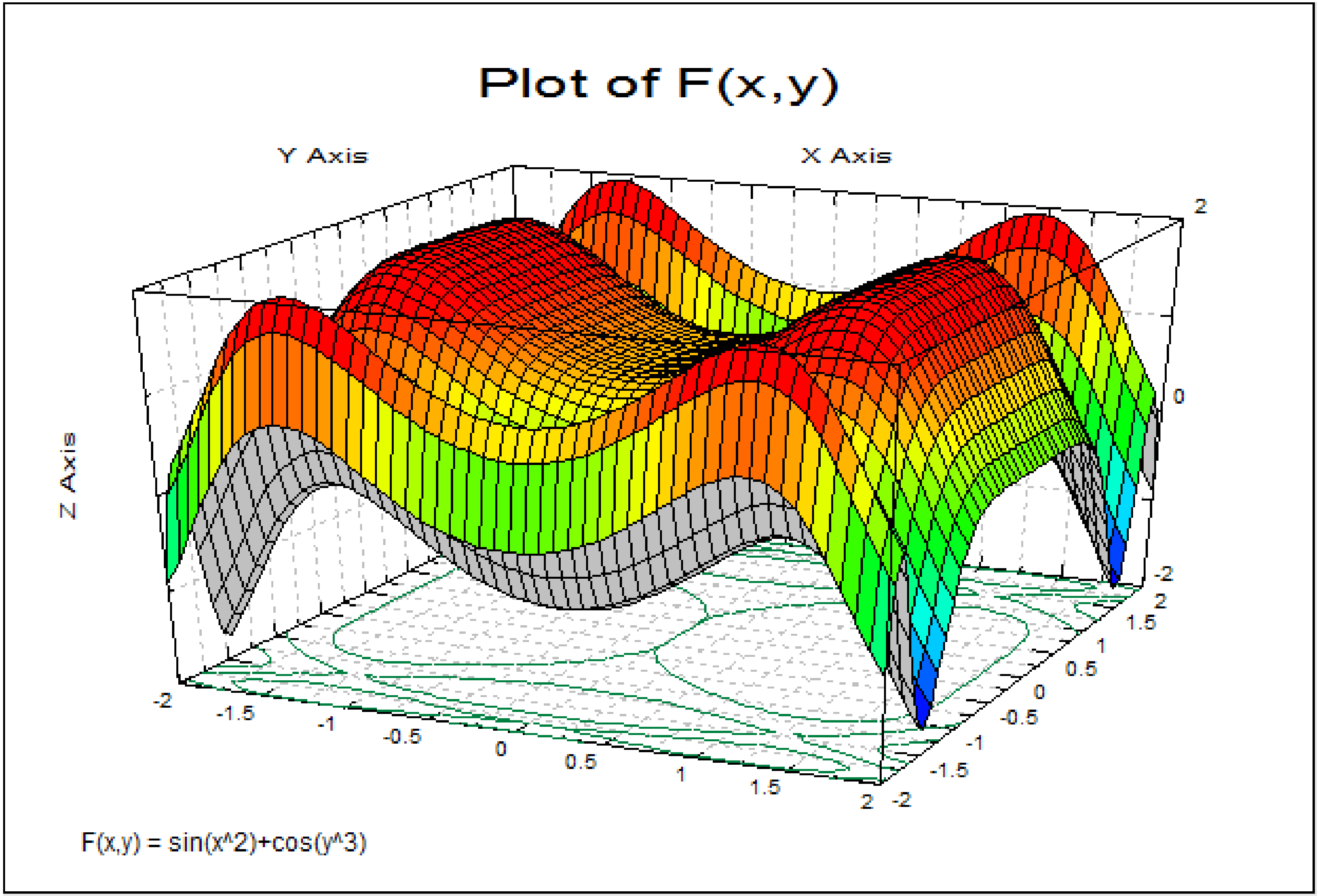
Mathematics B2: Newton



- Contents -

Parts of Chapter 14

(see the course schedule and the study guide for the relevant parts!)

Functions of several variables

Theme: Basic concepts

- Definition
- Plotting
- Contour lines

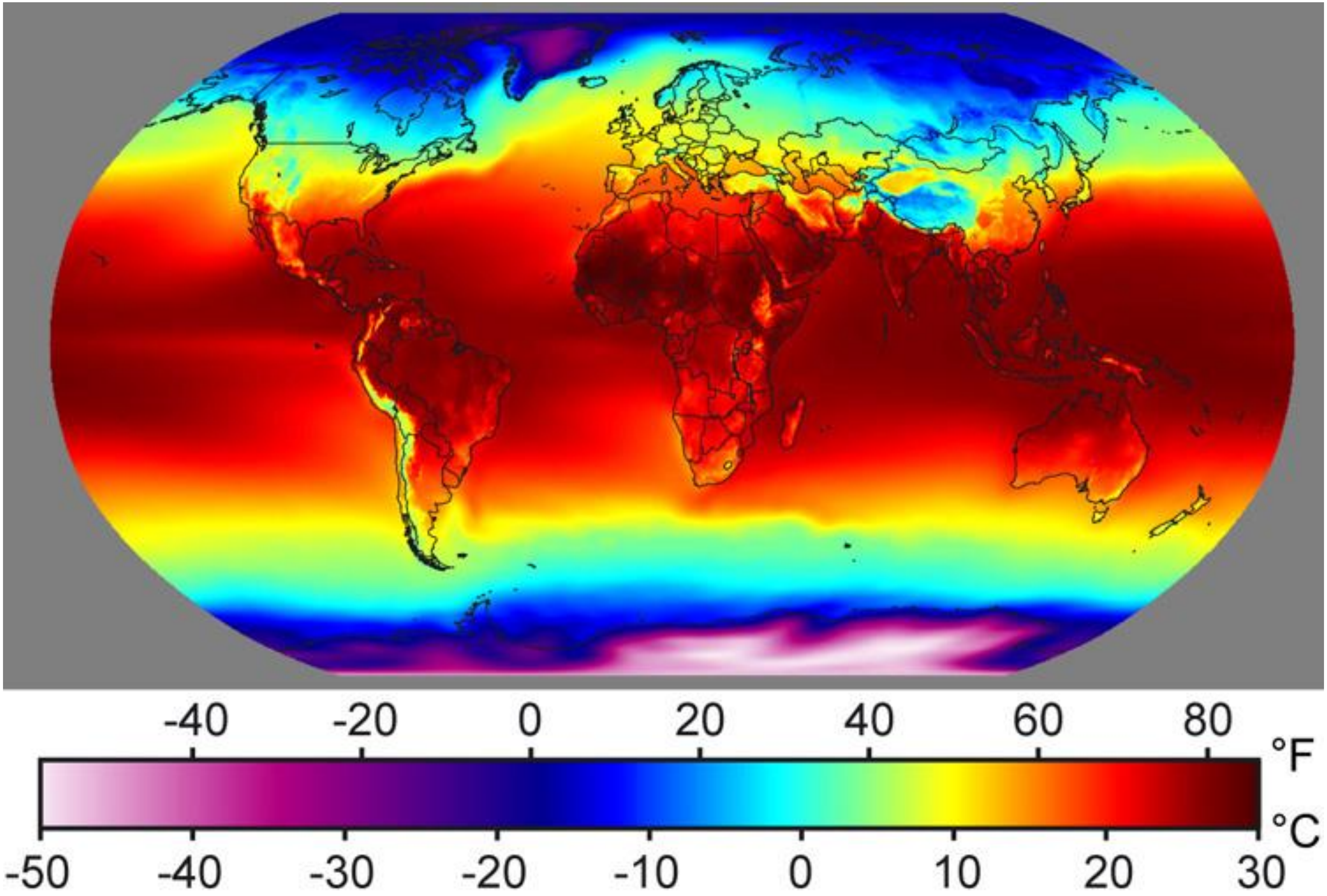
Theme: Continuity

Theme: Partial derivatives

Theme: Tangent plane

Definition of functions on $\mathbf{R}^2$

Functions of several variables



Definition of functions on $\mathbf{R}^n$

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

Definition of functions on $\mathbf{R}^2$

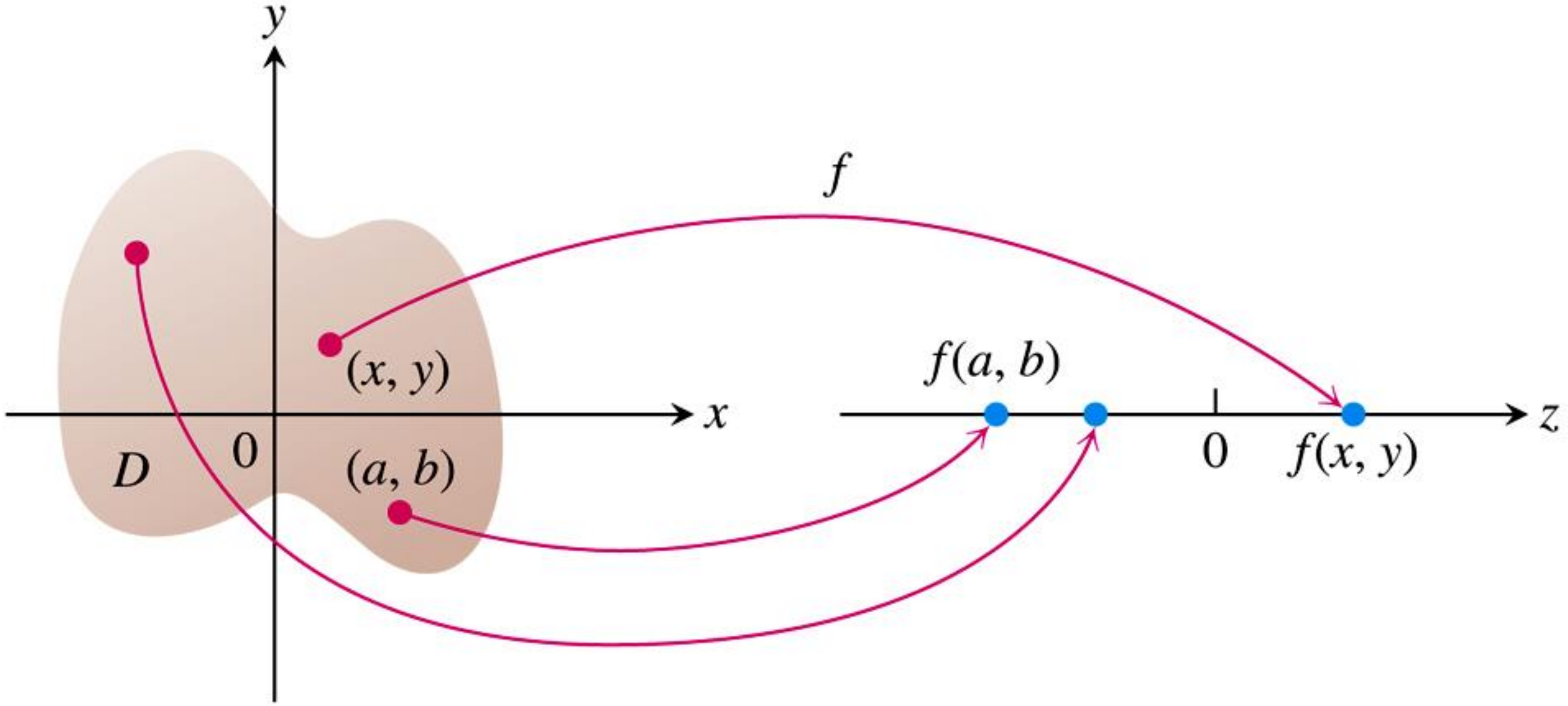


FIGURE 14.1 An arrow diagram for the function $z = f(x, y)$.

