# Solutions Chapter 4 (Testing Hypotheses) – Mathematical Statistics

### Exercise 1

**a.** $\overline{x} = 26.25$ ($s = 6.7$, is in this case superfluous information, since σ = 6 is known)

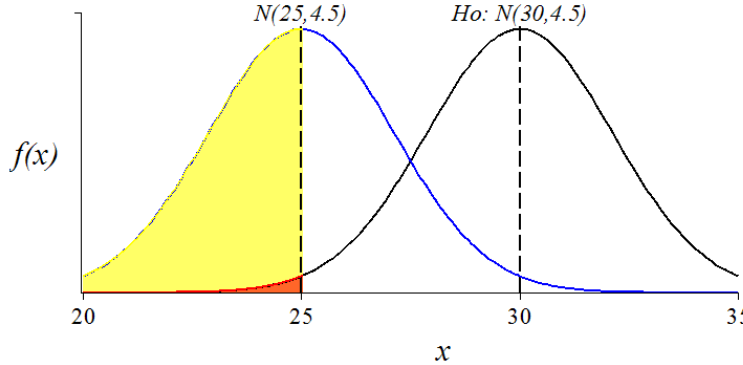
1. Model: the reductions in heating costs (in %) $X_1, \ldots, X_8$ are independent and $N(\mu, 6^2)$-distributed (with unknown mean reduction μ)
2. Hypotheses: we test $H_0: \mu = 30$ against $H_1: \mu < 30$ with α = 0.01
3. Test statistic $\overline{X}$
4. Distribution: $\overline{X}$ is under $H_0$ $N\left(30, \frac{36}{8}\right)$-distributed.
5. Observed value: $\overline{x} = 26.25$
6. This is a left-sided test: Reject $H_0$ if $\overline{X} \leq c$, $P(\overline{X} \leq c | H_0) = P(Z \leq \frac{c-30}{6/\sqrt{8}}) = 0.01$ (see the **red** area below the black $H_0$-graph of the density function below)
   So $\frac{c-30}{6/\sqrt{8}} = -2.33$, or $c = 30 - 2.33 \times 6/\sqrt{8} \approx 25.05$
7. $\overline{x} = 26.25$ is not contained in the Rejection Region ($\overline{X} \leq 25.05$), so we fail to reject $H_0$
8. From this sample we cannot conclude that the expected reduction in heating costs is smaller than 30%, at a 1% significance level.

**b.** **6.** Reject $H_0$ if $P(\overline{X} \leq \overline{x} | H_0) \leq \alpha = 0.01$,

where the p-value is $P(\overline{X} \leq 26.25 | H_0) = \Phi\left(\frac{26.25-30}{6/\sqrt{8}}\right) \approx \Phi(-1.77) = 1 - \Phi(1.77) = 3.84\%$

**7.** p-value = 3.84% > α, so we fail to reject $H_0$.
If $\alpha$ has a value of at least 3.84%, we can reject $H_0$.

**c.** The RR is the interval $(-\infty, 25.05]$




**d.** The power of the test for μ = 25 is the probability to reject $H_0$ if in reality μ = 25, so:
$P(\overline{X} \leq 25.05 | \mu = 25) = \Phi\left(\frac{25.05-25}{6/\sqrt{8}}\right) \approx \Phi(0.02) \approx 50.8\%$
*Note: the probability of a type II error for μ = 25 is the probability of not rejecting $H_0$ if in reality $\mu = 25$ (in $H_1$) is true, so:* $P(\overline{X} > 25.05 | \mu = 25) \approx 49.2\%$

### Exercise 2

**a.** The probability model: $X_1, \ldots, X_{100}$ are independent and all $N(\mu, 100^2)$-distributed.
The test statistic for testing $H_0: \mu = 475$ against $H_1: \mu > 475$ is $\overline{X}$.
$\overline{X}$ is under $H_0$ $N\left(475, \frac{\mathbf{100^2}}{\mathbf{100}}\right)$-verdeeld, so $\sigma_{\overline{X}} = \frac{100}{\sqrt{100}} = 10$
The value of $\overline{X}$ is 478, having z-score $\frac{478-475}{10} = 0.3$.
Therefore the p-value is $P(\overline{X} \geq 478) = P(Z \geq 0.3) = 1 - \Phi(0.3) = 0.38$. (38%)

Given a common choice of the level of significance (α between 1% and 10%), this result is "not statistically significant": we cannot reject $H_0$.

**b.** The probability model: $X_1, \ldots, X_{1000}$ are independent and all $N(\mu, 100^2)$-distributed.

The test statistic for testing $H_0: \mu = 475$ against $H_1: \mu > 475$ is $\overline{X}$.

$\overline{X}$ is under $H_0$ $N\left(475, \frac{100^2}{1000}\right)$-verdeeld, so $\sigma_{\overline{X}} = \frac{100}{\sqrt{1000}} = \sqrt{10}$

The value of $\overline{X}$ is 478, having z-score $\frac{478-475}{\sqrt{10}}$ = 0.949

Therefore the p-value is $P(\overline{X} \geq 478) \approx P(Z \geq 0.949) = 1 - \Phi(0.95) = 0.17$ (17%).

