

Summary Math B1

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Chapter 12 Vectors and the geometry of space

12.1 Three-Dimensional Coordinate Systems

To locate a point in space, we use three mutually perpendicular coordinate axes, arranged as the right-handed coordinate frame (when you hold your hand so that the fingers curl from the positive x-axis toward the positive y-axis, your thumb points along the positive z-axis).

The Cartesian coordinates (x, y, z) , or **rectangular coordinates** define points in space. These coordinates determine planes (**xy-plane**, **yz-plane**, **xz-plane**). These planes meet at the **origin**. The **coordinate planes** $x=0$, $y=0$ and $z=0$ divide space into eight cells, called **octants** (all positive \rightarrow first octant).

Perpendicular means 'loodrecht'

$z \geq 0$	The half-space consisting of the points on and above the xy-plane
$x = -3$	The plane perpendicular to the x-axis at $x = -3$. (parallel to yz-plane)
$z = 0, x < 0, y \geq 0$	The second quadrant of the xy-plane
$x \geq 0, y \geq 0, z \geq 0$	The first octant
$-1 \leq y \leq 1$	The slab between the planes $y = -1$ and $y = 1$ (planes included)
$y = -2, z = 2$	The line in which the planes $y = -2$ and $z = 2$ intersect
$x^2 + y^2 = 4$ and $z = 3$	The circle $x^2 + y^2 = 4$ in the plane $z = 3$

Distance between spheres $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ can be calculated with:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Standard equation for the Sphere of Radius a and Center (x_0, y_0, z_0) is:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

12.2 Vectors

A **vector** is a quantity such as force, displacement or velocity. It is represented by a **directed line segment**. A vector represented by the directed line segment \overrightarrow{AB} has **initial point A** and **terminal point B** and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction (so if their standard positions are the same). The **standard position** of a vector is the vector equal to the vector you want whose initial point is the origin.

If v is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) , then the **component form** of v is $v = \langle v_1, v_2 \rangle$.

If v is a **three-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the **component form** of $v = \langle v_1, v_2, v_3 \rangle$. The numbers v_1, v_2, v_3 are components of v .

The **magnitude or length** of a vector is given by the distance of the standard position of vector \overrightarrow{AB} .

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The only vector with length 0 is the zero vector $0 = \langle 0, 0 \rangle$ or $0 = \langle 0, 0, 0 \rangle$.

Two principal operations involving vectors are vector addition and scalar multiplication.

Addition: $u + v = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $ku = \langle ku_1, ku_2, ku_3 \rangle$

Addition of a vector is like adding to forces (remember the **parallelogram law**). The **resultant vector** is the diagonal of the parallelogram. Scalar multiplication is making the vector longer or shorter, without changing its angles (direction can change!). The difference $u-v$ of two vectors is defined by $u + (-v)$.

A vector v of length 1 is called the **unit vector**. The **standard unit vectors** are $i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$ and $k = \langle 0, 0, 1 \rangle$. Any vector v can be written as a linear combination of the standard unit vectors $v = \langle v_1i, v_2j, v_3k \rangle$, where v_1 is the **i-component**, v_2 the **j-component** and v_3 the **k-component**.

A vector can be written in terms of its two important features, length and direction, by writing

$v = |v| \frac{v}{|v|}$, where $|v|$ is the length of the vector and $\frac{v}{|v|}$ is its direction.

The **midpoint** of a vector can be found by averaging: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right)$

12.3 The Dot Product

The **angle** between two nonzero vectors u and v is given by: $\theta = \cos^{-1} \left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|u||v|} \right)$.

The **dot product** $u \cdot v$ is given by: $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$.

Using the notation of the dot product, the angle can also be given as $\theta = \cos^{-1} \left(\frac{u \cdot v}{|u||v|} \right)$.

Two nonzero vectors u and v are **perpendicular** or **orthogonal** if the angle between them is $\frac{\pi}{2}$, so if $u \cdot v = 0$.

$$u \cdot v = v \cdot u$$

$$(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$$

$$u \cdot (v + w) = u \cdot v + u \cdot w$$

$$u \cdot u = |u|^2$$

$$0 \cdot u = 0$$

The **vector projection** of u onto v is determined by dropping a perpendicular from the terminal point of u on the line through v . $proj_v u = (|u| \cos \theta) \frac{v}{|v|} = \left(\frac{u \cdot v}{|v|} \right) \frac{v}{|v|} = \left(\frac{u \cdot v}{|v|^2} \right) v$.

$|u| \cos \theta$ is called the **scalar component** of u in the direction of v . It can also be written as $\frac{u \cdot v}{|v|}$.

The equation $u = proj_v u + (u - proj_v u) = \left(\frac{u \cdot v}{|v|^2} \right) v + \left(u - \left(\frac{u \cdot v}{|v|^2} \right) v \right)$ expresses u as a sum of orthogonal vectors, where $\left(\frac{u \cdot v}{|v|^2} \right) v$ gives the projection parallel to v and $\left(u - \left(\frac{u \cdot v}{|v|^2} \right) v \right)$ gives the projection orthogonal to v .

