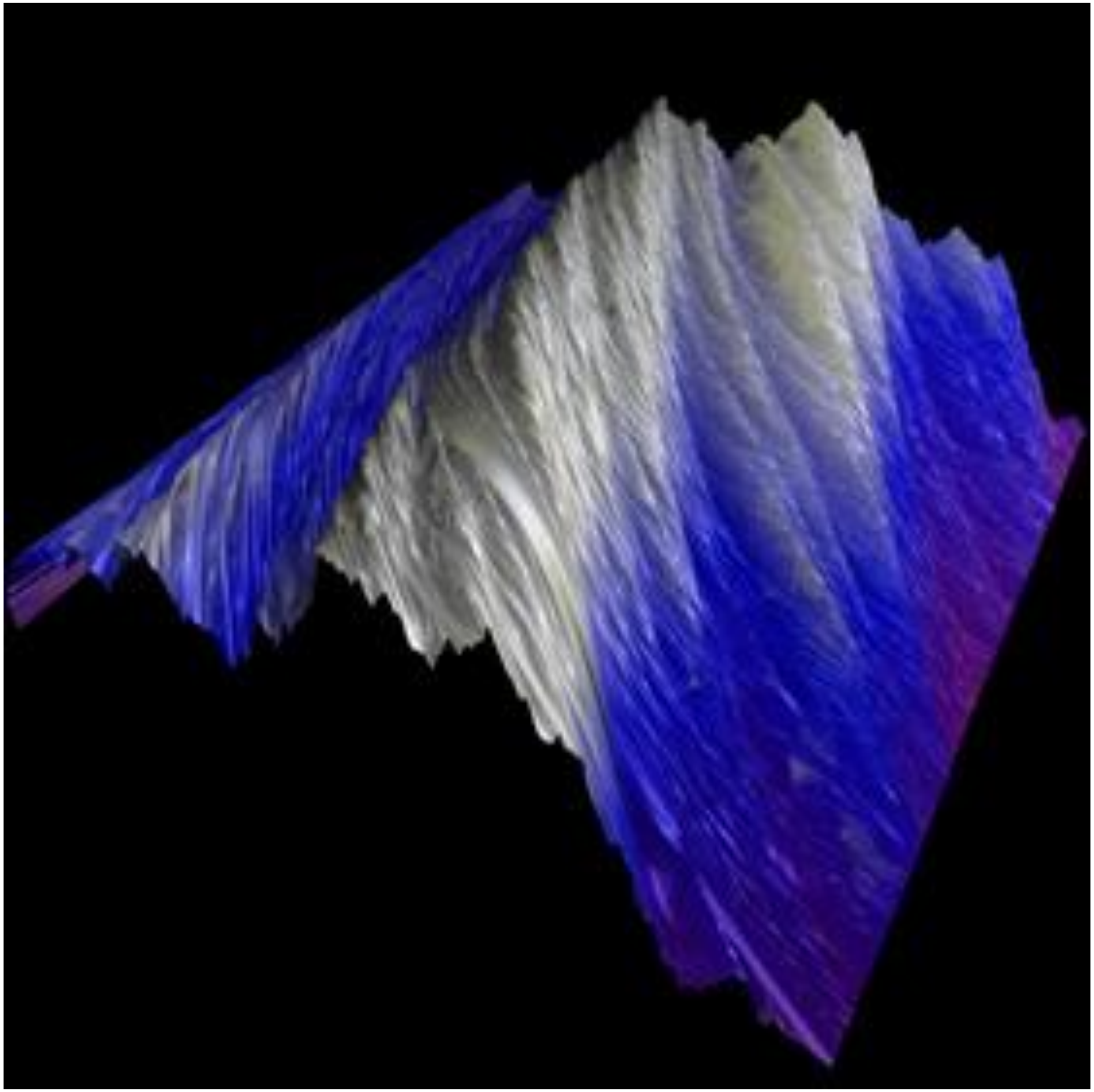
Mathematics B2: Newton



Mathematics B2: Newton

- Contents -

- ❑ Limits and continuity
- ❑ ✓ Derivatives and applications
- ❑ Functions of 2 variables

- ❑ Integrals
- ❑ Calculation techniques for integrals
- ❑ Power and Taylor series

Derivatives and applications

Based on ***parts of*** Chapters 3+4

(see the course schedule and the study guide for the relevant parts!)

Derivatives and applications

Theme: Defining derivatives
- Definition of derivative
- Borrowing money
- Differentiability and continuity

Theme: Calculating derivatives

Theme: Extrema of functions

Theme: L'Hôpital's rule

3.1

Definition of derivative

Definition of derivative

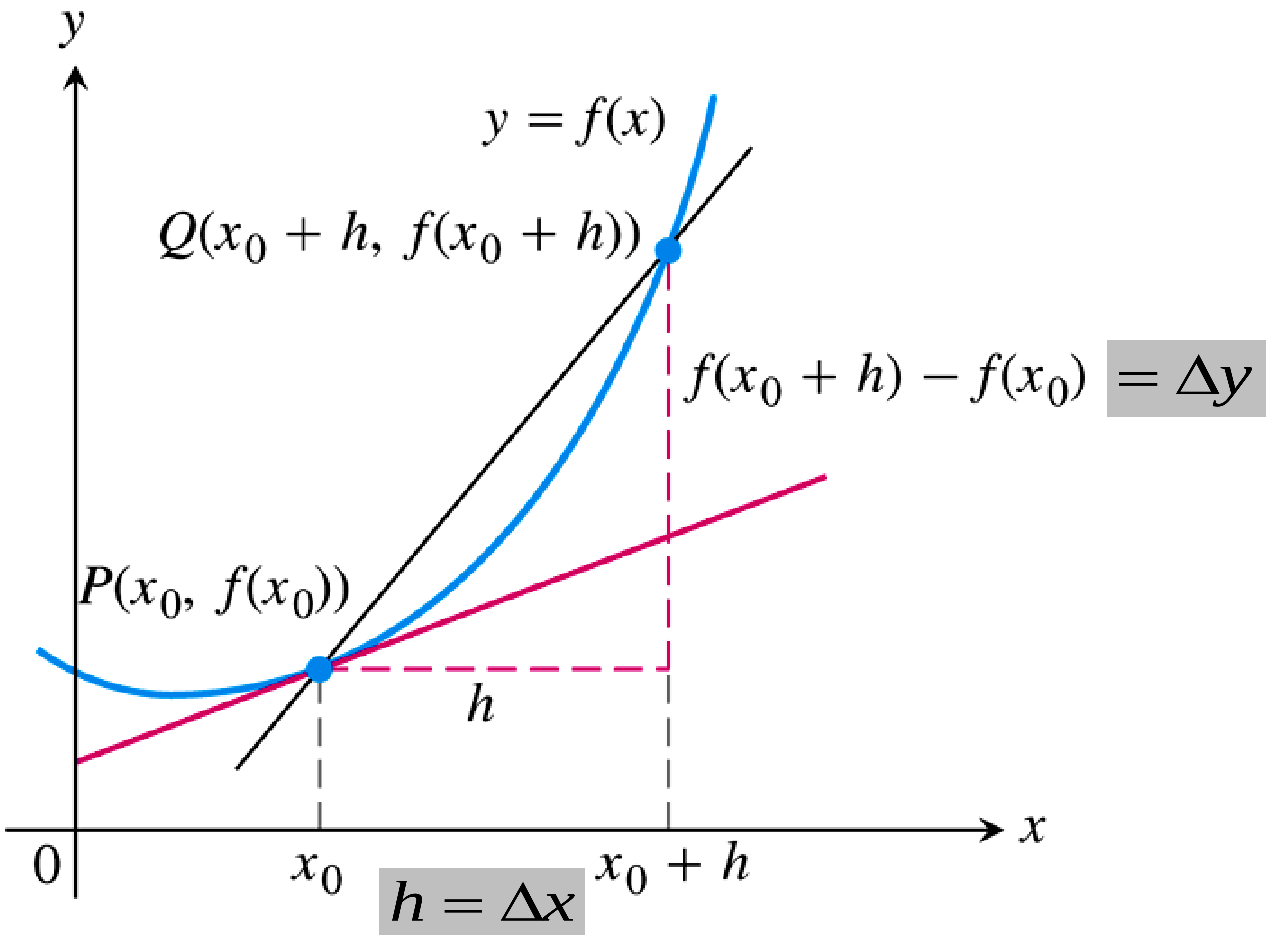


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

Definition of derivative

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

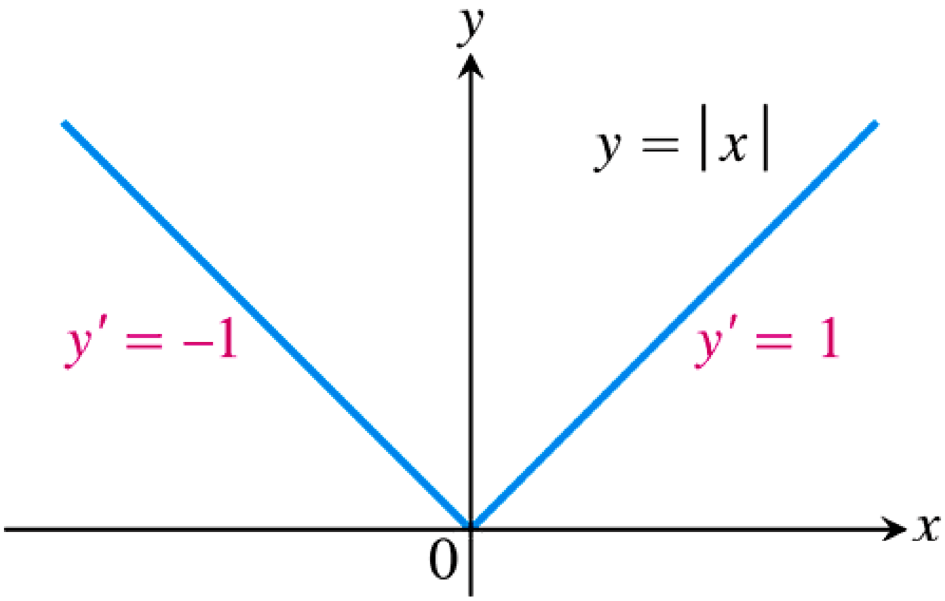
The **tangent line** to the curve at P is the line through P with this slope.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