Given a common choice of the level of significance (α between 1% and 10%), this result is "not statistically significant": we cannot reject $H_0$.

**c.** The probability model: $X_1, \ldots, X_{10000}$ are independent and all $N(\mu, 100^2)$-distributed.

The test statistic for testing $H_0: \mu = 475$ against $H_1: \mu > 475$ is $\overline{X}$.

$\overline{X}$ is under $H_0$ $N\left(475, \frac{100^2}{10000}\right)$-verdeeld, so $\sigma_{\overline{X}} = \frac{100}{\sqrt{10000}} = 1$

The value of $\overline{X}$ is 478, having a z-score $\frac{478-475}{1}$ = 3.

Therefore the p-value is $P(\overline{X} \geq 478) = P(Z \geq 3) = 1 - \Phi(3) = 0.001$. (0.1%)

Given a common choice of the level of significance (α between 1% and 10%), this result is "statistically significant": we can reject $H_0$.

*Observe that samples with increasing sample sizes can "prove" that small (insignificant) differences are "statistically significant". This why not only the test result should be considered, but also, for instance, confidence intervals should be computed to put the measured difference in perspective: is the shown difference meaningful?*

**d.** Apply the formula $\overline{X} \pm c\frac{\sigma}{\sqrt{n}}$, where $\overline{x} = 478, c = 2.575$ such that $\Phi(c) = 1 - \frac{1}{2}\alpha = 0.995$,

$n = 10000$ and σ = 100. This results in: (475.43, 480.58)

*Comment: we indeed see that 475 , at a sample size 10 000, is not included in the confidence interval, indicating a significant deviation, even at a large confidence level. Nevertheless we can see that the improvement in the mean results are less than 5 (1% of the mean score 475).*

**Exercise 3**

**a.** *Remark: μ and σ are* ***not*** *given, but the estimates* $\overline{x} = 2.57$ *and* $s^2 = 2.02$ (*so* $s \approx 1.421!$) *are!*

1. Model: the observed service times are a realization of a random sample $X_1, \ldots, X_{42}$ drawn from a normal distribution with unknown expected service time μ and unknown $\sigma^2$.
   (*It follows from this model directly that we can apply the 1 sample t-test on the mean μ*)
2. Test $H_0: \mu = 1.98$ against $H_1: \mu \neq 1.98$ with α = 0.05
3. $T = \frac{\overline{X}-\mu_0}{S/\sqrt{n}} = \frac{\overline{X}-1.98}{S/\sqrt{42}}$
4. $T$ is $t_{41}$-distributed if $H_0$ is true.
5. Observed value: $t = \frac{2.57-1.98}{1.4.21/\sqrt{42}} \approx 2.69$
6. It is a Two-sided test: if $T \leq -c$ or $T \geq c$, then reject $H_0$.
   For α = 0.05 is $c$ = 2.021.
   (*use an approximation with the* $t_{40}$*-table:* $P(T_{40} \geq c) = 0.025$)
7. $2.69 = t > c = 2.021$, so reject $H_0$.
8. A change in the expected service time is proven at a 5% significance level.

*Alternative, using the p-value:*

6. The **two-tailed p-value** is $2 \times P(T_{41} \geq 2.69) < 2 \times P(T_{40} \geq 2.423) = 2\%$
   Or reason: $P(T_{41} \geq 2.69)$ lies between 0.5% and 1%, so the p-value $2 \cdot P(T_{41} \geq 2.69)$ is between 1% and 2%.
   *(Here the p-value is the probability of a deviation of at least 2.69, in positive of negative direction!)*
7. So the p-value < $\alpha$ = 5%: reject $H_0$.

**b.** *A test on the standard deviation can be conducted in the same manner as a test on the variance:*

1. Model: the observed service times are a realization of a random sample $X_1, \ldots, X_{42}$ drawn from a normal distribution with unknown expected service time $\mu$ and unknown $\sigma^2$.
   (*It follows from this model directly that we can apply the Chi square test on the variance* $\sigma^2$)
2. We test $H_0: \sigma^2 \leq 1$ against $H_1: \sigma^2 > 1$ with $\alpha_0 = 0.10$.
3. Test statistic $S^2$
4. Distribution (in the bound of) $H_0$: $\frac{41S^2}{1}$ is $\chi^2_{41}$-distributed.
5. The observed value $s^2 = 2.02$.
6. It is a right-sided test: **Reject $H_0$ if $S^2 \geq c$.**
   $P(S^2 \geq c|H_0) = P(41S^2 \geq 41c|H_0) = P(\chi^2_{41} \geq 41c) \leq \alpha_0 = 0.10$,
   If we round the number of degrees of freedom to $df = 40$, we find: $41c = 51.81$ or $c \approx 1.26$.
   (Using "linear interpolation" between $df = 40$ and 50, we find $41c = 52.95$ or $c \approx 1.28$)
7. $s^2 = 2.02$ lies inside the Rejection Region (> 1.26), so reject $H_0$.
8. At a 5% level of significance we showed that the standard deviation exceeds 1.

