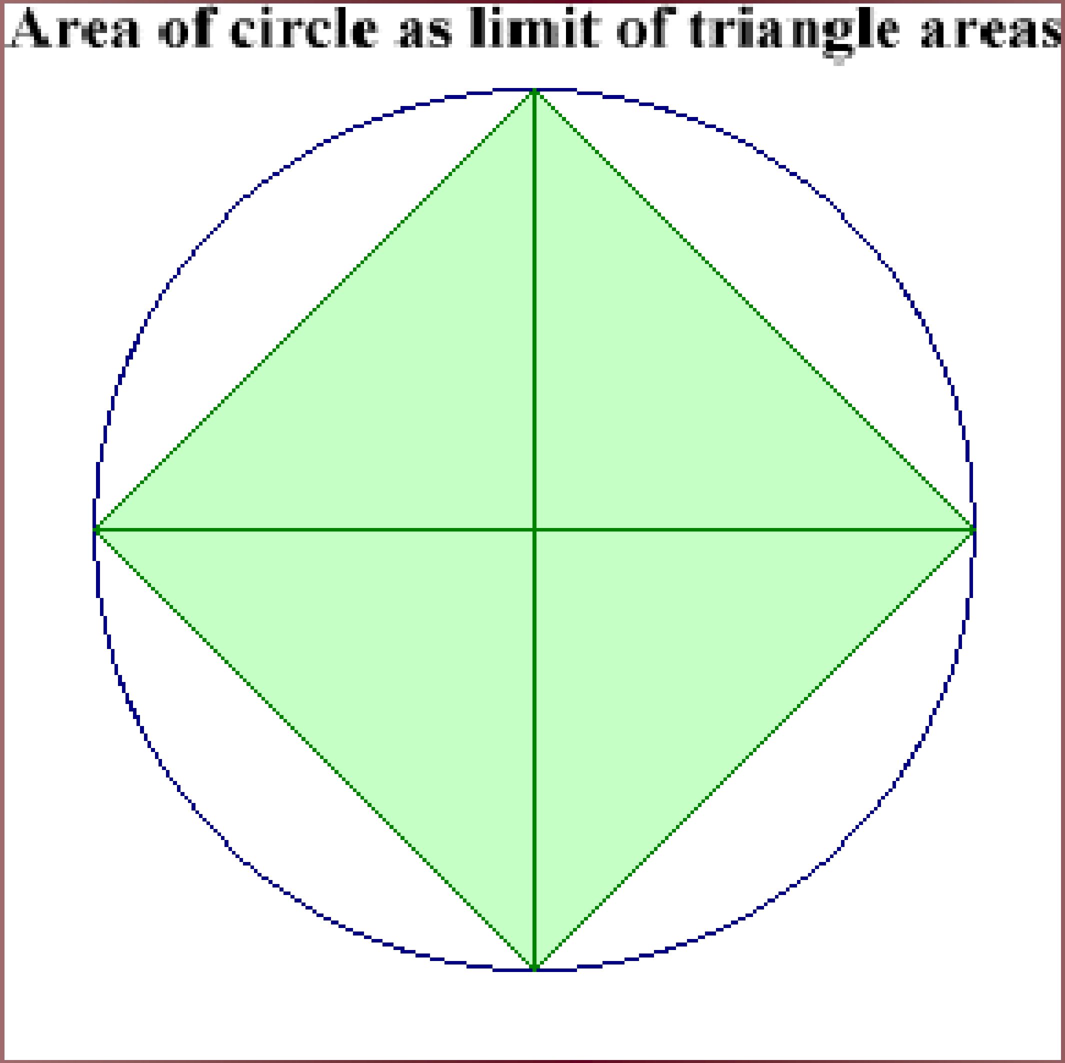
Mathematics B2: Newton



Mathematics B2: Newton

Lecturers: Bernard Geurts
Gerard Jeurnink

Mathematics B2: Newton

-Contents-

- [x] Limits and continuity
- [x] Derivatives and applications
- [x] Functions of 2 variables

- [x] Integrals
- [ ] Calculation techniques for integrals
- [ ] Power and Taylor series

Integrals

- Theme: Area
- Theme: Riemann Sum
- Theme: Fundamental Theorem
- Theme: Antiderivatives

Thomas' Calculus

5.1

Area and Estimating with Finite Sums

Area

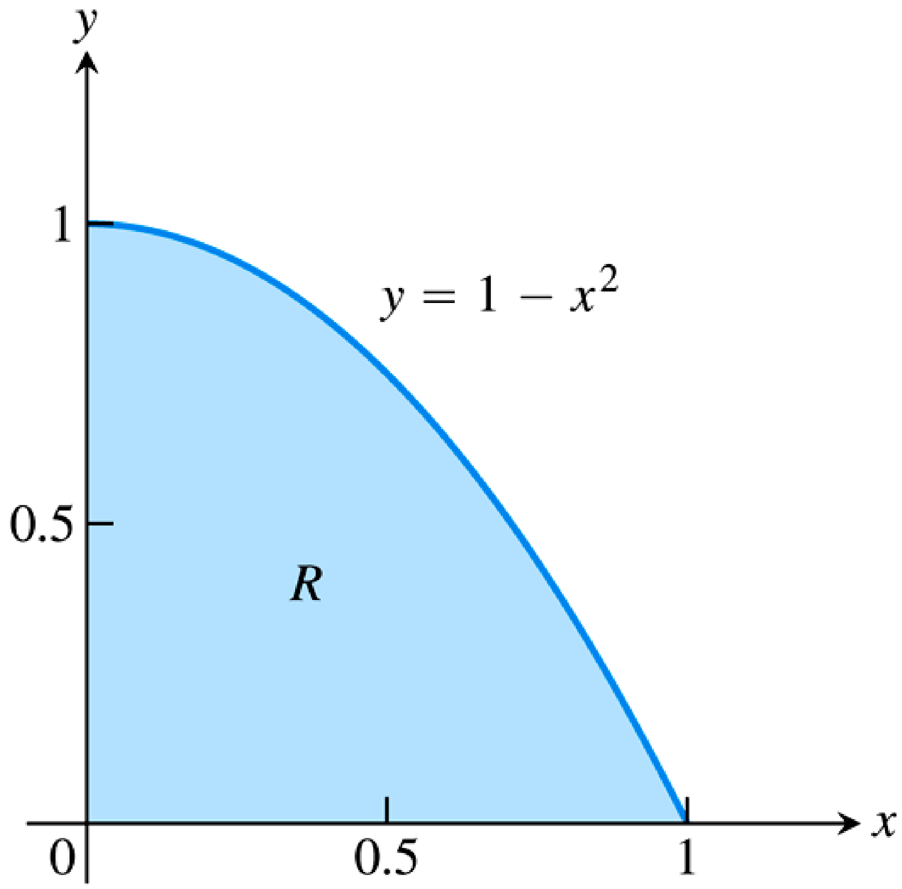


FIGURE 5.1 The area of the region *R* cannot be found by a simple formula.

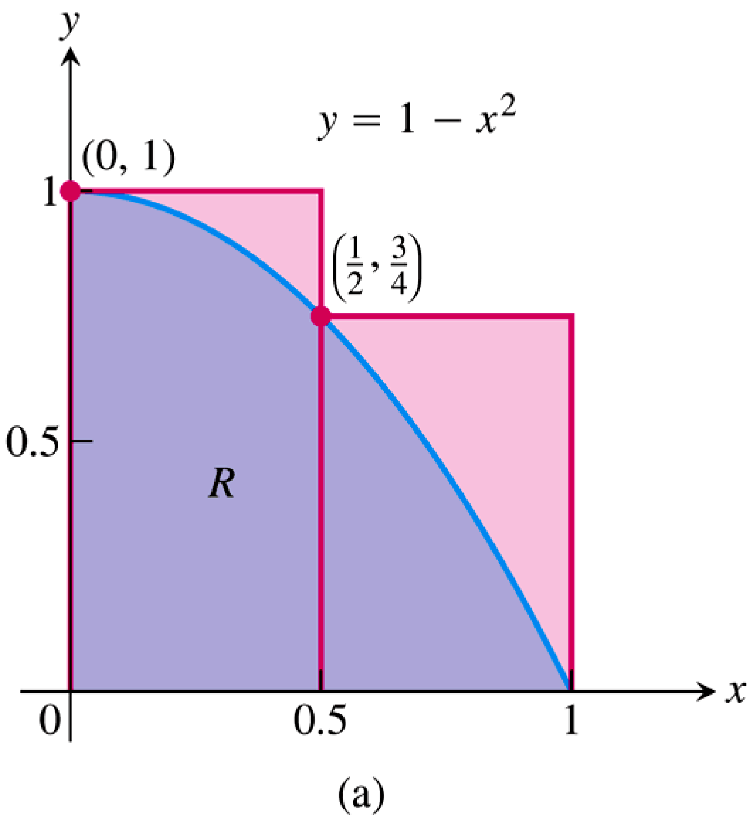


FIGURE 5.2 (a) We get an upper estimate of the area of *R* by using two rectangles containing *R*. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