Definition of functions on $\mathbf{R}^2$

Example: $f:\mathbf{R}^2 \to \mathbf{R}$

$$f:(x,y)\to\sqrt{y-x}\,\ln(y+x)$$

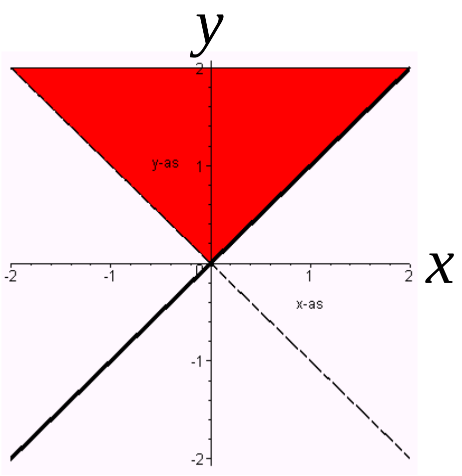
For example:

$$f(1,5)=\sqrt{5-1}\,\ln(5+1)=2\ln 6\approx 3{,}58$$

Domain:

$$\sqrt{y-x} \quad\to\quad y-x\geq 0$$

$$\ln(y+x) \quad\to\quad y+x>0$$



Domain of f (red)

Graphs of functions on $\mathbf{R}^2$

Graphs of functions on $\mathbf{R}^2$

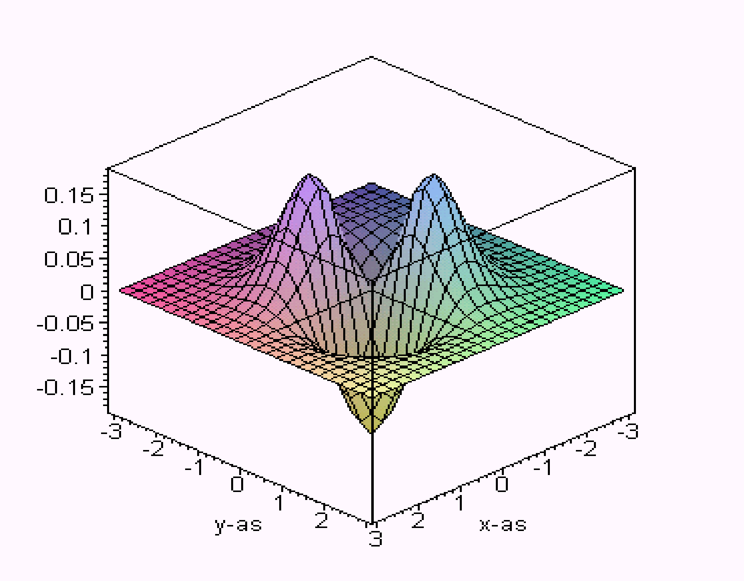
Graph of *f*: $\{(x, y, f(x,y)) \in \mathbf{R}^3 \mid (x,y) \in D\}$

(D = Domain of f)

Example:

$$f(x,y) = -xy \cdot e^{-(x^2+y^2)}$$

$D = \mathbf{R}^2$



Graph of f

Note: the graph of a function of 2 variables is a *surface* in $\mathbf{R}^3$

Graphs of functions on $\mathbf{R}^2$

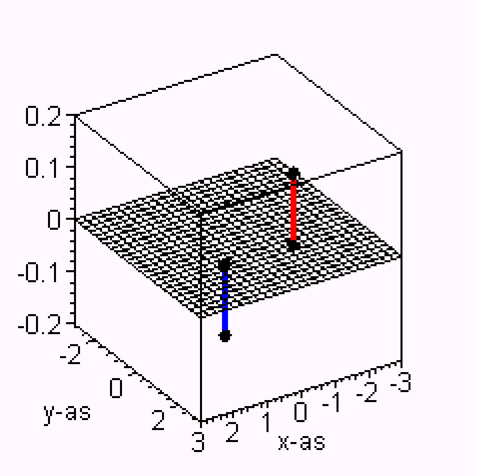
How is a graph built up?

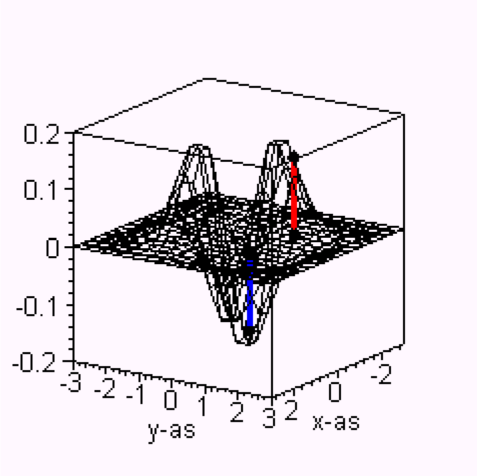
$$f(x,y) = -xy \cdot e^{-\left(x^2+y^2\right)}$$

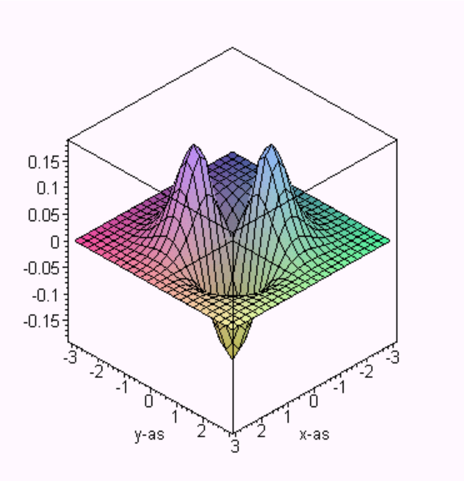
Domain: $D = \mathbf{R}^2$

For each $(x,y) \in \mathbf{R}^2$ ($z = 0$ plane),
the image (the z-value) should be calculated.

Examples: $f(1,1) = -e^{-2}$ (*blue*) en $f(-1,1) = e^{-2}$ (*red*)







Graph drawing by hand?

Only doable in ‘standard cases’:

balls, cones, cylinders, paraboloids, etc.

Method:

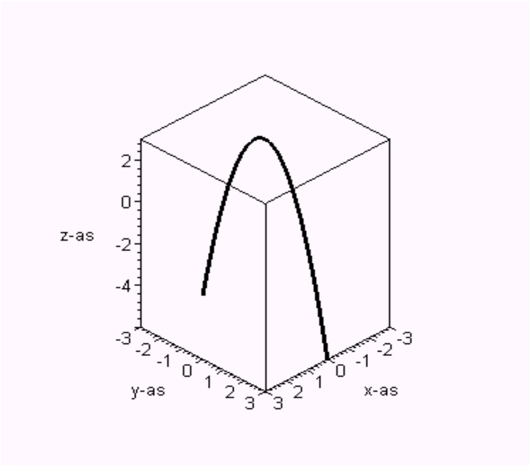
First plot the intersections of the graph with the $x = 0$, $y = 0$ and the $z = 0$ planes to get an impression.

Graphs of functions on $\mathbf{R}^2$

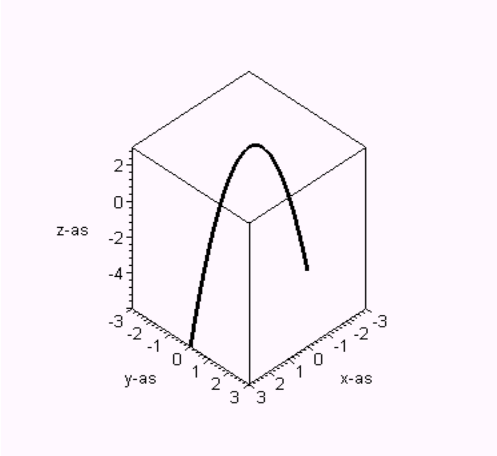
Example: $f(x,y) = 3 - x^2 - y^2$ $D = \mathbf{R}^2$

Intersections of the graph with:

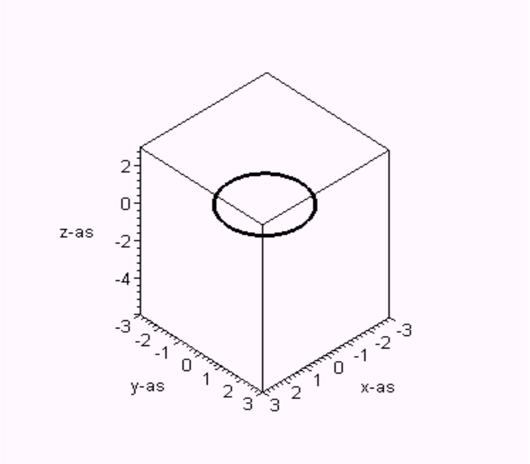
$x = 0$ plane:
$f(0,y) = 3 - y^2$



$y = 0$ plane:
$f(x,0) = 3 - x^2$



$z = 0$ plane: $f(x,y) = 0$
$x^2 + y^2 = 3$

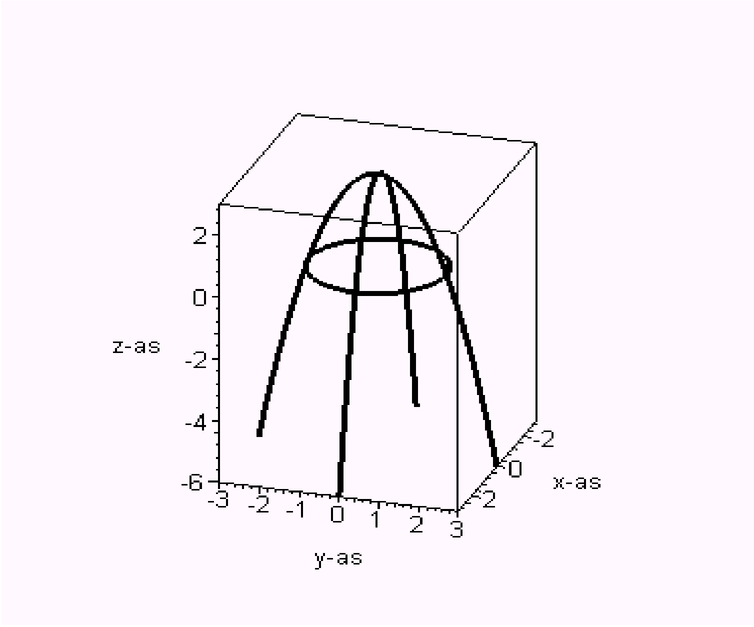


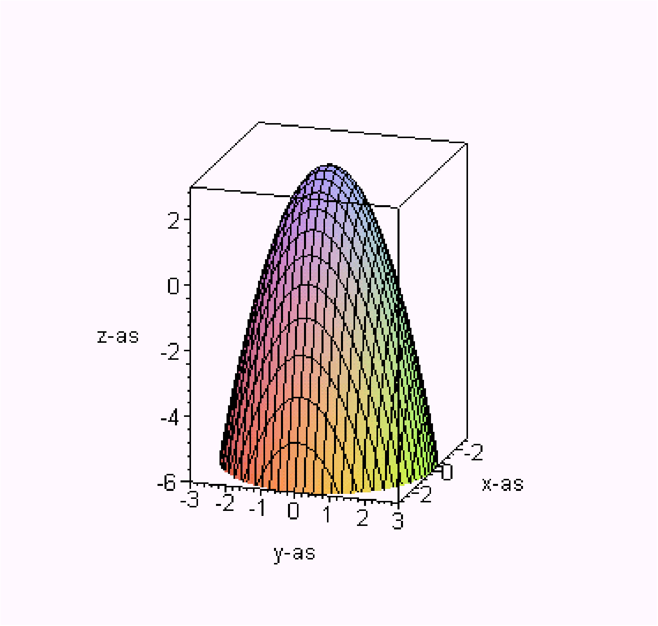
Example (continued):

$$f(x,y) = 3 - x^2 - y^2$$

$$D = \mathbf{R}^2$$

Intersections of the graph with the coordinate planes:





Level curves

"Andenes are terraces dug into the slopes of mountains for agricultural purposes. They were constructed and much used in the Andes mountain range to provide cultivable hillsides. The majority of these terraces were constructed and used by the pre-Hispanic cultures, and many can still be observed throughout the region."