In an alternative approach we can determine the upper-tailed p-value at the observed $s^2 = 2.02$:

$P(S^2 \geq 2.02|H_0) = P\left(\frac{41S^2}{1} \geq \frac{41 \cdot 2.02}{1}|H_0\right) = P(\chi^2_{41} \geq 82.8) < 0.005$,

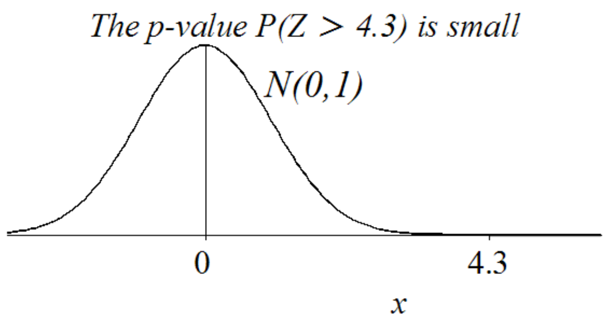
according to the $\chi^2$-table, since $P(\chi^2_{40} > 66.77) = 0.5\%$

The p-values asserts that we can reject the null hypothesis for any value of $\alpha$ , equal to or greater than 0.5%, so for $\alpha = 10\%$ as well.

**Exercise 4**

**a.** The 8 steps are given as follows:

1. If $X$ is the number of purchases decided by the mother among the 400 purchases in the sample, the $X$ has $B(400, p)$-distribution, where p is the probability that the mother decides.
2. Test $H_0: p = \frac{1}{2}$ against $H_0: p > \frac{1}{2}$ with $\alpha$ = 5%.
3. Test statistic: $X$
4. If $H_0$ is true, we have: $X \sim B\left(400, \frac{1}{2}\right)$,
   so approximately according to the CLT:
   $X \sim N(np_0, np_0(1 - p_0)) = N(200,100)$.
5. Observed value of the test statistic: $x = 243$.
   (*One can check that the alternative test statistic* $Z = \frac{X-200}{\sqrt{100}}$ *has the value 4.3 and that* $Z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
   *attains the value* $\frac{\frac{243}{400} - \frac{1}{2}}{\sqrt{\frac{0.5(1-0.5)}{400}}} = 4.3$ *as well*).




6. We reject $H_0$ if $X \geq c$. $P(X \geq c|H_0) \leq \alpha_0 = 0.05$.
   We will apply normal approximation with continuity correction:

$P(X \geq c|H_0) \overset{\text{c.c.}}{=} P(X \geq c - 0.5|H_0) \overset{\text{CLS}}{\approx} P\left(Z \geq \frac{c-0.5-200}{\sqrt{100}}\right) \leq 0.05$

and from $\Phi\left(\frac{c-0.5-200}{\sqrt{100}}\right) \geq 0.05$ it follows: $\frac{c-0.5-200}{\sqrt{100}} \geq 1.645$,

so $c \geq 200.5 + 10 \cdot 1.645 = 216.95$: $\boldsymbol{c = 217}$

7. The observed value 243 lies in the Rejection Region ($\geq 217$), so reject $H_0$.
8. We consider the statement "mothers decide on purchases in the majority of the cases" to be statistically confirmed at a level of significance 5%.

<u>Check with the p-value,</u> that is, with $P(X \geq 243|H_0) \approx P\left(Z \geq \frac{242.5-200}{10}\right) = 1 - \Phi(4.24)$
< 0.0001 (see graph) and much smaller than α.

(*the largest value in the table, at* $z = 3{,}69$, *has a tail probability* 1 - 0.9999 = 0.0001).

**b.** The probability of a type II for $p$ = 0.60 (computed with continuity correction):

$P(X < 217|p = 0.60) = P\left(Z \leq \frac{216.5-400\times0.60}{\sqrt{400\times0.6\times0.4}}\right) = P(Z \leq -2.40) = 1 - \Phi(2.40) = 0.82\%$

The power of the test for $p$ = 0.60 is $P(X \geq 217|p = 0.60) = 1 - 0.0082$ = 99.18%

**Exercise 5**

**a.** 1. Model: $X$ = "number of substandard products in a random sample of $n = 20$ products" ~ $B(20, p)$

2. Test $H_0: p \leq 0.1$ against $H_1: p > 0.1$ with $\alpha_0 = 0.05$
3. Test statistic is $X$.
4. Under $H_0$ $X$ is B(20, 0.1)-distributed.
5. $X = ?$
6. Reject $H_0$ if $X \geq c$, with $P(X \geq c|H_0) = P(B(20{,}0.1) \geq c) \leq \alpha_0 = 0.05$
   Since $c$ is integer, we have: $P(B(20{,}0.1) < c) = P(B(20{,}0.1) \leq c - 1) \geq 0.95$

From the $B(20{,}0.1)$-table we find: $P(X \leq 4) = 0.956$ and $P(X \leq 3)$ = 0.867, so $c - 1 = 4$ and $\boldsymbol{c} = \mathbf{5}$.