Area

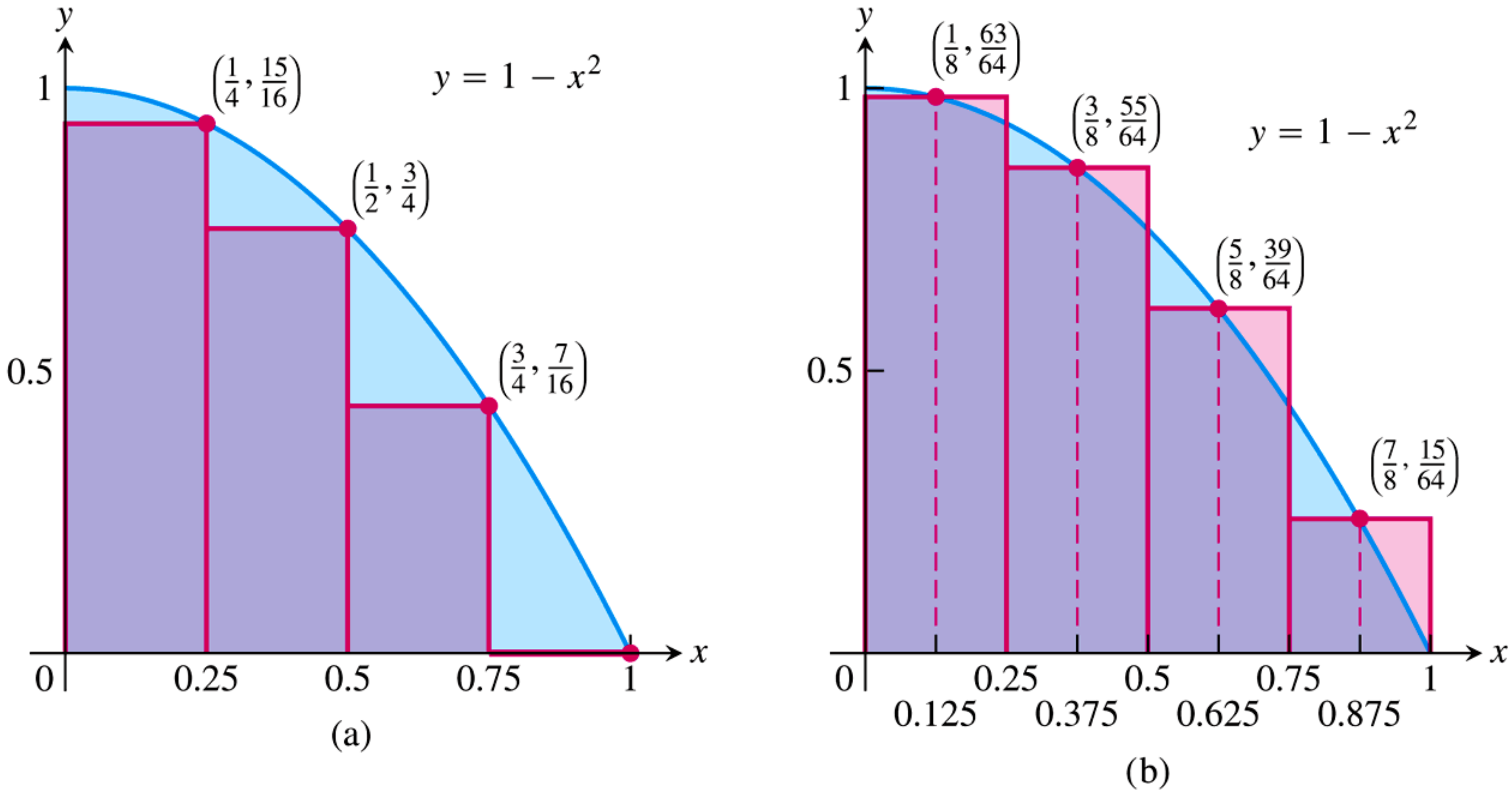
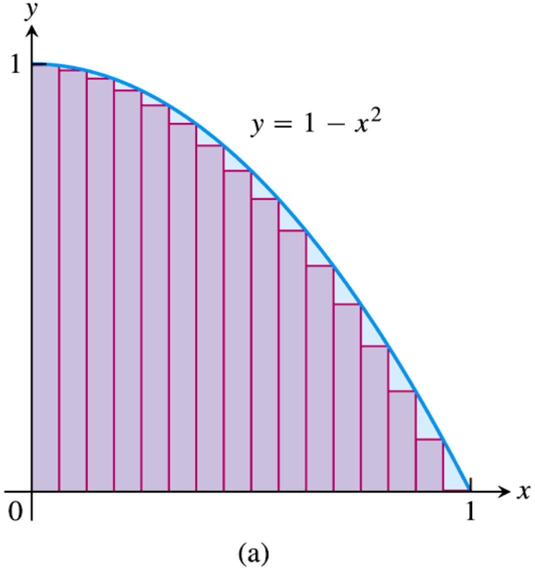


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.

Area



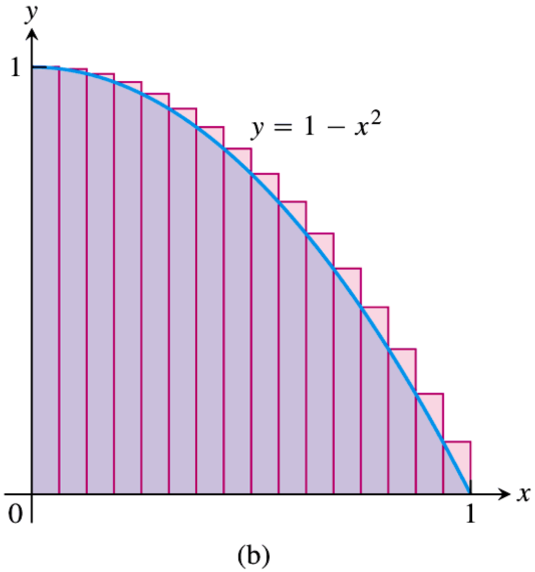


FIGURE 5.4 (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

Area

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

Upper and lower sums seem to converge to 0.6666… = 2/3 …

5.2

Sigma Notation and Riemann Sums

Σ-notation

$$\sum_{k=1}^{n} a_k$$

- The summation symbol (Greek letter sigma): $\sum$
- The index k ends at $k = n$.
- The index k starts at $k = 1$.
- a_k is a formula for the kth term.

Σ-notation

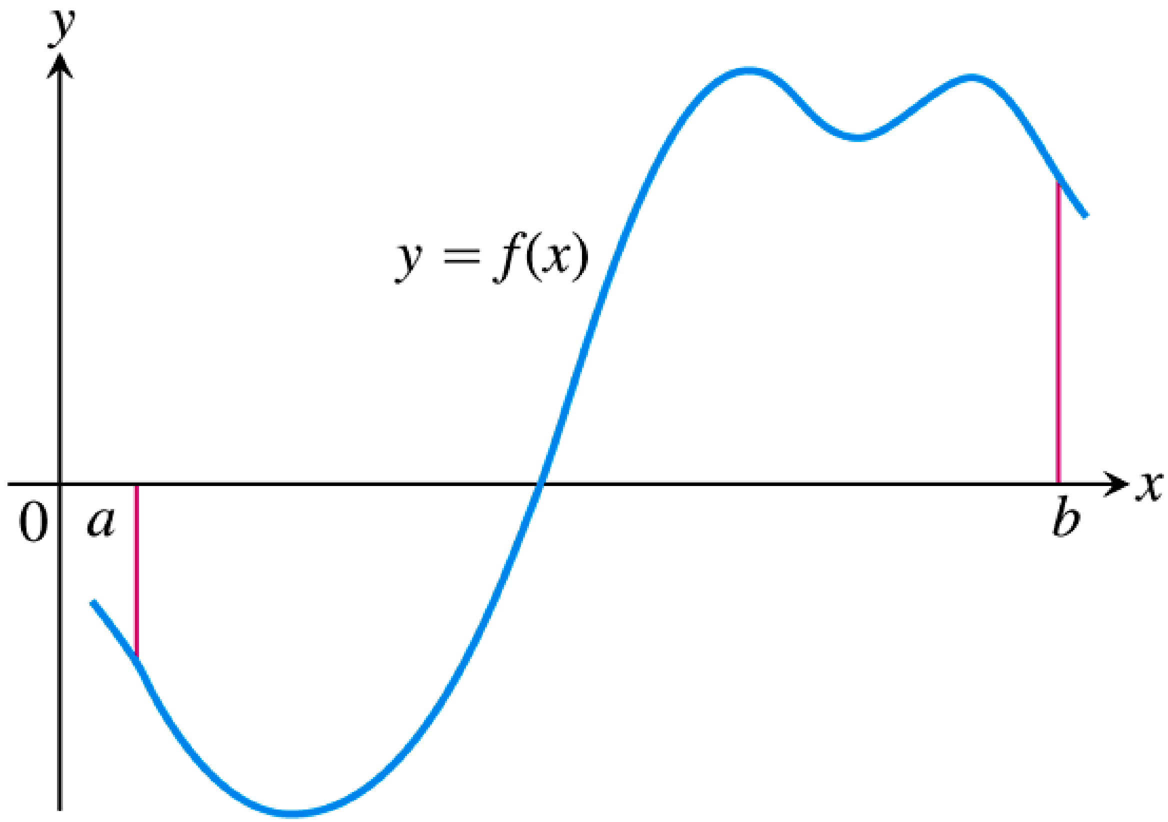
The sum in sigma notation	The sum written out, one term for each value of *k*	The value of the sum
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

$$\sum_{k=0}^{1000} 1 = 1001$$

Riemann sums

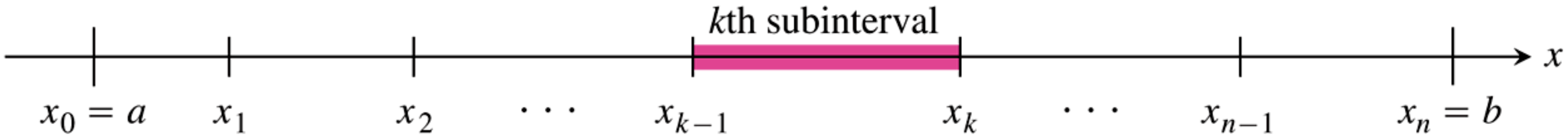
Georg Friedrich Bernhard Riemann (1826-1866)

Riemann sums

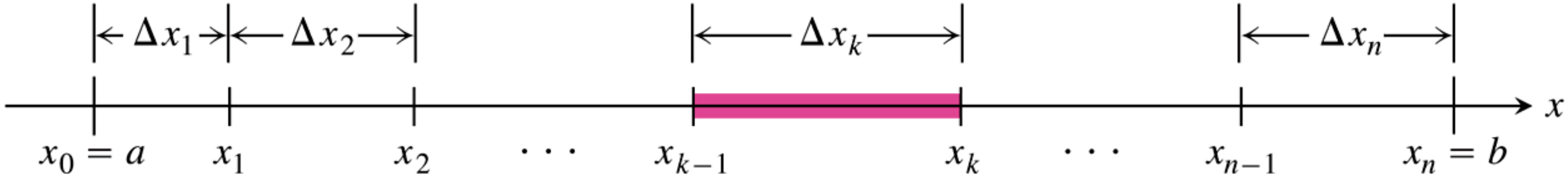


We want to approximate this "area", or integral.
First divide $[a,b]$ in n subintervals (*partition P*)