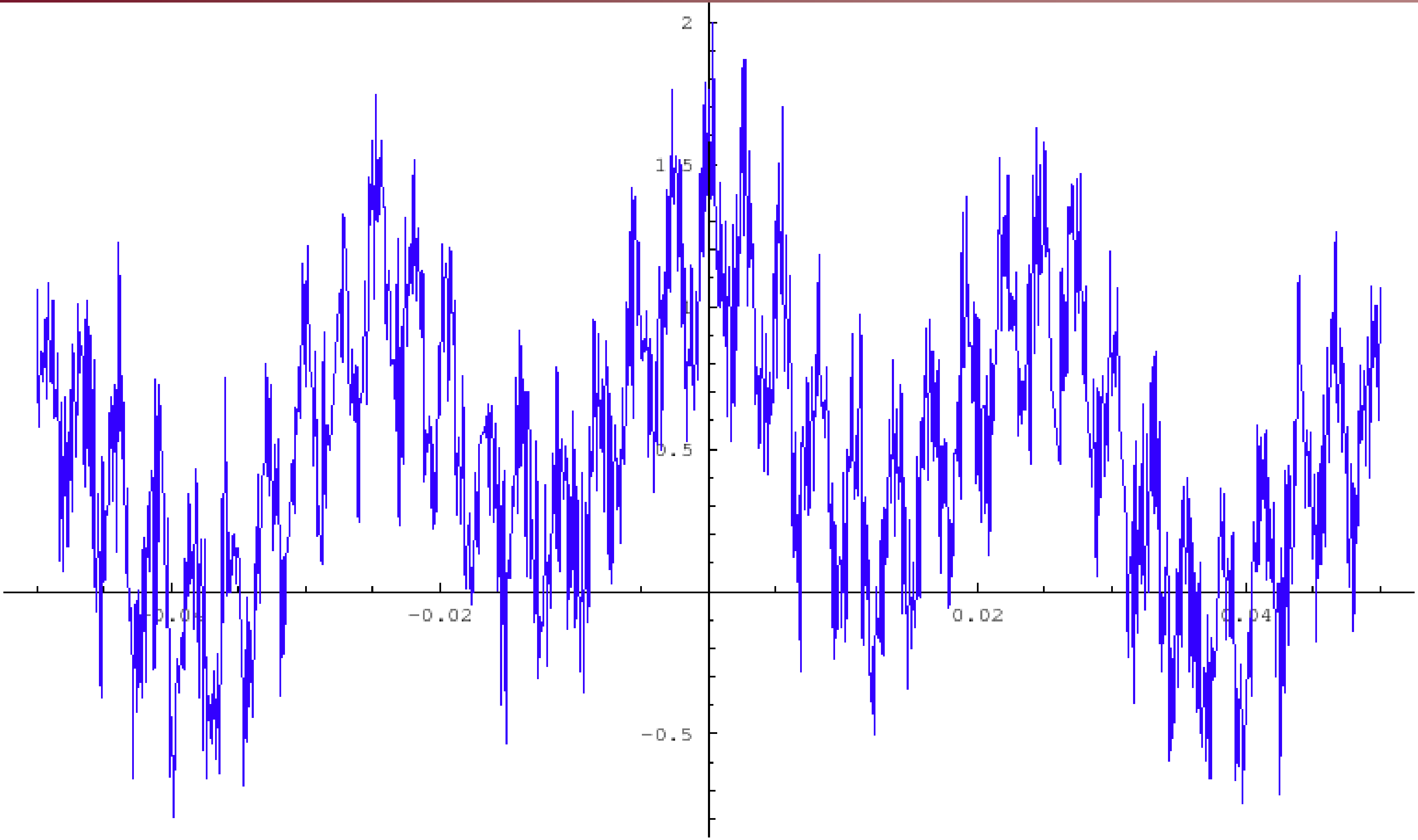
provided this limit exists.

Definition of derivative



Is the function $f(x) = |x|$ continuous?

The carnival function



There are functions that are continuous *everywhere,*

but nowhere differentiable …

Borrowing money

Borrowing money

Borrowing money from the maffia …

Unfortunately you are in desperate need of 1 million dollars for an investment with a pay-back of 5 million $ after a year.
You decide to ask your "friend" Liugi Lucardi to lend you this amount. He asks an interest rate of 100% per year which you reluctantly have to accept.

But then Liugi starts changing the terms of the arrangement…

Borrowing money

Borrowing money from the maffia …

Arrangement 1: 100% interest/year:

You pay back: 1 + 1 million.

Arrangement 2: 50% interest/half year:

You pay back:

$$1 + 0.5 + (1 + 0.5) \cdot 0.5 = (1 + 0.5)^2 = 2.25 \text{ million.}$$

Arrangement 3: 25 % interest/quarter:

You pay back:

$$(1 + 0.25)^4 = 2.44 \text{ million.}$$

You fear a new arrangement (per month/per day/…) which will cost you even more. Is there an end?

Borrowing money

We need to calculate: $\lim_{x \to 0} (1+x)^{1/x}$

$$\lim_{x \to 0} (1+x)^{1/x} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\frac{\ln(1+x)}{x}}$$

We will prove on the next slide that: $\lim_{x \to 0} \left(\frac{\ln(1+x)}{x} \right) = 1$

Then:

$$\lim_{x \to 0} (1+x)^{1/x} = \lim_{x \to 0} e^{\frac{\ln(1+x)}{x}} = e^{\lim_{x \to 0} \left(\frac{\ln(1+x)}{x} \right)} = e^{1} = e$$

Borrowing money

Left to prove: $\lim_{x \to 0} \left(\frac{\ln(1+x)}{x} \right) = 1$

If $f(x) = \ln(1+x)$ then $f'(0) = 1$

Also: $f'(0) = \lim_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h} = \lim_{x \to 0} \frac{\ln(1+x)}{x}$

So indeed: $\lim_{x \to 0} \left(\frac{\ln(1+x)}{x} \right) = 1$

Borrowing money

Borrowing money from the maffia …

Arrangement n: 100/n % interest/nth part of the year:
You pay back:
$(1 + 1/n)^n < 2.7183$ million.

You have some profit left…

Differentiability and continuity

Differentiability implies continuity

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof: Suppose f is differentiable in c, which means:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

Then:

$$\lim_{x \to c} \left(f(x) - f(c)\right) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) =$$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} \left(x - c\right) = f'(c).0 = 0$$

So indeed: $\lim_{x \to c} f(x) = f(c).$ $\square$

Question

$$f(x) = \begin{cases} x & \text{als} \quad x < 1 \\ 2\sqrt{x} & \text{als} \quad x \geq 1 \end{cases}$$

Question

(a) f is continuous, but ***not*** differentiable at 1.

(b) f is continuous and also differentiable at 1.

(c) f is ***not*** continuous and ***not*** differentiable at 1.

(d) f is differentiable, but ***not*** continuous at 1.

Diff. implies cont..

$$f(x) = \begin{cases} x & \text{als} \quad x < 1 \\ 2\sqrt{x} & \text{als} \quad x \geq 1 \end{cases}$$

Solution:

If $x < 1$ then: $f'(x) = 1$ So: $\lim\limits_{x \to 1^-} f'(x) = 1$

If $x > 1$ then:

$$f'(x) = \left(2x^{\frac{1}{2}}\right)' = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$$

Hence: $\lim\limits_{x \to 1^+} f'(x) = 1$

So f is differentiable in $x = 1$? **No !!!**

Differentiability implies continuity

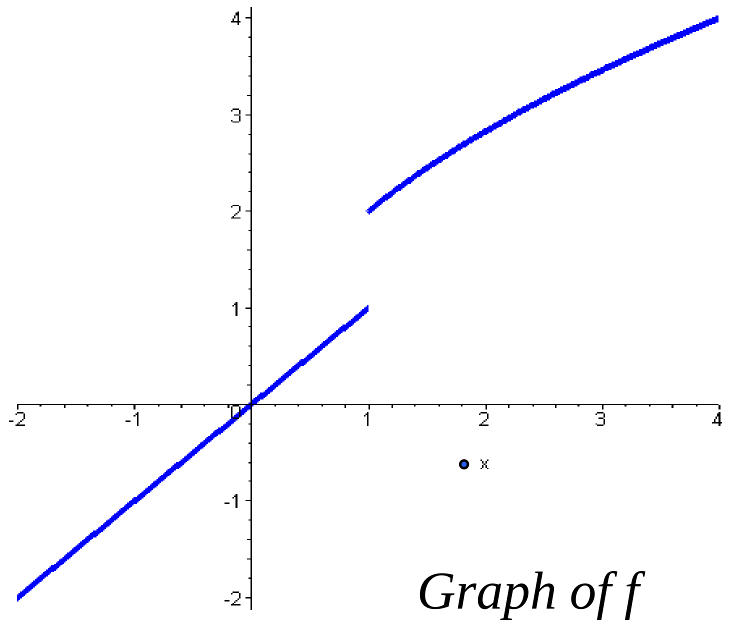
Solution (continued):

$$f(x) = \begin{cases} x & \text{als} \quad x < 1 \\ 2\sqrt{x} & \text{als} \quad x \geq 1 \end{cases}$$

The left-hand and right-hand limits of $f'(x)$ for $x = 1$, give no information on $f'(1)$!!

f is not continuous in $x = 1$, so surely not differentiable !!

