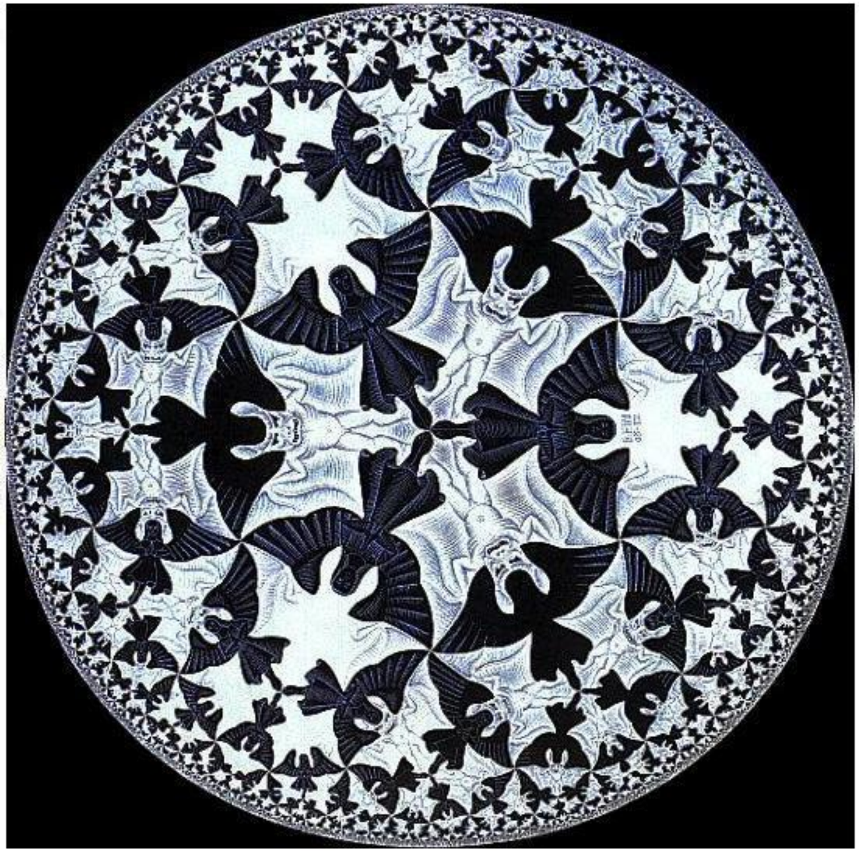
Mathematics B2: Newton



Lecturers:

Bernard Geurts and Gerard Jeurnink

Mathematics B2: Newton
- Contents -

- ✓ Limits and continuity
- Derivatives and applications
- Functions of 2 variables

- Integrals
- Calculation techniques for integrals
- Power and Taylor series

Limits and continuity

Theme: defining limits

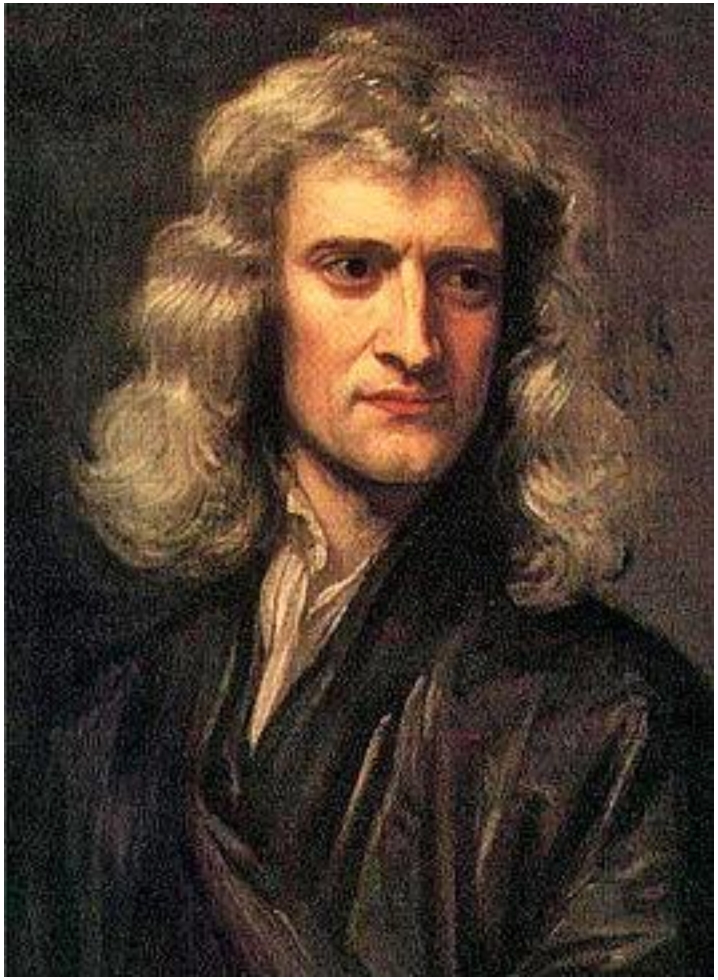
- Newton
- Limit of a sequence
- Limit of function values

Theme: evaluating limits

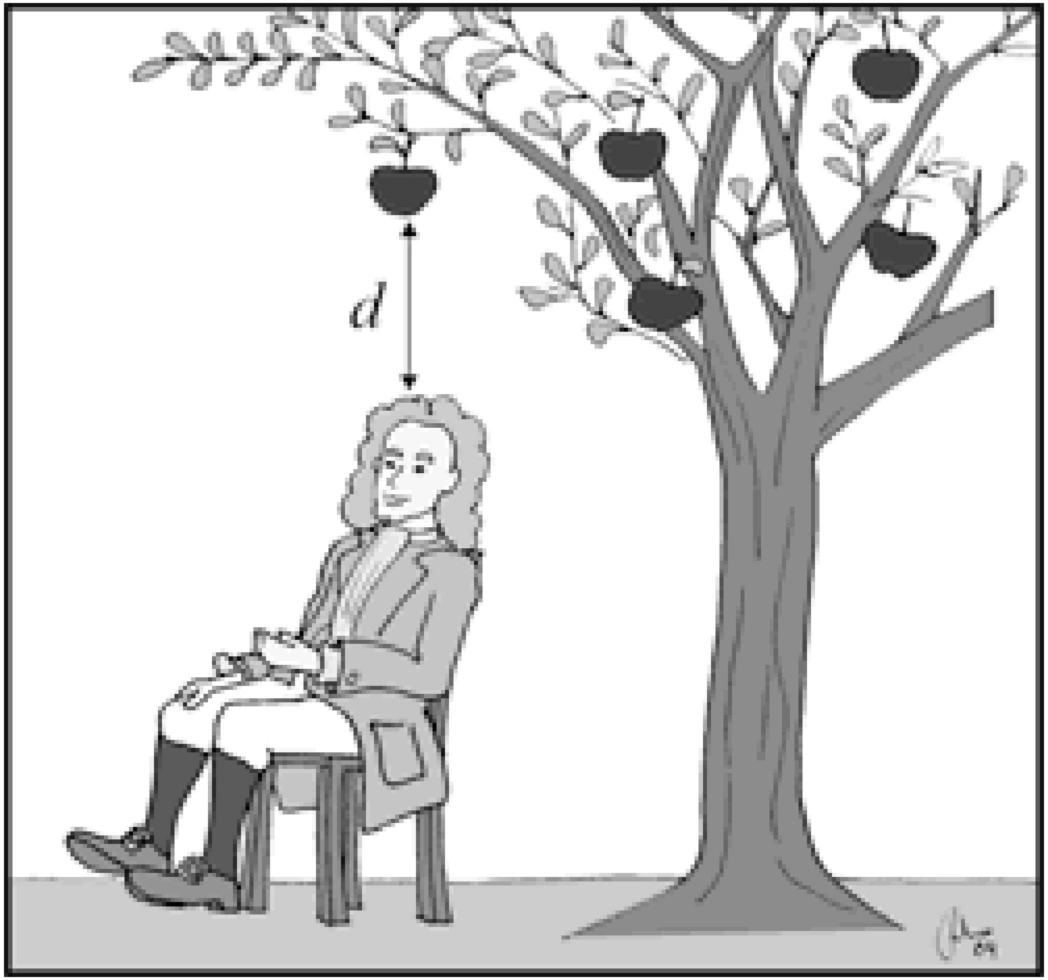
Theme: continuity

Newton

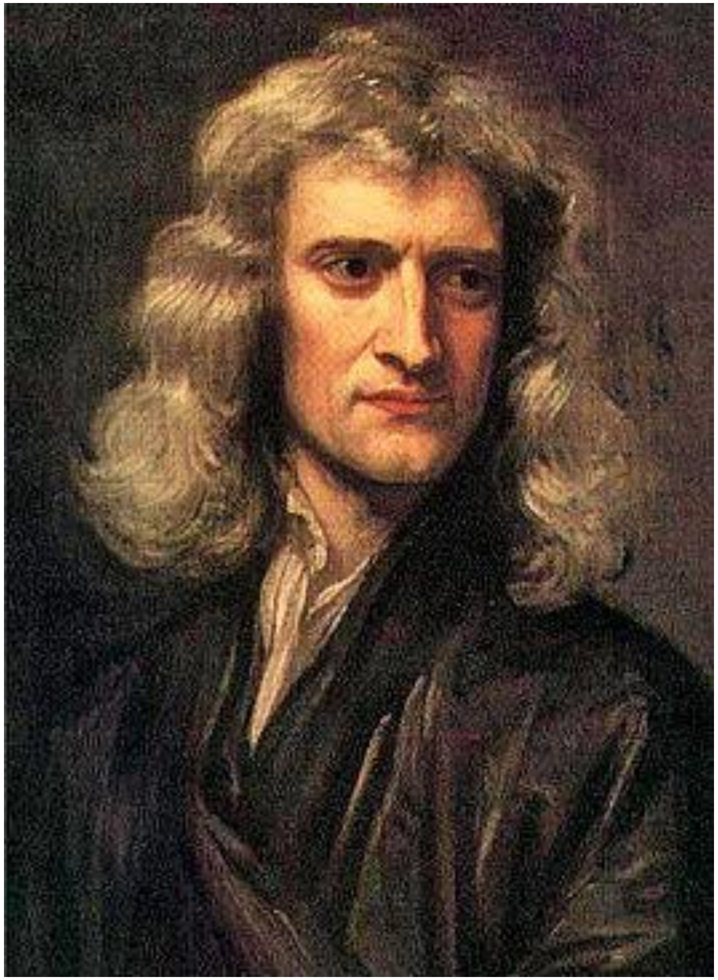
Newton



Isaac Newton (1643 – 1727)