**b.** $\alpha = P(X \geq 5|H_0) = 1 - P(X \leq 4|H_0) = 1 - 0.956 = 4.4\%$

*(Note that in this computation the condition $H_0$ implies that we use its boundary value 0.1: for this value of p the probability of a type I error attains its largest value.).*

*So we have, for instance:* $P(X \geq 5|p = 0.05) = 1 - P(X \leq 4|p = 0.05) = 0.3\% < 4.4\%$)

**c.** Computation of the power of the test, $P(X \geq 5|H_1)$, for $p = 0.2, 0.3, 0.4$:

$\beta(0.2) = P(X \geq 5|p = 0.2) = 1 - P(X \leq 4|p = 0.2) = 1 - 0.630 = 27.0\%$

$\beta(0.3) = P(X \geq 5|p = 0.3) = 1 - P(X \leq 4|p = 0.3) = 1 - 0.238 = 76.2\%$

$\beta(0.4) = P(X \geq 5|p = 0.4) = 1 - P(X \leq 4|p = 0.4) = 1 - 0.051 = 94.9\%$

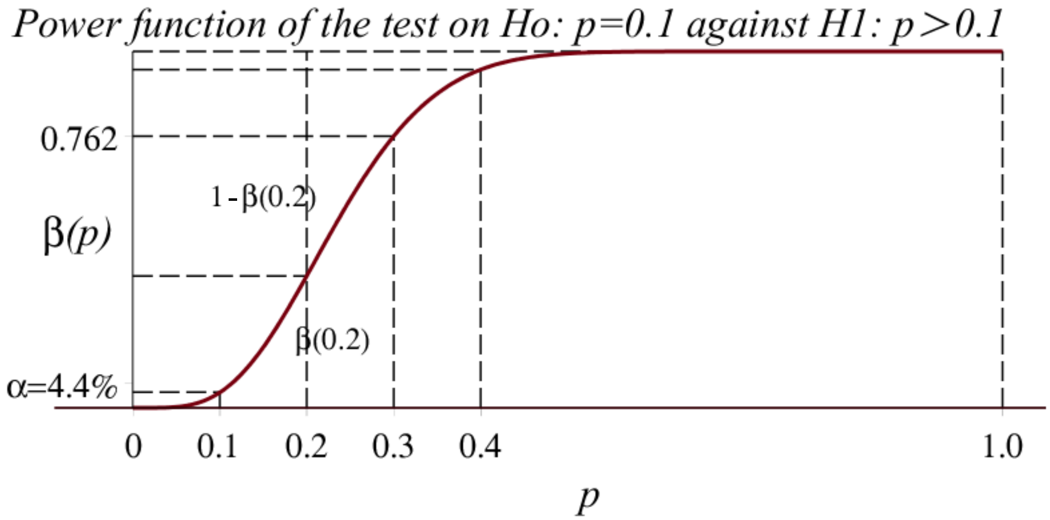
**d.** $\beta(0.05) = P(X \geq 5|p = 0.05) = 1 - P(X \leq 4|p = 0.05) = 1 - 0.997 = 0.3\%$

**e.** See the graph below.

**f.** The upper tailed p-value is $P(X \geq 4|p = 0.1) = 1 - P(X \leq 3|p = 0.1) = 1 - 0.867 = 13.3\%$

p-value = $13.3\% > 5\% = \alpha$, so we cannot reject $H_0$.

At a level of significance 5% we did not succeed to prove that the proportion of substandard products is greater than 10%.




**Exercise 6**

**a.** **(***Assumptions of Normality and independence*)
The lifetimes $X_1, \ldots, X_{20}$ of the 20 tires are independent and $N(\mu, \sigma^2)$-distributed with unknown μ and $\sigma^2$

**b.** **1.** Model: see a.
2. Test $H_0: \mu = 45000$ against $H_1: \mu < 45000$ with α = 0.01
3. $T = \frac{\overline{X}-\mu_0}{S/\sqrt{n}} = \frac{\overline{X}-45000}{S/\sqrt{20}}$
4. $T$ is $t_{19}$-distributed if $H_0$ is true.
5. Observed: $t = \frac{41000-45000}{4000/\sqrt{20}} \approx -4.47$
6. Left-sided test: if $T \leq c$, then reject $H_0$.
For α = 0.01 is $c$ = -2.539, such that $P(T_{19} \geq 2.539) = 0.01$
7. $-4.47 = t < c = -2.539$, so reject $H_0$.
8. The test showed that the expected lifetime of the tires is less than 45000 km, at 5% level of significance.

**c.** $\sigma = 1500$ *is equivalent to* $\sigma^2 = 1500^2$
1. Model: see a.
2. We tests $H_0: \sigma^2 = 1500^2$ against $H_1: \sigma^2 \neq 1500^2$ with $\alpha_0 = 0.05$.
3. Test statistic is $S^2$
4. Distribution under (the bound of) $H_0$: $\frac{19S^2}{1500^2}$ is $\chi^2_{19}$-verdeeld.
5. Observed: $s^2 = 4000^2 = 1.6 \times 10^7$.
6. This s a two-sided test: **reject $H_0$ if $S^2 \leq c_1$** or **$S^2 \geq c_2$**
$P(S^2 \geq c_2|H_0) = P\left(\chi^2_{19} \geq \frac{19c_2}{1500^2}\right) \leq \frac{\alpha}{2} = 0.025$, so $\frac{19c_2}{1500^2} = 32.85$.
Then $c_2 = \frac{32.85}{19} \cdot 1500^2 \approx 3.89 \times 10^6$.
Similarly we find $c_1 = \frac{8.91}{19} \cdot 1500^2 \approx 1.05 \times 10^6$.
7. $s^2 = 1.6 \times 10^7$is greater than $c_2$, so reject $H_0$.
8. At a 5% significance level we showed that the standard deviation of the lifetimes deviates from 4000.