Level curves

First use of level curves: Charles Hutton (1774)

Level curves

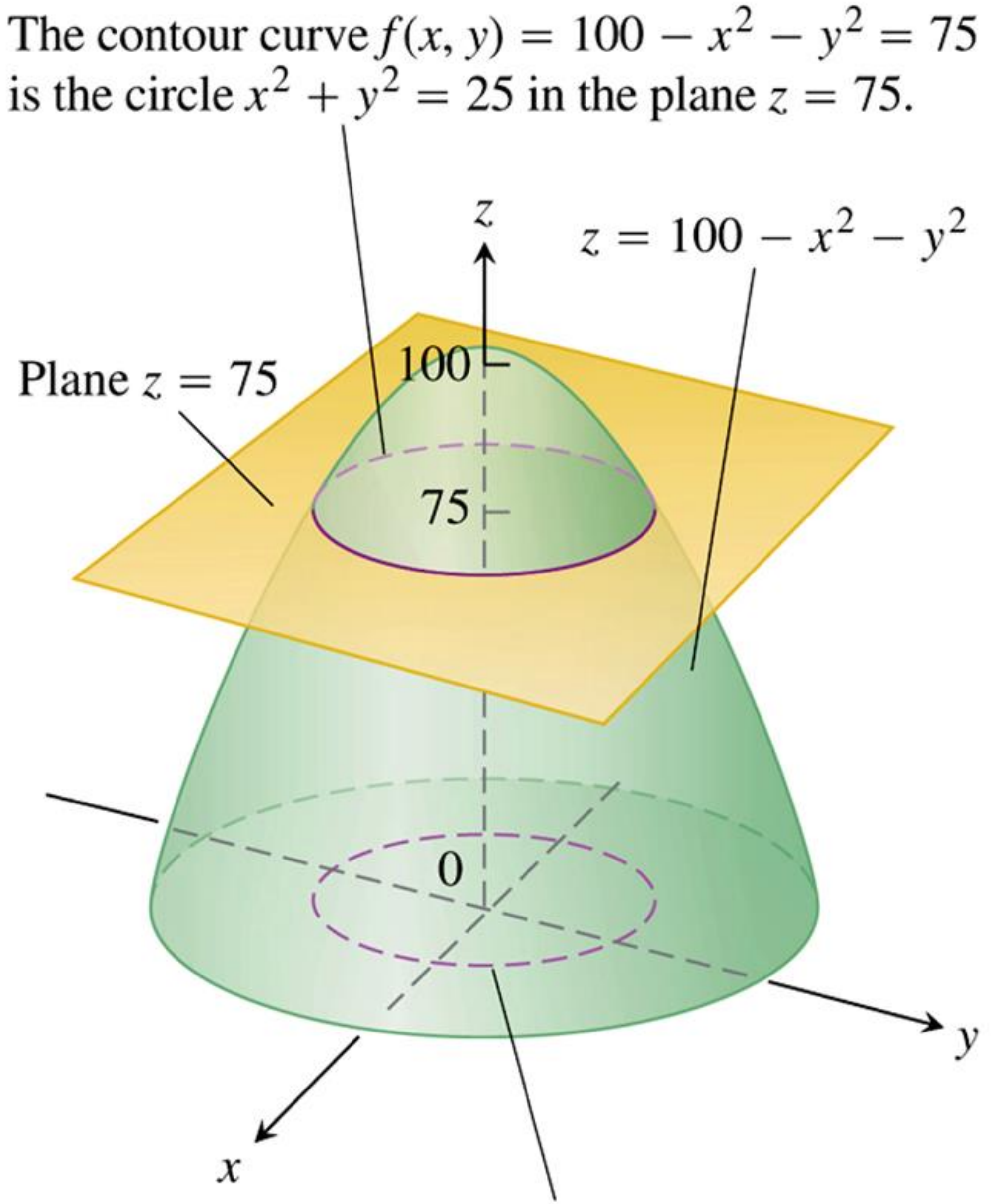
Definition:

A level curve of a function f: $\mathbf{R}^2 \to \mathbf{R}$ is a set:

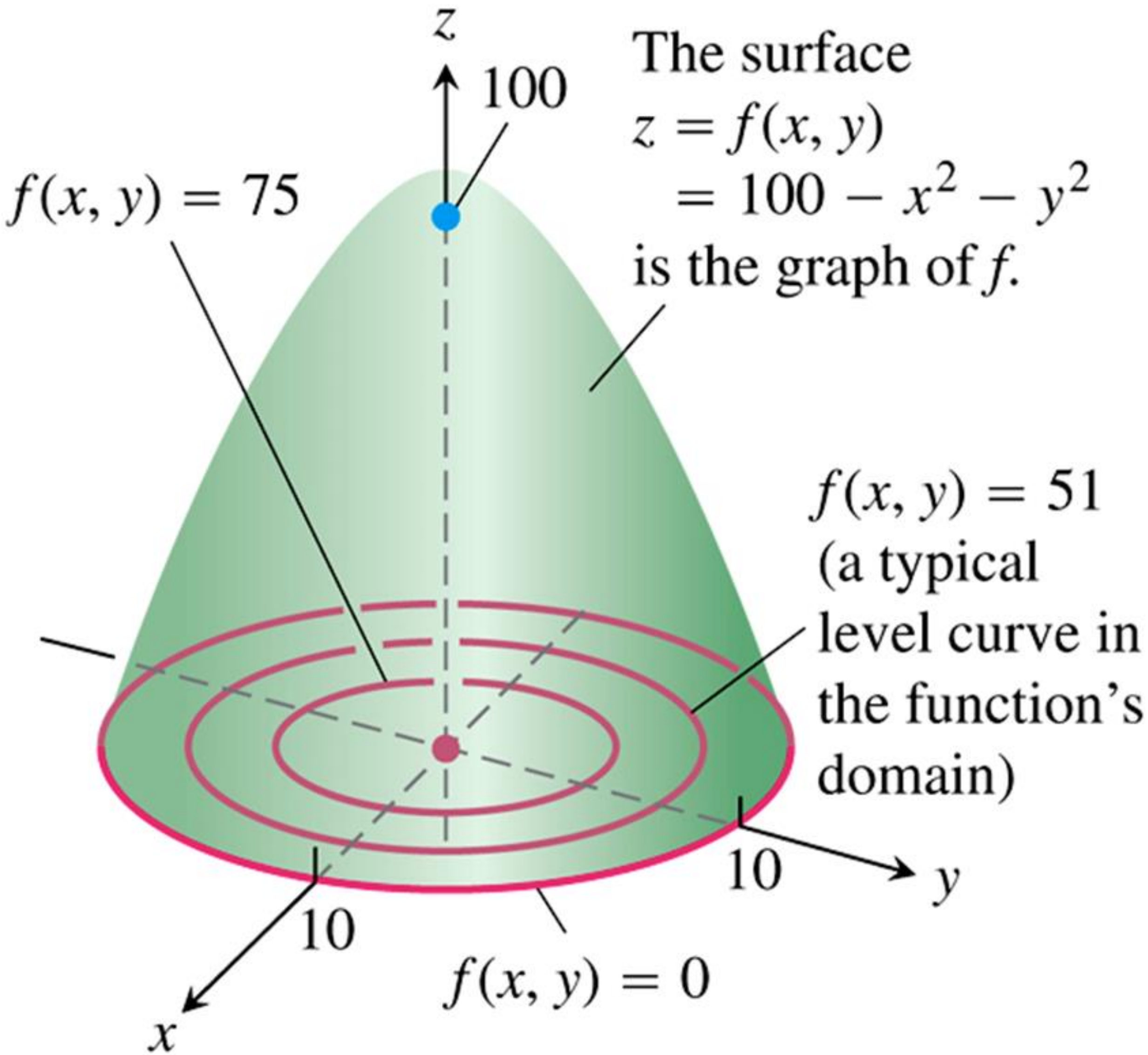
$$\left\{(x, y) \in \mathbf{R}^2 \mid f(x, y) = c\right\} \quad \text{where } c \in \mathbf{R}$$

It is possible that the equation $f(x,y) = c$ has no solutions; then the level curve (for level c) does not exist.

Level curves



Level curves



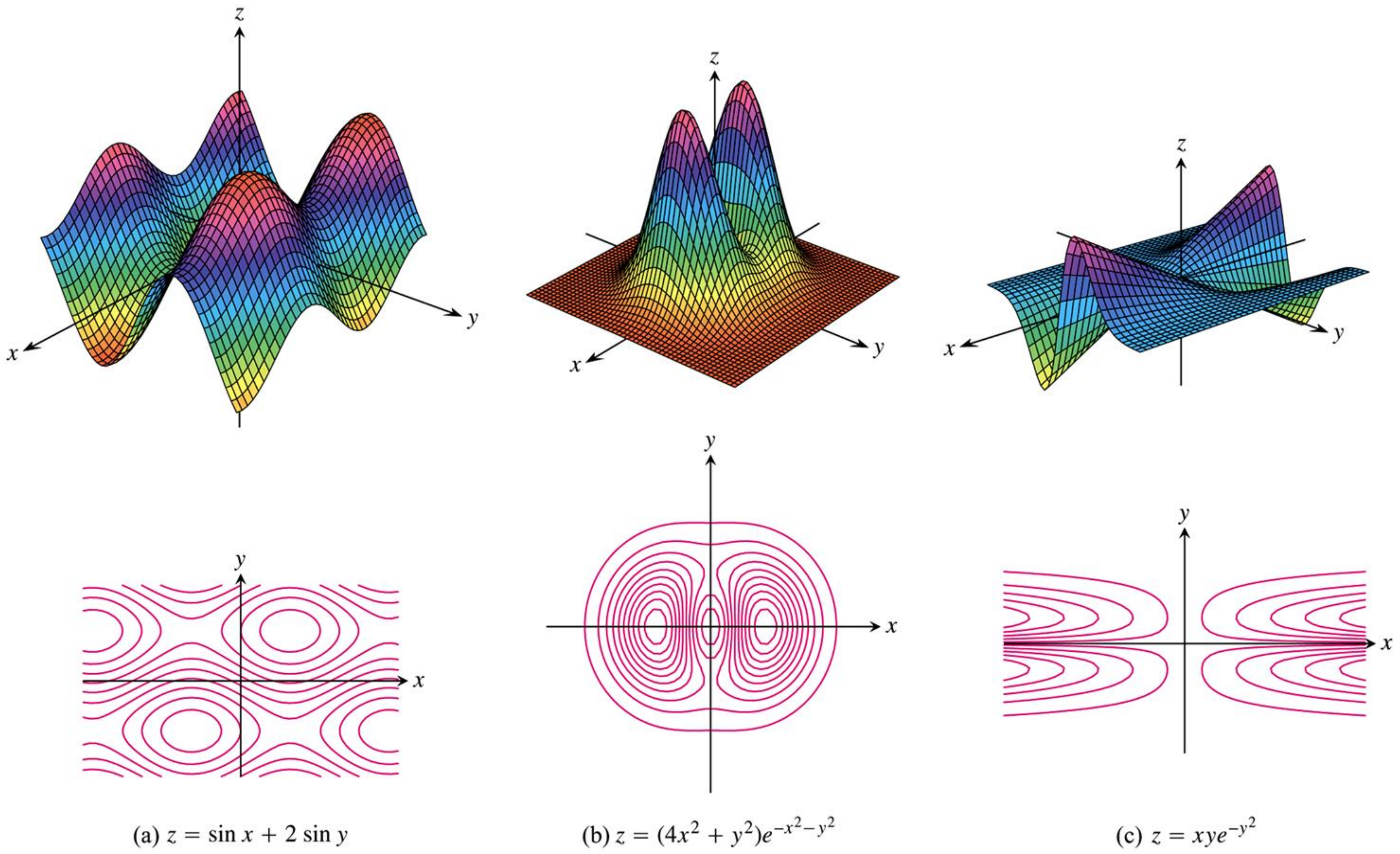


FIGURE 14.11 Computer-generated graphs and level curves of typical functions of two variables.