Newton



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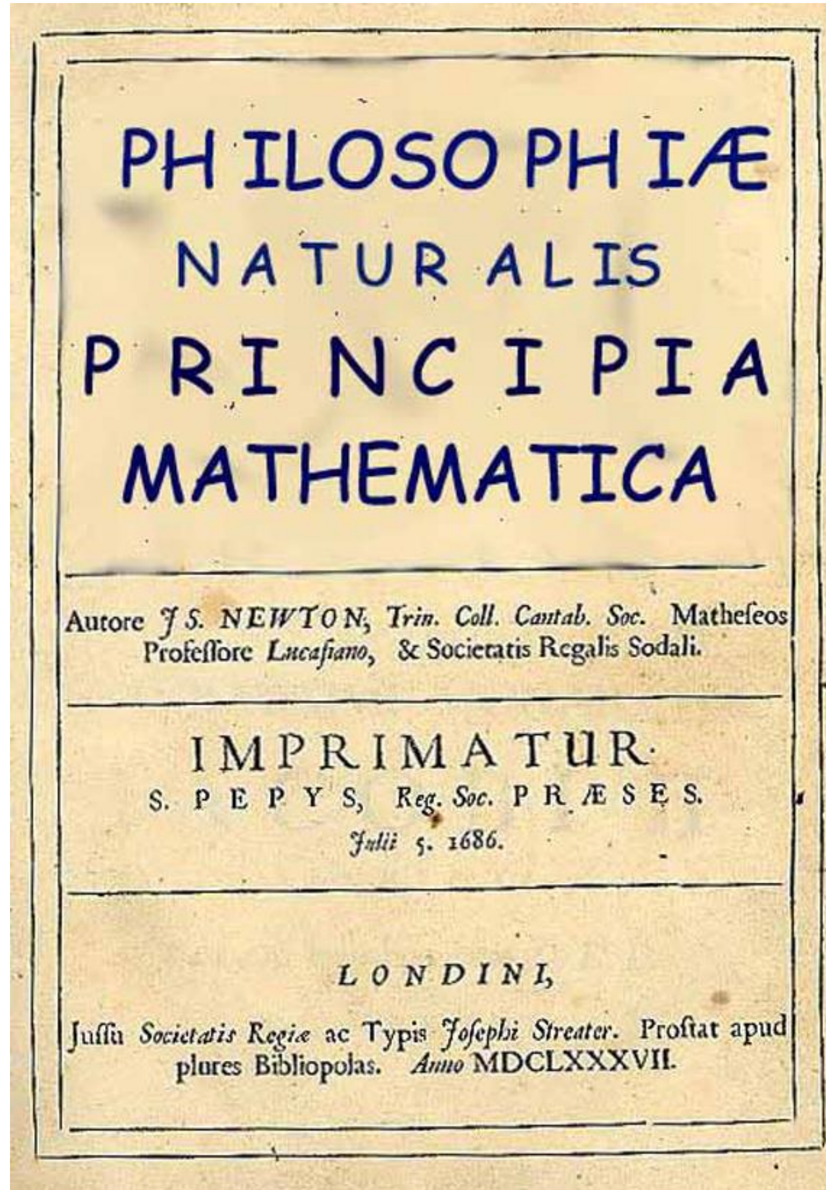


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore *J S.* NEWTON, *Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

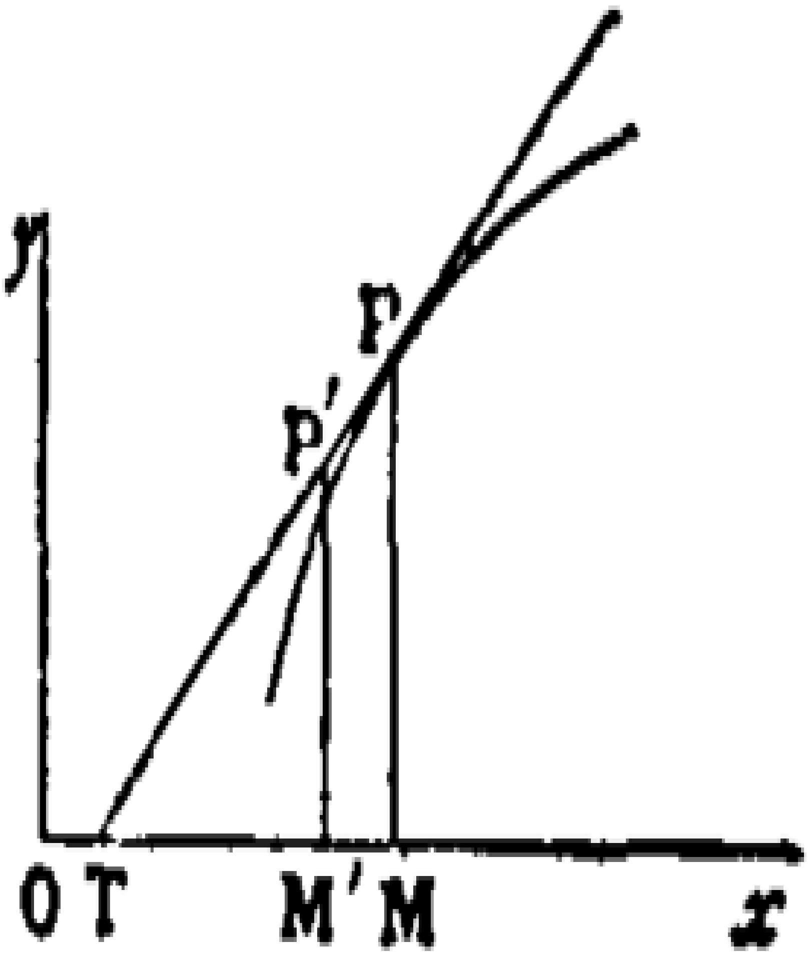
IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater.* Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

Newton

Newton's innovation:
Introducing calculus



2.2

What is a limit of a sequence of numbers?

Limit of a sequence

Suppose $x_1, x_2, x_3, \ldots$ is a sequence of numbers.

What is the meaning of: $\lim_{n \to \infty} x_n$?

Can we give a meaning to: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots$

Can we give a meaning to: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots$

Limit of a sequence; does program complete?

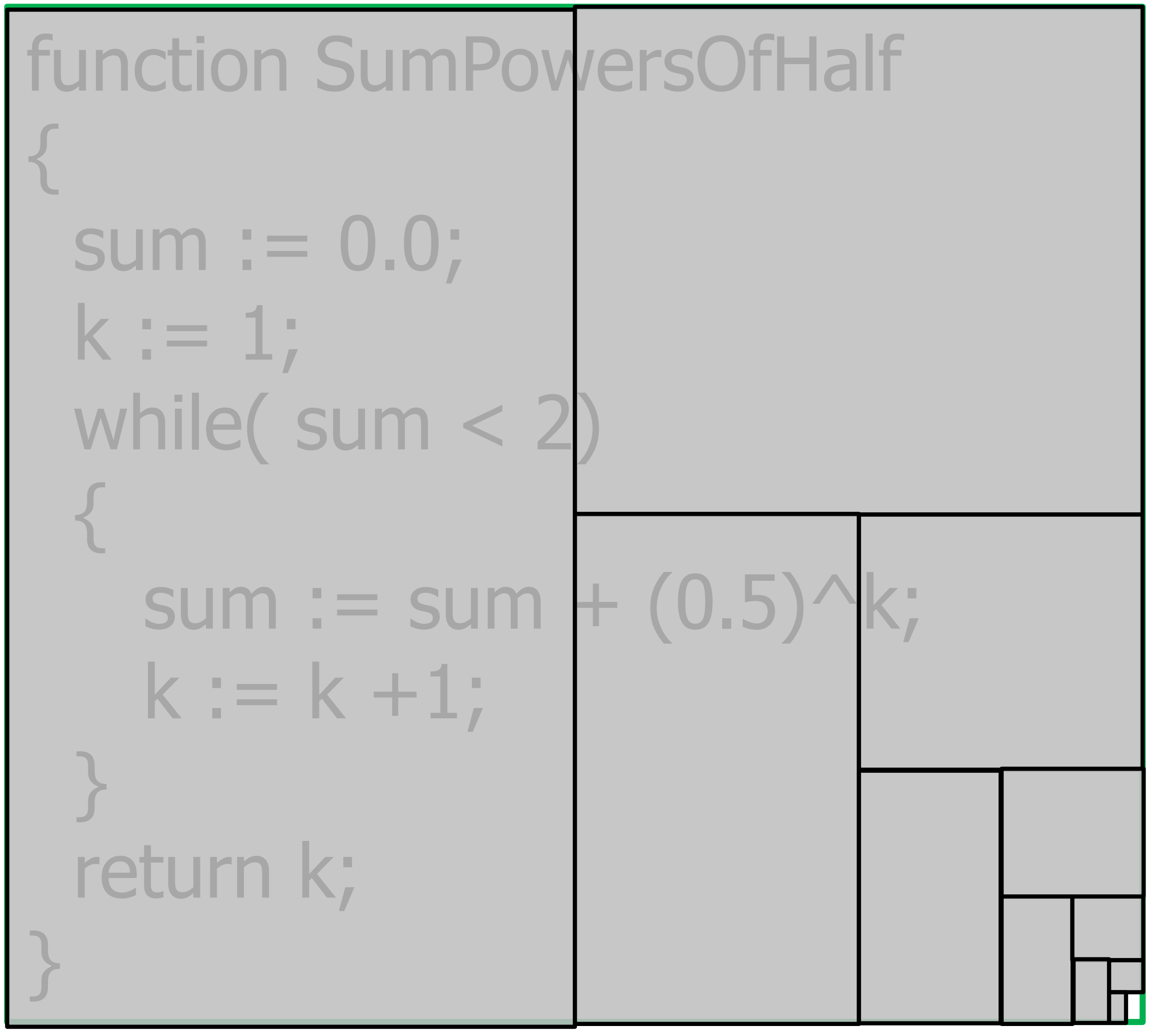
```
function SumPowersOfHalf
{
  sum := 0.0;
  k := 1;
  while( sum < 2)
  {
    sum := sum + (0.5)^k;
    k := k +1;
  }
  return k;
}
```