Work (also known as the term of physics "arbeid") is given by the formula

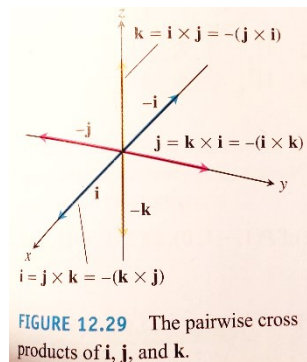
$$W = (\text{scalar component of } F \text{ in the direction of } D)(\text{length of } D) = (|F| \cos \theta)|D| = F \cdot d.$$

12.4 The Cross Product

If you have two nonzero vectors u and v which are not parallel, they determine a plane. The unit vector n perpendicular to the plane by the **right-hand rule** is given by the **cross product** $u \times v$:

$$u \times v = (|u||v| \sin \theta) n$$

Nonzero vectors are **parallel** if and only if $u \times v = 0$



$$\begin{aligned} (ru) \times (sv) &= (rs)(u \times v) \\ u \times (v + w) &= u \times v + u \times w \\ v \times u &= -(u \times v) \\ (v + w) \times u &= v \times u + w \times u \\ 0 \times u &= 0 \\ u \times (v \times w) &= (u \cdot w)v - (u \cdot v)w \\ i \times j &= -(j \times i) = k \\ j \times k &= -(k \times j) = i \\ k \times i &= -(i \times k) = j \end{aligned}$$

Because n is a unit vector, the magnitude of $n (u \times v)$ is (this can also be described as the area of the parallelogram determined by u and v):

$$|u \times v| = |u||v||\sin \theta||n| = |u||v| \sin \theta$$

$u \times v$ can be calculated with:

$$\begin{aligned} u \times v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} \\ v \times u &= -(u \times v) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \end{aligned}$$

The **torque vector** is given by $\text{torque vector} = (|r||F| \sin \theta)n$.

The magnitude of the torque vector is given by $|\text{torque vector}| = |r||F| \sin \theta$

The **triple scalar product** $((u \times v) \cdot w)$ of u , v and w is also the volume of a parallelepiped:

$|(u \times v) \cdot w| = |u \times v||w||\cos \theta|$ This can also be calculated with the determinant scheme as shown in the picture above (with an extra row under it).

12.5 Lines and Line Segments in Space

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector v . Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to v .

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to v is (Where r is the position vector of a point $P(x, y, z)$ on L and r_0 is the position vector of $P_0(x_0, y_0, z_0)$)

$$r(t) = r_0 + tv, -\infty < t < \infty$$

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $v = v_1i + v_2j + v_3k$ is

$$x = x_0 + tv_1, y = y_0 + tv_2, z = z_0 + tv_3, -\infty < t < \infty$$

The last formula can be rewritten as $r(t) = r_0 + tv = r_0 + t|v|\frac{v}{|v|}$. Where r_0 is the initial position, t is the time, $|v|$ is the speed and $\frac{v}{|v|}$ is the direction.

A plane in space is determined by knowing a point on the plane and its **tilt** or orientation. This tilt is defined by specifying a vector that is perpendicular or normal to the plane. The plane through $P_0(x_0, y_0, z_0)$ normal to $n = Ai + Bj + Ck$ has:

Vector equation: $n \cdot \overrightarrow{P_0P} = 0$
Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component equation simplified: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

Chapter 9 First-Order Differential Equations

9.1 Solutions, Slope Fields, and Euler's Method

A **First order differentials equation** is an equation $\frac{dy}{dx} = f(x, y)$. A **solution** of this equation is a differentiable function $y = y(x)$ such that $\frac{d}{dx}y(x) = f(x, y(x))$. The general solution is a solution that contains all possible solutions.

An antiderivative is called "primitieve" in Dutch. To find an antiderivative, we need a **particular solution**.

A **first-order initial value problem** is a differential equation $y' = f(x, y)$, whose solution must satisfy an initial condition $y(x_0) = y_0$. Each time we specify such an initial condition, a **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. These slopes can be pictured graphically in a **slope field/ direction field**.

9.2 First-Order Linear Equations

A first-order **linear** differential equation is one that can be written in the form $\frac{dy}{dx} + P(x)y = Q(x)$ where P and Q are continuous functions of x . Equations can be written as this **standard form**. This equation is linear, because y and its derivative dy/dx only occur to the first power.

