Software Testing & Risk Assessment
Assignment 3

Problem 1: You are waiting for a package to arrive from the webshop Wish. People have figured out that, for this webshop, the delivery time X is exponentially distributed with parameter λ=1/15 days.

Questions

1. What is the expected amount you have to wait?
2. What is the probability that you wait
   A. less than 5 days?
   B. more than 10 days?
   C. exactly 10 days?
   D. between 10 and 25 days?
3. So far, you have been waiting for 15 days.
   A. What is the probability that you have to wait less than 5 more days?
   B. What is the probability that you have to wait more than 10 more days?
   C. exactly 10 more days?
   D. between 10 and 25 more days?

   Hint: use that the exponential distribution is memoryless, i.e., $\mathbf{P}[X > t+t' \mid X > t'] = \mathbf{P}[X > t]$.

Solution:

1. The expected time is $\mathbf{E}[X] = 1/\lambda = 15$ days.
2. Respectively:
   a. $\mathbf{P}[X < 5] = 1\text{-}e^{-5/15} = 28\%$
   b. $\mathbf{P}[X > 10] = e^{-11/15} = 48\%$
   c. The answer here depends on the definition of 'exactly'. If we mean 'the waiting time is 10 days, up to infinitesimal accuracy', then this is $\mathbf{P}[X = 10]$, which is 0. If we mean 'the package arrives on the 10th day', then the answer is

   $$\begin{aligned}\mathbf{P}[9 < X < 10] &= \mathbf{P}[X < 10] - \mathbf{P}[X < 9] \\ &= (1\text{-}e^{-10/15}) - (1\text{-}e^{-9/15}) \\ &= 3.5\%\end{aligned}$$

   d. Similar to the above,

   $$\begin{aligned}\mathbf{P}[10 < X < 25] &= \mathbf{P}[X < 25] - \mathbf{P}[X < 10] \\ &= (1\text{-}e^{-25/15}) - (1\text{-}e^{-10/15}) \\ &= 32\%\end{aligned}$$

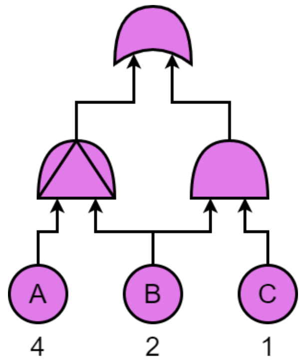
   We can also calculate this via the memoryless property:

   $$\begin{aligned}\mathbf{P}[10 < X < 25] &= \mathbf{P}[X > 10] * \mathbf{P}[X < 25 \mid X > 10] \\ &= \mathbf{P}[X > 10] * (1 - \mathbf{P}[X > 25 \mid X > 10]) \\ &= \mathbf{P}[X > 10] * (1 - \mathbf{P}[X > 15]) \\ &= e^{-10/15} * (1\text{-}e^{-15/15}) \\ &= 32\%\end{aligned}$$

Problem 2 (homework):

Consider the dynamic fault tree to the right.