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots$$

```
function SumOfReciprocals
{
  sum := 0.0;
  k := 1;
  while( sum < 2)
  {
    sum := sum + 1/k;
    k := k +1;
  }
  return k;
}
```

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$$

Limit of a sequence



```
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```

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots$$

Limit of a sequence

```
function SumPowersOfHalf
{
  sum := 0.0;
  k := 1;
  while( sum < 2 )
  {
    sum := sum + (0.5)^k;
    k := k +1;
  }
  return k;
}
```

$$x_1 = \frac{1}{2}$$

$$x_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$x_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$x_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

So: $$\lim_{n \to \infty} x_n = 1$$

Limit of a sequence

```
function SumOfReciprocals
{
  sum := 0.0;
  k := 1;
  while( sum < 2 )
  {
     sum := sum + 1/k;
     k := k +1;
  }
  return k;
}
```

$$x_1 = 1$$

$$x_2 = 1 + \frac{1}{2} = 1.5$$

$$x_3 = 1 + \frac{1}{2} + \frac{1}{3} \approx 1.8333$$

$$x_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \approx 2.0833$$

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

Limit of a sequence

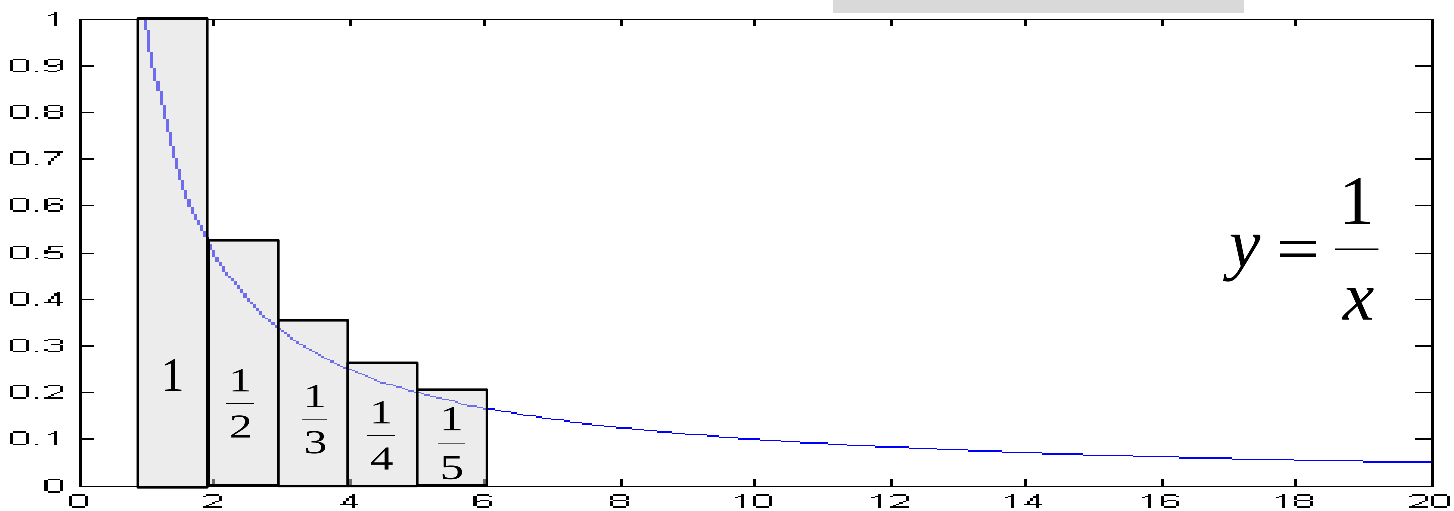
```
function SumOfReciprocals
{
  sum := 0.0;
  k := 1;
  while( sum < 2 )
  {
     sum := sum + 1/k;
     k := k +1;
  }
  return k;
}
```

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

$$x_n \; 3 \; \log(n+1)$$

(integrate (1/x) from 1 to n+1)

So: $$\lim_{n \to \infty} x_n = \infty$$



Limit of a sequence

Definition: We say that the sequence $x_1, x_2, x_3, \ldots$ ***tends to*** *a if* $|x_n - a|$ becomes very small for n very large.

We write:

$$\lim_{n\to\infty} x_n = a$$

Remark 1: $|x_n - a|$ is the ***distance*** from x_n to a.

Remark 2: ∞ is NOT a number.