Functions of several variables

Theme: Basic concepts

Theme: Continuity

- Limits
- Continuity
- Polar coordinates

Theme: Partial derivatives

Theme: Tangent plane

Limits in $\mathbf{R}^2$

Limits in $\mathbf{R}^2$

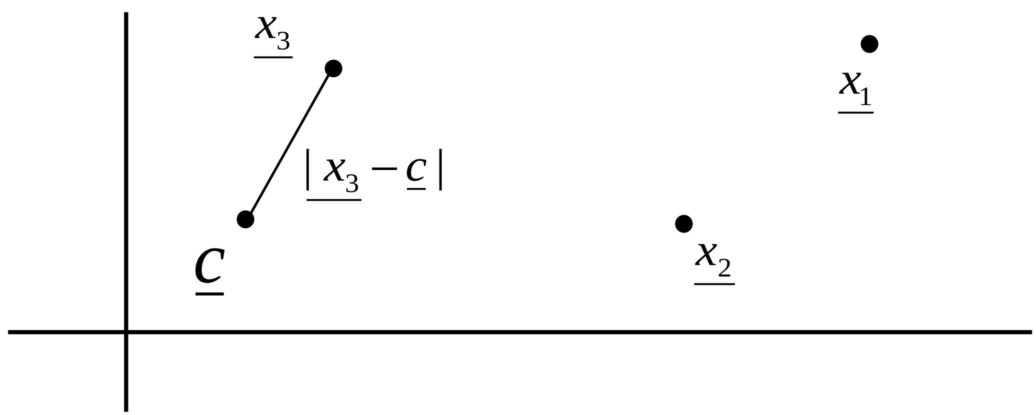
Definition: We say that the sequence of vectors

$\underline{\mathbf{x}}_1 = (x_1, y_1)$, $\underline{\mathbf{x}}_2 = (x_2, y_2)$, $\underline{\mathbf{x}}_3 = (x_3, y_3)$, … ***tends to***

$\underline{\mathbf{c}} = (c_1, c_2)$ *if* $|\underline{\mathbf{x}}_n - \underline{\mathbf{c}}|$ becomes very small for n very large.

We write: $$\lim_{n \to \infty} \underline{x}_n = \underline{c}$$

Remark: $|\underline{\mathbf{x}}_n - \underline{\mathbf{c}}|$ is the ***distance*** from $\underline{\mathbf{x}}_n$ to $\underline{\mathbf{c}}$.



Limits in $\mathbf{R}^2$

Definition: We say that $\lim_{\underline{x} \to \underline{c}} f(\underline{x}) = L$

if for every sequence $\underline{\boldsymbol{x}}_1, \underline{\boldsymbol{x}}_2, \underline{\boldsymbol{x}}_3, \ldots,$ **different from** $\underline{\boldsymbol{c}}$, holds:

$\underline{\boldsymbol{x}}_1, \underline{\boldsymbol{x}}_2, \underline{\boldsymbol{x}}_3, \ldots$ tends to $\underline{\boldsymbol{c}}$

implies

$z_1, z_2, z_3, \ldots$ tends to L

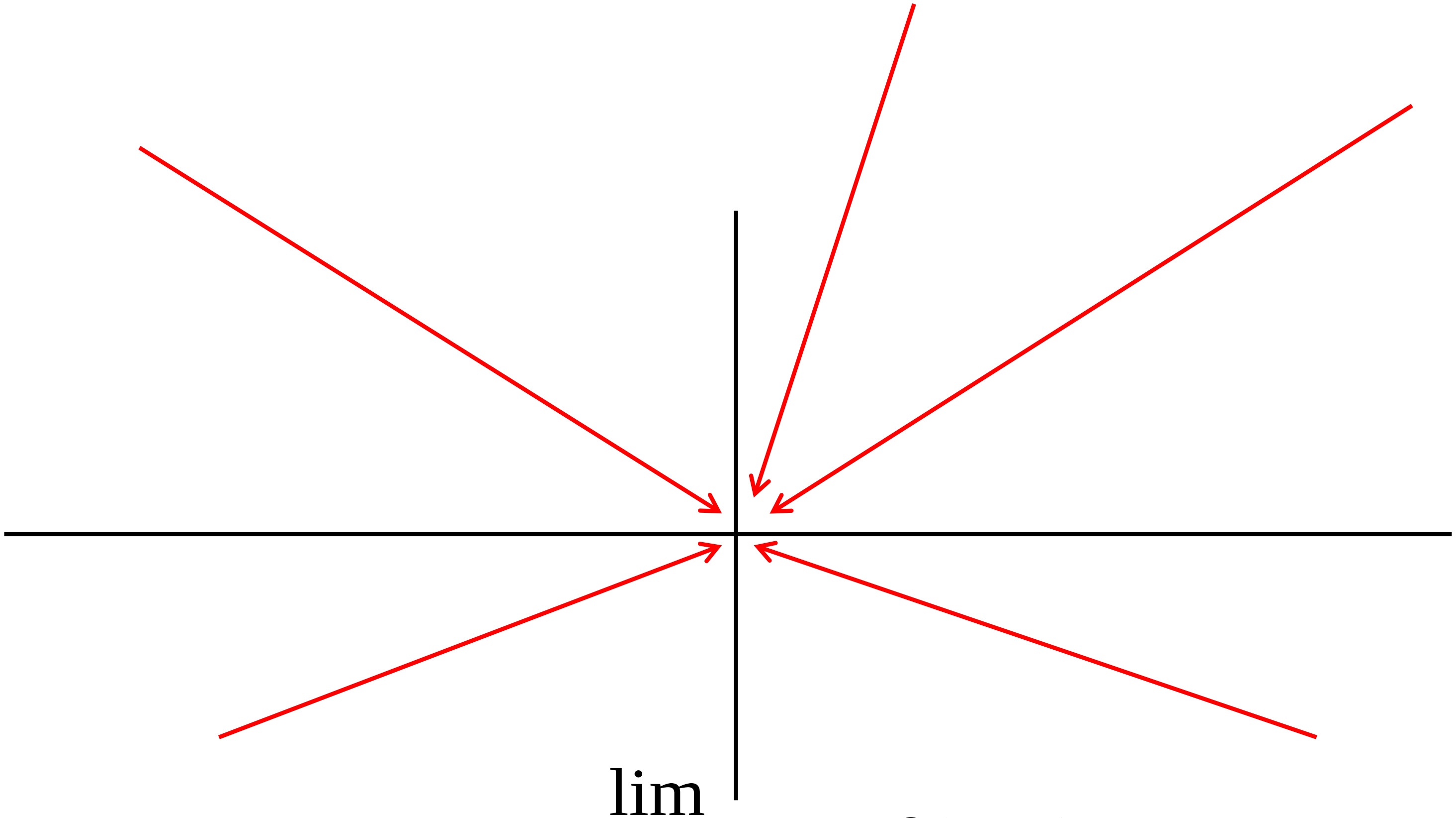
$\underline{\boldsymbol{x}}_i = (x_i, y_i)$

Here: $z_1 = f(x_1, y_1)$, $z_2 = f(x_2, y_2)$, $z_3 = f(x_3, y_3), \ldots$

Remark: We need that $\underline{\boldsymbol{x}}_1, \underline{\boldsymbol{x}}_2, \underline{\boldsymbol{x}}_3, \ldots$ are in the domain of f.

Limits in $\boldsymbol{R}^2$

Does having the same limit along all straight half lines approaching (0,0) imply that the limit exists at (0,0)?



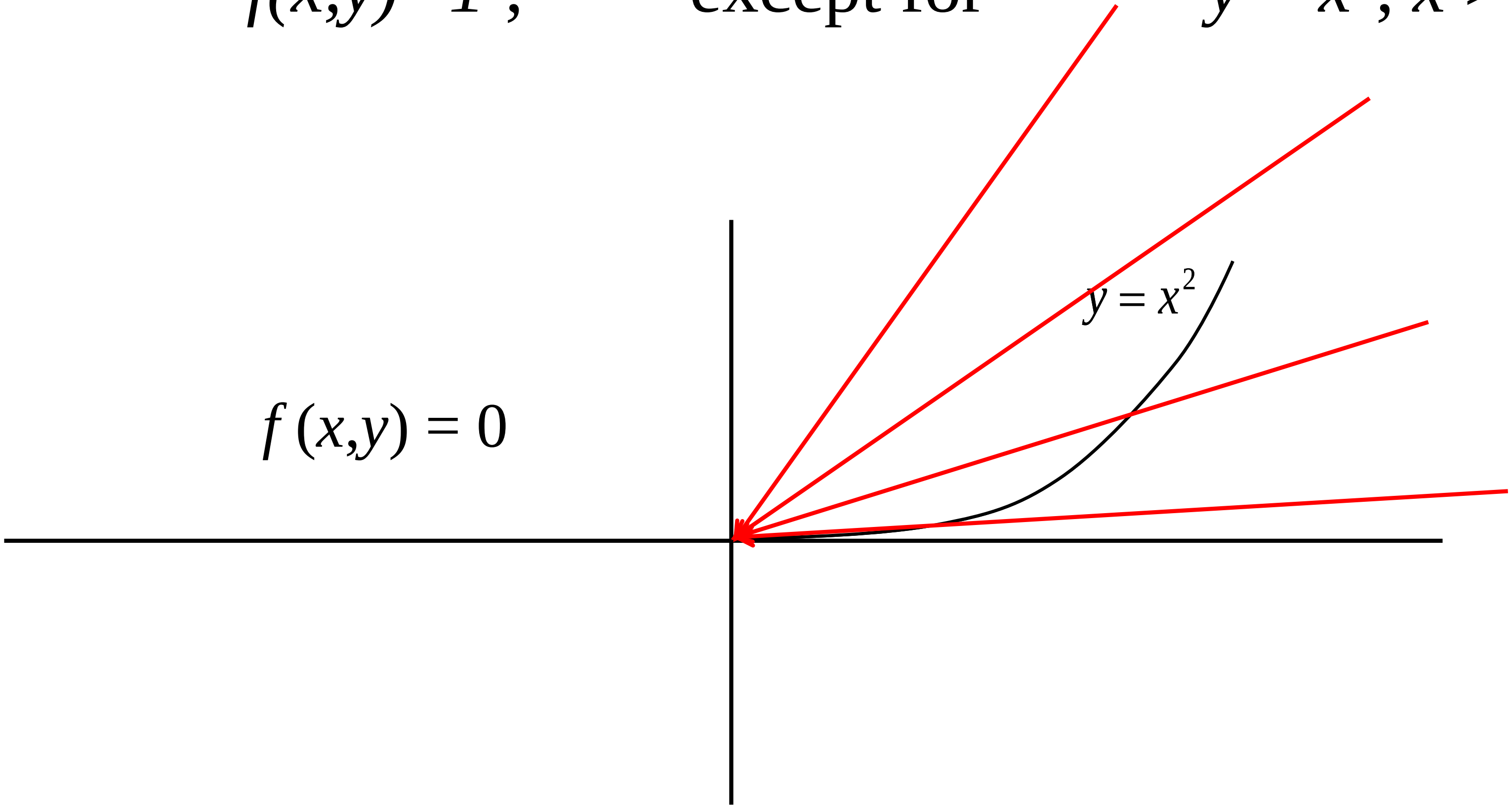
$$\lim_{(x,y)\to(0,0)} f(x,y) = ???$$

Limits in $\boldsymbol{R}^2$

Having the same limit along all half lines approaching (x_0,y_0) does not imply a limit exists at (x_0,y_0).