(Theorem 1: Diff -> Cont, so,
not Cont -> not Diff)



Graph of f

Derivatives and applications

Theme: Defining derivatives

Theme: Calculating derivatives

- Differentiation rules
- Linear approximation
- Differentials

Theme: Extrema of functions

Theme: L'Hôpital's rule

The chain rule

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

The chain rule

Example Calculate: $\frac{d}{dx}(x^x)$

Solution: $F(x) = x^x = e^{x\ln x}$

Now: $F(x) = (f \circ g)(x)$

with $g(x) = x\ln x$ and $f(x) = e^x$

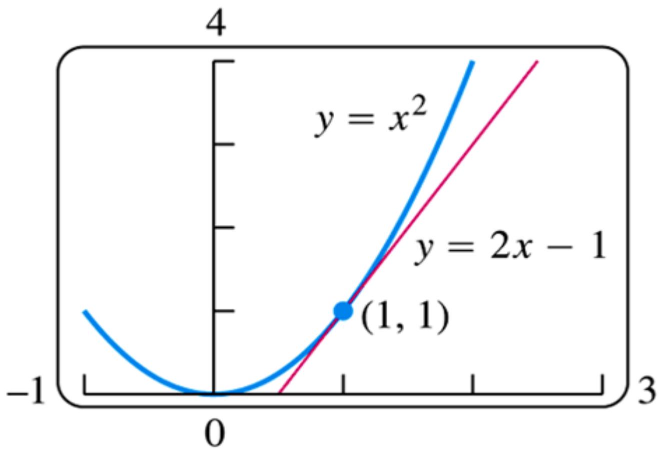
So: $\frac{d}{dx}(F(x)) = f'(g(x)) \cdot g'(x)$

$$= e^{x\ln x} \cdot (\ln x + x \cdot \frac{1}{x}) = (1 + \ln x)x^x$$

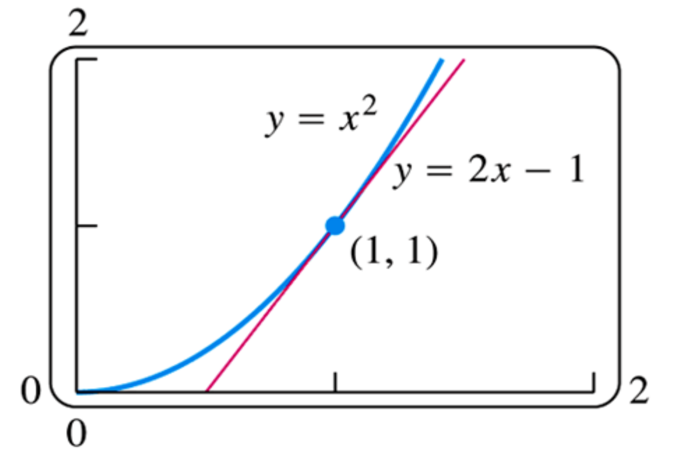
3.11

Linearization and Differentials

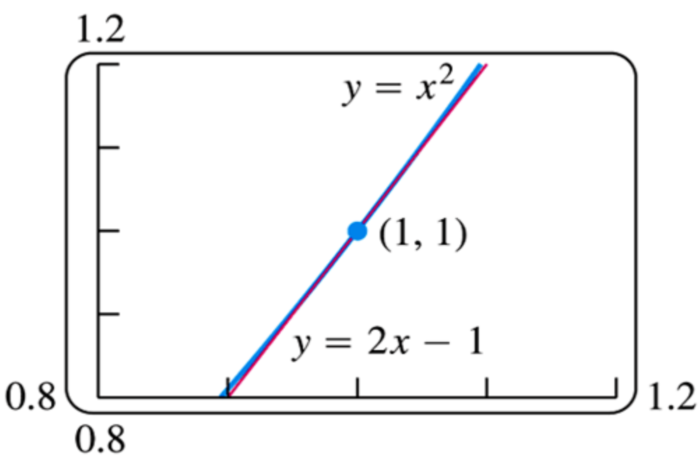
Linearization and Differentials



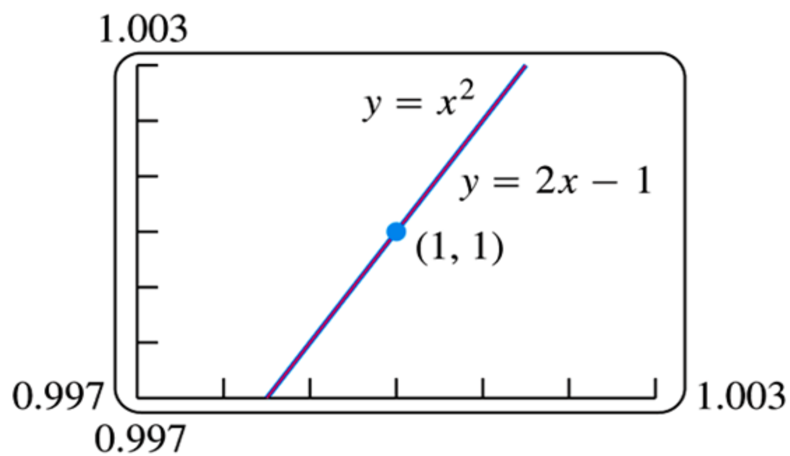
$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).



Tangent and curve very close near (1, 1).



Tangent and curve very close throughout entire x-interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

Definition

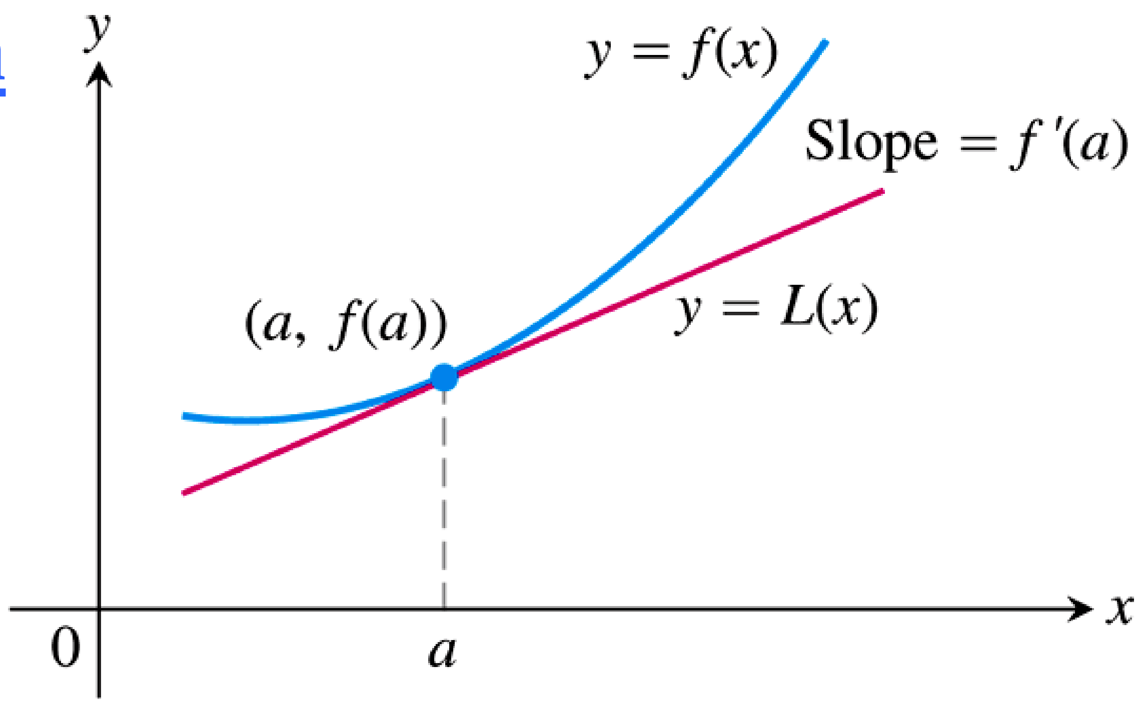


FIGURE 3.50 The tangent to the curve $y = f(x)$ at $x = a$ is the graph of the function $L(x) = f(a) + f'(a)(x - a)$.

L is called the ***linearization*** of f at $x = a$.