It represents growth beyond any bound.

2.2

Definition of $\lim_{x \to c} f(x)$

Limit of function values

Definition: We say that $\lim_{x \to c} f(x) = L$

if for every sequence $x_1, x_2, x_3, \ldots,$ **different from** c, holds:

$x_1, x_2, x_3, \ldots$ tends to c

implies

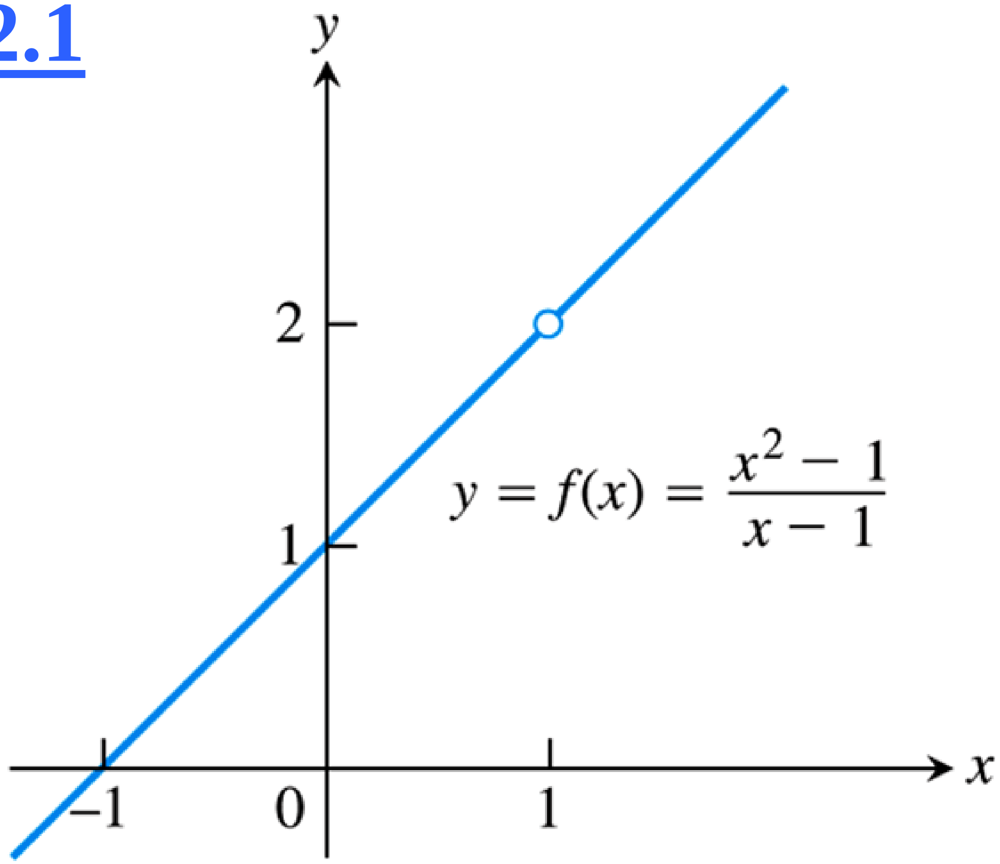
$y_1, y_2, y_3, \ldots$ tends to L

Here: $y_1 = f(x_1), y_2 = f(x_2)$, $y_3 = f(x_3), \ldots$

Remark: We need that $x_1, x_2, x_3, \ldots$ are in the domain of f.

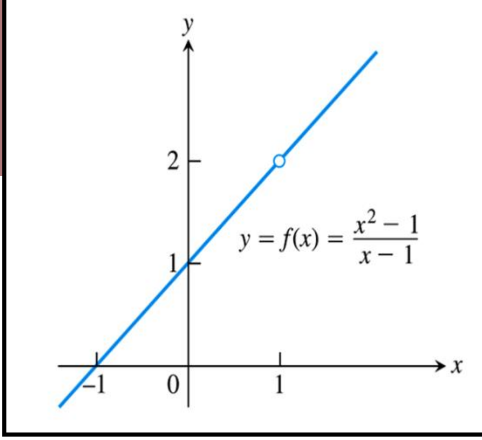
Limit of function values

Example 2.2.1



$$\lim_{x \to 1} f(x) = ???$$

Limit of function values



Example (continued)

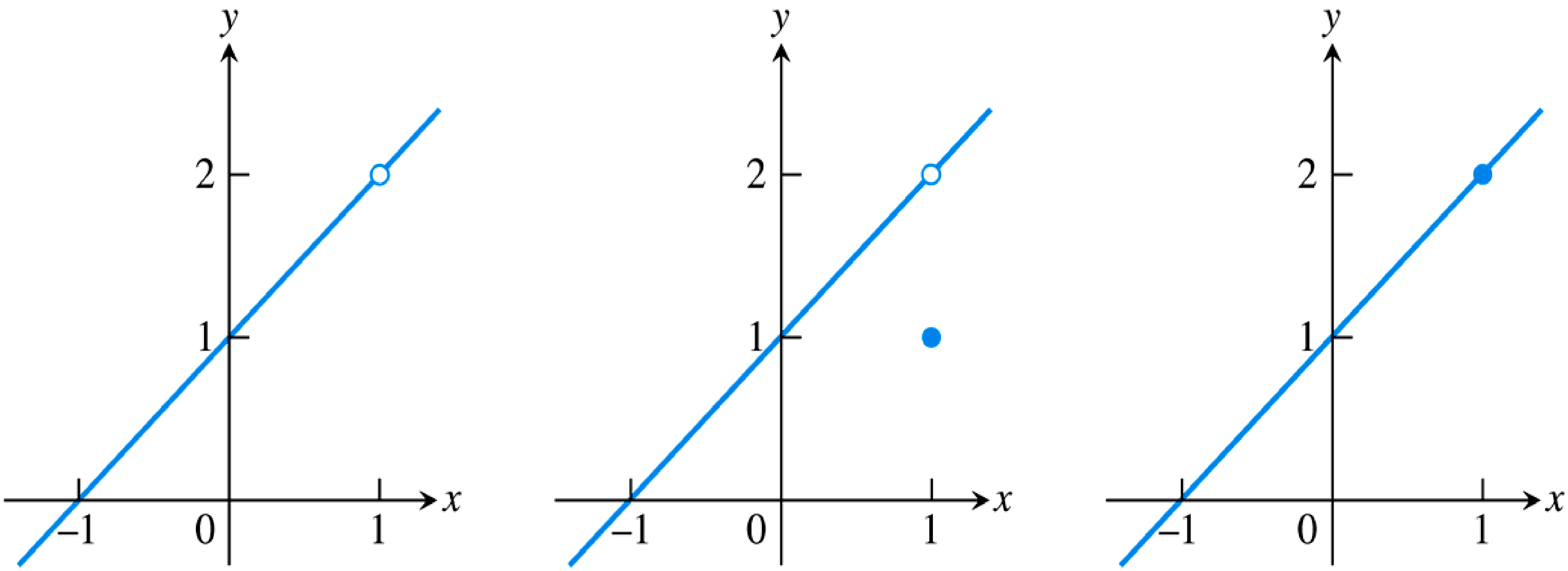
$$\lim_{x \to 1} f(x) = 2$$

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Limit of function values

Example 2.2.2



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

Limit of function values

Example 2.2.3

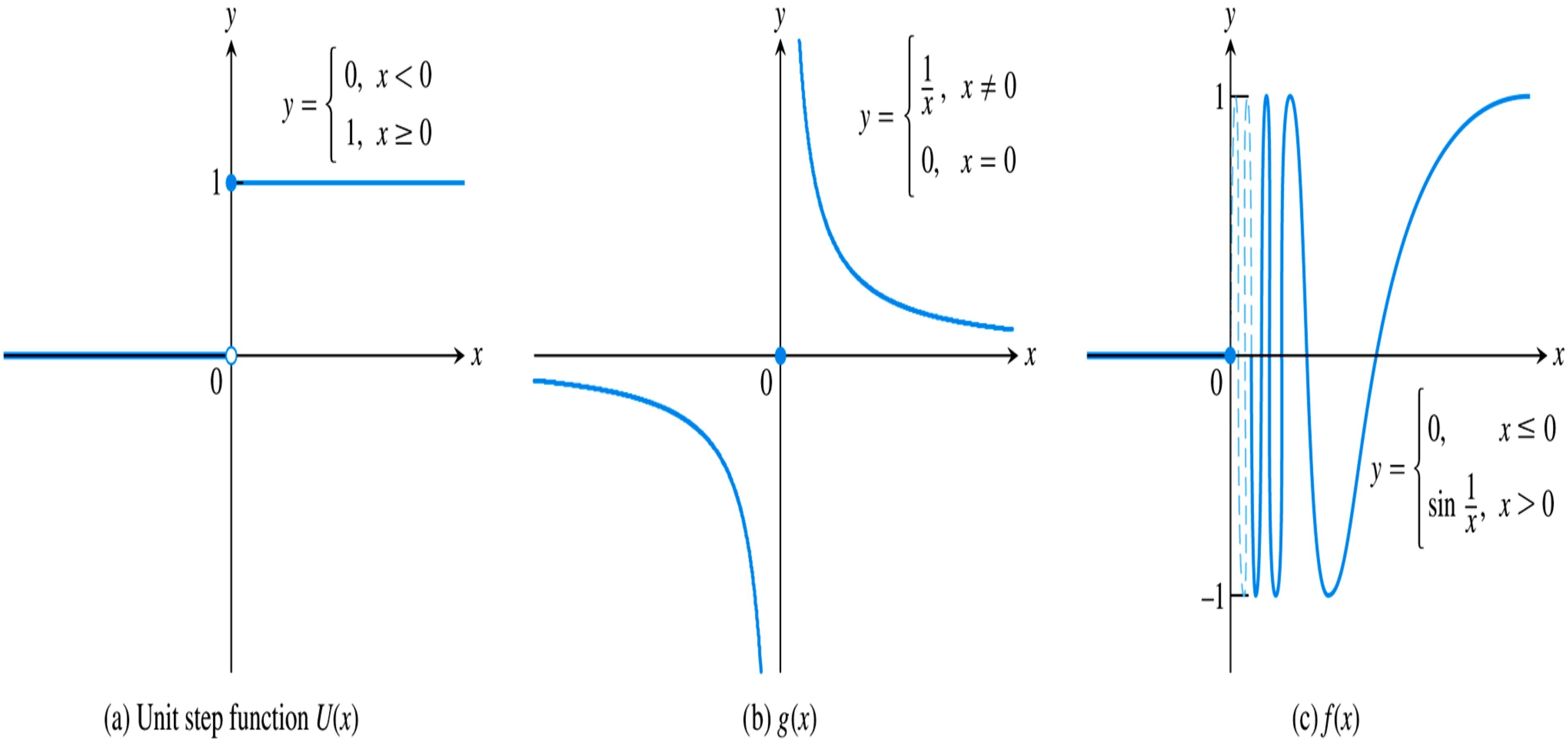


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

Limits and continuity

Theme: defining limits

Theme: evaluating limits

- Handling limits
- Calculating limits

Theme: continuity

2.2

Handling limits

Handling limits

Direct substitution theorem:

if f is a "nice formula-function" in which the point c is "without problem", then the limit is obtained by "direct" substitution:

$$\lim_{x \to c} f(x) = f(c)$$

Example:

$$\lim_{x \to 2} x^3 - 7x + 10 = 8 - 7 \cdot 2 + 10 = 4$$

Handling limits

Example

$$\lim_{x \to \infty} \frac{x^2}{x^4} = \frac{"\infty"}{"\infty"} = \textit{take care!}$$

Improved solution:

$$\lim_{x \to \infty} \frac{x^2}{x^4} = \lim_{x \to \infty} \frac{1}{x^2} = 0$$

Handling limits

Example:

$$\lim_{x \to 0} \frac{3.1415x}{x} = \frac{"0"}{"0"} = take\ care!$$

Improved solution:

$$\lim_{x \to 0} \frac{3.1415x}{x} = \lim_{x \to 0} 3.1415 = 3.1415$$

