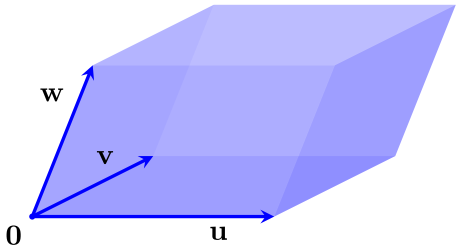
Parallelepipeds



Definition

A **parallelepiped** is a polyhedron who's faces are parallelograms.

Parallelepipeds spanned by three vectors



Definition

The **parallelepiped spanned by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$** is the set $\mathcal{P}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ defined by

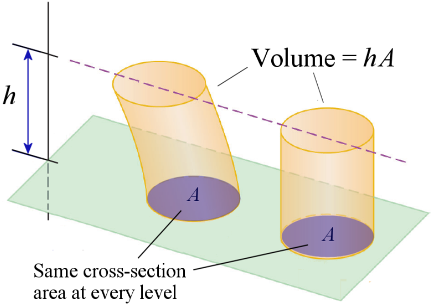
$$\mathcal{P}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \{s\mathbf{u} + t\mathbf{v} + r\mathbf{w} \,|\, 0 \leq s \leq 1, 0 \leq t \leq 1, 0 \leq r \leq 1\}$$

Cavalieri's principle

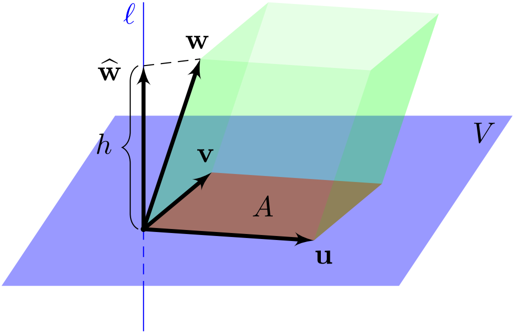
Theorem

Two solids with equal altitude and identical cross-section areas have the same volume.

- The volume of the solid is hA.

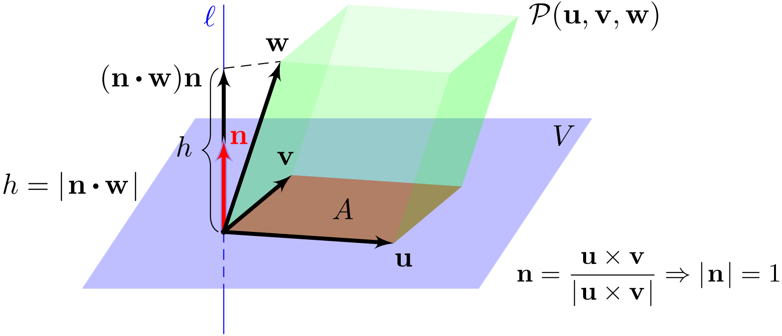


The volume of a parallelepiped



- The line ℓ is orthogonal to the plane V containing $\mathbf{u}$ and $\mathbf{v}$.
- The projection of $\mathbf{w}$ onto ℓ is $\widehat{\mathbf{w}}$, and $h = |\widehat{\mathbf{w}}|$.
- The volume of the parallelepiped is hA.
- The surface area of the bottom A is $|\mathbf{u} \times \mathbf{v}|$.

The Triple Scalar product



- The surface area of the bottom A is $|\mathbf{u} \times \mathbf{v}|$.
- The volume of the parallelepiped is

$$hA = \left| \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} \cdot \mathbf{w} \right| |\mathbf{u} \times \mathbf{v}| = \boxed{|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|}$$