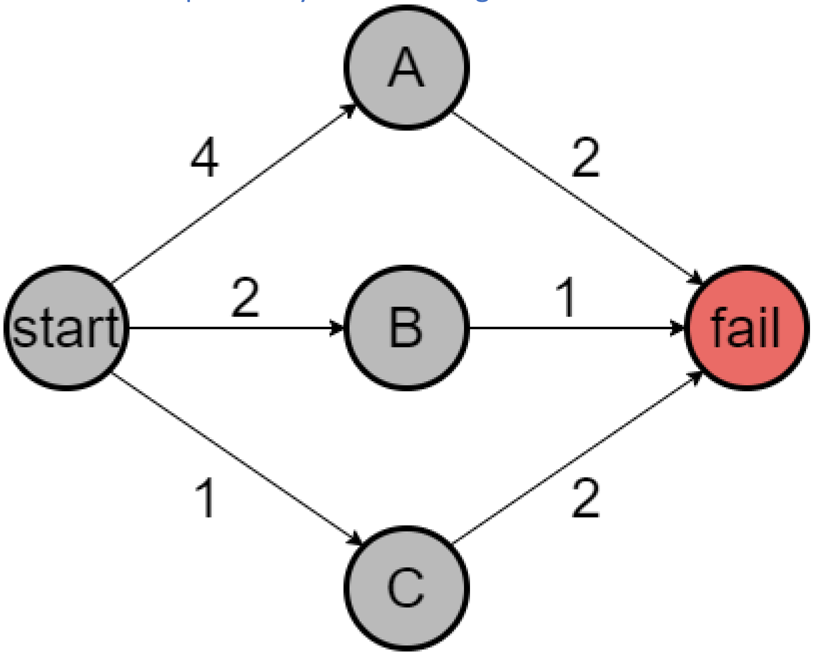
1. Represent this DFT as a Markov chain with 4 or 5 states.
2. Give the transition matrix of this Markov chain.
3. Calculate probability that this DFT fails before $t = 0.4$.



Solution:

1. Observe the following:
   a. If A has failed, C is irrelevant and B is needed for the system to fail;
   b. If B has failed, A is irrelevant and C is needed for the system to fail;
   c. If C has failed, A is irrelevant and B is needed for the system to fail.

   This can be captured by the following Markov chain.



2. If we order the states *start, A, B, C, fail*, we get the following transition matrix (other orders are fine as well):

$$A = \begin{pmatrix} -7 & 4 & 2 & 1 & 0 \\ 0 & -2 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3. In this case,

$$\begin{aligned} p(t) &= p(0) \cdot e^{tA} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.061 & 0.311 & 0.203 & 0.078 & 0.348 \\ 0 & 0.449 & 0 & 0 & 0.551 \\ 0 & 0 & 0.670 & 0 & 0.330 \\ 0 & 0 & 0 & 0.449 & 0.551 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.061 & 0.311 & 0.203 & 0.078 & 0.348 \end{pmatrix} \end{aligned}$$

Since failure corresponds to the last entry, we conclude that the failure probability at t=0.4 is equal to 0.348.

Alternative solution:
We can merge A and C to get the following transition matrix:

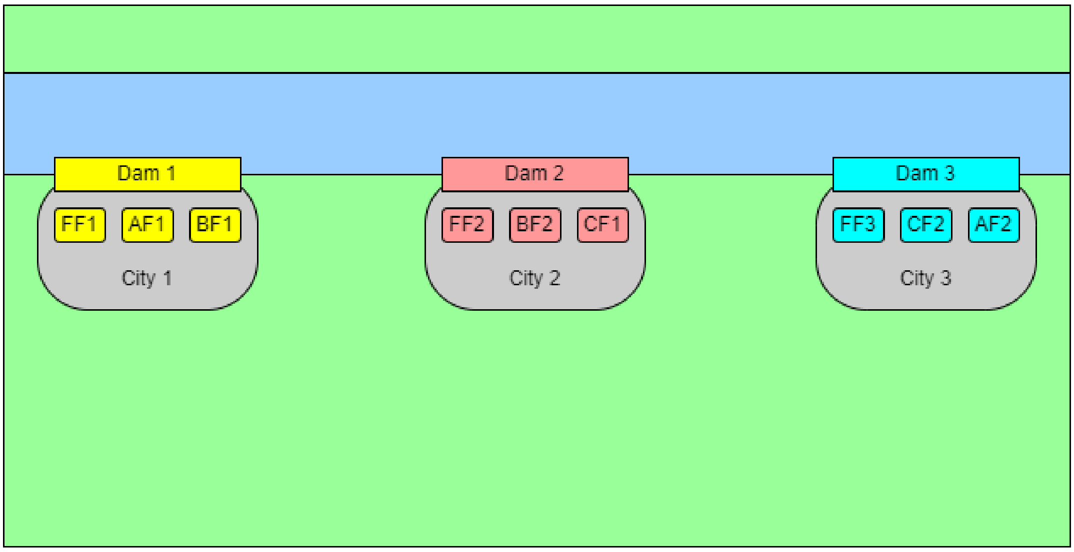
$$A = \begin{pmatrix} -7 & 5 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And