**Exercise 7**

$X$ has a geometric distribution with probability function $P(X = x|p) = (1-p)^{x-1}p$, 0 < p < 1 and $x$ = 1, 2, …
The MP-test is the test, that rejects $H_0$ for **small**

$$r(x) = \frac{P\left(X = x|p = \frac{1}{6}\right)}{P\left(X = x|\, p = \frac{1}{3}\right)} = \frac{\left(\frac{5}{6}\right)^{x-1}\frac{1}{6}}{\left(\frac{2}{3}\right)^{x-1}\frac{1}{3}} = \frac{1}{2}\cdot\left(\frac{5}{6}\times\frac{3}{2}\right)^{x-1} = \frac{1}{2}\left(\frac{5}{4}\right)^{x-1}$$

sof or small $x$: reject $H_0$ if $X \leq c$.
Since under $H_0$ we have $P\left(X = 1|p = \frac{1}{6}\right) = \frac{1}{6}$ and $P\left(X = 2|p = \frac{1}{6}\right) = \frac{5}{36}$, so
$P(X \leq 1) = \frac{1}{6} < \alpha_0 = 0.20$ and $P(X \leq 2) = \frac{11}{36} > \alpha_0$, the MP-test is: reject $H_0$ if $X \leq 1$.
*Remark: we might fulfil the condition that* $\alpha = 0.20$ exactly, *by applying randomization: if* $X = 2$ *is observed (probability 5/36), then we should reject* $H_0$ with probability $\frac{0.2-1/6}{5/36}$. *In practice we will not apply randomization: it has only theoretical implications.*

**Exercise 8**

**a.** The density function is $f(x|\sigma^2) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2\sigma^2}$

Neymann-Pearson's ratio $r(x_1, \dots, x_{10}) = \frac{\prod f(x_i \mid 1)}{\prod f(x_i \mid 2)} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^{10} e^{-\frac{1}{2}\sum {x_i}^2}}{\left(\frac{1}{\sqrt{4\pi}}\right)^{10} e^{-\frac{1}{4}\sum {x_i}^2}} = 2^5 e^{-\frac{1}{4}\sum x_i^2}$ decreases as $\sum x_i^2$ increases. The **Most Powerful test** rejects $H_0: \sigma^2 = 1$ in favour of $H_1: \sigma^2 = 2$, if $\sum X_i^2 \geq c$
Under $H_0$ $X_i \sim N(0,1)$, so $Y = \sum_{i=1}^{10} X_i^2 \sim \chi^2(10)$.
Here $\alpha = 5\%$, so $P(\sum X_i^2 \geq c \,|\, \sigma^2 = 1) = 5\%$, if $c = 18.31$.

( $\sum X_i^2 \geq 18.31$ is equivalent to $r(X_1, \dots, X_{10}) = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^{10} e^{-\frac{1}{2}\sum {X_i}^2}}{\left(\frac{1}{\sqrt{8\pi}}\right)^{10} e^{-\frac{1}{8}\sum {X_i}^2}} \geq 2^5 e^{-\frac{1}{4}\cdot 18.31} = k$)

**b.** If $H_1: \sigma^2 = 4$, then $r(x_1, \dots, x_{10}) = 4^5 e^{-\frac{1}{8}\sum x_i^2}$ , so the conclusion (the test with statistic and accompanying rejection region) remains the same.

**c.** If $H_1: \sigma^2 = \sigma_0^2$, for any $\sigma_0^2 > 1$), then $r(x_1, \dots, x_{10}) = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^{10} e^{-\frac{1}{2}\sum {x_i}^2}}{\left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^{10} e^{-\frac{1}{2\sigma_0^2}\sum {x_i}^2}} = (\sigma_0^2)^5 e^{-\frac{1}{2}\left(1-\frac{1}{\sigma_0^2}\right)\sum x_i^2}$ is a decreasing function in $\sum x_i^2$ for all $\sigma_0^2 > 1$, so the conclusion is that for any $\sigma_0^2 > 1$, the test "Reject $H_0$ if $\sum_{i=1}^{10} X_i^2 \geq 18.31$" is most powerful.
Then "Reject $H_0$ if $\sum_{i=1}^{10} X_i^2 \geq 18.31$" is the **UMP test** for $H_0: \sigma^2 = 1$ against $H_1: \sigma^2 > 1$ with α = 5%.