Linear functions can be solved following these steps:

Write in standard form	$y' + Py = Q$
Find the integrating factor $v(x)$	$v = e^{\int P(x) dx}$
Multiply with v	$vy' + vPy = vQ$
Use $v' = vP$	$vy' + v'y = vQ$
Product rule	$(vy)' = vQ$
Integrate	$vy = \int v(x)Q(x) dx$
Divide by v	$y = \frac{1}{v} \int v(x)Q(x) dx$

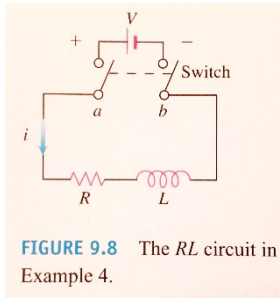


FIGURE 9.8 The RL circuit in Example 4.

This is a RL circuit, where the total resistance is a constant R ohms and whose self-inductance is L henries (a constant). Ohm's law ($V=RI$) has to be adapted for this circuit, the correct equation is: $L \frac{di}{dt} + Ri = V$ where I is the current ("stroom") (A) and t the time (s)

9.3 Applications

These first-order linear equations can be used to analyze an object moving along a straight line while subject to a force opposing its motion, making a model of population growth, or analyzing chemical concentrations entering and leaving a container.

Appendix 7 Complex Numbers

Complex numbers are expressions in the form $a + ib$ where a and b are real numbers and i is a symbol for $\sqrt{-1}$.

There are different kinds of numbers: **natural numbers (counting numbers/positive integers), integers, rational numbers, complex numbers.**

The **complex number system** consists of the set of all ordered pairs of real numbers (a, b) or $a+ib$, where a is the **real part** and b the **imaginary part**.

Equality: $a + ib = c + id$ if and only if $a = c$ and $b = d$

Addition: $(a + ib) + (c + id) = (a + c) + i(b + d)$

Multiplication: $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$ or $c(a + ib) = ac + i(bc)$

$a - ib$ is called the **complex conjugate** of $a + ib$ (also denoted by \bar{z}).

There are two geometric representations of the complex number $z = x + iy$ (where the x -axis is called the **real axis** and the y -axis is the **imaginary axis**):

1. As the point $P(x,y)$ in the xy -plane
2. As the vector \overrightarrow{OP} from the origin to P .

Both representations are **Argand diagrams** for $x + iy$ ($x = r \cos \theta$, $y = r \sin \theta$ and $z = x + iy = r(\cos \theta + i \sin \theta)$). The **absolute value** of a complex number $x + iy$ is the length r of a vector \overrightarrow{OP} : $|x + iy| = \sqrt{x^2 + y^2}$.

If we always choose the polar coordinates r and θ so that r is nonnegative, then $r = |x + iy|$. The polar angle θ is called the **argument** of z and is written $\theta = \arg z$.

Euler's formula is: $e^{i\theta} = \cos \theta + i \sin \theta$. With Euler's formula, z can also be written as: $z = re^{i\theta}$. Where r is the distance between the point z and the origin and θ is the angle. To multiply two complex numbers, we multiply their absolute values and add their angles:

$$z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2} \quad \text{where } |z_1| = r_1, \arg z_1 = \theta_1 \text{ and } |z_2| = r_2, \arg z_2 = \theta_2$$

$$\text{then } z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$|z_1 z_2| = r_1 r_2$$

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

De Moivre's Theorem is: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

If $z = re^{i\theta}$ is a complex number, different from zero and n is a positive integer, then there are precisely n different complex numbers w that are n th roots of z . All those n th roots are given by:

$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r} e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)}, k = 0, \pm 1, \pm 2$. All those n th roots lie on a circle centered at the origin and having radius equal to the real positive n th root of r .

$$\sqrt[4]{16 \exp i\pi} = 2 \exp i\left(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right)$$

$$w_0 = 2 \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right] = \sqrt{2}(1 + i)$$

$$w_1 = 2 \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right] = \sqrt{2}(-1 + i)$$

$$w_2 = 2 \left[\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right] = \sqrt{2}(-1 - i)$$

$$w_3 = 2 \left[\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right] = \sqrt{2}(1 - i)$$

The **Fundamental Theorem of Algebra** says that with the introduction of the complex numbers we now have enough numbers to factor every polynomial into a product of linear factor factors and so enough numbers to solve every possible polynomial equation.

Chapter 17 Second-Order Differential Equations

17.1 Second-Order Linear Equations

Suppose you have an **second-order equation** of the form $ay'' + by' + cy = d$. If $d = 0$, it is a **homogeneous** differential equation. If d does not equal 0, it is an **nonhomogeneous** equation. The **auxiliary equation / characteristic equation** of the second-order homogeneous equation is: $ar^2 + br + c = 0$ where $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$. If r_1 and r_2 are real and unequal roots, then $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is the **general solution** to that second-order homogeneous equation. If there is only one root, then in the solution, the same r is used twice.

If the discriminant is smaller than zero, you get $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ as complex roots. The general solution is then: $y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$.

The general solution contains two arbitrary constants, therefore it is necessary to specify two conditions. One way of doing this is to specify $y(x_0) = y_0$ and $y'(x_0) = y_1$. These conditions are called the **initial conditions**.