Example: $f(x,y) = 0$, everywhere

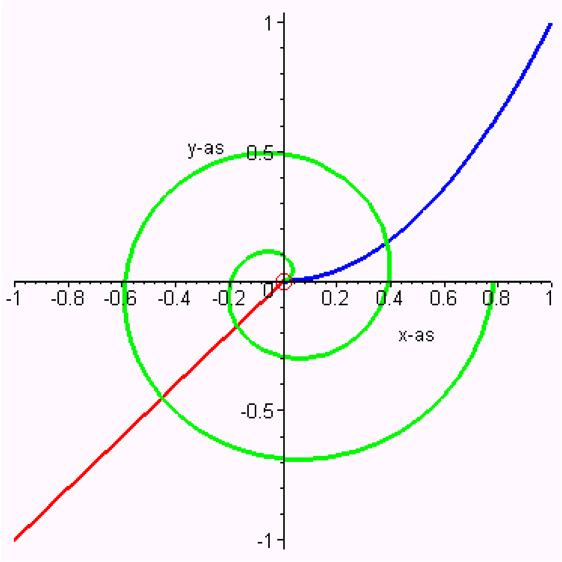
$f(x,y)=1$, except for $y = x^2$, $x > 0$



Concluding:

- For all ***paths*** in $\mathbf{R}^2$ towards the limit point (a,b), the function values should tend to the same value L.

- **Hence:** if two paths lead to different values, then the limit does not exist.



Example:

Three different paths to (0,0)

Continuity in $\mathbf{R}^2$

Continuity in $\boldsymbol{R}^2$

Definition:

A function f: $\mathbf{R}^2 \to \mathbf{R}$ is called <u>continuous</u> at (x_0, y_0) if:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Continuity in $\boldsymbol{R}^2$

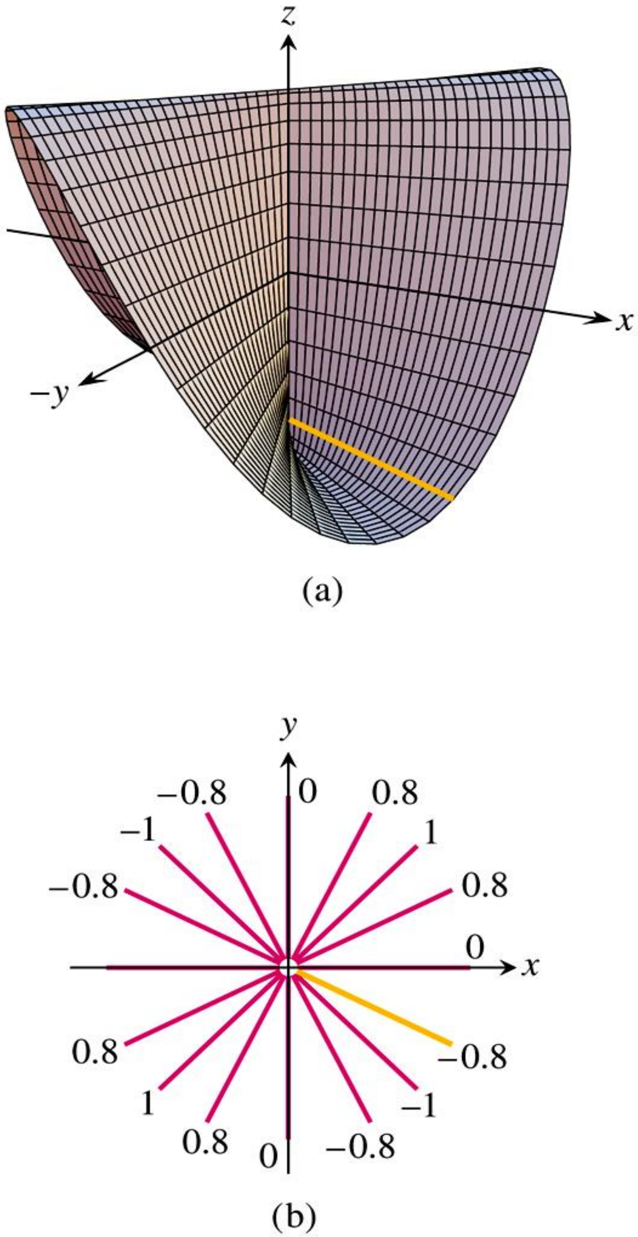


FIGURE 14.13 (a) The graph of

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

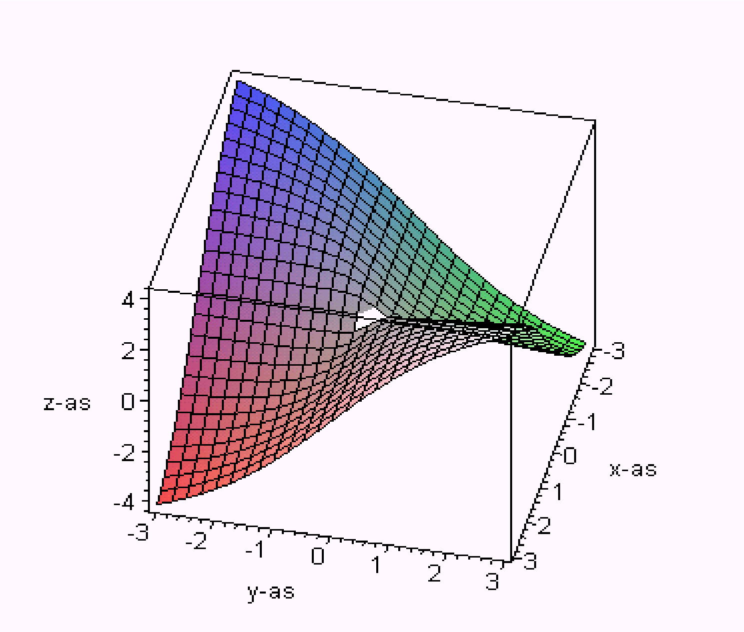
The function is continuous at every point except the origin. (b) The values of f are different constants along each line $y = mx, x \neq 0$ (Example 5).

Polar coordinates

Polar coordinates

Example: calculate $$\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}}$$

(if the limit exists)



Graph of $f(x,y) = \dfrac{2xy}{\sqrt{x^2+y^2}}$

Polar coordinates

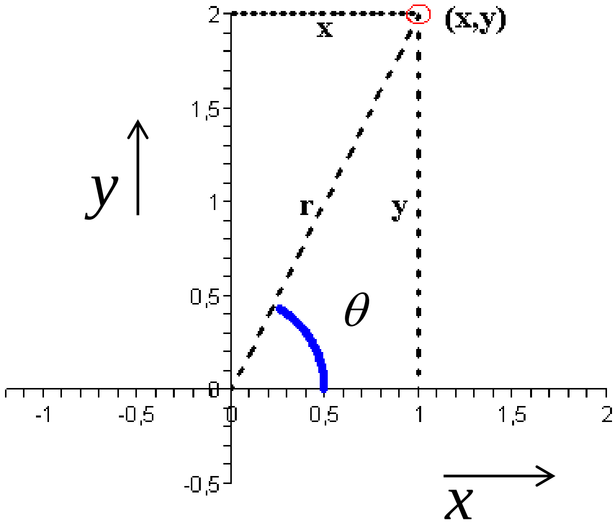
$$\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}}$$

Solution: use ***polar coordinates***:

r = distance to (0,0)

So: $r = \sqrt{x^2+y^2}$

θ = angle with the positive x-axis



So: $x = r \cdot \cos(\theta), \quad y = r \cdot \sin(\theta)$

If $(x,y)\to(0,0)$

Then: $r \to 0^+$

So: $$\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{2r\cos\theta \cdot r\sin\theta}{r}$$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}}$$

Solution: (continued)

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{2r\cos\theta \cdot r\sin\theta}{r}$$

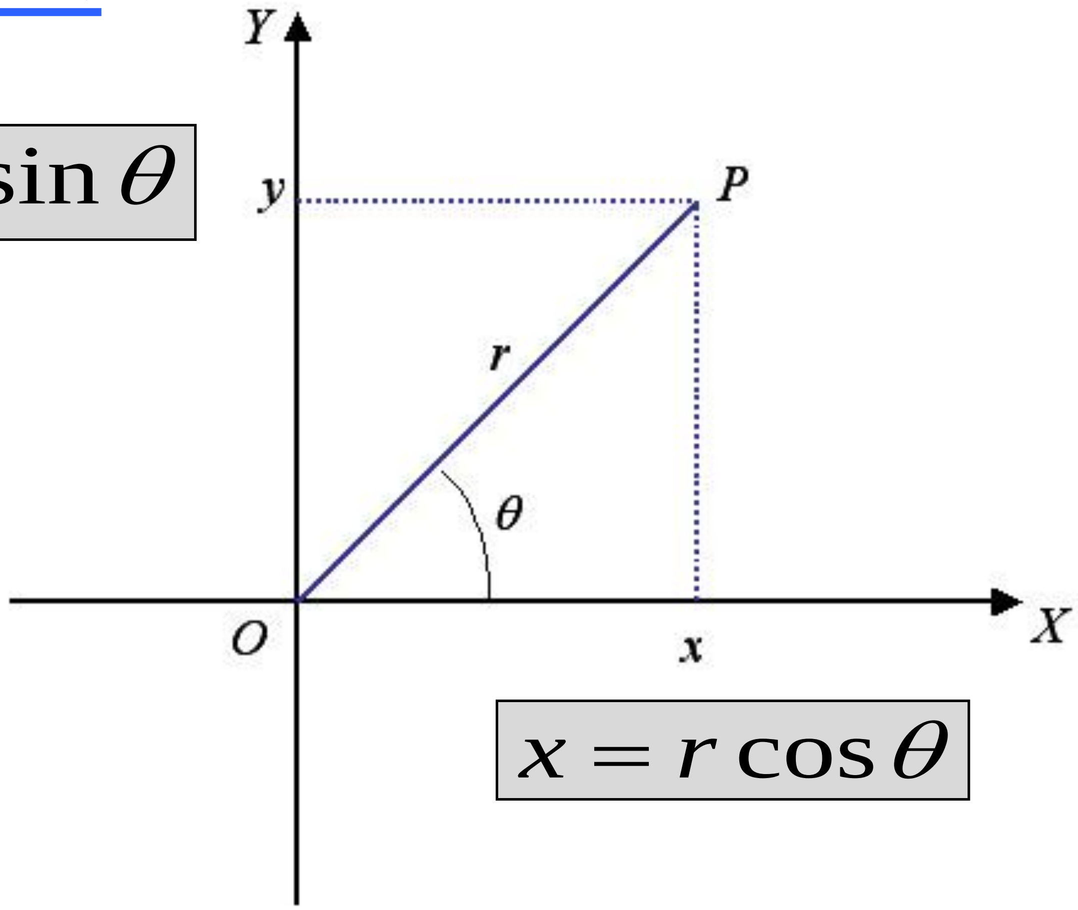
$$= \lim_{r\to 0^+} 2r\cos\theta\sin\theta = 0 \qquad \text{(independent of } \theta\text{)}$$