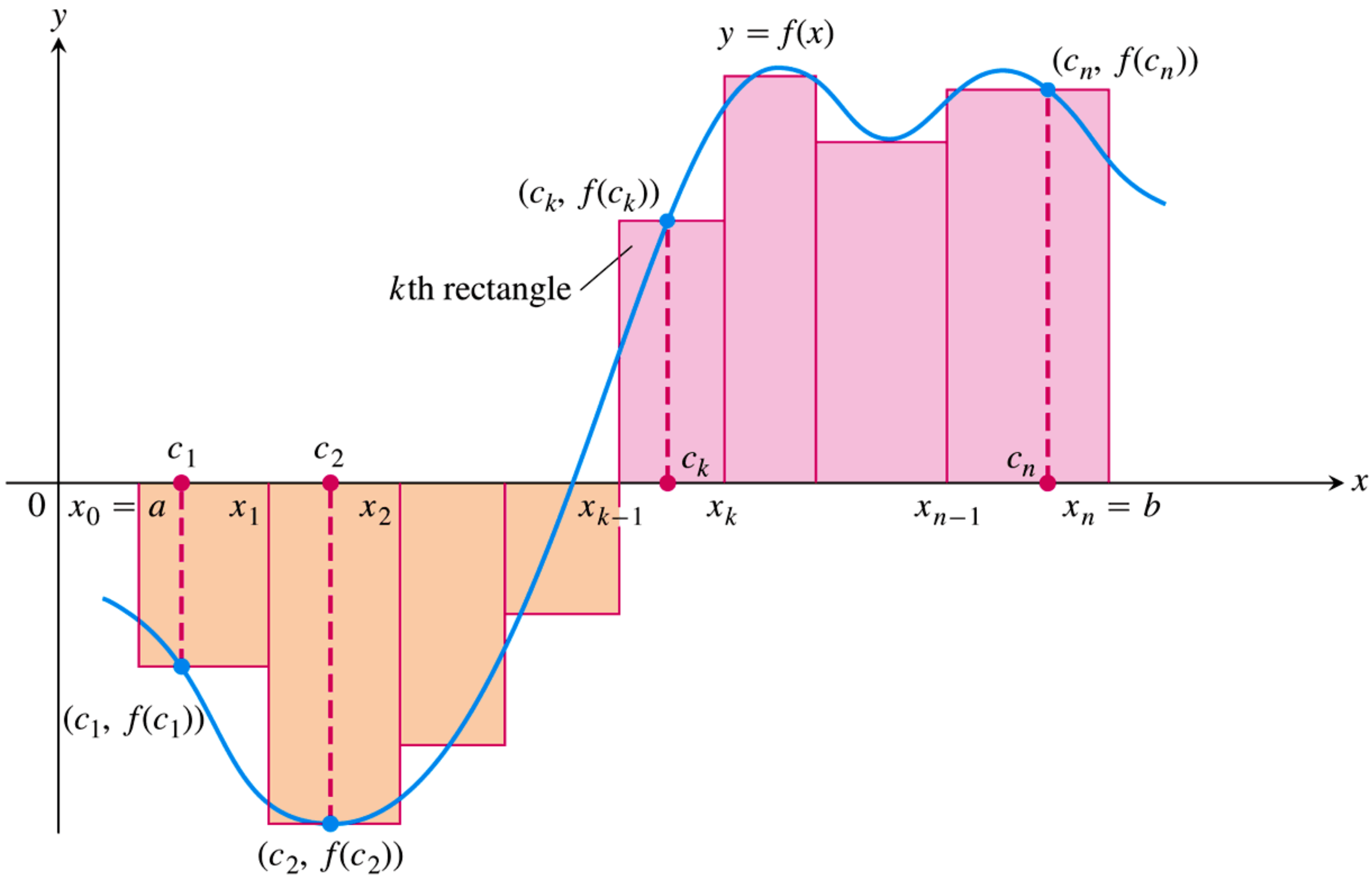
The k^{th} subinterval of P is $[x_{k-1}, x_k]$, for k an integer between 1 and n



The width of the k^{th} interval $\Delta x_k = x_k - x_{k-1}$, If all n subintervals have equal width, then the common width Δx_k is equal to $(b - a)/n$.

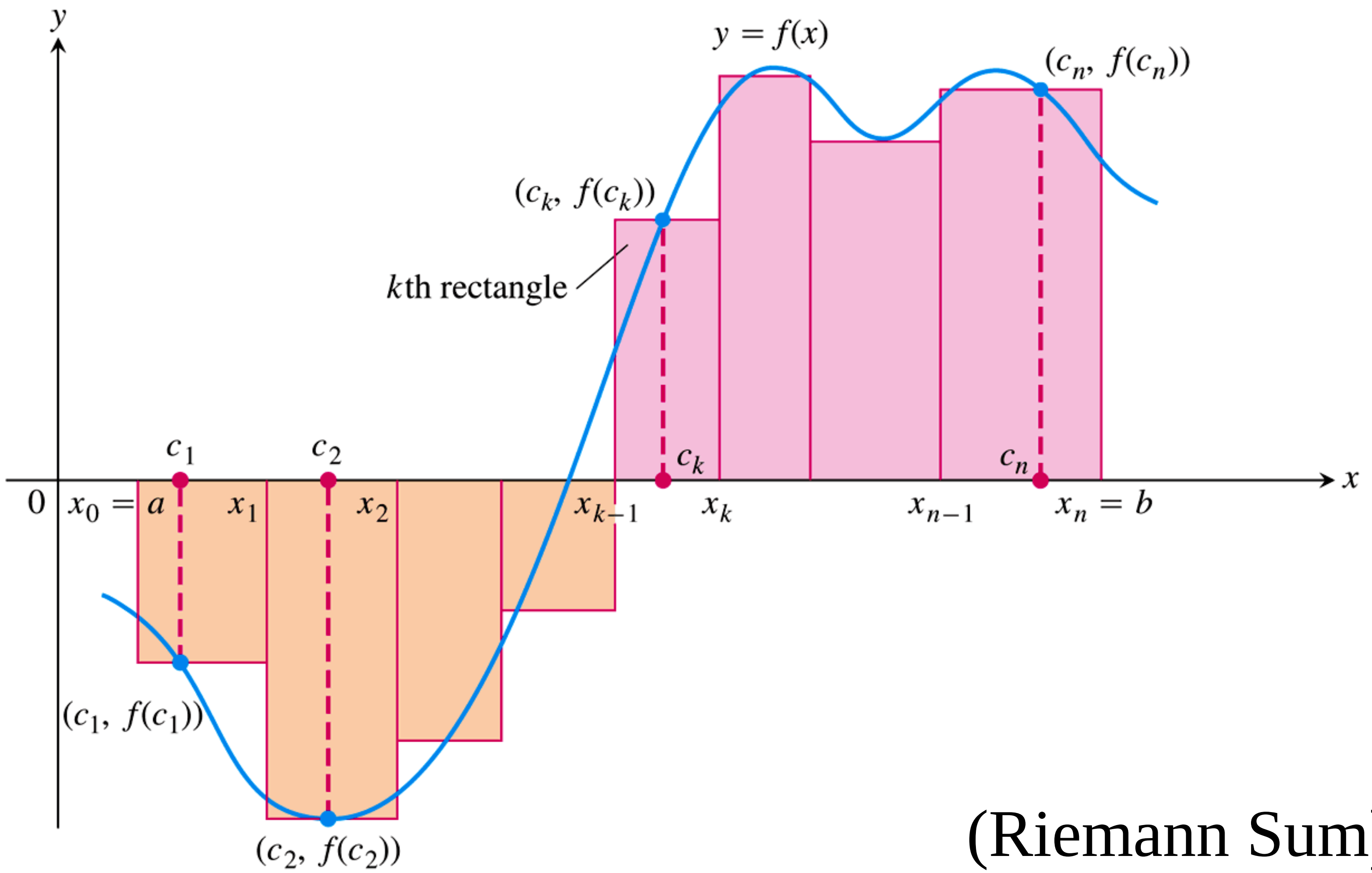


Riemann sums



In each subinterval $[x_{k-1}, x_k]$ choose an element c_k and erect the rectangle with signed area $f(c_k) \cdot (x_k - x_{k-1})$ …

Riemann sums



(Riemann Sum)

$$\text{Integral} \approx \sum_{k=1}^{n} f\left(c_k\right)\cdot\left(x_k - x_{k-1}\right) = \sum_{k=1}^{n} f\left(c_k\right)\cdot \Delta x_k$$