Example

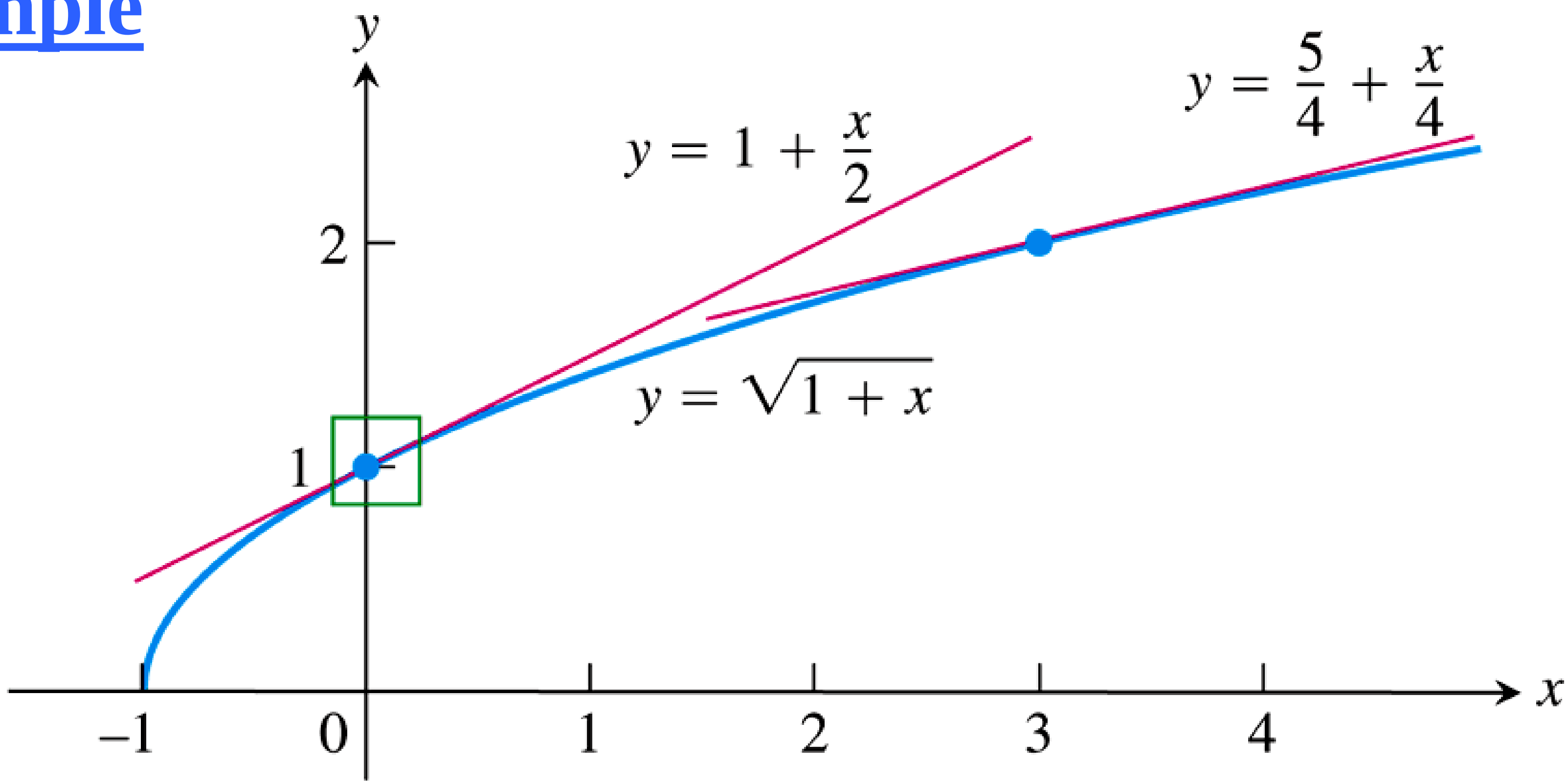


FIGURE 3.51 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.52 shows a magnified view of the small window about 1 on the y-axis.

Example
(continued)

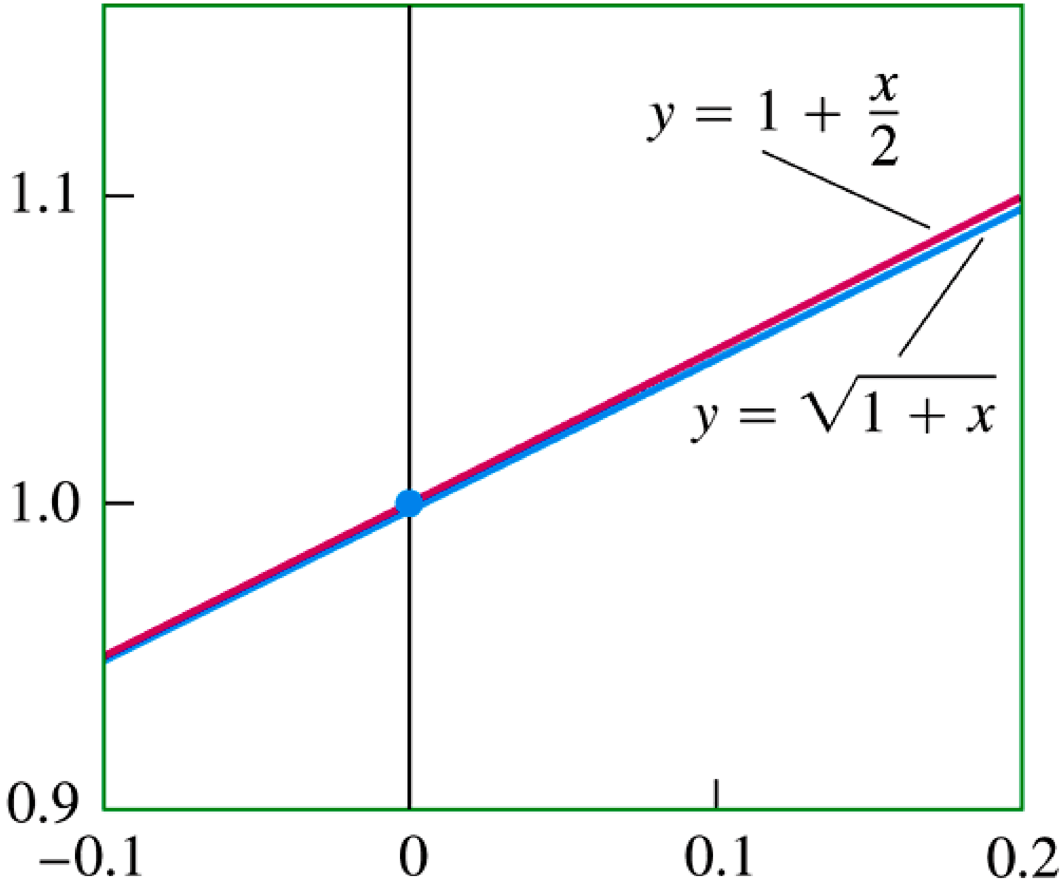


FIGURE 3.52 Magnified view of the window in Figure 3.51.

Example (continued)

The linearisation can be used for approximating the function!

$$f(x) = \sqrt{1+x}$$

$$L(x) = 1 + \tfrac{1}{2} x$$

Approximation	True value	$\lvert \text{True value} - \text{approximation} \rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

Linearization and Differentials

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(x) - f(a) \approx f'(a)(x-a)$$

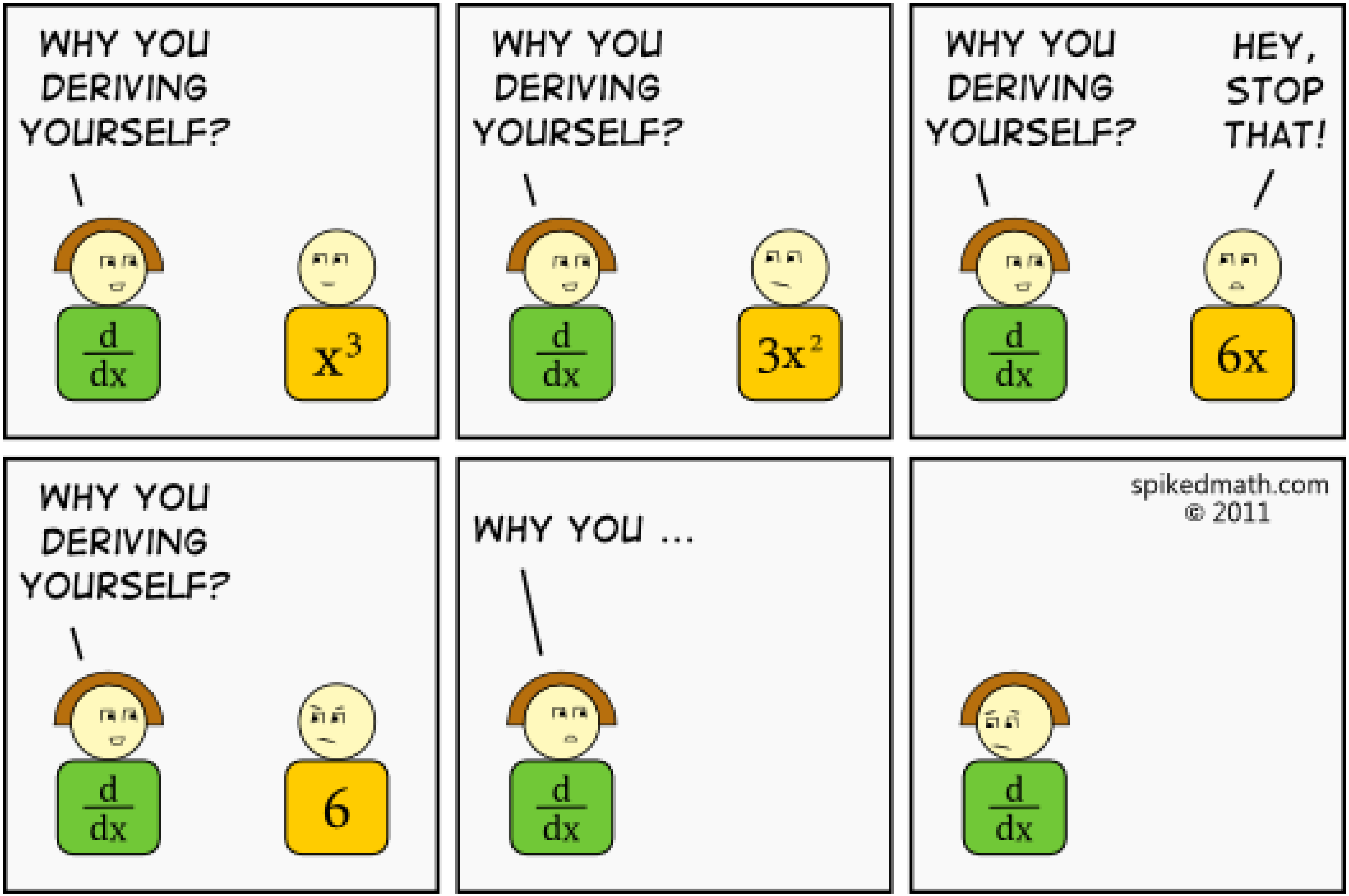
$$\Delta f(a) \approx f'(a)\Delta x(a)$$

$$\Delta f \approx f'\Delta x$$

Definition The ***differential*** of f is given by

$$df = f'dx$$

Break


WHY YOU DERIVING YOURSELF?
d/dx
x^3
WHY YOU DERIVING YOURSELF?
d/dx
$3x^2$
WHY YOU DERIVING YOURSELF?
HEY, STOP THAT!
d/dx
6x
WHY YOU DERIVING YOURSELF?
d/dx
6
WHY YOU ...
d/dx
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d/dx

Derivatives and applications

Theme: Defining derivatives

Theme: Calculating derivatives

Theme: Extrema of functions

- Absolute extrema
- Extreme value theorem
- Local extrema
- How to find absolute extrema?