Conclusion: $\displaystyle \lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = 0$

Polar coordinates (see Thomas, page 781)

Remember:

$$y = r\sin\theta$$



$$x = r\cos\theta$$

Useful for limits of the form:

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{(x^2+y^2)^k}$$

Break

Functions of several variables

Theme: Basic concepts

Theme: Continuity

Theme: Partial derivatives

- Definitions
- How to calculate?
- Higher order derivatives

Theme: Tangent plane

Definition of partial derivatives

Definition of partial derivatives

Definition:

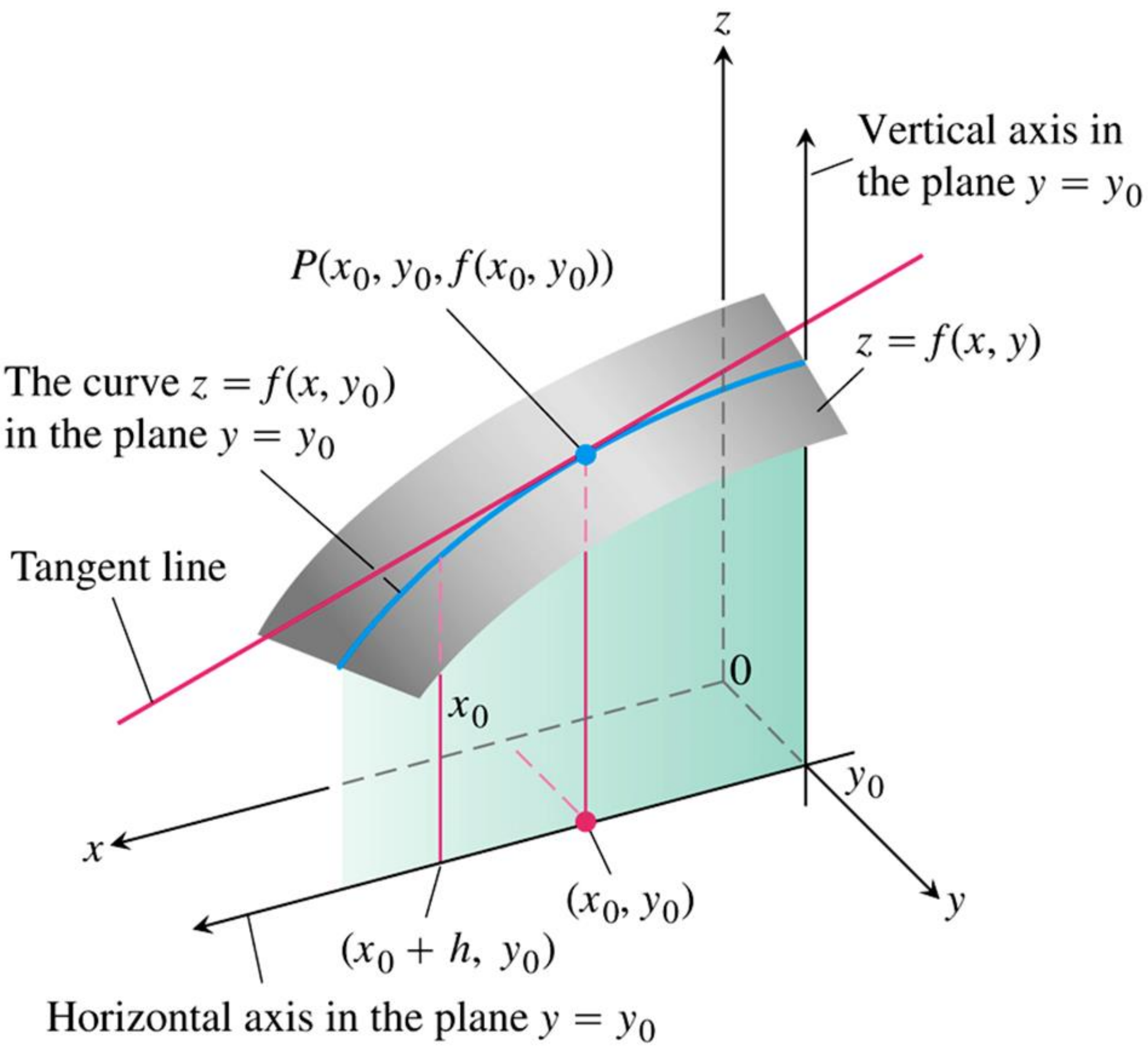
The partial derivative with respect to x of the function $f:\mathbf{R}^2 \to \mathbf{R}$, at the point (x_0, y_0) is given by:

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

Compare: the definition of derivate $f:\mathbf{R} \to \mathbf{R}$:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Definition of partial derivatives



Definition of partial derivatives

Interpretations

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$f_x(x_0, y_0) =$

- the rate of change of $f(x_0, y_0)$ if x changes ($x = x_0 \;\rightarrow\; x = x_0 + h$) and y remains constant ($y = y_0$).

- the slope of the tangent line to the graph of f in the point $(x_0, y_0, f(x_0, y_0))$ “in the x-direction”.

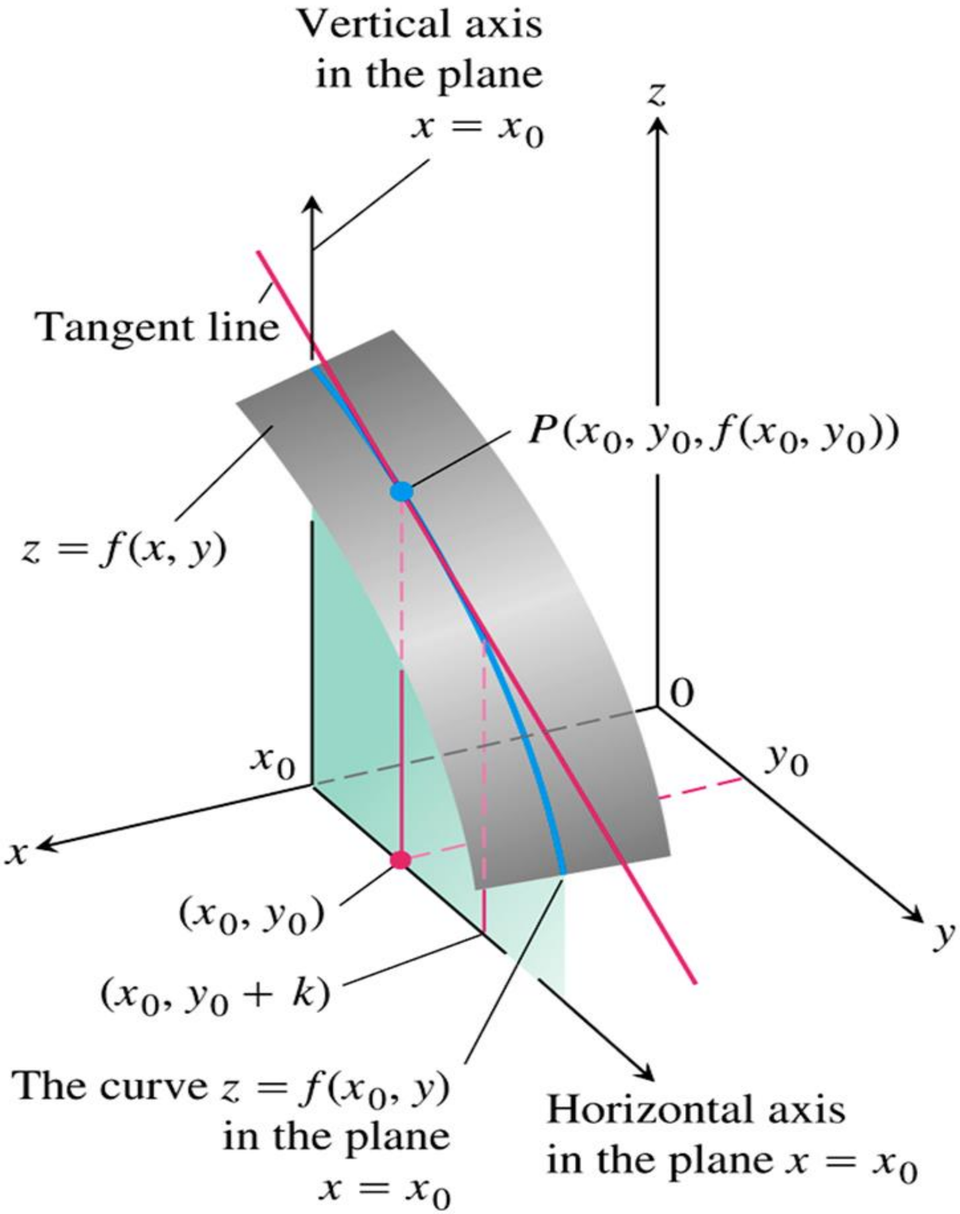
Definition of partial derivatives

Definition:

The partial derivative with respect to y of the function $f : \mathbf{R}^2 \to \mathbf{R}$, at the point (x_0, y_0) is given by:

$$f_y(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

Definition of partial derivatives



Definition of partial derivatives

Interpretations

$$f_y(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

$f_y(x_0, y_0) =$

- the rate of change of $f(x_0, y_0)$ if y changes ($y = y_0 \;\to\; y = y_0 + k$) and x remains constant ($x = x_0$).

- the slope of the tangent line to the graph of f in the point $(x_0, y_0, f(x_0, y_0))$ “in the y-direction”.

Definition of partial derivatives

Notations

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Also: $$f_y(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

How to calculate partial derivatives?

How to calculate partial derivatives?