Riemann sums

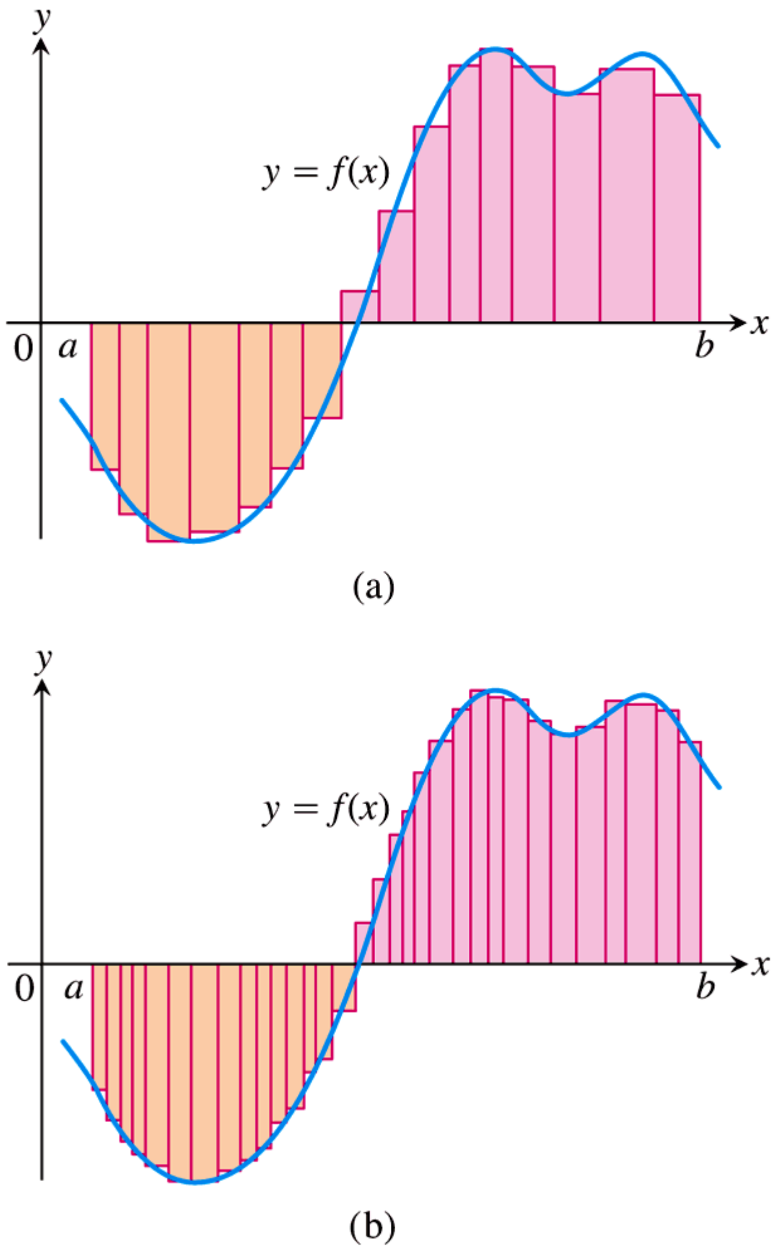


FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

Definite Integral

$$\int_a^b f(x)dx = \underset{n\to\infty}{\text{"lim"}} \left(\sum_{i=1}^{n} f(t_i) \cdot \Delta x_i \right)$$

The definite integral exists if all Riemann sums converges to the same number.

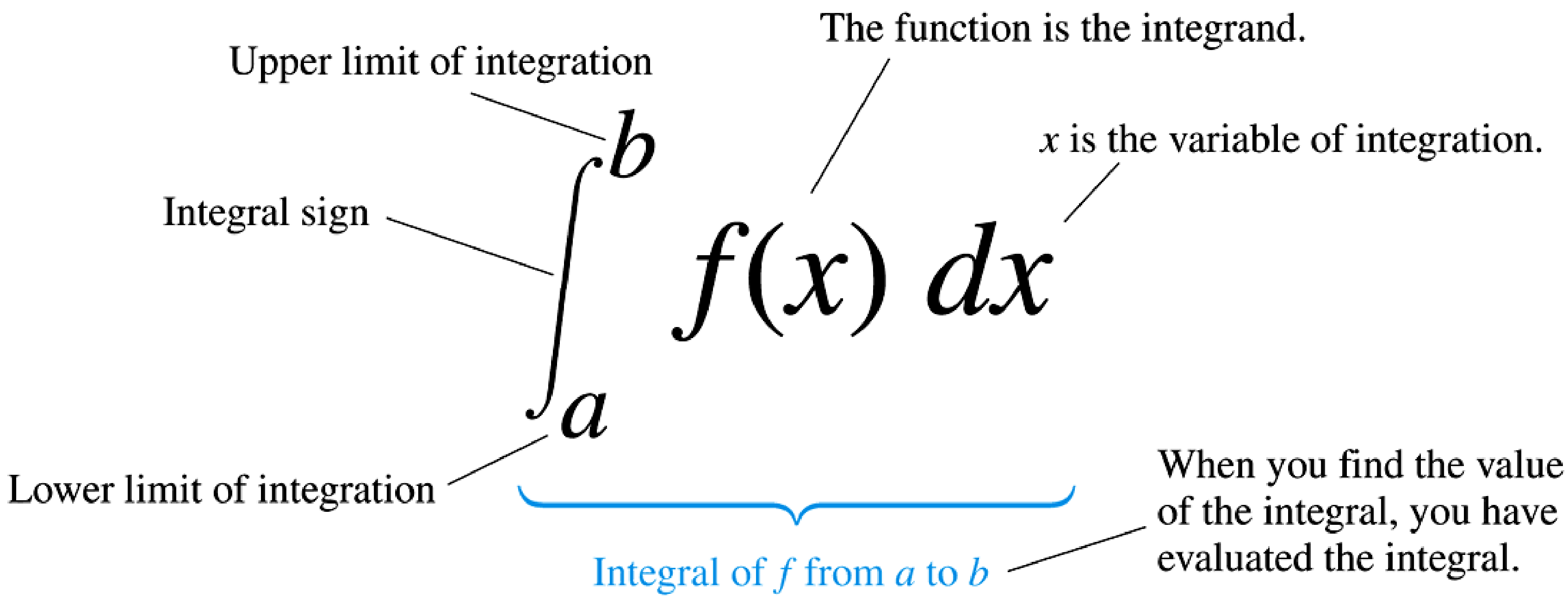
Area and mean

$$\text{an integral } \int = \text{limit of Riemann sums}$$

5.3

The Definite Integral

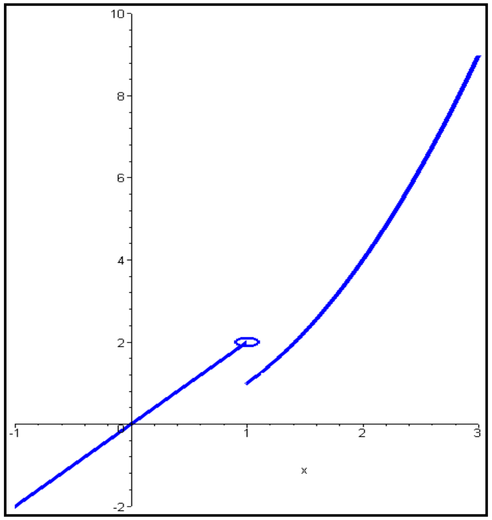
Definite Integral



Definite Integral

THEOREM 1—Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

A jump discontinuity in *a* means that *f* is *not* continuous, but the left-hand and right-hand limits of *f* in *a* exist.



A jump discontinuity

Rules for integrals

Rules for integrals

TABLE 5.4 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A Definition when $f(a)$ exists
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then	

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)

Area, mean, volume, …

Area and mean

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\, dx.$$

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Example:

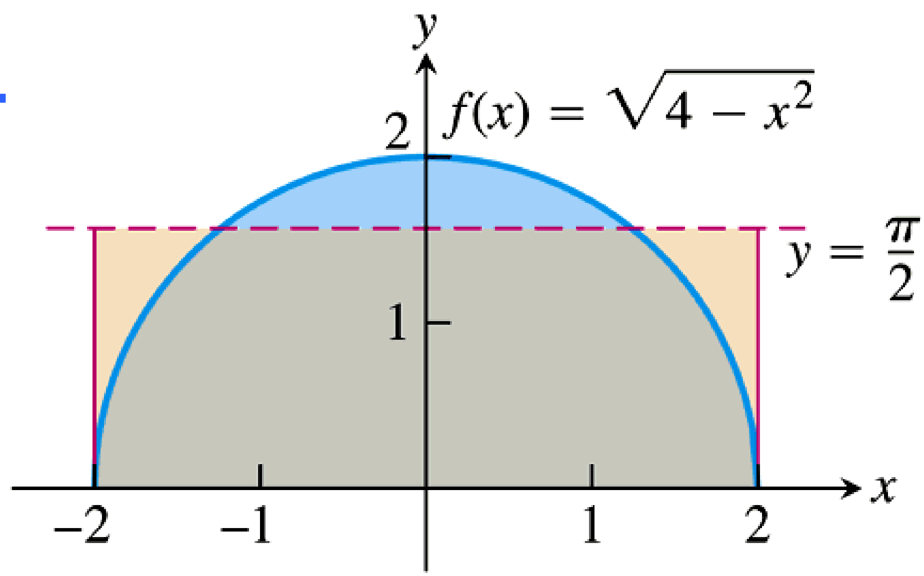


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$

Because: the area of a half circle with radius 2 is 2π and the width $(b - a)$ is 4, leading to mean $\frac{1}{2}\pi$.