Theme: L'Hôpital's rule

Absolute extrema

Absolute extrema

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Absolute extrema

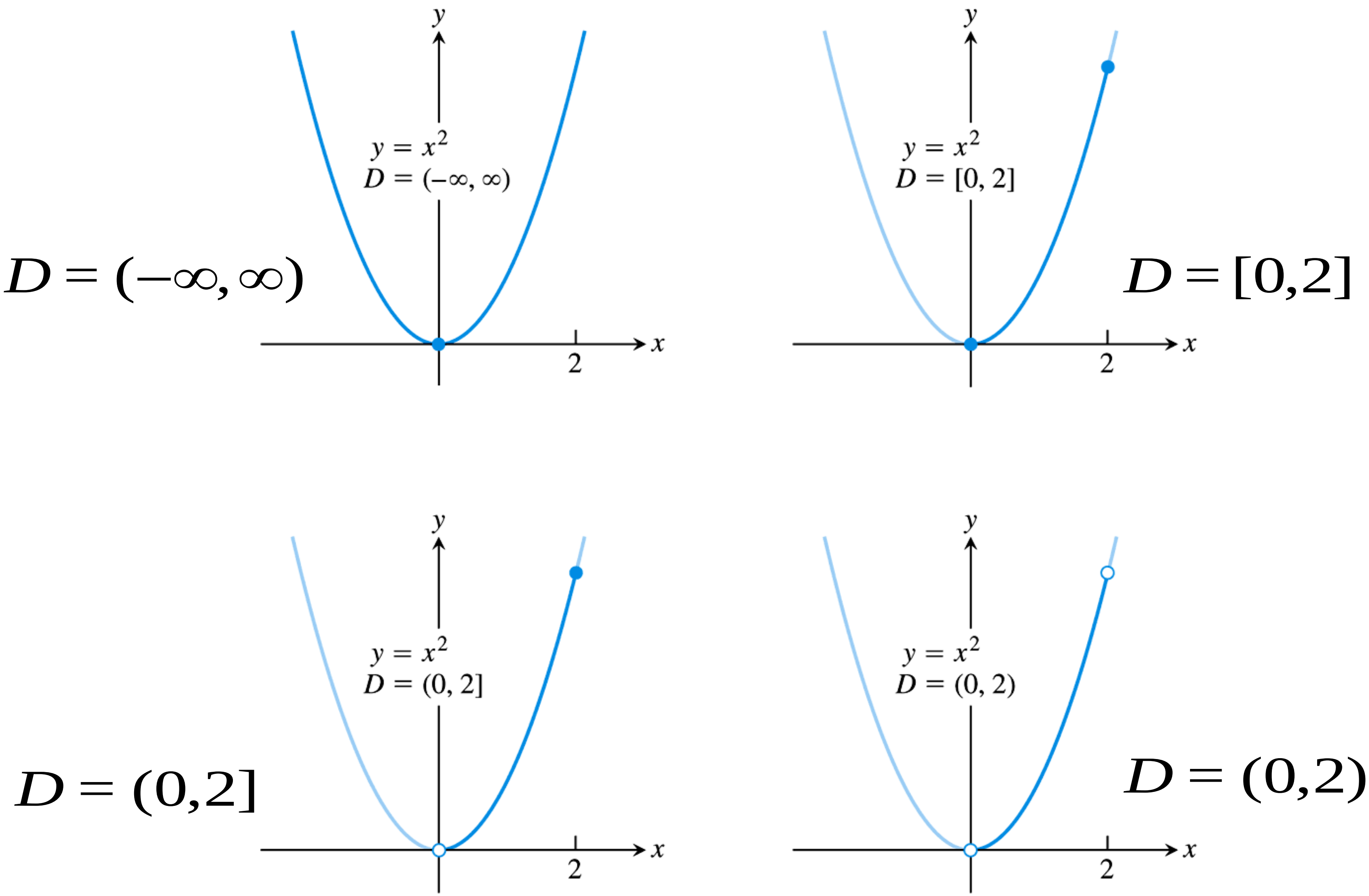


FIGURE 4.2 Graphs for Example 1.

The Extreme Value theorem

The Extreme Value theorem

Theorem 1 – The Extreme Value Theorem
If f is continuous on a closed interval $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m.

Remark: Due to the intermediate value theorem the ***range*** of f is $[m, M]$.

For which x are these absolute extrema attained?

Many situations possible …

The Extreme Value theorem

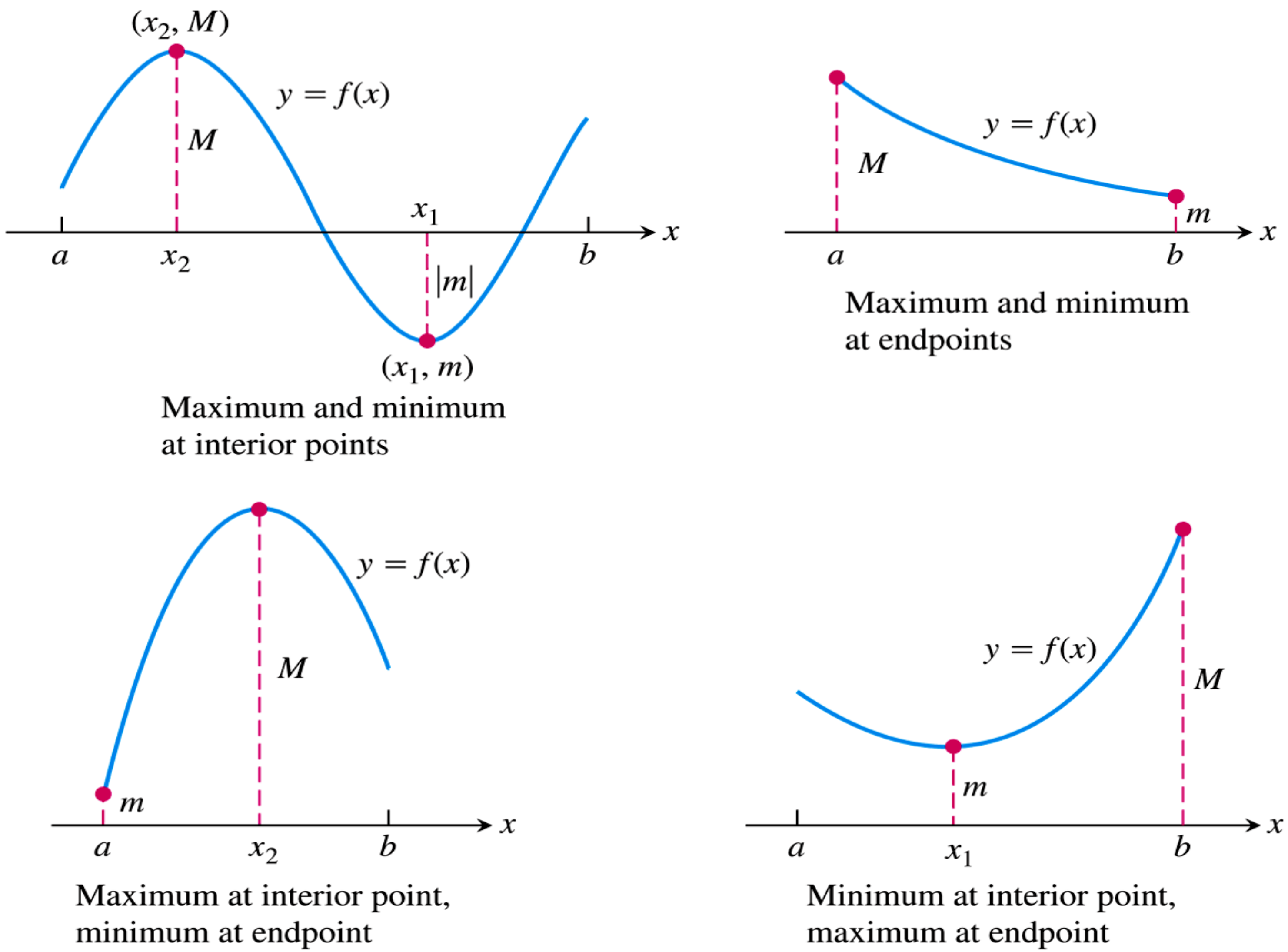


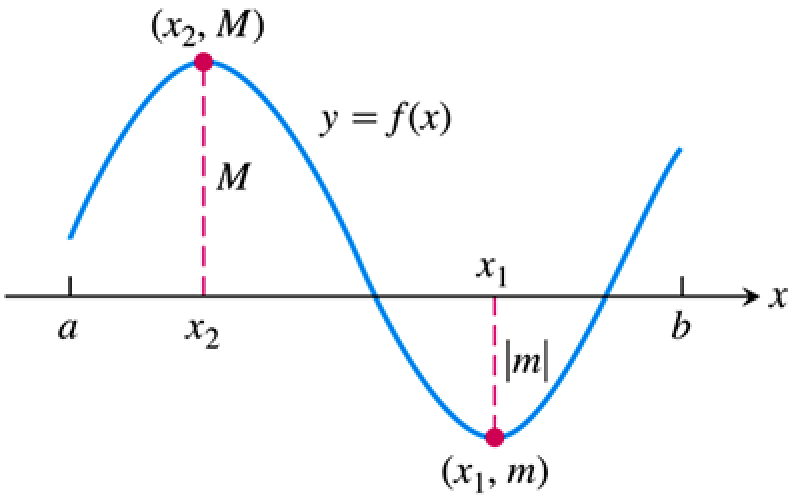
FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

Local extrema

Local extrema

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.