**Exercise 9**

Test $H_0: \theta = 0.1$ against $H_0: \theta = 0.5$ if $f(x_i \mid \theta) = \frac{\theta^{x_i}e^{-\theta}}{x_i!}$, where $x_i = 0, 1, 2, \dots$.

$$r(x_1, \dots, x_n) = \frac{\prod f(x_i \mid 0.1)}{\prod f(x_i \mid 0.5)} = \frac{\frac{0.1^{\sum x_i}e^{-0.1n}}{x_1!\cdot\ldots\cdot x_n!}}{\frac{0.5^{\sum x_i}e^{-0.5n}}{x_1!\cdot\ldots\cdot x_n!}} = e^{0.4n}\cdot\left(\frac{0.1}{0.5}\right)^{\sum x_i},$$

which is an decreasing function in $\sum x_i$ (and $n = 10$). We will reject $H_0$ for small values of $r$:

$r(x_1, \dots, x_n) \le k \Leftrightarrow \ e^{0.4n} \cdot \left(\frac{1}{5}\right)^{\sum x_i} \le k \Leftrightarrow \ 0.4n + \sum x_i \ln\left(\frac{1}{5}\right) \le \ln(k) \ \Leftrightarrow \ \ \sum x_i \ge \frac{\ln(k)+0.4n}{\ln(5)} = c$

If $n = 10, \sum X_i \sim Poisson\ (\mu = 10\theta)$ and rejection region is $X \ge 3$, then:

$\alpha = P(\sum X_i \ge 3|\theta = 0.1) = 1 - P(X \le 2|\mu = 1) = 1 - 0.920 = 8.0\%$ (Poisson table)

$\beta(0.5) = P(\sum X_i \ge 3|\theta = 0.5) = 1 - P(X \le 2|\mu = 5) = 1 - 0.125 = 87.5\%$

**Exercise 10**

**a.** $f(x_i \mid \mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu)^2}$ : the maximum likelihood estimator ($mle$) of $\mu$ is $\overline{X}$.

$$\Lambda(x_1, \dots, x_n) = \frac{\prod_{i=1}^n f(x_i \mid 0)}{\sup\limits_{\mu \ge 0} \prod_{i=1}^n f(x_i \mid \mu)} = \frac{\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2}}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\overline{x})^2}} = \prod_{i=1}^n e^{-\overline{x}\cdot x_i + \frac{1}{2}\overline{x}^2} = e^{-\overline{x}\cdot\sum x_i + \frac{n}{2}\overline{x}^2}$$

$$= e^{-\overline{x}\cdot n\overline{x} + \frac{n}{2}\overline{x}^2} = e^{-\frac{n}{2}\cdot\overline{x}^2}$$

*Since supremum is determined for $\mu \ge 0$ we should use $max(0, \overline{x})$ as estimate. As a simplification we use $\overline{x}$ instead, assuming that $\overline{x} \ge 0$. Not that if $\overline{x} < 0$, then 0 is the mle of $\mu$ and $\Lambda$ attains its maximum value 1.*

The likelihood ratio test rejects $H_0$ for small values of $\Lambda$, so for large values of $\overline{x}^2$, and since $\Lambda$ is large if $\overline{x} < 0$, we will reject for large $\overline{x}$. The likelihood ratio test a right-sided rejection region "$\overline{X} \ge c$", such that $P(\overline{X} \ge c|\mu = 0) = \alpha_0$.

**b.** Determine $n$ such that $P(\overline{X} \ge c|\mu = 0) = \alpha_0 = 0.05$.

$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(0, \frac{1}{n}\right)$ under $H_0$, $P(\overline{X} \le c|\mu = 0) = \Phi\left(\frac{c}{1/\sqrt{n}}\right) = \Phi(c\sqrt{n}) = 1 - \alpha_0 = 0.95$, if $c\sqrt{n} = 1.645$ or $c = \frac{1.645}{\sqrt{n}}$.

**c.** Choose $\mu_0 < 0$, the probability of a type I error is:

$$P(\overline{X} \ge c|\mu = \mu_0) = 1 - P\left(\overline{X} \le \frac{1.645}{\sqrt{n}}\middle|\mu = \mu_0\right) = 1 - \Phi\left(\frac{\frac{1.645}{\sqrt{n}} - \mu_0}{\frac{1}{\sqrt{n}}}\right) = 1 - \Phi(1.645 - \mu_0\sqrt{n})$$

The more negative $\mu_0 < 0$ , the larger $1.645 - \mu_0\sqrt{n}$ en $\Phi(1.645 - \mu_0\sqrt{n})$ gets and the smaller the probability of a type I error, with a maximum bound $P(\overline{X} \ge c|\mu = 0) = \alpha_0 = 0.05$.

**d.** $\beta_n(1) = P\left(\overline{X} \ge \frac{1.645}{\sqrt{n}}\middle|\mu = 1\right) = P\left(Z \ge \frac{\frac{1.645}{\sqrt{n}} - 1}{\frac{1}{\sqrt{n}}}\right) = 1 - \Phi(1.645 - \sqrt{n}) \ge 0.99,$

if $\Phi(1.645 - \sqrt{n}) \le 0.01$, so if $1.645 - \sqrt{n} \le -2.33$, or $n \ge (1.645 + 2.33)^2 \approx 15.8$

The $n$ should be chosen (at least) 16.

**e.** For $n = 16$ the RR is $\overline{X} \ge \frac{1.645}{\sqrt{16}} = 0.41125$ and $\overline{X}$ has under $H_0$ a $N\left(0, \frac{1}{16}\right)$–distribution.