Area and mean

Example:

The area under the line y = x over [0,b] is $\frac{1}{2}b^2$,
therefore we can write:

$$\int_0^b x\,dx = \frac{1}{2}b^2$$

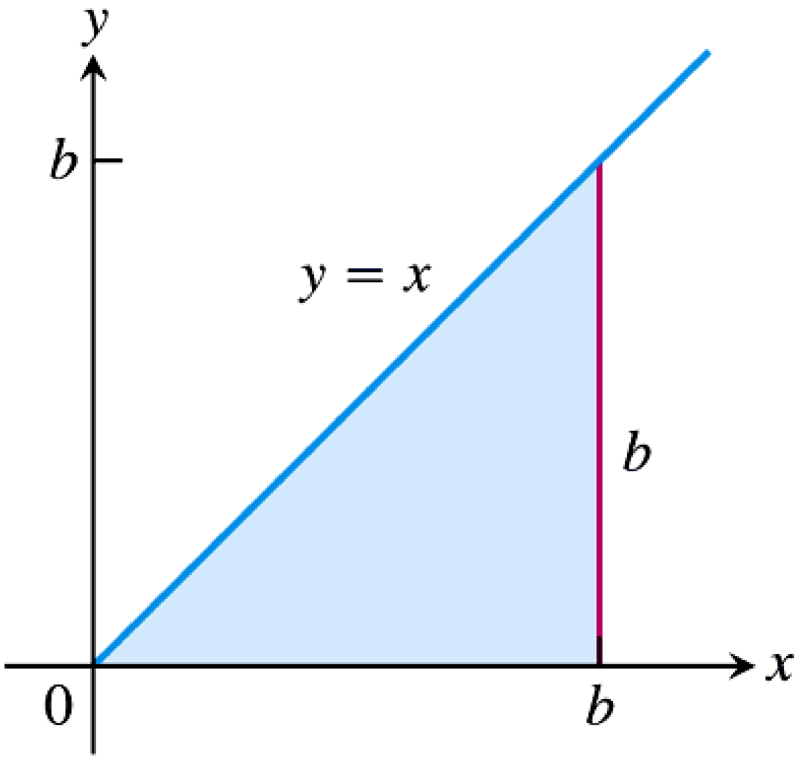


FIGURE 5.12 The region in Example 4 is a triangle.

$$\text{an integral } \int = \text{limit of Riemann sums}$$

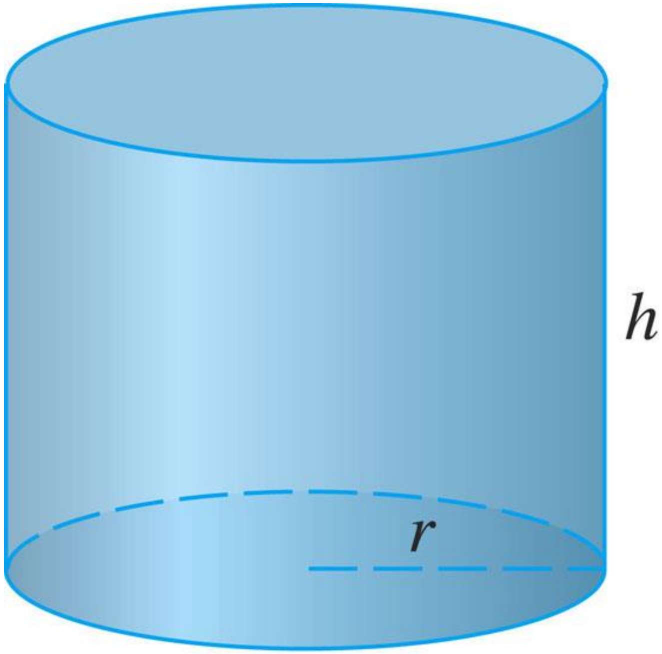
Volume

Volume

Example:

If the base is a circle with radius *r*, then circular cylinder with height *h* has volume:

$$V = \pi r^2 h$$



Circular cylinder
$V = \pi r^2 h$

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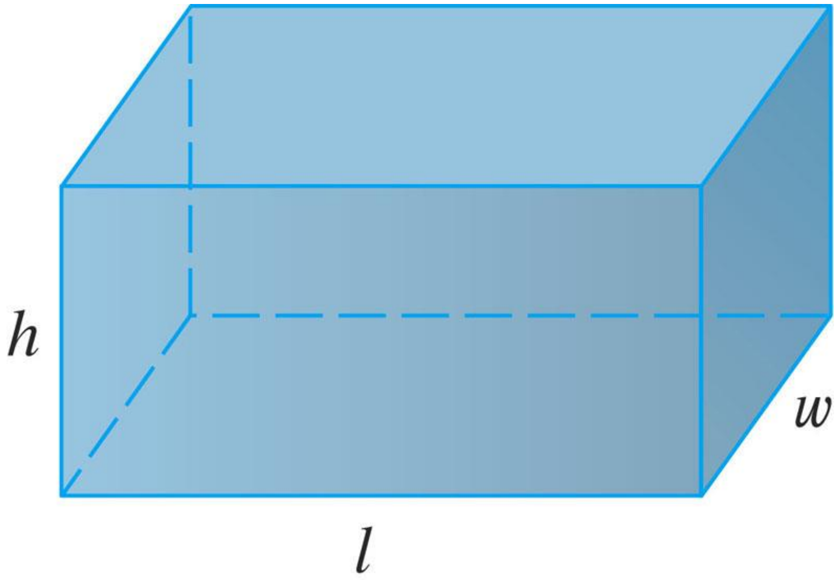
Volume

Example:

If the base is a rectangle with length *l* and width *w* then the rectangular parallelepiped with height *h* has volume:

$$V = lwh$$



Rectangular box

$V = lwh$

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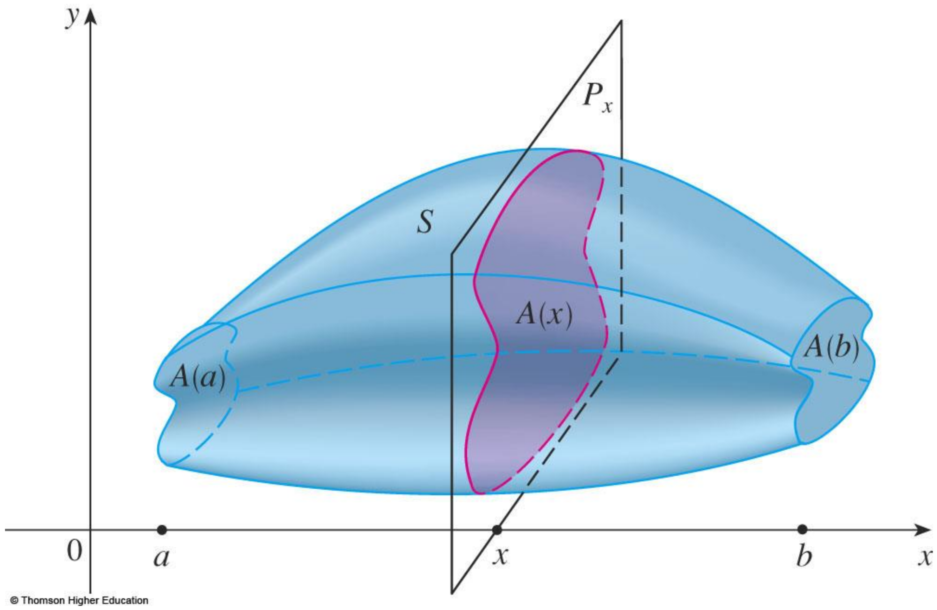
Volume by cross-sections

Volume by cross-sections

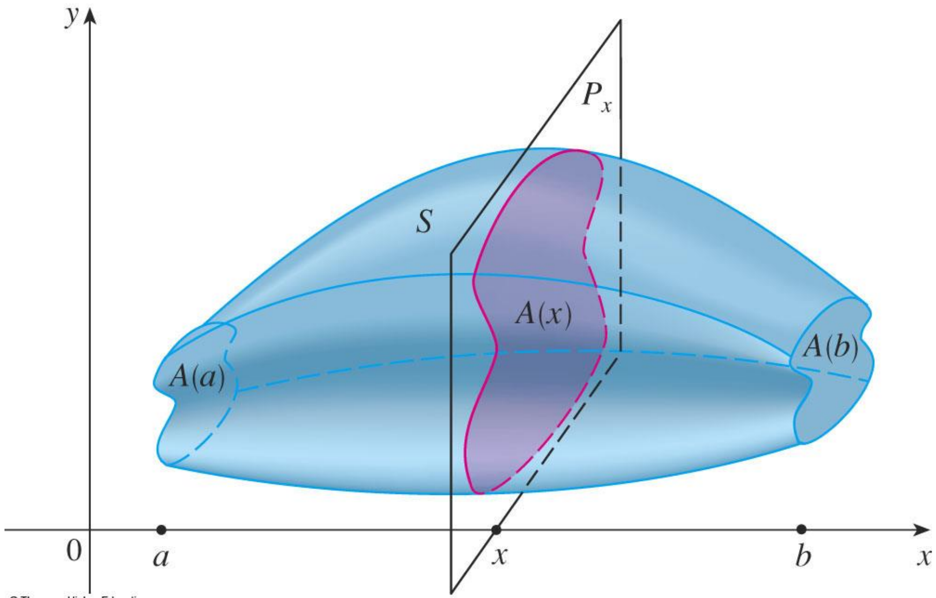
We start by intersecting *S* with a plane and obtaining a plan region that is called a *cross-section* of *S*



Volume by cross-sections

Let *A(x)* be the area of the cross-section of *S* in a plan P_x perpendicular to the *x*-axis and passing through the point *x*, where $a \le x \le b$.

Think of slicing *S* with a knife through *x* and computing the area of this slice.



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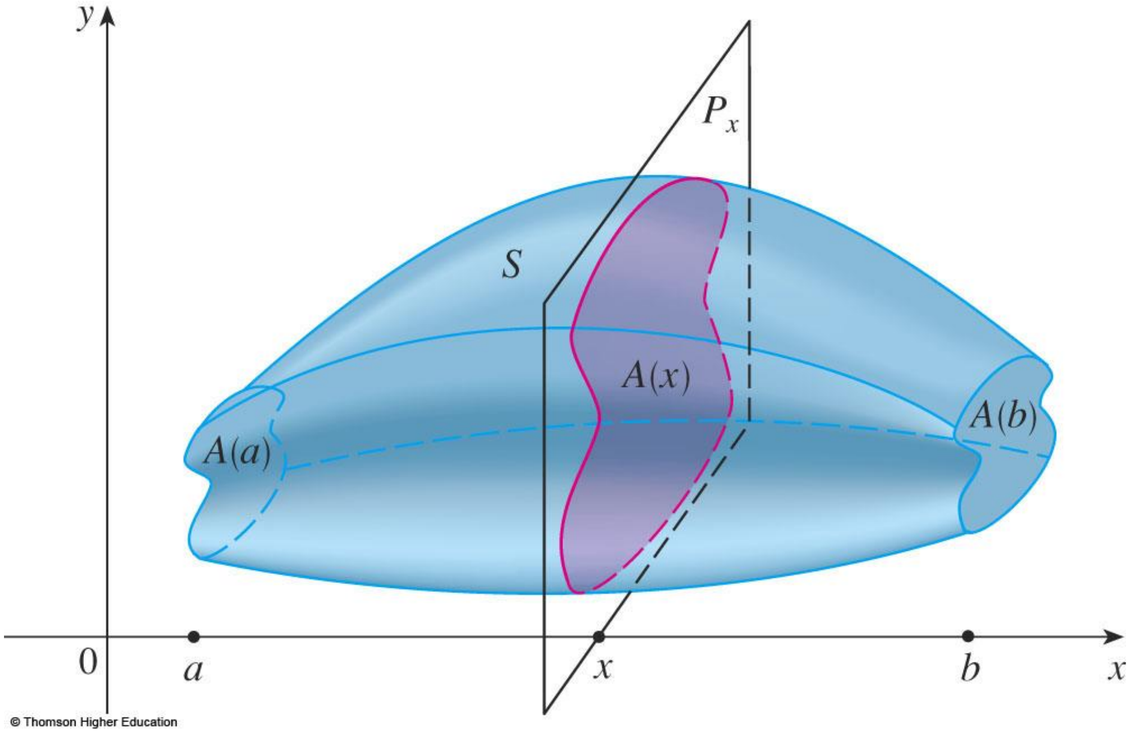
Volume by cross-sections

The cross-sectional area *A(x)* will vary as *x* increases from *a* to *b*.