$$p(t) = p(0) \cdot e^{tA}$$
$$= (1 \quad 0 \quad 0 \quad 0) \begin{pmatrix} 0.061 & 0.389 & 0.203 & 0.348 \\ 0 & 0.449 & 0 & 0.551 \\ 0 & 0 & 0.670 & 0.330 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= (0.061 \quad 0.389 \quad 0.203 \quad 0.348)$$

Which also returns failure probability 0.348

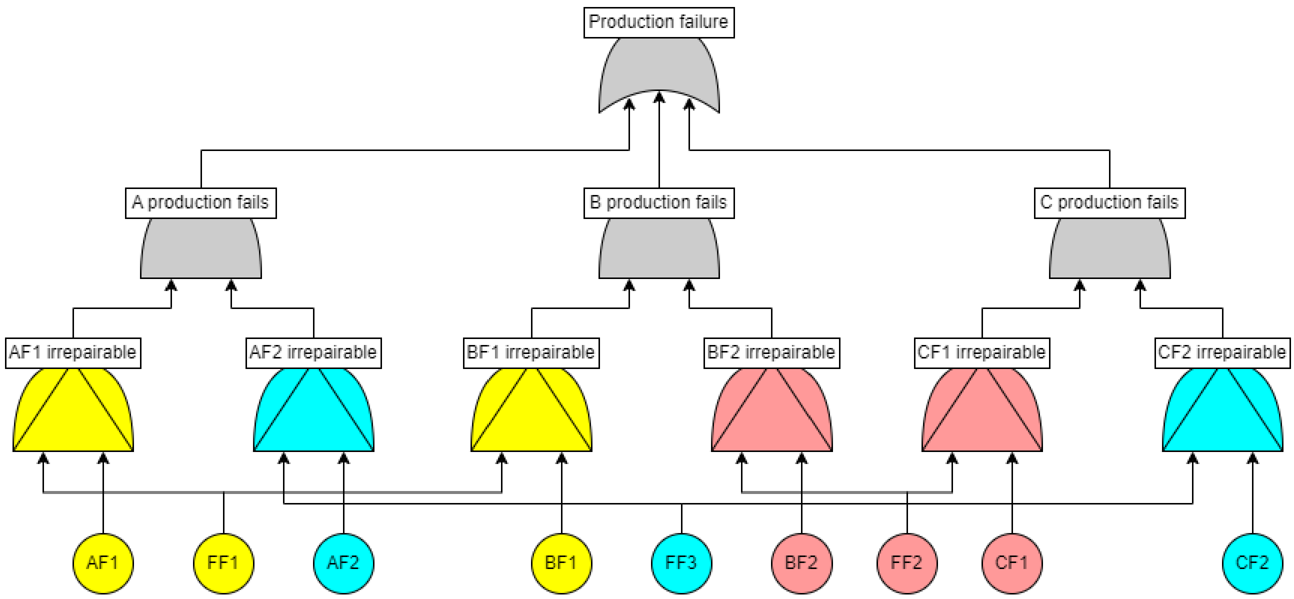
Problem 3 Consider the marvel of modern art below.