Maximum and minimum
at interior points

Local minima at x_1 ***and*** a
Local maxima at x_2 ***and*** b

Local extrema

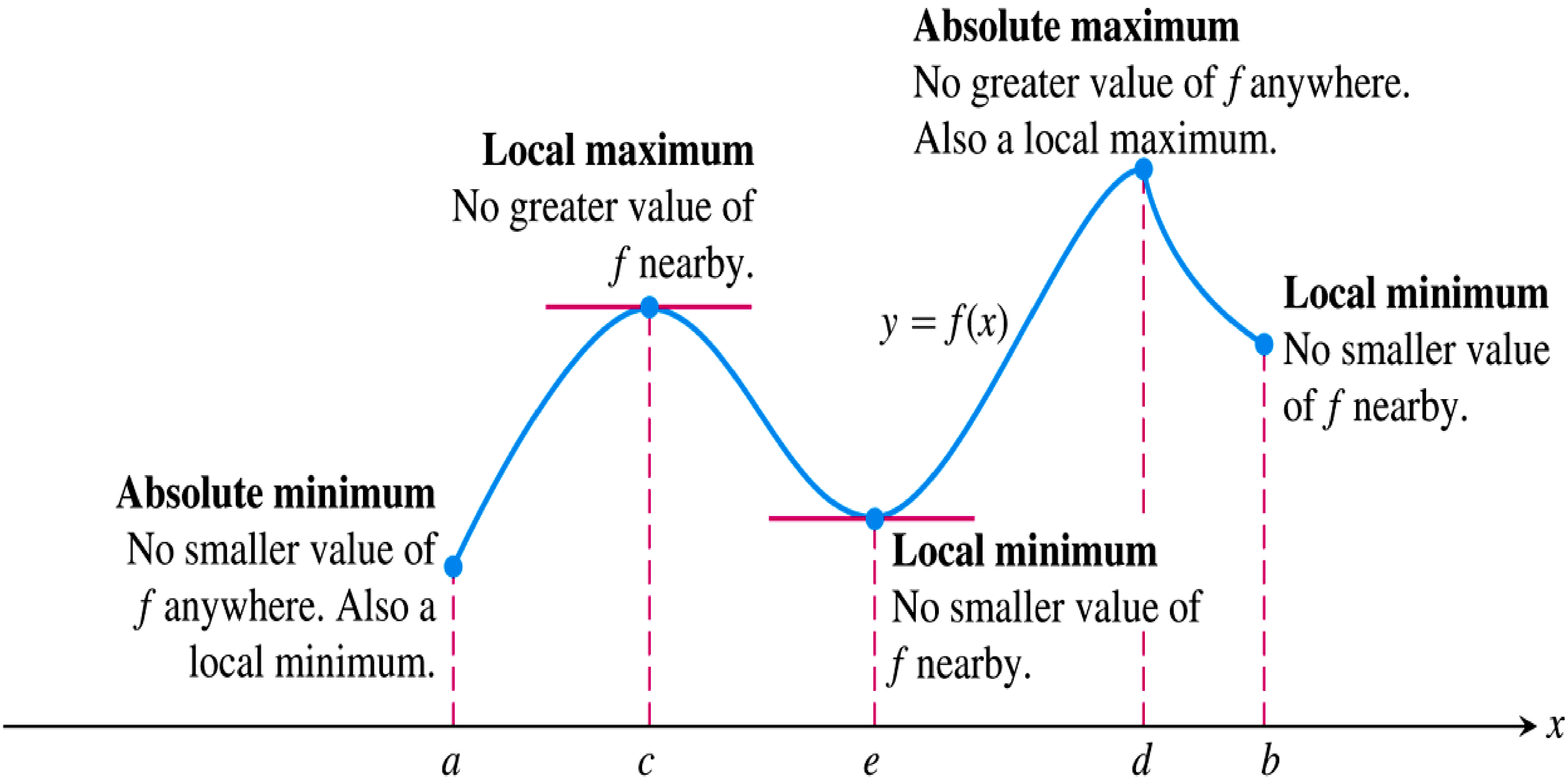


FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

Local extrema

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

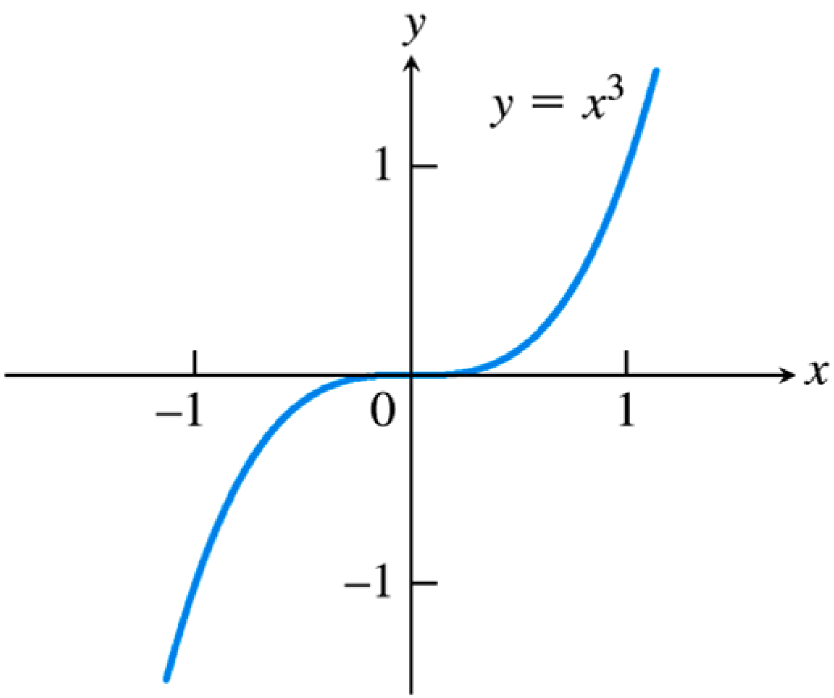
Pierre de Fermat (1601-1665)

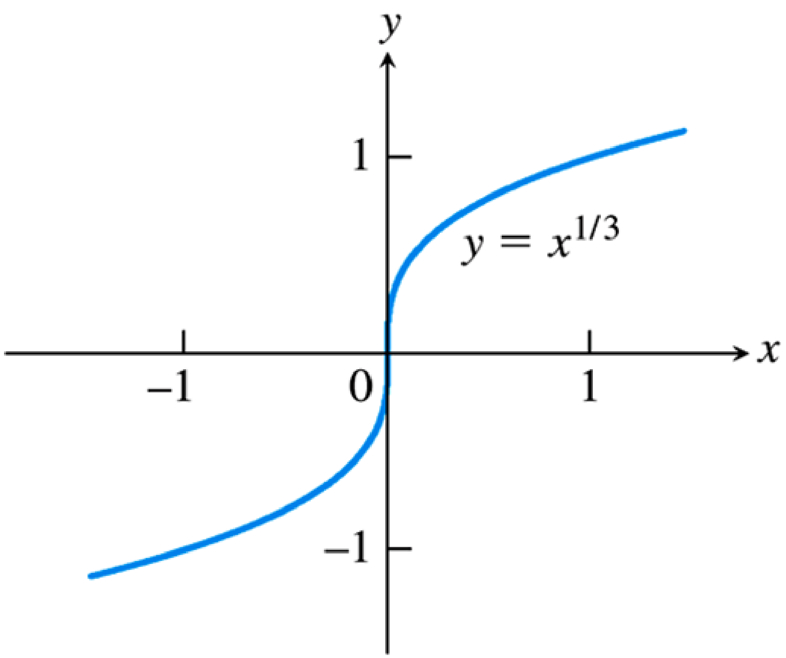
How to find (absolute) extrema on [a,b] ?

How to find extrema ?

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

A critical point need ***not*** to be a local maximum or minimum.





$f'(0) = 0$ (point of inflection)

$f'(0)$ doesn't exist

How to find extrema ?

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval $[a,b]$

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

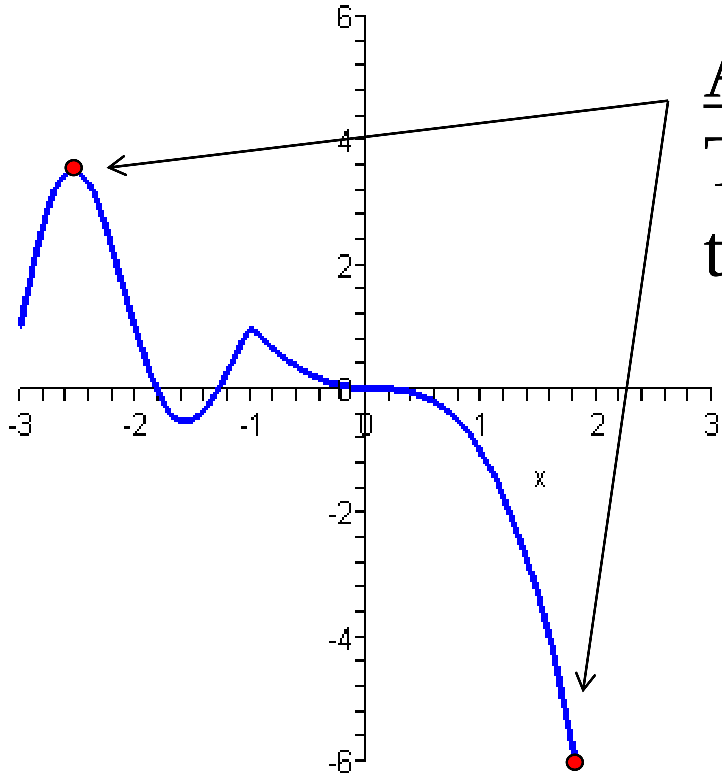
Procedure (more detailed)

- Find all points in (a, b) where f' does not exist.
- Find all $c \in (a, b)$ where $f'(c) = 0$.
- Calculate the function values in the previous points and a and b.
- The largest value is the absolute maximum.
- The smallest value is the absolute minimum.