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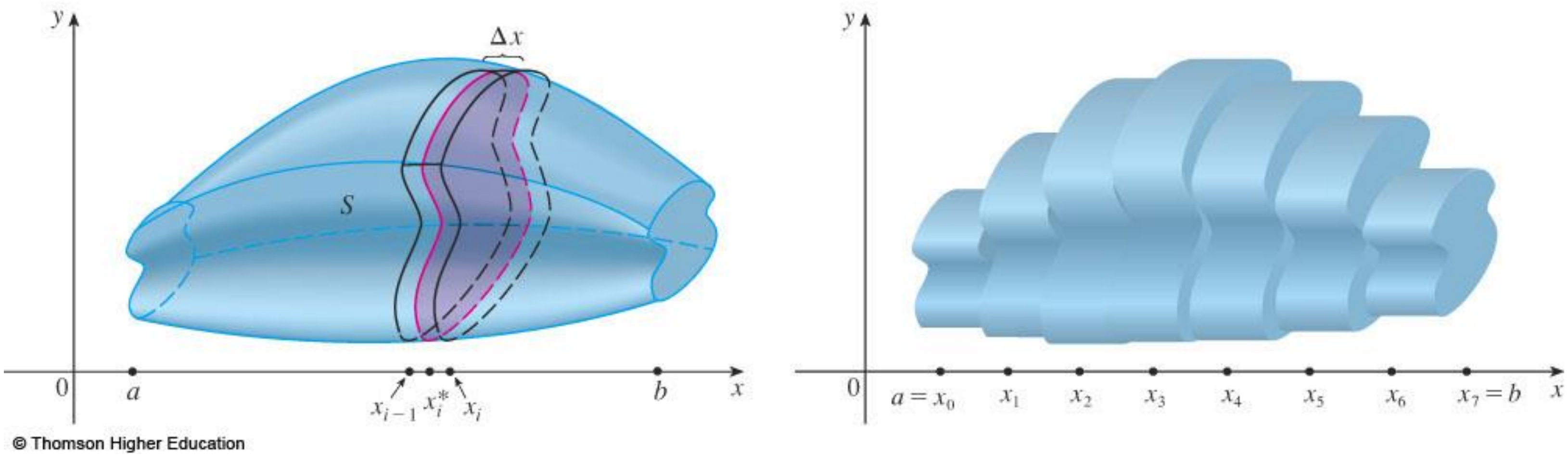
We divide S into n ‘slabs’ of equal width Δx using the planes P_{x1}, P_{x2}, … to slice the solid

Think of slicing a loaf of bread



Volume by cross-sections

So, an approximation to our intuitive conception of the volume of the *i*-th slab S_i is:

$$V(S_i) \approx A(x_i)\Delta x$$

Volume by cross-sections

Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x$$

- This approximation appears to become better and better as $n \to \infty$
- Think of the slices as becoming thinner and thinner

Volume by cross-sections

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x$$

Therefore, we define the volume as the limit of these sums as $n \longrightarrow \infty$.

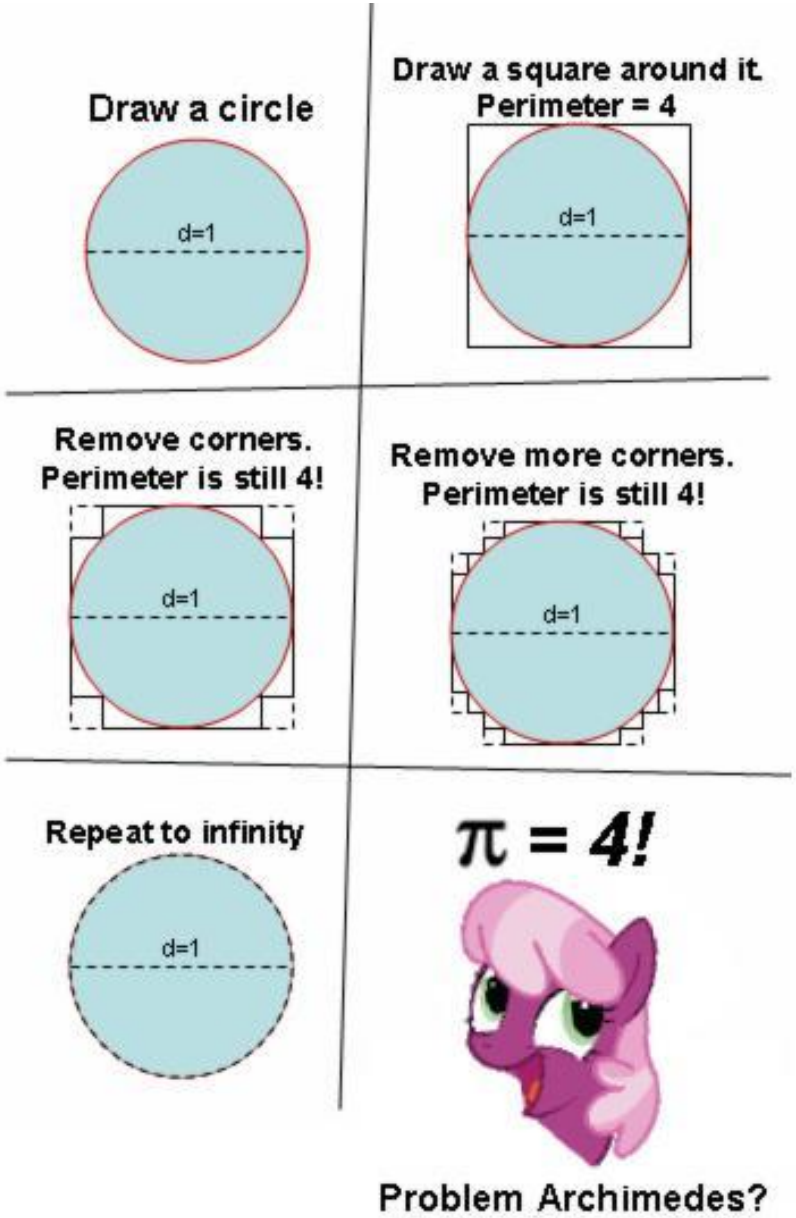
However, we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

Definition of Volume:

Let *S* be a solid that lies between $x = a$ and $x = b$.
If the cross-sectional area of *S* in the plane P_x through *x* and perpendicular to the x-axis, is *A(x)* then the volume of *S* is:

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)dx$$

Break


Draw a circle
d=1
Draw a square around it
Perimeter = 4
d=1
Remove corners.
Perimeter is still 4!
d=1
Remove more corners.
Perimeter is still 4!
d=1
Repeat to infinity
d=1
π = 4!
Problem Archimedes?

5.4

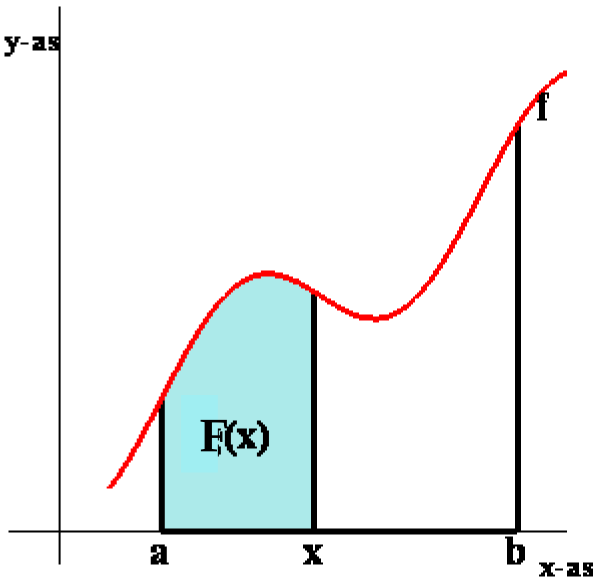
The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

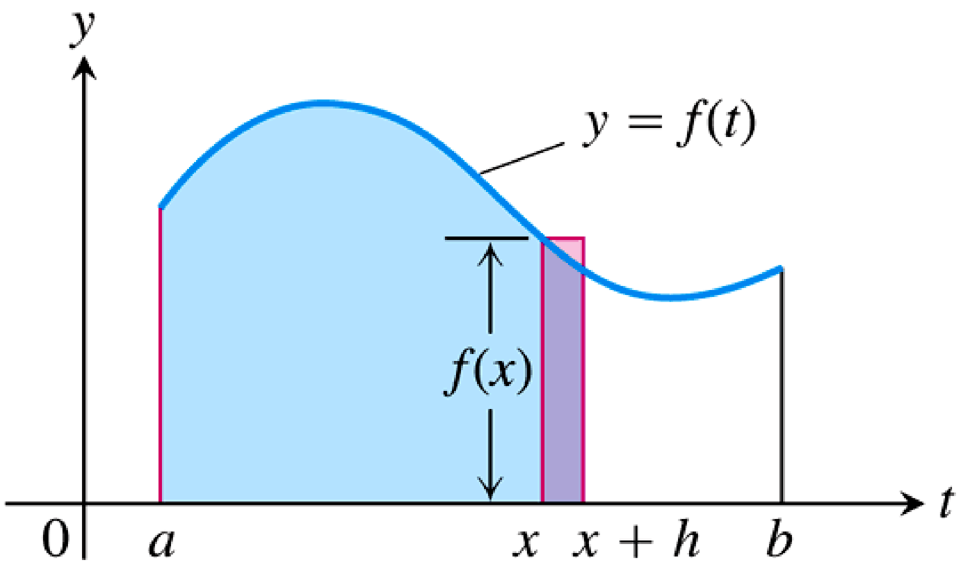
Given is a continuous function *f* on the interval [*a*, *b*].
Define the function *F* by:

$$F(x) = \int_a^x f(t)dt$$

The interpretation of *F* is:
the area under the graph of *f*
on the interval from *a* to *x*.