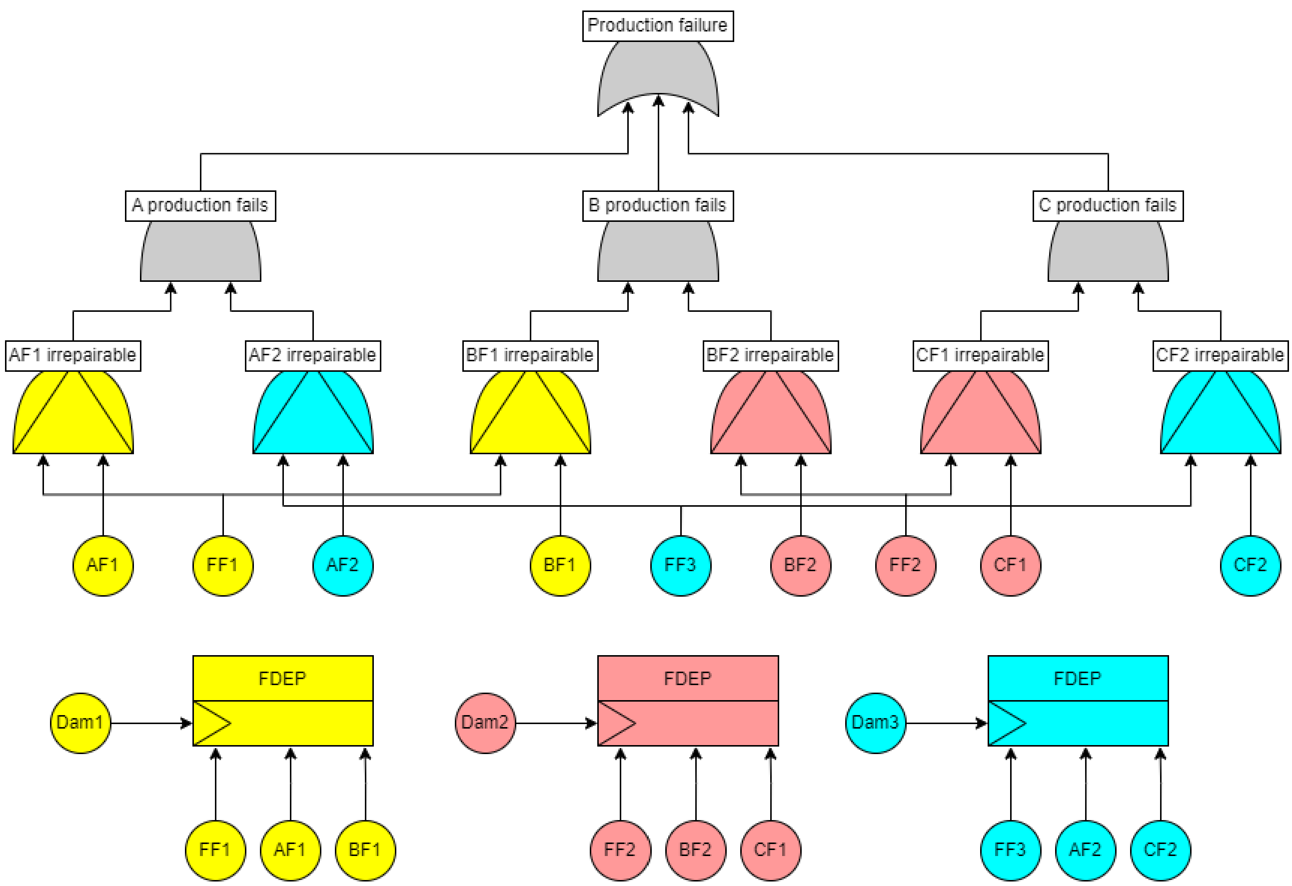
- Three types of components (A,B,C) are being produced in three cities: A in A Factory 1 (AF1) and AF2 in Cities 1 and 3, B in BF1 and BF2 in Cities 1 and 2, and C in CF1 and CF2 in Cities 2 and 3.
- At least one factory of each component must be operational.
- All factories are operational from the start.
- Each city contains a factory factory (FF1-3): as long as the factory factory a city is operational, each factory that fails in that town is immediately repaired.
- A factory repaired by a factory factory can never fail again.
- Factory factories cannot repair factories in other cities.
- Each city is protected from the river by a dam. If a dam fails, all factories in that city (including the factory factory) are immediately destroyed.