How to find extrema ?

Procedure:

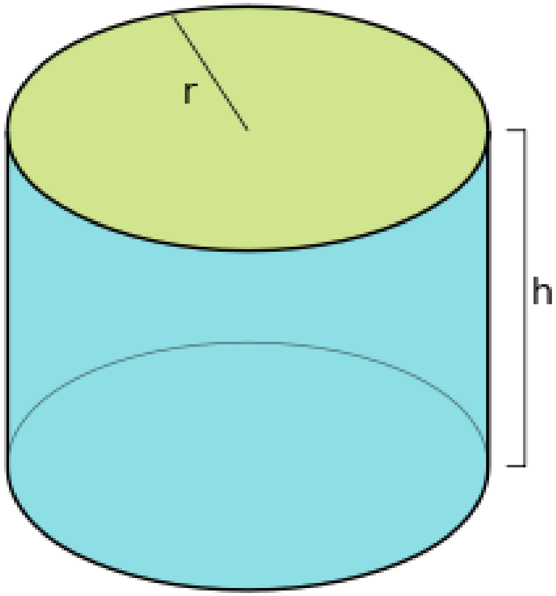
- ***f*** **'(*c*) does not exist:** *c* = -1.
- ***f*** **'(*c*) = 0:** *c* = -2.5, *c* = -1.6, *c* = 0.
- **Endpoints:** *a* = -3 and *b* = 2.



Absolute extrema:
Take largest and smallest of these 6 function values.

Design problem

You have designed a cylinder with volume 1 liter. The material is expensive, so you want to minimize the surface material used.



Volume: $V=\pi r^2 h$

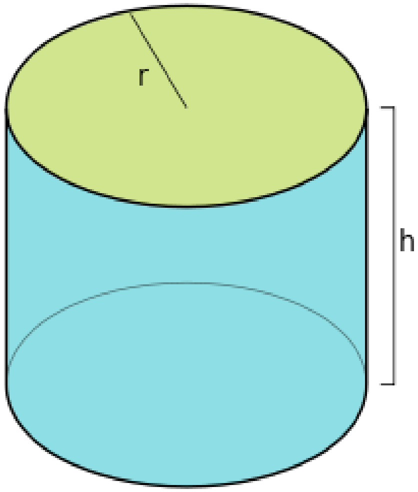
Surface: $S=2\pi rh+2\pi r^2$

Material cost: *k* euro / m^2

Since the volume must be 1, we have: $h = \dfrac{1}{\pi r^2}$

Hence the cost is: $c(r) = k(2\pi rh+2\pi r^2)=k(2/r+2\pi r^2)$

Design problem



$$c(r) = k(2/r + 2\pi r^2)$$

Minimize the cost c, hence derivative is 0:

$$c'(r) = k(-2/r^2 + 4\pi r) = 0$$

So $c'(r) = 0$ if: $4\pi r = \dfrac{2}{r^2}$ or $r^3 = \dfrac{1}{2\pi}$

For $r > 0$ there is a unique solution:

$$r^* = \left(\frac{1}{2\pi}\right)^{1/3}$$

Further investigation: $c(r)$ attains the absolute minimum for $r = r^*$.

Derivatives and applications

Theme: defining derivatives

Theme: Calculating derivatives

Theme: Extrema of functions

Theme: L'Hôpital's rule

- Type 0/0
- Type ∞/ ∞
- Type 0 ·∞
- Type ∞ – ∞
- Power types

4.5

Indeterminate Forms and L′ Hôpital’s Rule

A method for calculating limits of types
(“indeterminate forms”):

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

Limits of type $\frac{0}{0}$

L'Hôpital's rule

THEOREM 6— L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

N.B. It is allowed that $a = \infty$ or $a = -\infty$

Marquis de L'Hôpital (1661-1704)

L'Hôpital's rule

Example: calculate $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

Solution: Substituting $x = 0$ yields: $\frac{0}{0}$

Hence we can apply L'Hôpital rule:

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x}$$

Substituting $x = 0$ again gives: $\frac{0}{0}$

Hence we can apply L'Hôpital rule once more:

$$\lim_{x \to 0} \frac{1-\cos x}{x^2}$$

Example **(continued):**

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

First application of L'Hôpital

Second application of L'Hôpital

L'Hôpital's rule

Limits of type $\dfrac{\infty}{\infty}$

Theorem:

Suppose that f en g are differentiable and

$g'(x) \neq 0$ in the neighborhood of $x = a$

Moreover assume that:

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(if the latter limit exists)

Example Calculate: $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

Solution:

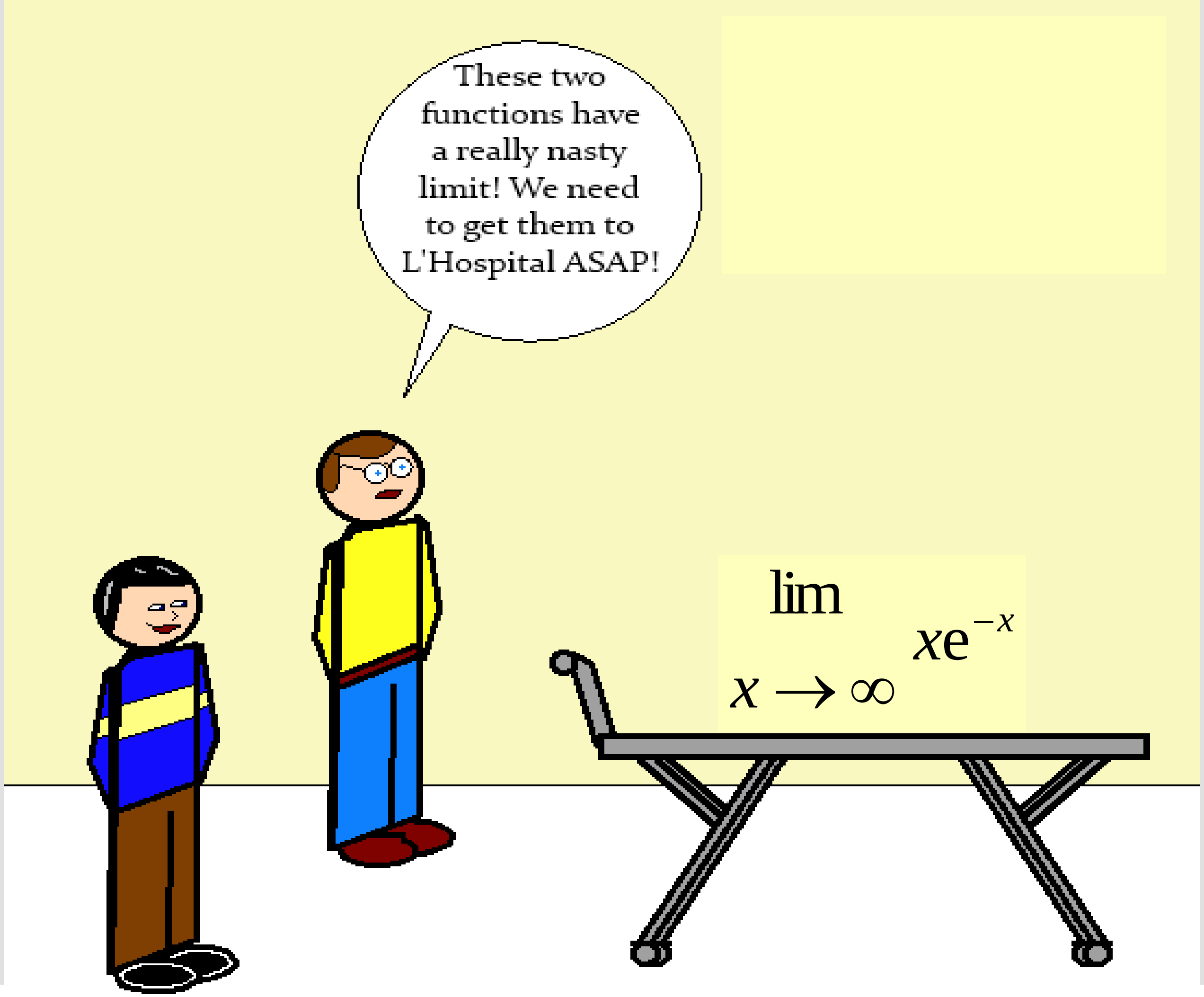
"Substituting" $x = \infty$ gives: $\frac{\infty}{\infty}$

Hence we can apply L'Hôpital's rule:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0$$

So: $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = 0$

Limits of type $0 \cdot \infty$



L'Hôpital's rule

Strategy:

Limits of type $\lim_{x \to a} f(x) \cdot g(x)$ that on substitution give:

$0 \cdot \infty$, should be written as fraction, to apply L' Hôpital's rule

So either as: $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}}$

or as $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}}$

Which one, depends on the situation (whatever is easier).