The power $\beta_{n=16}(\mu) = P(\overline{X} \ge 0.41125 \mid \mu) = 1 - \Phi\left(\frac{0.41125-\mu}{0.25}\right) = 1 - \Phi(1.645 - 4\mu)$

$\mu = -0.5:\ \beta_{n=16}(-0.5) = 1 - \Phi(1.645 + 2) \approx 1 - 0.9999 = 0.0001$

$\mu = 0:\quad \beta_{n=16}(0) = 1 - \Phi(1.645 + 0) = 0.05 = \alpha_0 \qquad (= 0.05)$

$\mu = +0.5:\ \beta_{n=16}(0.5) = 1 - \Phi(1.645 - 2) = \Phi(0.355) = 0.6387$

$\mu = 1:\ \beta_{n=16}(1) = 1 - \Phi(1.645 - 4) = \Phi(2.355) = 0.9908$

$\mu = 1.5:\ \beta_{n=16}(1.5) = 1 - \Phi(1.645 - 6) = \Phi(5.355) \approx 1.0000$

**f.** $\beta_n(\mu) = P\left(\overline{X} \ge \frac{1.645}{\sqrt{n}}\middle|\mu\right) = P\left(Z \ge \frac{\frac{1.645}{\sqrt{n}} - \mu}{\frac{1}{\sqrt{n}}}\right) = 1 - \Phi(1.645 - \mu\sqrt{n})$

If $\mu$ is positive we have: $1.645 - \mu\sqrt{n}$ is strongly negative for large $n$ sterk:

$$\lim_{n\to\infty} \beta_n(\mu) = 1 - \lim_{n\to\infty} \Phi(1.645 - \mu\sqrt{n}) = 1 - 0 = 1$$

If μ = 0 : $\lim_{n\to\infty} \beta_n(0) = \lim_{n\to\infty} [1 - \Phi(1.645)] = 0.05 = \alpha_0$.

If $\mu$ is negative, $1.645 - \mu\sqrt{n}$ attains for large $n$ large positive values:

$$\lim_{n\to\infty} \beta_n(\mu) = 1 - \lim_{n\to\infty} \Phi\big(1.645 - \mu\sqrt{n}\big) = 1 - 1 = 0$$

**Exercise 11**

**a.** $\int_{-\infty}^{\infty} f(x)dx = 1$, so

$\int_{-1}^{0} c(1+\theta x)dx + \int_{0}^{1} c(1-\theta x)dx = \left[c(x+\frac{1}{2}\theta x^2)\right]_{-1}^{0} + \left[c(x-\frac{1}{2}\theta x^2)\right]_{0}^{1} = c(1-\frac{1}{2}\theta+1-\frac{1}{2}\theta) = 1$,

if $c = \frac{1}{2-\theta}$.

For $\theta = 0$ $f(x;0) = \frac{1}{2}$, if $|x| \le 1$ (uniform on $[-1,1]$) and $f(x \mid 1) = 1 - |x|$ if $|x| \le 1$.

**b.** $r(x) = \frac{f(x\mid 0)}{f(x\mid 1)} = \frac{1/2}{1-|x|} = \frac{1}{2-2\cdot|x|}$, so the smaller $|x|$, the smaller $r(x)$: the MP-test rejects $H_0$ in favour of $H_1$ if $|X| \le c$, where $c$ depends on the value of $\alpha_0$:

$$\alpha_0 = P(|X| \le c | H_0 : \theta = 0) = 2 \cdot \frac{1}{2} c, \qquad \text{so } c = \alpha_0.$$

**c.** If $\alpha_0 = 0.25 = c$, $x = 0.75$ does not lie in the rejection region $|X| \le 0.25$: we fail to reject $H_0$.

**Exercise 12**

**a.** $L(x_1, \dots, x_n) = f(x_1 \mid \theta) \cdot \dots \cdot f(x_n \mid \theta) = e^{n\theta - (x_1 + .. + x_n)}$, if $x_1 \ge \theta, \; \dots\dots\; x_n \ge \theta$.

(Otherwise $L(x_1, \dots, x_n) = 0$) So $\theta \le \min(x_1, \dots, x_n)$

The maximum value of $\theta$ ($\min(x_1, \dots, x_n)$) results in a maximum of $l$, since $\frac{d}{d\theta} L > 0$.

Therefore $\hat{\theta} = \min(X_1, \dots, X_n)$ is the maximum likelihood estimator of $\theta$.

**b.** The distribution function of $\hat{\theta} = \min(X_1, \dots, X_n)$ can be derived as follows:

$P(\hat{\theta} \le x) = 1 - P(\hat{\theta} \ge x) = 1 - P(X_1 \ge x \text{ and} \dots \text{and } X_n \ge x)$

$= 1 - P(X_1 \ge x)^n = 1 - \left(e^{-(x-\theta)}\right)^n = 1 - e^{-n(x-\theta)}$ $(x \ge \theta)$

or: $f_{\hat{\theta}}(x) = ne^{-n(x-\theta)}$ $(x \ge \theta)$.