Handling limits

Be careful with limits where term(s) *tend* to 0 or infinity !

Clear:

$$\frac{\infty}{6} = \infty \qquad \frac{2}{\infty} = 0$$

$$\frac{\infty}{-6} = -\infty \qquad \frac{2}{-\infty} = 0$$

$$\infty \cdot \infty = \infty$$

$$\infty \cdot -\infty = -\infty$$

$$-\infty \cdot -\infty = \infty$$

$$\infty + \infty = \infty$$

Unclear:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \frac{-\infty}{\infty} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

Handling limits

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

Handling limits

Limit laws (product):

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist, then:

$$\lim_{x \to c} \left(f(x) \cdot g(x) \right) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$$

"If *f(x)* approaches *L* and *g(x)* approaches *M*, then *f(x)* · *g(x)* approaches *L* · *M*."

Example

Suppose: $f(x) = x - 1$ and $g(x) = \dfrac{7}{x-1}$

Since: $\lim_{x \to 1} f(x) = 0$ we have: ~~$\lim_{x \to 1} f(x) \cdot g(x) = 0$~~

The answer 0 is wrong!

Correct is: $$\lim_{x \to 1} f(x) \cdot g(x) = \lim_{x \to 1} (x-1) \cdot \frac{7}{(x-1)} = \lim_{x \to 1} 7 = 7$$

Handling limits

Question: Is this correct ?

$$\lim_{h \to 0} \frac{0}{h} = undecided$$

Correct is: $$\lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

N.B. Note the difference between

being 0 and ***tending*** to 0.

Handling limits

Which solution is correct ? $(h > 0)$

$$\lim_{h \to 0} (1+h)^{1/h} = 1$$

(I)

$$\lim_{h \to 0} (1+h)^{1/h} = \infty$$

(II)

(a) (I) is correct, (II) is ***not*** correct.

(b) (I) is ***not*** correct, (II) is correct.

(c) Both solution are **in**correct.

(d) Both solution are correct.

answers

Calculating limits

Calculating limits

Methods

- direct substitution (“no problems”).