Fundamental Theorem of Calculus



We want to find $F'(x)$:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

Fundamental Theorem of Calculus

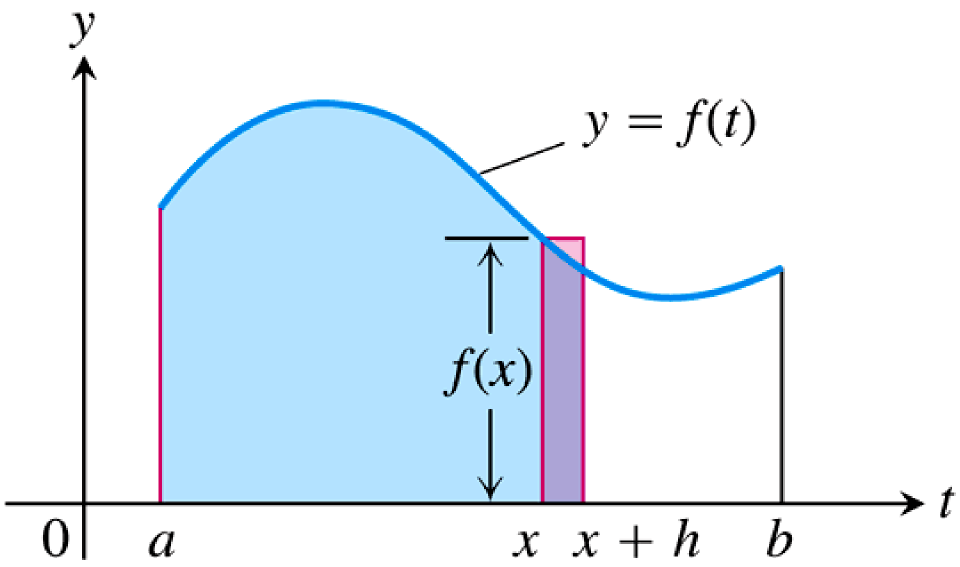


FIGURE 5.19 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

The Fundamental Theorem of Calculus, Part 1

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (2)$$

Fundamental Theorem of Calculus

A function with derivative f is called an *antiderivative* of f. We see that the integral function:

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f, independent of the starting point a.

4.8

Antiderivatives

Anti-derivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Anti-derivatives

TABLE 4.2 Antiderivative formulas, k a nonzero constant

	Function	General antiderivative		Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	**8.**	e^{kx}	$\frac{1}{k}e^{kx} + C$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$	**9.**	$\frac{1}{x}$	$\ln\lvert x\rvert + C, \quad x \neq 0$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$	**10.**	$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	**11.**	$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	**12.**	$\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, \quad kx > 1$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	**13.**	a^{kx}	$\left(\frac{1}{k \ln a}\right)a^{kx} + C, \quad a > 0, \ a \neq 1$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$			

Fundamental Theorem of Calculus

THEOREM The Fundamental Theorem of Calculus, Part 2 If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This result is independent of the *antiderivative* of f. If $G(x)$ is another antiderivative, i.e. also $G' = f$, then

$$F(x) = G(x) + C$$

But then:

$$F(b) - F(a) = (G(b) + C) - (G(a) + C) = G(b) - G(a)$$

Fundamental Theorem of Calculus

THEOREM **The Net Change Theorem** The net change in a function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx.$$

Remember the differential: $dF = F'(x)dx$

Then: $$F(b) - F(a) = \int_a^b dF$$

Variations on the Fundamental Theorem

Variations on the Fundamental Theorem

Define:

$$g(x) = \int_{x}^{b} f(t)dt$$

where f is continuous is on $[a, b]$. Then g is differentiable on (a, b), and

$$g'(x) = -f(x)$$

Indeed:

$$g(x) = -\int_{b}^{x} f(t)dt$$

Variations on the Fundamental Theorem

Examples:

$$g(x)=\int_{2}^{x}\sqrt{1+t^{4}}\,dt \quad\rightarrow\quad g'(x)=\sqrt{1+x^{4}}$$

$$h(s)=\int_{0}^{s}\sqrt{3+\sin x}\;dx \quad\rightarrow\quad h'(s)=\sqrt{3+\sin s}$$

$$k(t)=\int_{t}^{0}\ln\left(1+s^{2}\right)ds \quad\rightarrow\quad k'(t)=-\ln\left(1+t^{2}\right)$$

Variations on the Fundamental Theorem

Example:

$$f(x) = \int_0^{x^2} \sqrt{1+t}\,dt$$

Then $f(x) = g(x^2)$, where

$$g(u) = \int_0^{u} \sqrt{1+t}\,dt$$

Then

$$f'(x) = g'(x^2) \cdot 2x = \sqrt{1+x^2} \cdot 2x$$

Quiz

Given $$h(x) = \int_{x}^{x^2} \tan(t)dt$$

Then $h'(x)$ equals

a) $\tan(x^2)$

b) $\tan(x^2) - \tan(x)$

c) $2x\tan(x^2) - \tan(x)$

d) $2x\tan(x^2) - 2x\tan(x)$

Quiz

Given $$h(x) = \int_x^{x^2} \tan(t)dt$$

Then $h'(x)$ equals

a)

b)

c) $2x\tan(x^2) - \tan(x)$

d)

Indefinite integrals

Indefinite integrals

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Examples:

$$\int\left(\frac{3}{x}-\frac{2}{1+x^2}\right)dx \quad = 3\ln|x| - 2\tan^{-1}x + C$$

Check by differentiation!

$$\int\left(2^x - 2\sqrt{x}\right)dx \quad = \int\left(2^x - 2x^{\frac{1}{2}}\right)dx \quad = \frac{2^x}{\ln 2} - \frac{4}{3}x^{\frac{3}{2}} + C \quad = \frac{2^x}{\ln 2} - \frac{4}{3}x\sqrt{x} + C$$

Check by differentiation!

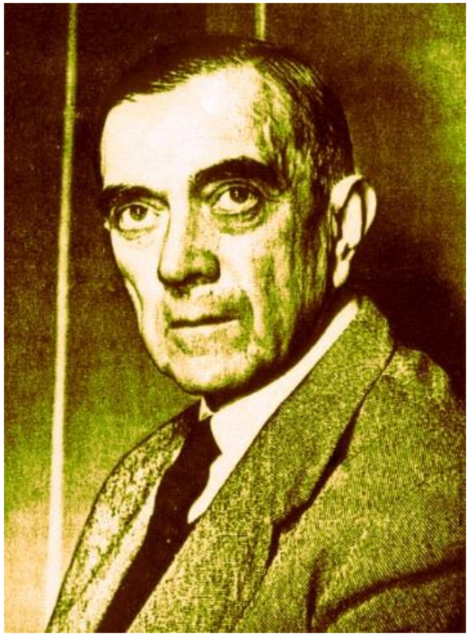
$$\int e^{-x^2}dx$$

It exists, but is not expressible in known functions. It creates a new function Erf.

The Gini index is an integral

The Gini index is an integral

How can we measure the distribution of income among the inhabitants of a country?



Corrado Gini (1884 – 1965)