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Higher order partial derivatives

Higher order partial derivatives

EXAMPLE 9 If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Higher order partial derivatives

THEOREM 2—The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Functions of several variables

Theme: Basic concepts

Theme: Continuity

Theme: Partial derivatives

Theme: Tangent plane

- Tangent plane
- Linearization

The tangent plane

Remember:

f: $\mathbf{R}^2 \to \mathbf{R}$ leads to a surface in $\mathbf{R}^3$, the ***graph*** of f:

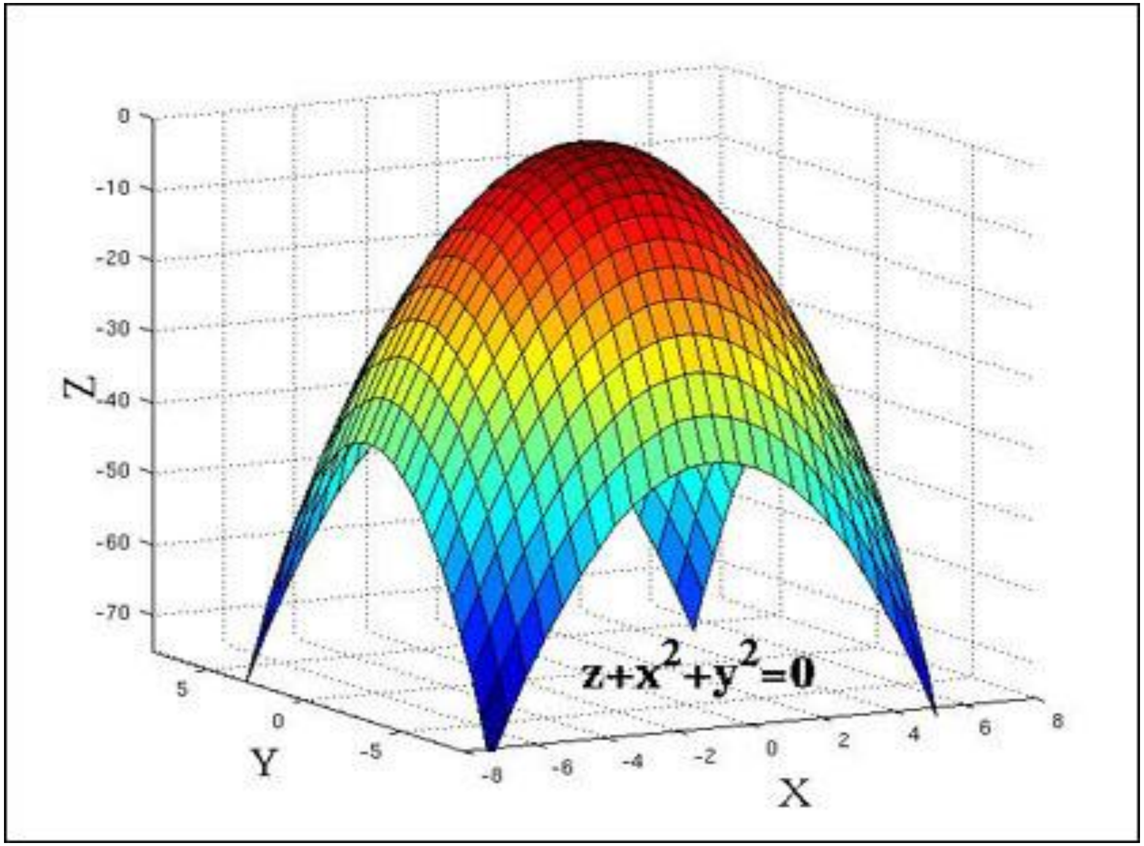
$$z = f(x,y)$$

Or: if $F(x,y,z) = z - f(x,y)$,
then graph of f is given by: $F(x,y,z) = 0$

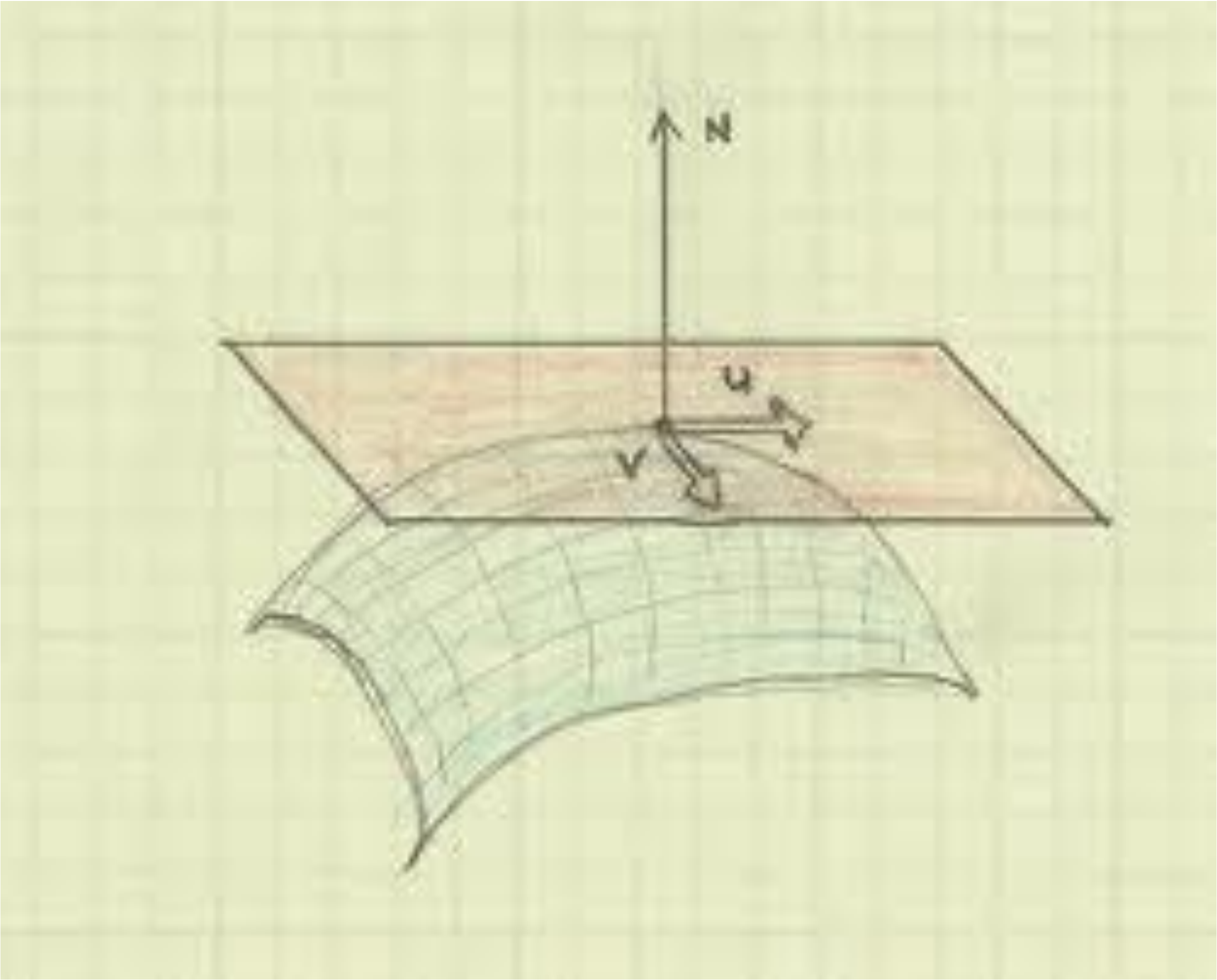
The tangent plane

Example: $f(x,y) = -x^2 - y^2$

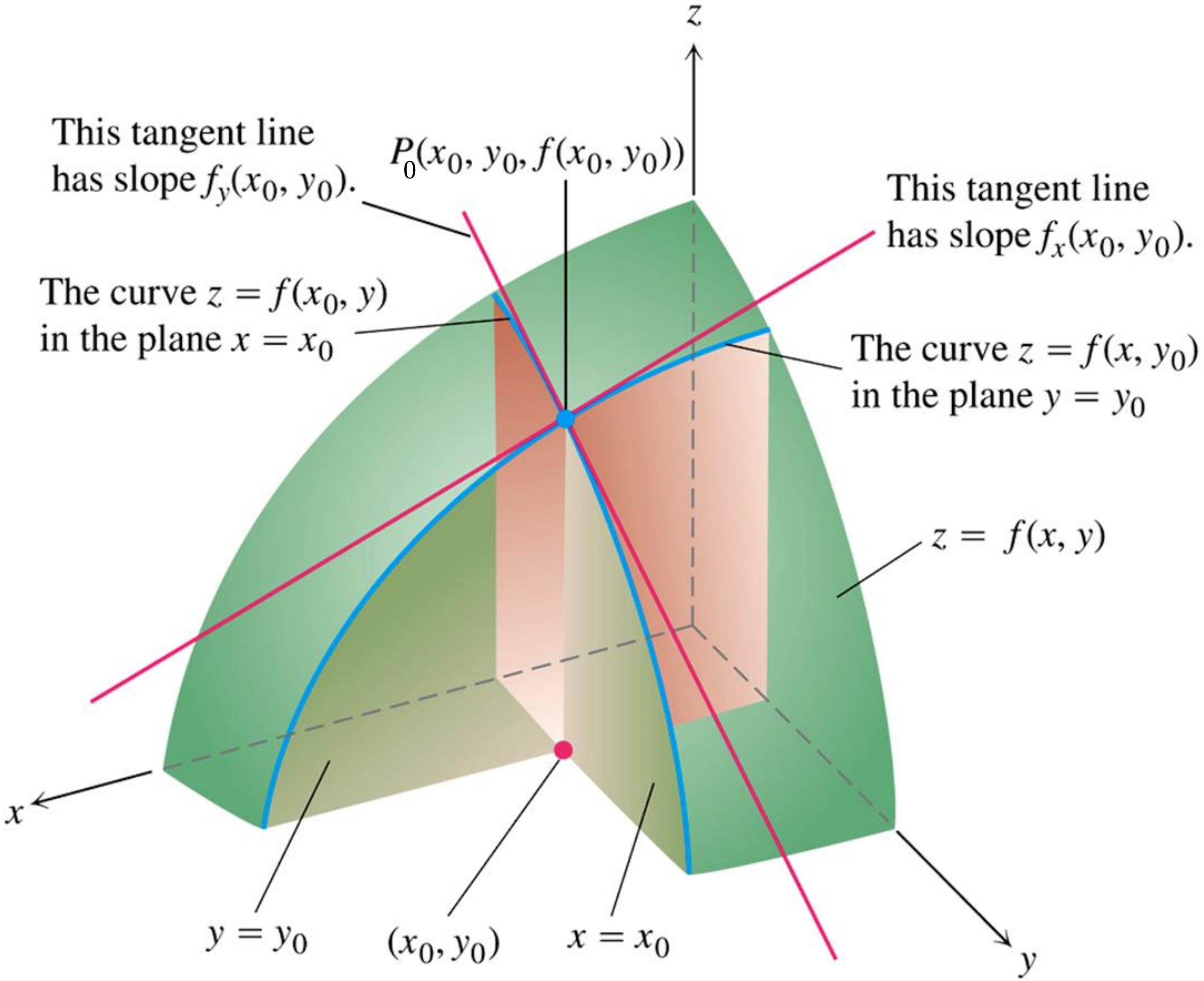
$z = f(x,y)$ is the graph $f: z + x^2 + y^2 = 0$