- split in left-hand and right-hand limit.

- divide out common zero-making factor.
- divide out highest power at infinity.

- use the square root trick.
- use the sandwich theorem.

Calculating limits

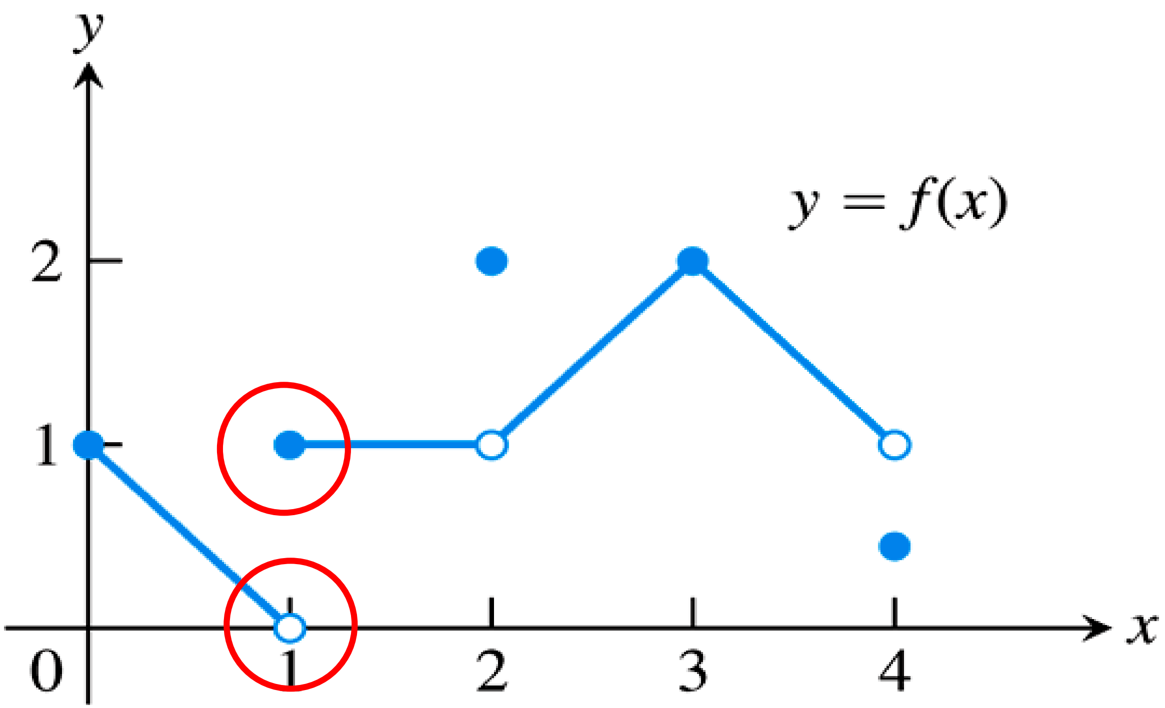


FIGURE 2.27 Graph of the function in Example 2.

$$\lim_{x \to 1^-} f(x) = 0$$

$$\lim_{x \to 1^+} f(x) = 1$$

Calculating limits

Calculating limits:
(split in left and right)

Calculating limits

Example: calculate $\lim_{x \to 0} \frac{x}{|x|}$ Type: $\frac{0}{0}$

Solution

Right-hand limit: $\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 1^+} \frac{x}{x} = 1$

Left-hand limit: $\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 1^-} \frac{x}{-x} = -1$

Hence: $\lim_{x \to 0} \frac{x}{|x|}$ does not exist !

Calculating limits

Example (continued): graph

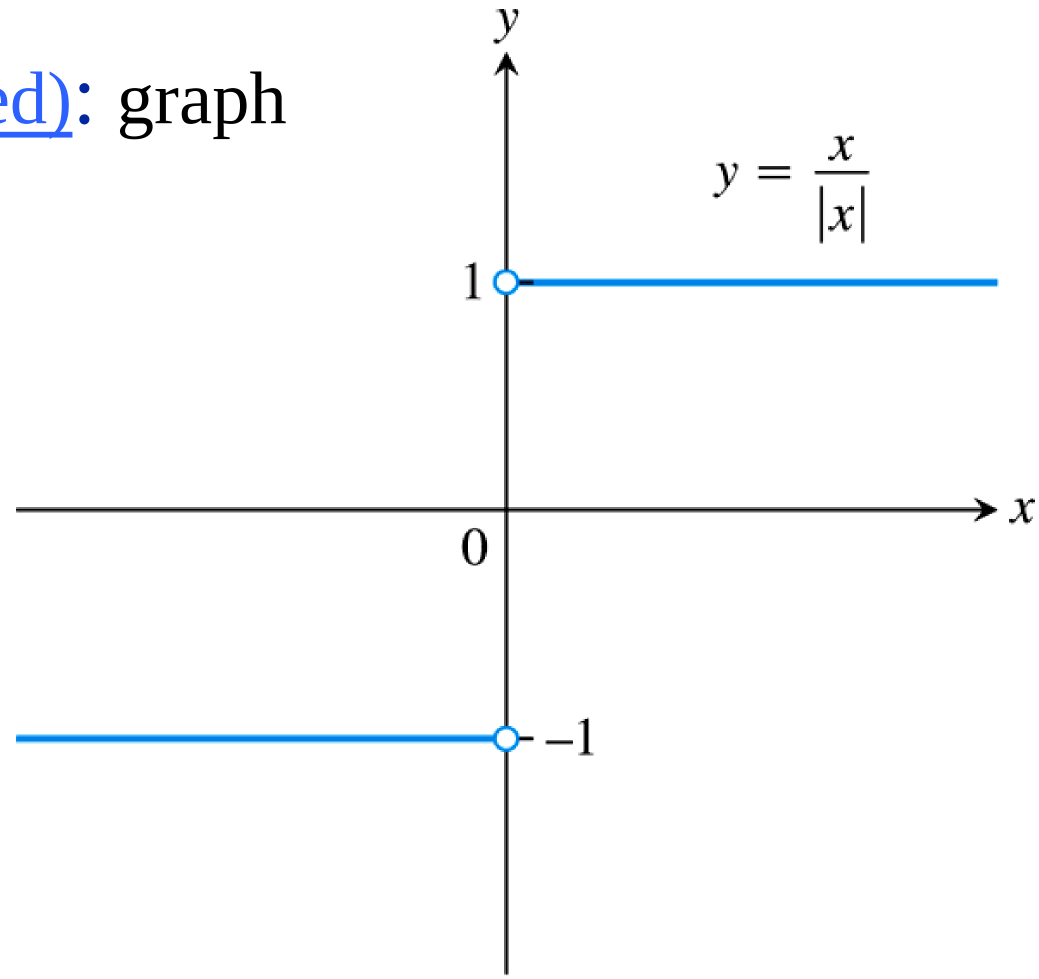
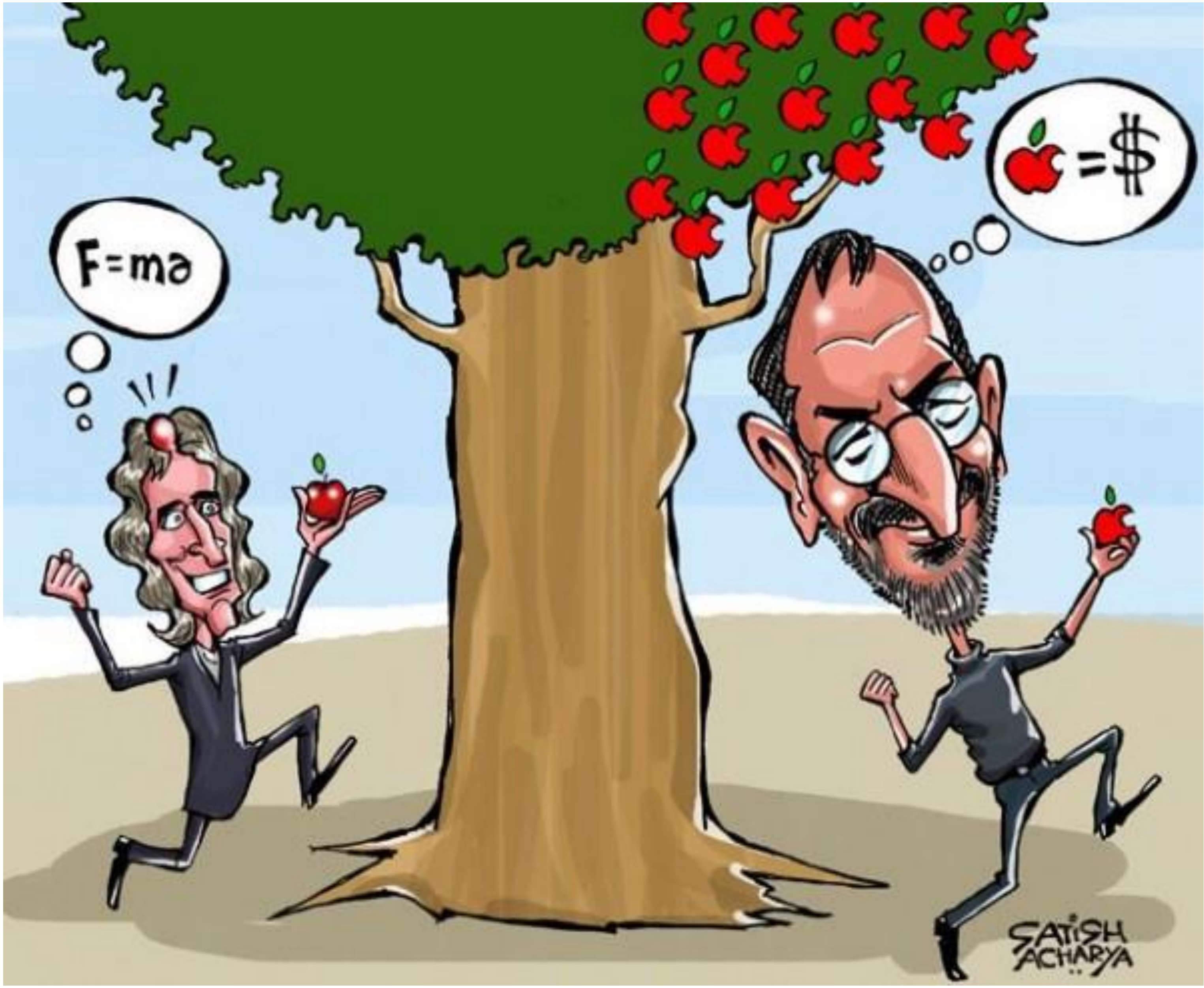


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

Break



Calculating limits

Dividing out factors

Calculating limits

Examples

$$\lim_{x \to \infty} \frac{2x^2 - 6x}{3x^2 + 2x} =$$

[For $x \to \infty$ the **highest** power of x will dominate!

Divide by $\underline{x^2}$.]

$$= \lim_{x \to \infty} \frac{2 - \frac{6}{x}}{3 + \frac{2}{x}} = \frac{2}{3}$$

$$\lim_{x \to 0} \frac{2x^2 - 6x}{3x^2 + 2x} =$$

[For $x \to 0$ the **lowest** power of x will dominate!

Divide by $\underline{x}$ (remove factor).]

$$= \lim_{x \to 0} \frac{2x - 6}{3x + 2} = -3$$

Calculating limits

“The” trick with square roots

Calculating limits

Example: $\lim_{x \to 7} \dfrac{\sqrt{x+2}-3}{x-7}$

Type: $\dfrac{0}{0}$

Remove $\sqrt{}$ in numerator:

$$\sqrt{a}-b = \frac{\sqrt{a}+b}{\sqrt{a}+b}(\sqrt{a}-b) = \frac{a-b^2}{\sqrt{a}+b}$$

Calculating limits

$$\lim_{x \to 7} \frac{\sqrt{x+2}-3}{x-7}$$