The Gini index is an integral

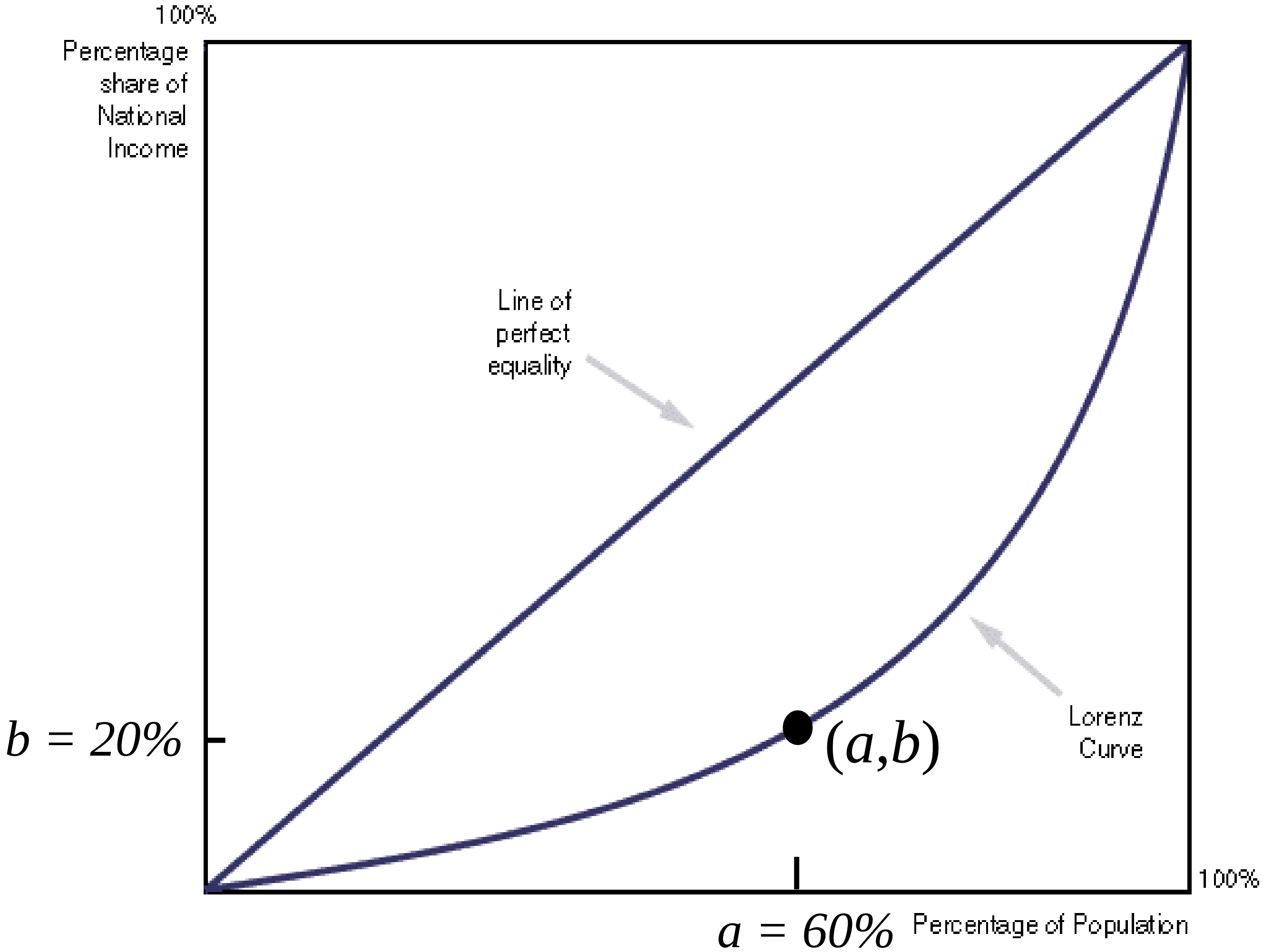
- First, rank all households by income.
- Then compute the percentage of households whose income is at most a given percentage of the country's total income.

Lorenz curve $y = L(x)$ on the interval [0,1].

- If the bottom a % of households receive b % of the total income, then plot the point (a,b) on the Lorenz curve.

The Gini index is an integral

If the bottom *a* % of households earn *b* % of the total income, then plot the point (*a*,*b*) on the Lorenz curve.



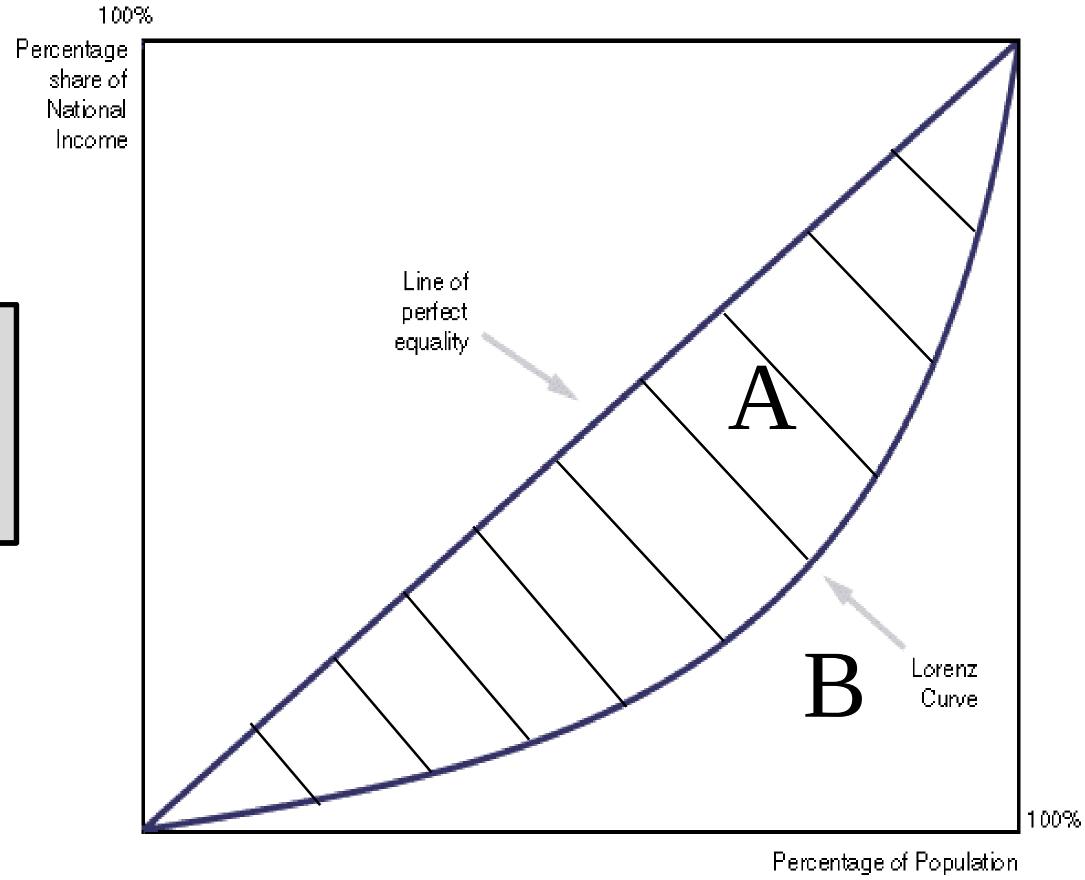
The Gini index is an integral

Definition: The Gini index ***G*** is $A / (A + B)$

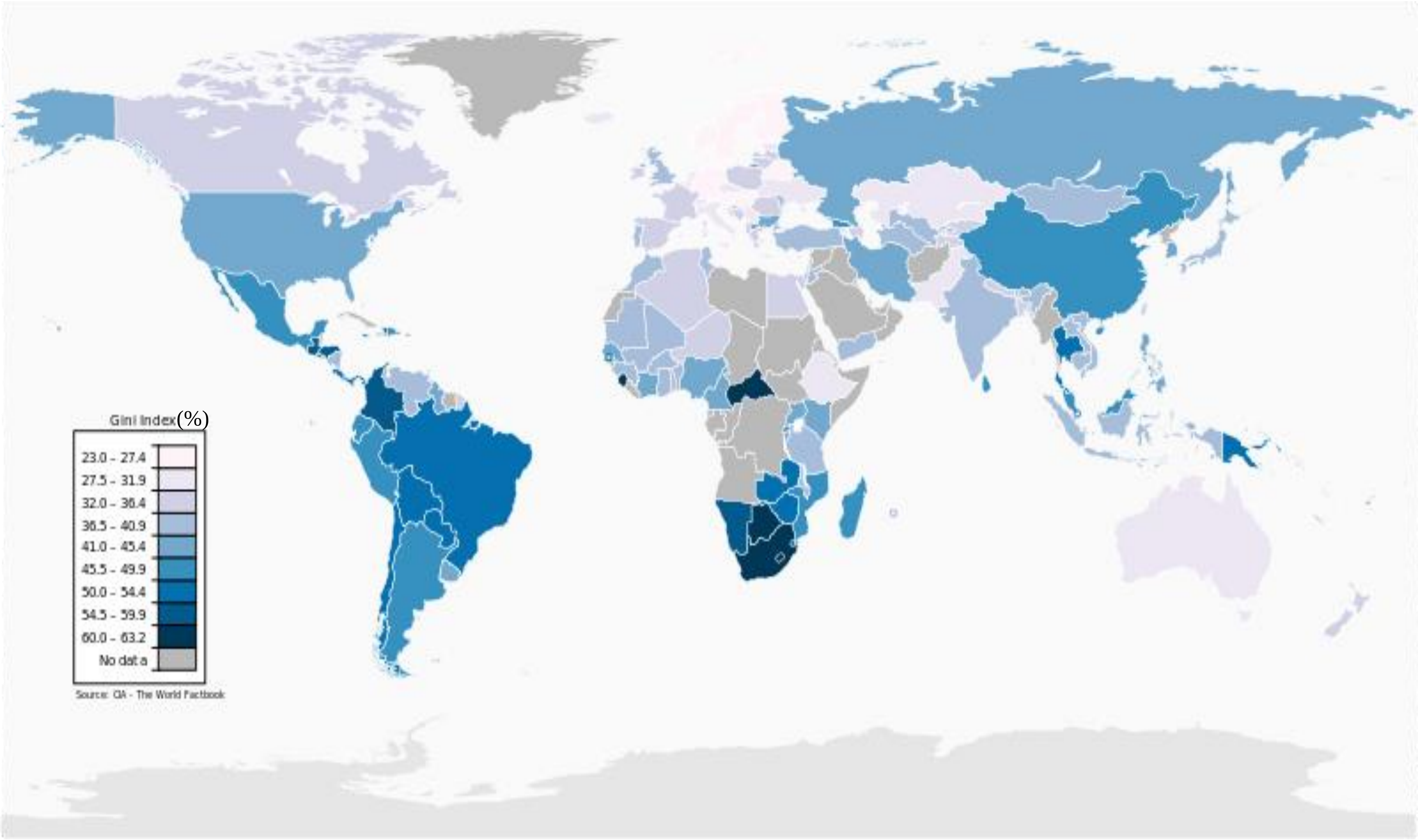
Now: $A + B = ½$

So: $G = 2A$

$$G = 2\int_0^1 (x - L(x))dx$$



The Gini index is an integral



Netherlands: $G = 0.29$

Integrals

- Theme: Area
- Theme: Riemann Sum
- Theme: Fundamental Theorem
- Theme: Antiderivatives

Summarizing Exercise

Determine dy/dx in case y is given by

$$y(x) = x \int_1^x \frac{t}{1+t^4} \, dt$$

-Contents-

- ☑ Limits and continuity
- ☑ Derivatives and applications
- ☑ Functions of 2 variables

- ☑ Integrals
- ❑ Calculation techniques for integrals
- ❑ Power and Taylor series