L'Hôpital's rule

Example: Calculate $\lim_{x \to \infty} x e^{-x}$

Solution:

"Substituting" $x = \infty$ gives: $\infty \cdot 0$

Rewrite as a fraction:

$$\lim_{x \to \infty} x e^{-x} = \lim_{x \to \infty} \frac{x}{e^{x}}$$

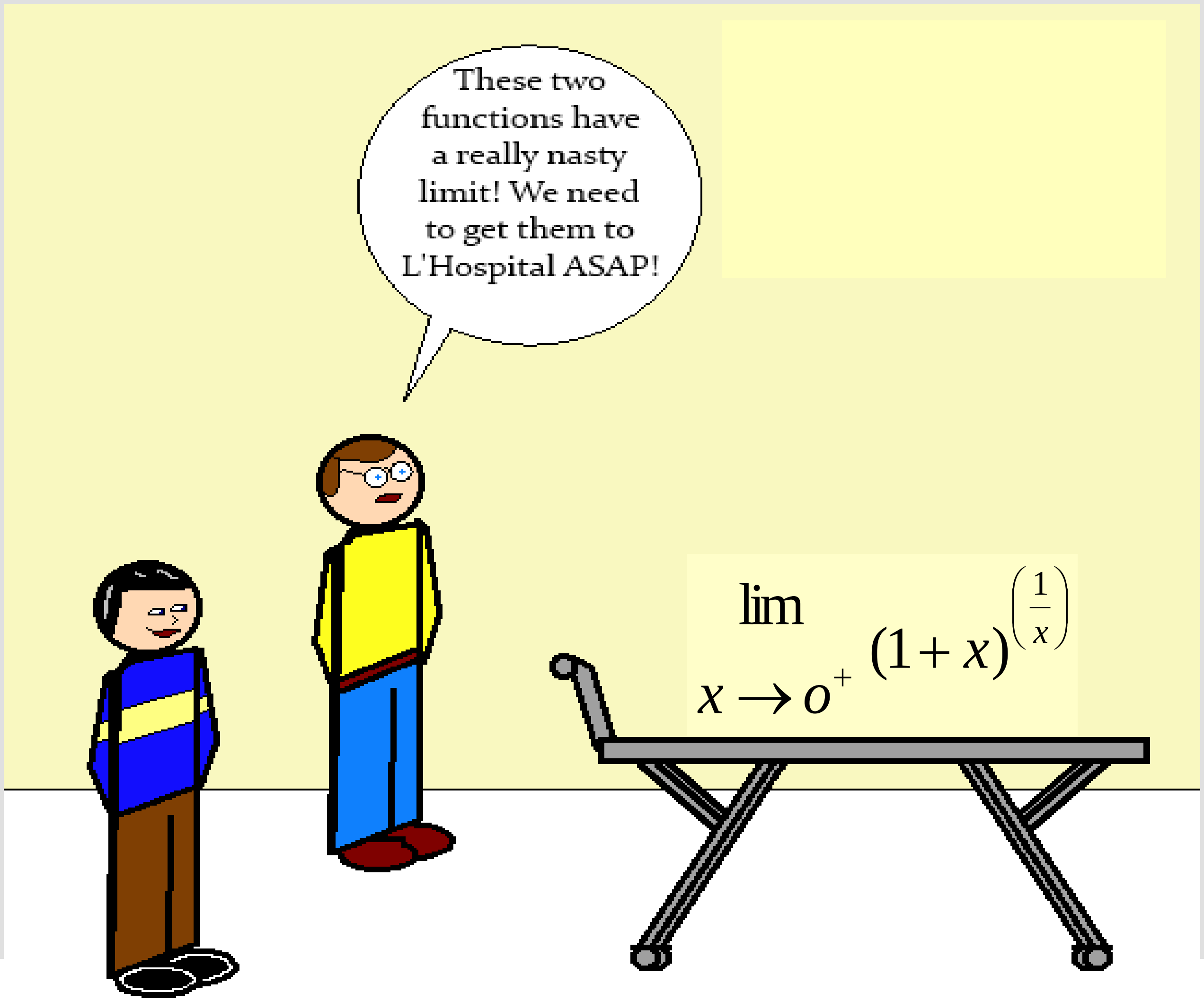
Type: $\frac{\infty}{\infty}$

Now we can apply L'Hôpital's rule:

$$\lim_{x \to \infty} x e^{-x} = \lim_{x \to \infty} \frac{1}{e^{x}} = 0$$

L’Hôpital’s rule

Limits with indeterminate powers



Strategy:

Limits of type $\lim_{x \to a} f(x)^{g(x)}$

that after substitution are of one of the types

0^0 ∞^0 1^∞

can be simplified by using: $f(x)^{g(x)} = e^{g(x)\ln f(x)}$

First calculate: $\lim_{x \to a} g(x)\ln f(x) = L$

then: $\lim_{x \to a} f(x)^{g(x)} = e^L$

L'Hôpital's rule

Example: Calculate $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$

Solution: $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)}$

First calculate: $\lim_{x \to 0} \frac{\ln(1+x)}{x}$ Type: $\frac{0}{0}$

Hence we can apply L' Hôpital's rule:

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\left(\frac{1}{x+1}\right)}{1} = 1$$

So: $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e^1 = e$

Derivatives and applications

Theme: Defining derivatives

Theme: Calculating derivatives

Theme: Extrema of functions

Theme: L'Hôpital's rule

Summarizing Excercise

The function $f : [1,4] \to R$ is given by

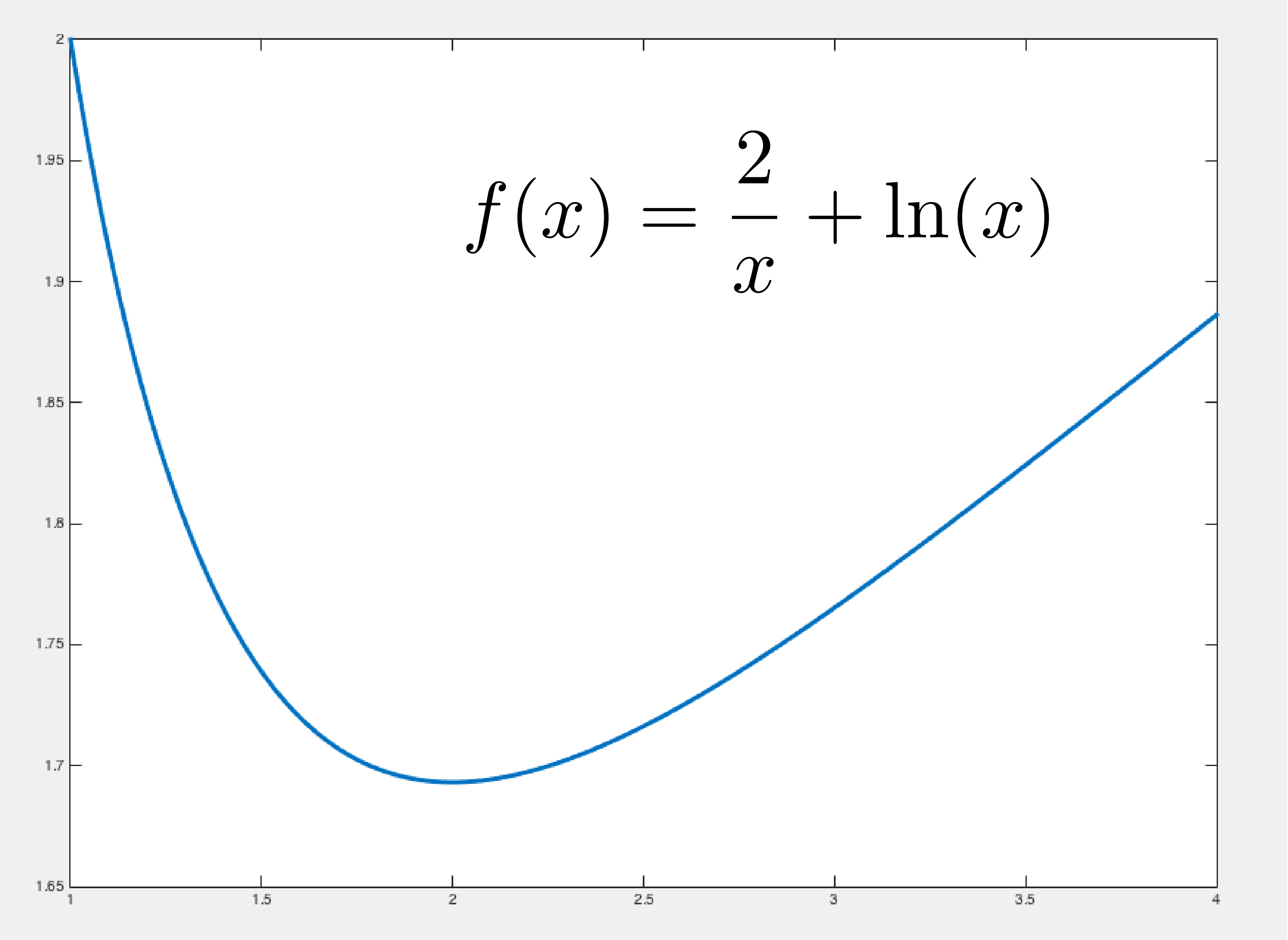
$$f(x) = \frac{2}{x} + \ln(x)$$

(a) Determine the absolute extrema of *f(x)* on [1,4]

(b) Compute the linearization of *f(x)* in *x=3*

Hint: $0.6 < \ln(2) < 0.7$

Summarizing Excercise



$$f(x) = \frac{2}{x} + \ln(x)$$

- Contents -

- ❑ Limits and continuity
- √ Derivatives and applications
- ❑ Functions of 2 variables

- ❑ Integrals
- ❑ Calculation techniques for integrals
- ❑ Power and Taylor series

See you next week!