$$\lim_{x \to 7} \frac{\sqrt{x+2}-3}{x-7} = \lim_{x \to 7} \frac{\sqrt{x+2}-3}{x-7} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} =$$

$$= \lim_{x \to 7} \frac{(x+2)-3\sqrt{x+2}+3\sqrt{x+2}-9}{(x-7)(\sqrt{x+2}+3)}$$

$$= \lim_{x \to 7} \frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x \to 7} \frac{x-7}{(x-7)(\sqrt{x+2}+3)}$$

$$= \lim_{x \to 7} \frac{1}{\sqrt{x+2}+3} = \frac{1}{6}$$

Calculating limits

Sandwich theorem

Calculating limits

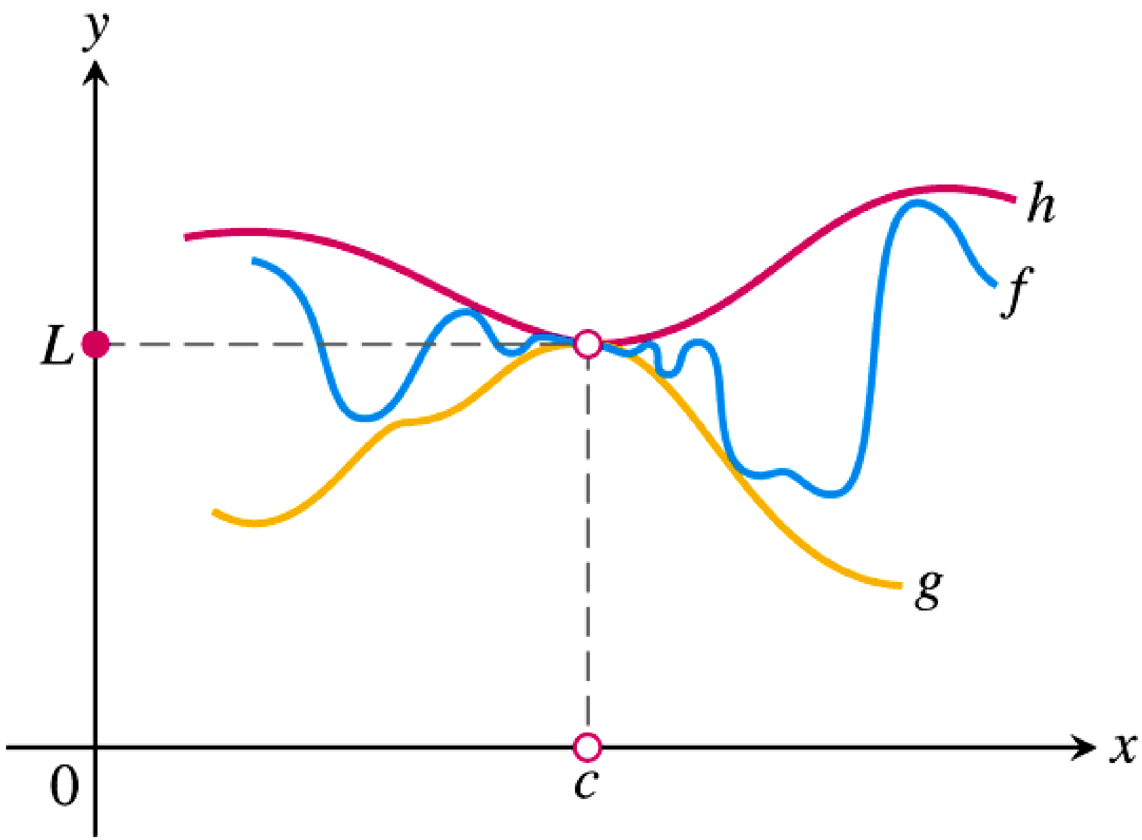


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

Calculating limits

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

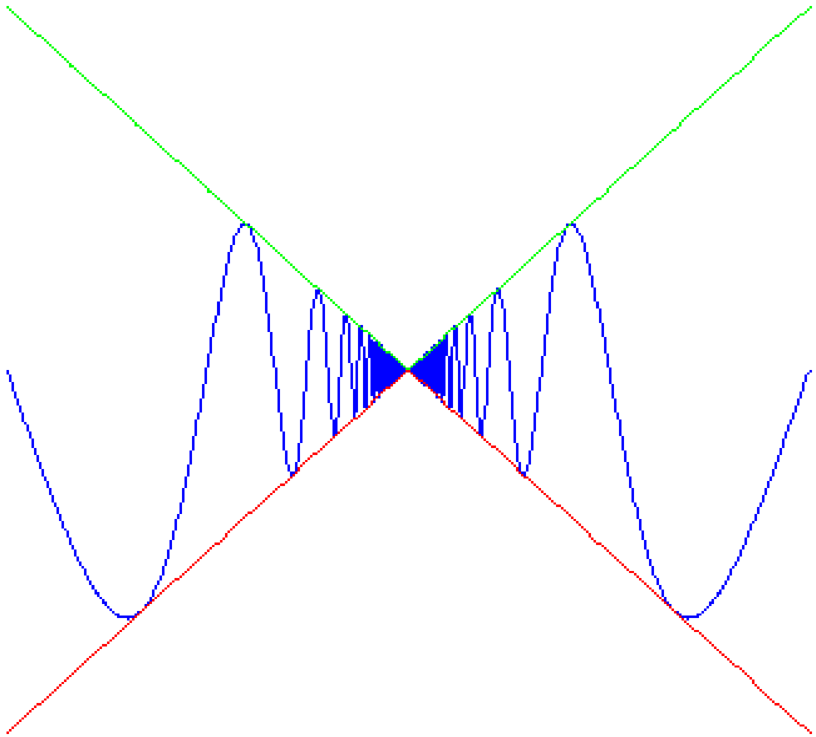
$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Calculating limits

Example: Calculate $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

Solution:



$$-|x| \le x \sin\left(\frac{1}{x}\right) \le |x|$$ for all $x \neq 0$.

Calculating limits

$$-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$$

Clearly: $\lim_{x \to 0} -|x| = 0$ and $\lim_{x \to 0} |x| = 0$

So (by the sandwich theorem):

$$\lim_{x \to 0} x\sin\left(\frac{1}{x}\right) = 0$$

Limits and continuity

Theme: defining limits

Theme: evaluating limits

Theme: continuity

- Definitions
- Intermediate value theorem
- Asymptotes

2.5

Continuity

Defining continuity

Domain of $f : [a, b]$

DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

Defining continuity

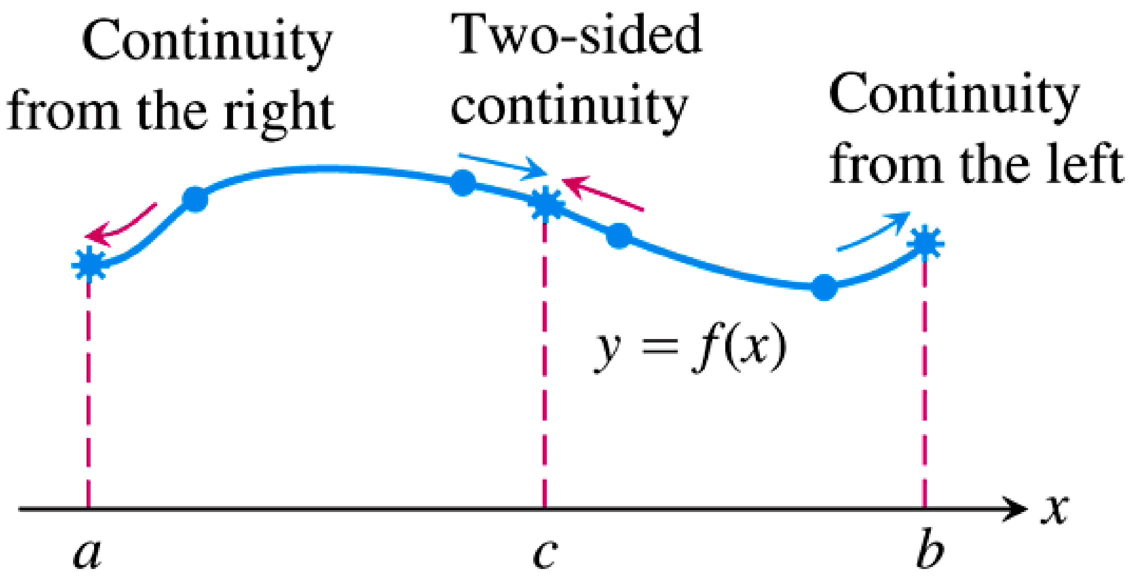


FIGURE 2.36 Continuity at points a, b, and c.

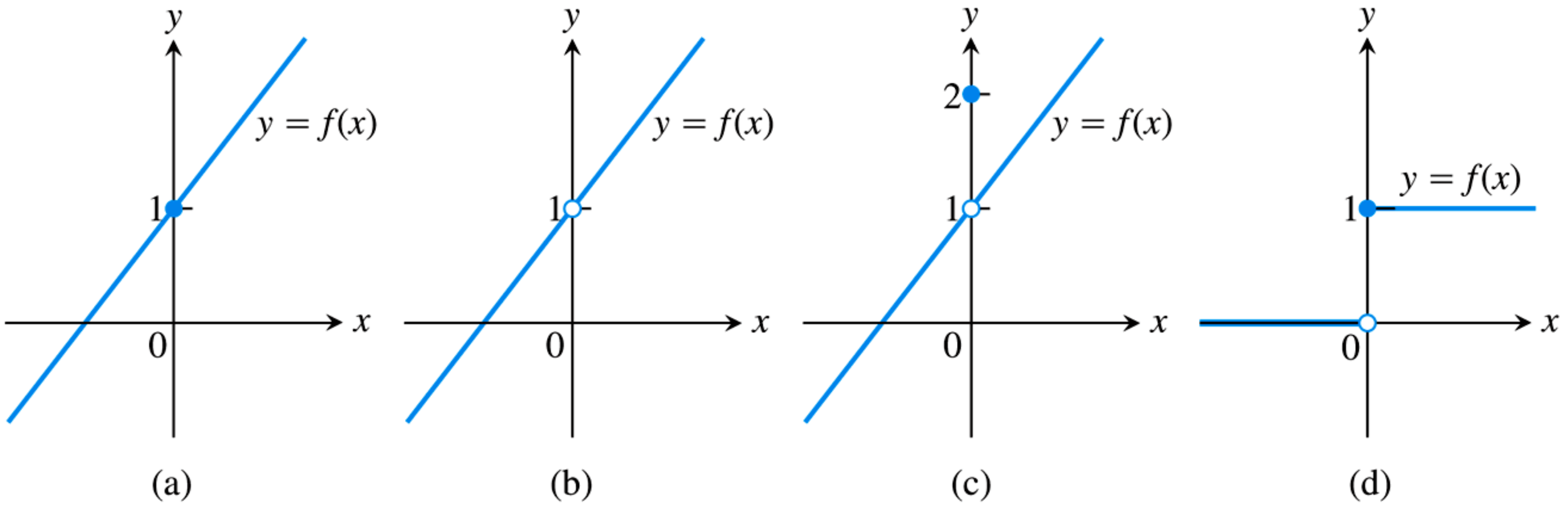
Defining continuity

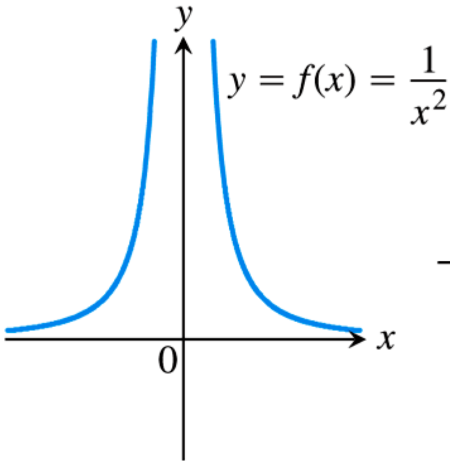
Continuity Test

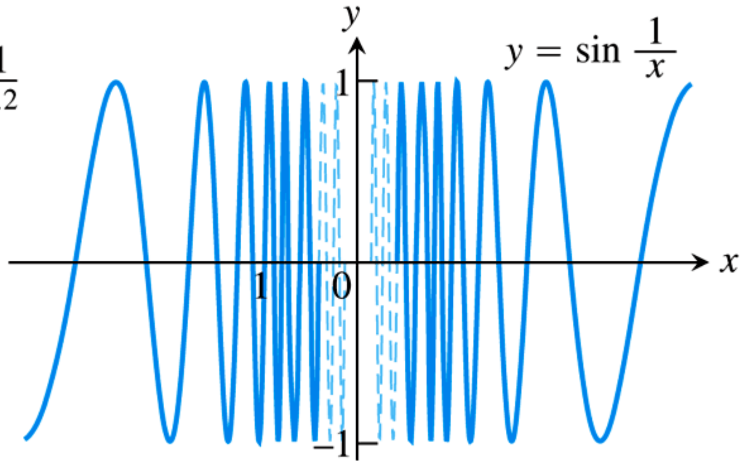
A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1.	$f(c)$ exists	(c lies in the domain of f).
2.	$\lim_{x \to c} f(x)$ exists	(f has a limit as $x \to c$).