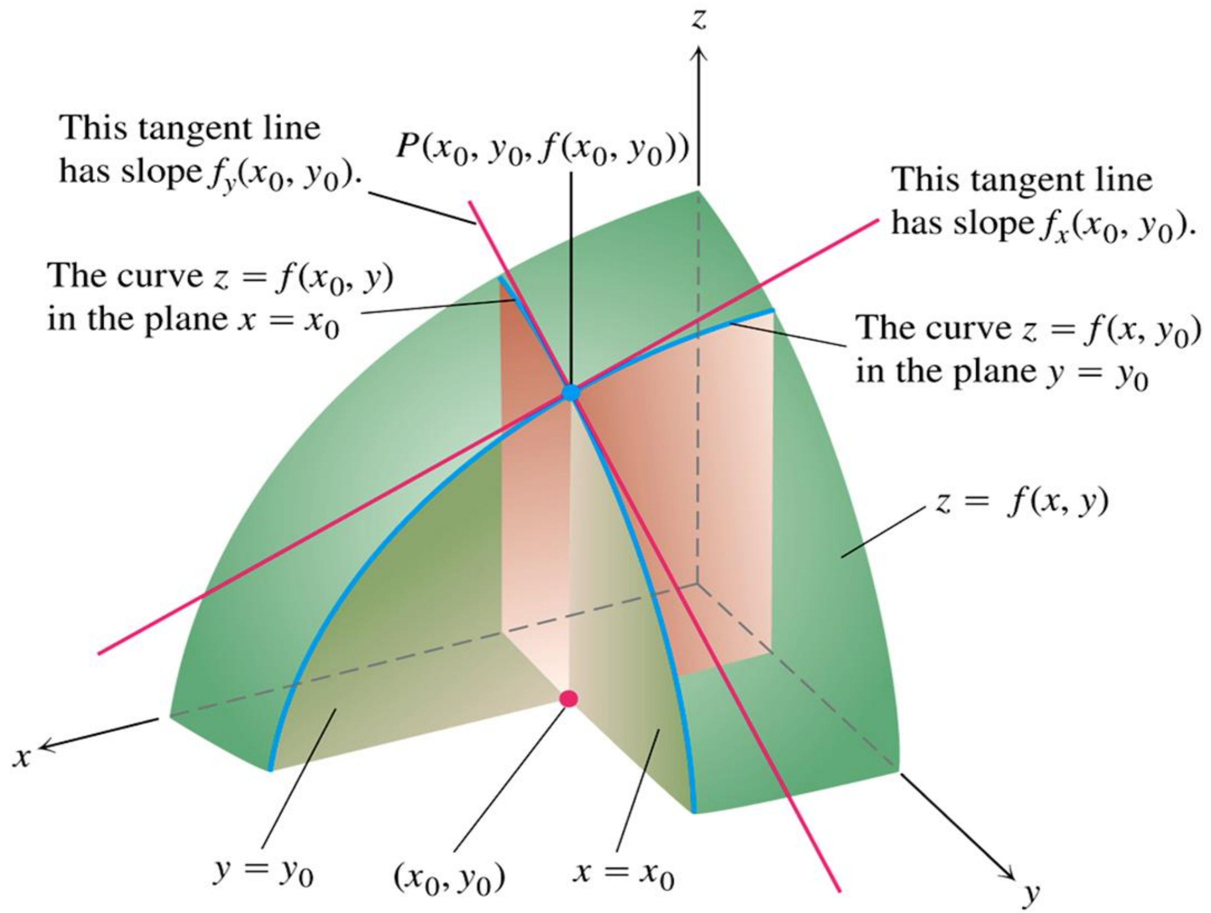
What is the tangent plane?



The tangent plane at $\mathbf{P}_0 = (x_0, y_0, z_0)$



The tangent plane at **P** = (x_0, y_0, z_0)



The tangent plane is spanned by:

$$\mathbf{i} + f_x(x_0, y_0)\mathbf{k} \quad \text{and} \quad \mathbf{j} + f_y(x_0, y_0)\mathbf{k}$$

The tangent plane at $\mathbf{P_0} = (x_0,y_0,z_0)$

The vectors **u** and **v** span the tangent plane at :

$$\mathbf{u} = \mathbf{i} + f_x(x_0, y_0)\mathbf{k} \quad \text{and} \quad \mathbf{v} = \mathbf{j} + f_y(x_0, y_0)\mathbf{k}$$

Hence: **n** = **v** × **u** is orthogonal to the tangent plane:

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$$

Hence $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ is in the tangent plane if:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{P_0}) = 0$$

Or:

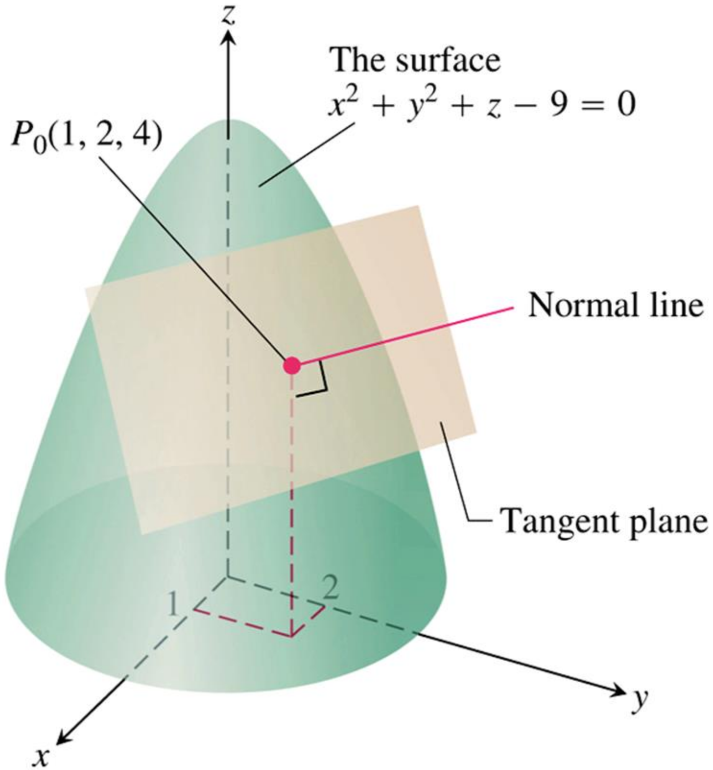
$$f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0) - (z - z_0) = 0$$

The tangent plane

Theorem: The equation for the tangent plane to the graph of f at the point $\mathbf{P_0}=(x_0, y_0, z_0)$ is:

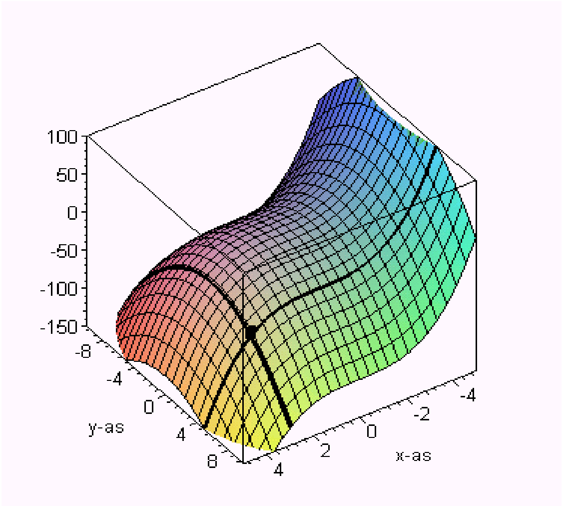
$$z - z_0 = f_x(x_0, y_0)\cdot(x - x_0) + f_y(x_0, y_0)\cdot(y - y_0)$$

Here $z_0 = f(x_0, y_0)$.

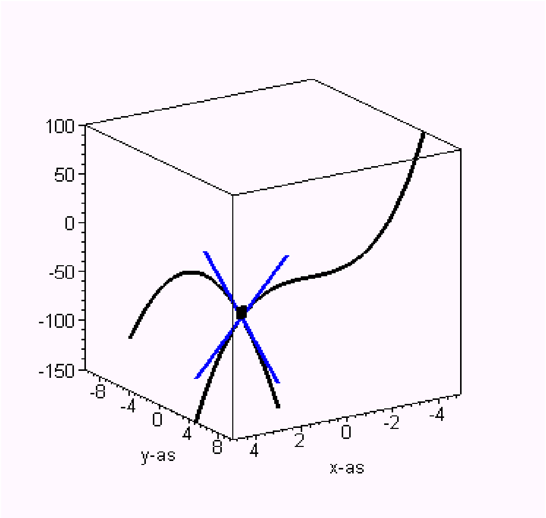


Example

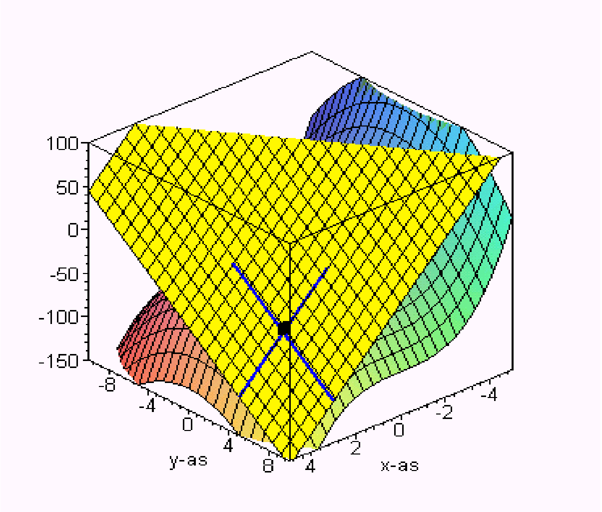
Find the equation for the tangent plane to the graph of the function $f(x,y) = -x^3 - y^2$ at the point (x_0,y_0,z_0) = (3,5,–52).



Graph of *f* and the point (3,5,–52)



Partial derivative at the point (3,5)



Tangent plane at (3,5,–52)

The tangent plane

$$f(x,y) = -x^3 - y^2$$

$$(x_0, y_0, z_0) = (3, 5 - 52)$$

Solution: Equation tangent plane:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

with: $x_0 = 3 \quad y_0 = 5 \quad z_0 = f(x_0, y_0) = f(3,5) = -3^3 - 5^2 = -52$

and: $f_x(x,y) = -3x^2$ so: $f_x(x_0, y_0) = f_x(3,5) = -3 \cdot 3^2 = -27$

$f_y(x,y) = -2y$ so: $f_y(x_0, y_0) = f_y(3,5) = -2 \cdot 5 = -10$

Hence: $$z + 52 = -27(x - 3) - 10(y - 5)$$

or: $$z = -27x - 10y + 79$$

Linearization

Linearization

Equation: for the tangent plane to the surface $z = f(x,y)$:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The tangent plane is the graph of the function $L(x, y)$:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The function $L(x, y)$ is called the *linearization* of f at (x_0, y_0).

Around (x_0, y_0) the linearization is an approximation of f :

$$f(x, y) \approx L(x, y)$$

Linearization

Example: $f(x,y) = -x^3 - y^2$ and $(x_0, y_0) = (3,5)$

Tangent plane in (3,5,–52) is given by:

$$z + 52 = -27(x-3) - 10(y-5)$$

So: $L(x,y) = -52 - 27(x-3) - 10(y-5)$

The linear approximation of $f(3.1, 4.95)$:

$$L(3.1, 4.95) \quad = -52 - 27(3.1-3) - 10(4.95-5) \quad = -54.2$$

Exact value:

$$f(3.1, 4.95) \quad = -(3.1)^3 - (4.95)^2 \quad = -54.2935$$

Functions of several variables

Theme: Basic concepts

Theme: Continuity

Theme: Partial derivatives

Theme: Tangent plane

Summarizing Excercise

The function $f : R^2 \to R$ is given by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

(a) Investigate continuity of *f* in *(0,0)*

(b) Determine the tangent plane to the graph of *f* in *(1,2,2/5)*

Mathematics B2: Newton

- Contents -

- ❑ Limits and continuity
- ❑ Derivatives and applications
- ☑ Functions of several variables

- ❑ Integrals
- ❑ Calculation techniques for integrals
- ❑ Power and Taylor series