Note that $\hat{\theta} - \theta$ is exponentially distributed with parameter $\lambda = n$, so:

1. $E(\hat{\theta} - \theta) = \frac{1}{n}$ and $E(\hat{\theta}) = \theta + \frac{1}{n}$. $\lim_{n\to\infty} E(\hat{\theta}) = \theta$, so $\hat{\theta}$ is an asymptotically unbiased estimator.
2. $var(\hat{\theta} - \theta) = var(\hat{\theta}) = \frac{1}{n^2}$ $\quad \to \infty$ for $n \to \infty$

From 1. and 2 it follows that $\hat{\theta}$ is a consistent estimator of $\theta$.

**c.** $\Lambda(x_1, \dots, x_n) = \frac{\prod_{i=1}^{n} f(x_i \mid 0)}{\sup_{\theta \ge 0} \prod_{i=1}^{n} f(x_i \mid \theta)} = \frac{e^{0 - (x_1 + \dots + x_n)}}{e^{n\hat{\theta} - (x_1 + \dots + x_n)}} = e^{-n\hat{\theta}}$ is a decreasing function of $\hat{\theta}$.

We reject $H_0$ for small values of $\Lambda$, which is equivalent to large values of $\hat{\theta}$.

The likelihood ratio test for $H_0 : \theta = 0$ against $H_1 : \theta > 0$: **reject $\boldsymbol{H_0}$ if $\boldsymbol{\hat{\theta} = \min(X_1, .., X_n) \ge c}$** where $c$ is such that $P(\min(X_1, \dots, X_n) \ge c | H_0) \le \alpha_0$,

so $P(X_1 \ge c \text{ and} \dots \text{ and } X_n \ge c | H_0) = P(X_1 \ge c \,|H_0)^n = (e^{-c})^n = \alpha_0 \quad \Leftrightarrow c = -\frac{\ln(\alpha_0)}{n}$

**Exercise 13**

Note that $X$ has an exponential distribution (the parameter is $\frac{1}{\lambda}$ instead of λ), so $E(X) = \lambda$, having a $mle\ \hat{\lambda} = \overline{X}$

We apply the likelihood ratio formula (and use that $\sum x_i = n \cdot \overline{x}$):

$$\Lambda(x_1, \ldots, x_n) = \frac{\prod_{i=1}^{n} f(x_i \mid \lambda_0)}{\sup\limits_{\lambda>0} \prod_{i=1}^{n} f(x_i \mid \lambda)} = \frac{\left(\frac{1}{\lambda_0}\right)^n e^{-\frac{1}{\lambda_0}\sum x_i}}{\left(1/\overline{x}\right)^n e^{-\sum x_i/\overline{x}}} = \left(\frac{\overline{x}}{\lambda_0}\right)^n e^{-n\left(\frac{\overline{x}}{\lambda_0}-1\right)}$$

Considering $\Lambda$ to be a function of $y = \frac{\overline{x}}{\lambda_0}$, Λ is a function having a shape $f(y) = y^n e^{-ny+n}$

$f(y)$ is positive for $y > 0$ and decreases to 0 if $y$ approaches 0 and if $y \to \infty$ *(see the graph).*

So: small values $\Lambda(X_1, \ldots, X_n)$ coincide with small (positive) values and with vary large values of $\overline{X}$: if the critical values for $\overline{X}$ are chosen such that $P(\Lambda \leq \lambda | H_0) = \alpha_0$ , the test is the likelihood ratio test.
This test is two-sided: **If $\overline{X} \leq c_1$ or $\overline{X} \geq c_2$, reject $H_0$.**

The condition is: $P(\overline{X} \leq c_1 \text{ or } \overline{X} \geq c_2 | H_0) = \alpha_0$.

***Remark**: determining the critical value λ of Λ is often complicated (like in this case).*
*In practice we will solve this by taking both $P(\overline{X} \leq c_1)$ and $P(\overline{X} \geq c_2)$ equal $\frac{\alpha_0}{2}$.*
*Then we can determine $c_1$ and $c_2$ directly from the distribution of $\overline{X}$ (for large n we can apply the CLT!).*

In this case the CLT delivers approximations of $c_1$ and $c_2$ for large $n$. ($\overline{X}$ is appr. $N\left(1, \frac{1}{n}\right)$ under $H_0$).
For $\alpha_0 = 5\%$ we have:
$\left|\overline{X} - 1\right| \geq \frac{1.96}{\sqrt{n}}$, so $\overline{X} - 1 \leq -\frac{1.96}{\sqrt{n}}$ or: $\overline{X} - 1 \geq \frac{1.96}{\sqrt{n}}$, or: $\overline{X} \leq 1 - \frac{1.96}{\sqrt{n}} = c_1$ or: $\overline{X} \geq 1 + \frac{1.96}{\sqrt{n}} = c_2$

For example, if $n = 25$ we have $c_1 = 0.608$ and $c_2 = 1.392$.
If we use these values in $\Lambda(x_1, \ldots, x_{25})$, we find: $\Lambda = f(0.608) = 0.071$ and $\Lambda = f(1.392) = 0.216$:
the (real) critical $\lambda$ of $\Lambda$ is obviously between these values $0.071$ and $0.216$ (see the graph below).

**The critical value $\lambda$ of the likelihood ratio Λ**