3.	$\lim_{x \to c} f(x) = f(c)$	(the limit equals the function value).

Defining continuity





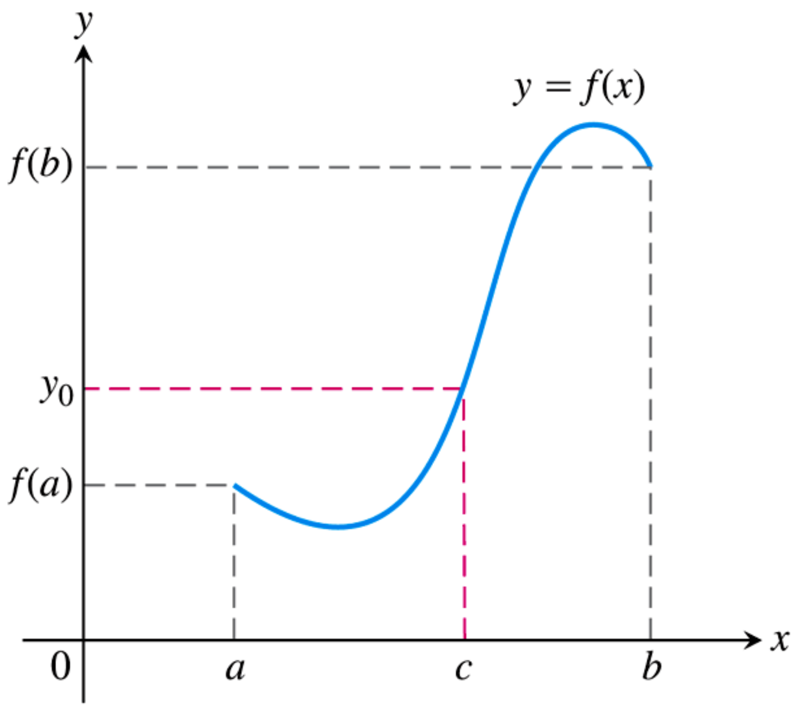


How many of these are continuous at 0 ?

Intermediate value theorem

Intermediate value theorem

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Intermediate value theorem: *f* must be continuous

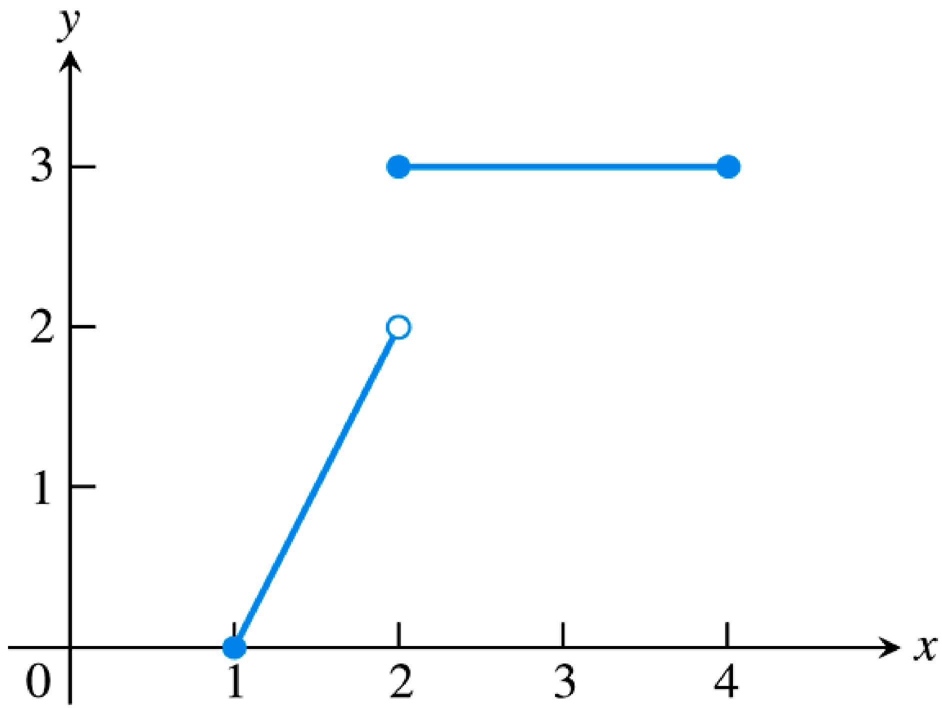


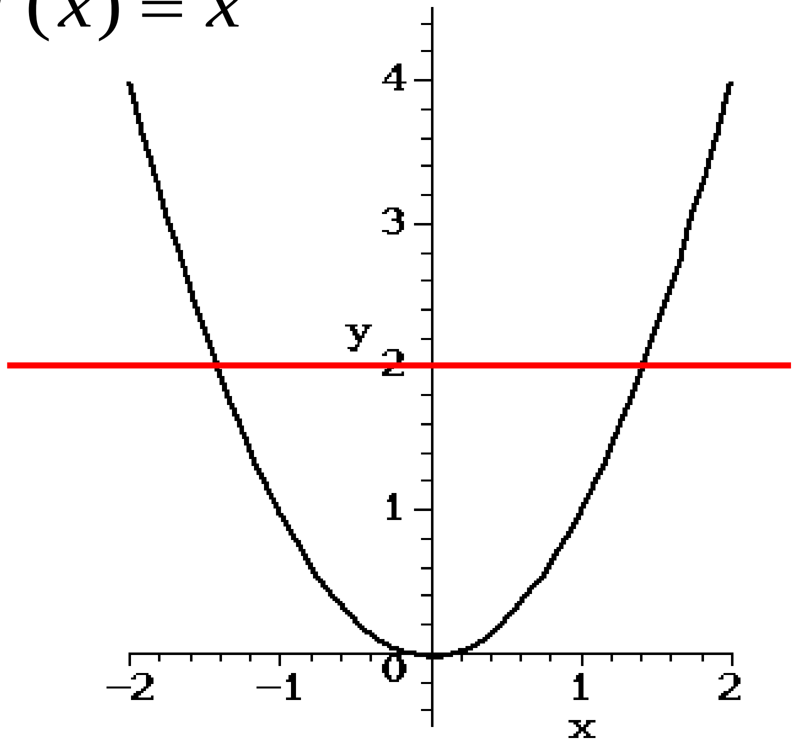
FIGURE 2.46 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Intermediate value theorem: not on Q

$f(x) = x^2$



$y = 2$

The domain of f is $\mathbb{Q}$, the set of fractions.

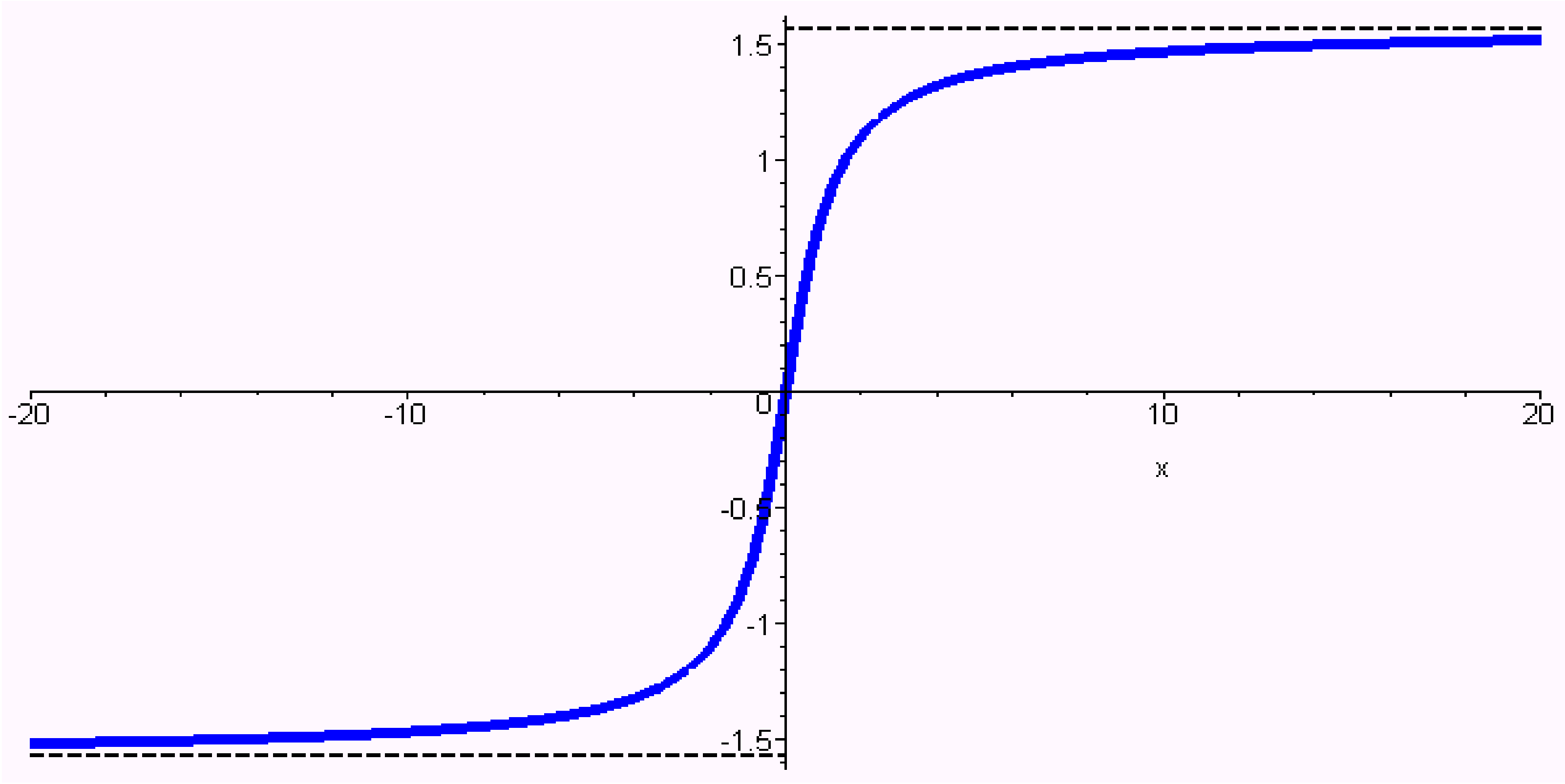
There is no x in $\mathbb{Q}$, such that f(x) = 2.

2.6

Limits Involving Infinity; Asymptotes of Graphs

Asymptotes (horizontal)

$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$$



$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$

Asymptotes (horizontal)

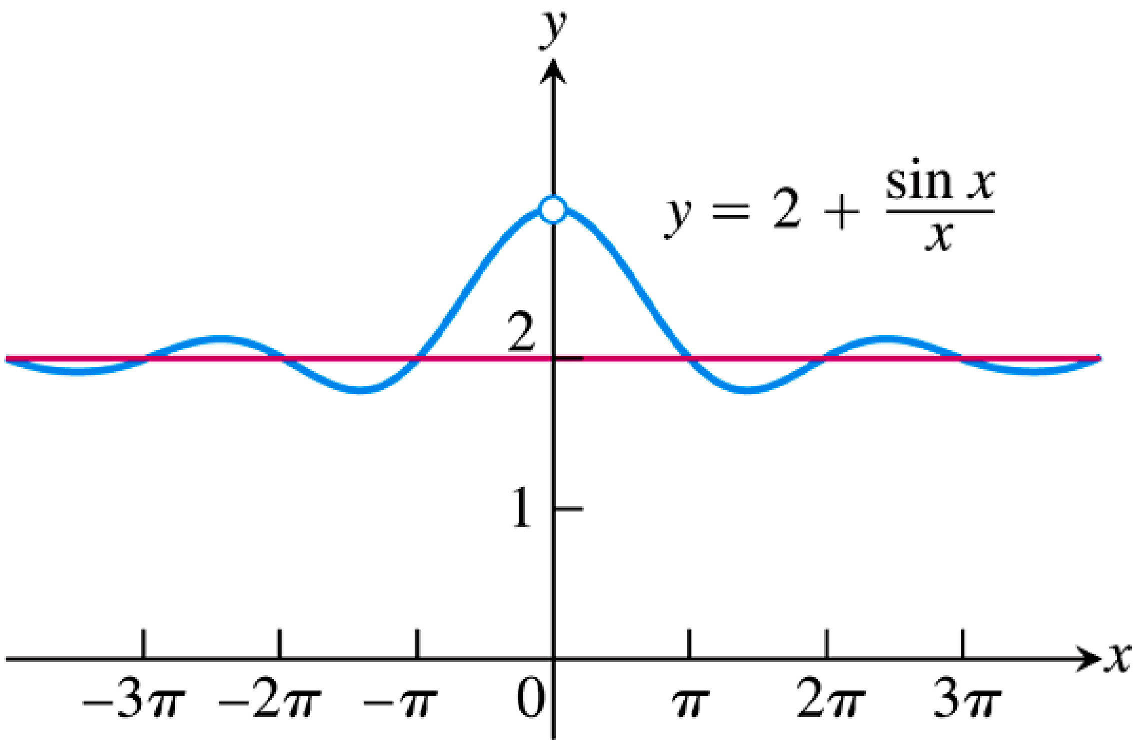
Definition: $\lim_{x \to \infty} f(x) = b$ will mean:

If x grows very large,
then $f(x)$ will approach L as close as you want.

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Asymptotes (horizontal)



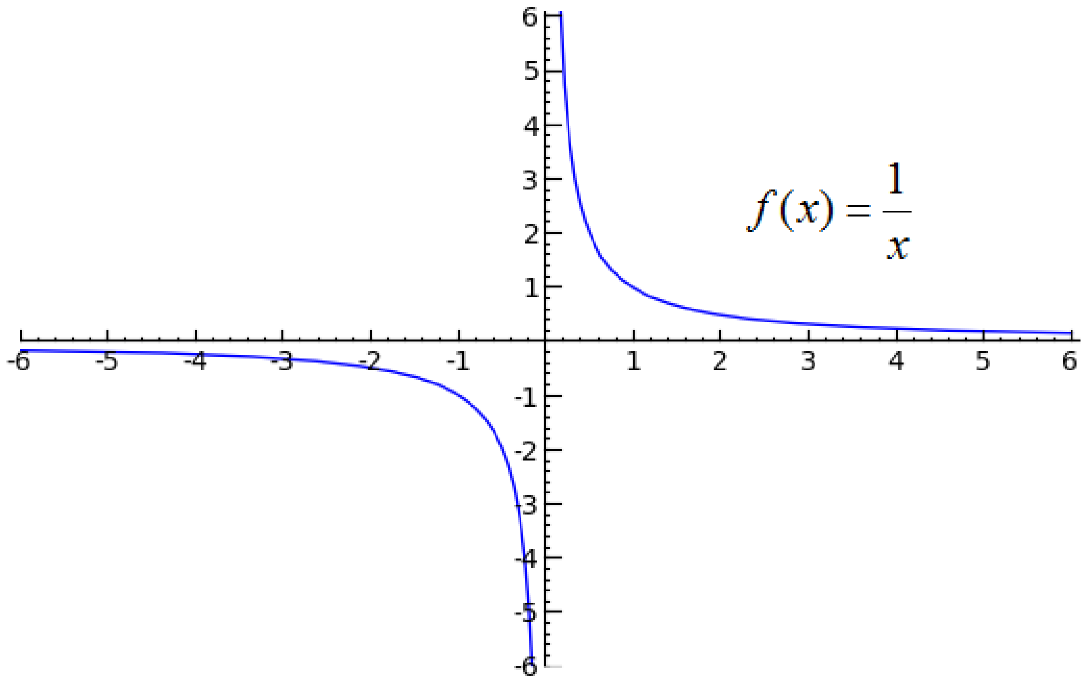
Is the line y=2 a horizontal asymptote?

Asymptotes (vertical)

Example:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$



Limits and continuity

Theme: defining limits

Theme: evaluating limits

Theme: continuity

Summarizing Excercise

$$f(x) = \frac{x}{x^2+1} \quad ; \quad x < 0$$

$$= x + 2\sqrt{x} \quad ; \quad x \geq 0$$

(a) Show that *f(x)* is continuous in *x=0*

(b) Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

Mathematics B2: Newton

- Contents -

- ✓ Limits and continuity
- ❑ Derivatives and applications
- ❑ Functions of 2 variables

- ❑ Integrals
- ❑ Calculation techniques for integrals
- ❑ Power and Taylor series