Create a DFT for this scenario.

Solution: First consider AF1. The only way for it to permanently fail is if first FF1 fails, and only then AF1 fails. This can be modeled by PAND(FF1,AF1). Furthermore, A-production fails only when both AF1 and AF2 fail permanently, and production as a whole fails when production of either A, B, or C fails. This yields the following DFT (colours are not mandatory but convenient):



However, we have not yet taken the dams into account. These can be modeled as FDEP gates. This yields the following DFT:



Problem 4:

In the DFT of problem 2, suppose that B has failed (and no other basic event). What is the probability that the DFT fails within t = 0.5 time units?

1. **Solution:** In this case $p(0) = (0 \quad 0 \quad 1 \quad 0 \quad 0)$, so

$$p(t) = p(0) \cdot e^{tA}$$
$$= (0 \quad 0 \quad 1 \quad 0 \quad 0) \begin{pmatrix} 0.030 & 0.270 & 0.192 & 0.068 & 0.440 \\ 0 & 0.368 & 0 & 0 & 0.632 \\ 0 & 0 & 0.607 & 0 & 0.393 \\ 0 & 0 & 0 & 0.368 & 0.632 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= (0 \quad 0 \quad 0.607 \quad 0 \quad 0.393)$$

So the failure probability is 0.393.

Problem 5:

1. In the lecture, you have been told that the exponential probability distribution is memoryless, i.e., that **P**[$X > t{+}t'$ | $X > t'$] = **P**[$X > t$]. Prove this. *Hint*: recall that the conditional probability of an event A given an event B is given by **P**[$A|B$] = **P**[$A \wedge B$]/**P**[B].
2. **For those of you with a background in calculus:* In the lecture, you have been told that the expected value of an exponentially distributed random variable with parameter λ is equal to 1/λ. Prove this. *Hint*: recall that the expected value of a continuous random variable with cumulative density function P is equal to $\int_0^\infty P'(x)x\,\mathrm{d}x$, where P' is the derivative of P.

Solution:

1. We have

P[$X > t{+}t'$ | $X > t'$] = **P**[$X > t{+}t' \wedge X > t'$]/**P**[$X > t'$]
= **P**[$X > t{+}t'$]/**P**[$X > t'$]
= $e^{-\lambda(t+t')}/e^{-\lambda t'}$
= $e^{-\lambda t}$
= **P**[X > t]

2. First note that $\mathrm{P}'(\mathrm{x}) = \lambda \mathrm{e}^{-\lambda \mathrm{x}}$, so the expected value is expressed by the integral $E[X] = \int_0^\infty \lambda x e^{-\lambda x} dx$. We solve this integral via integration by parts: Let $u(x) = x$ and $v(x) = -e^{-\lambda x}$, then $E[x] = \int_0^\infty u(x)v'(x)dx = [u(x)v(x)]_0^\infty - \int_0^\infty u'(x)v(x)dx$. To calculate this, we start with the left term, which is equal to $[u(x)v(x)]_0^\infty = \lim_{x\to\infty} -xe^{-\lambda x} = \lim_{x\to\infty} \frac{f(x)}{g(x)}$, where $f(x) = -x$ and $g(x) = e^{\lambda x}$. Since both f and g go to ∞, we can use l'Hôpital's rule and find that $[u(x)v(x)]_0^\infty = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{-1}{\lambda e^{\lambda x}} = 0$.

Now, we evaluate the integral $\int_0^\infty u'(x)v(x)dx = \int_0^\infty -e^{-\lambda x}\,dx = \left[\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty = -\frac{1}{\lambda}$. We conclude that **E**[X] = 1/λ.