Exercises Linear Optimization

- This year, these exercises are used for the second time. If you notice a mistake, please notify the tutor.
- **Challenges** are the most difficult exercises. These exercises are not necessary to pass the course, but are there for students for whom the other exercises are too easy. Don’t waste too much time on these exercises!

Session 1

1. Exercise 1.14 from BT
2. Consider the following LP:
   $\max x - y$
   $s.t.\, y - x \leq 2$
   $2y + x \geq 8$
   $x \leq 6$
   a. Visualize the feasible region graphically.
   b. Draw three level curves of the objective function and find the optimum graphically.
   c. Suppose we were to minimize $x - y$ over the feasible region instead. Graphically find the total number of optimal solutions.
   d. Formally prove (not graphically) that the solutions found in question c) are optimal.
3. Consider the following LP:
   $\max x + 2z$
   $s.t\; x + 2y + 3z = 6$
   $x \geq 0$
   $y \geq 0$
   $z \geq 0$
   a. Visualize the feasible region graphically.
   b. Draw three level curves of the objective function
   c. Find the optimum graphically.
4. Suppose $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are variables, and $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are parameters. Which of the following are linear constraints?
   a. $3x = 5$
   b. $3x + 3 \geq 5$
   c. $3 \leq 5$
   d. $3 \geq 5$
   e. $3 \geq x^2$
   f. $3 \geq x + y$
   g. $3 \geq 5xy$
   h. $3 \geq \min(x, y)$
   i. $3 \geq a^2$
   j. $3 \geq ax - by$
   k. $3 \geq \min(x, a)$
   l. $3 \geq \min(a, b)$
5. Consider the communication network problem on page 12 and 13 of BT. For each of the following, state whether it is a variable, parameter, index or set:
   a. $\mathcal{A}$
   b. k
   c. ℓ
   d. c_{ij}
   e. $x_{ij}^{k\ell}$
   f. $b_i^{k\ell}$
   g. u_{ij}
6. Exercises 1.11, and 1.15 from BT. Note that in these exercises, you only need to construct the LP, you do not need to solve it!

Session 2

1. Consider the following LP:

   $\max x + 3y$

   $s.t\ 2x + y \geq 2$

   $5x + 3y = 5$

   $3x - 4y \leq 9$

   $x \geq 0$

   a. Write this LP in general form.
   b. Write this LP in standard form.
2. Exercises 1.4, 1.6, 1.5, and 1.8 from BT.
3. Are the following statements true or false?
   a. We can rewrite any problem as an LP.
   b. We can rewrite any LP in standard form
   c. We can rewrite any LP in general form
   d. We can rewrite any problem that is linear except for absolute values, as an LP.
   e. Let A denote an LP in general form, and B an equivalent LP rewritten in standard form. A has the same feasible region as B.
   f. Let A denote an LP in general form, and B an equivalent LP rewritten in standard form. A has the same optimal value as B, if it exists.
   g. A practical way to solve large LP's is to visualize them and graphically find optima.

Session 3

1. Consider the following functions:
    i. $f(x) = \sin(x)$
    ii. $g(x) = 1 + x^2 + x^4$
    iii. $h(x) = \sqrt{x}$
    iv. $i(x, y) = -x^2 - y^2$
    v. $j(x, y) = x^2 - y^2$
    a. For each of these functions, state whether it is convex. If it is not convex, prove it.
    b. For each of these functions, state whether it is concave. If it is not concave, prove it.
2. Prove or give a counterexample: The union of two convex sets is a convex set.
3. Consider the following alternate proof of Theorem 2.1 (c). In this proof correct? If not, where does it fail?
   Consider a convex set S and let $\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n} \in S$. Let $\boldsymbol{x}' = \lambda_1 \boldsymbol{x^1} + \lambda_2 \boldsymbol{x^2} + \cdots + \lambda_n \boldsymbol{x^n}$, i.e. $\boldsymbol{x}'$ is a convex combination of a finite set of vectors in S. We now show that $\boldsymbol{x}'$ is also in S:
   Choose $\epsilon \in \mathbb{R}$ such that $(\lambda_1 + \epsilon)\boldsymbol{x^1} + (\lambda_2 + \epsilon)\boldsymbol{x^2} + \cdots + (\lambda_n + \epsilon)\boldsymbol{x^n} \in S$ and $(\lambda_1 - \epsilon)\boldsymbol{x^1} + (\lambda_2 - \epsilon)\boldsymbol{x^2} + \cdots + (\lambda_n - \epsilon)\boldsymbol{x^n} \in S$. Now
   $$\boldsymbol{x}' = \frac{1}{2}\big((\lambda_1 + \epsilon)\boldsymbol{x^1} + (\lambda_2 + \epsilon)\boldsymbol{x^2} + \cdots + (\lambda_n + \epsilon)\boldsymbol{x^n}\big) + \frac{1}{2}\big((\lambda_1 - \epsilon)\boldsymbol{x^1} + (\lambda_2 - \epsilon)\boldsymbol{x^2} + \cdots + (\lambda_n - \epsilon)\boldsymbol{x^n}\big).$$
   Therefore $\boldsymbol{x}' \in S$ as well, thereby proving the theorem.
4. Are the following statements true or false?
    a. There exist functions that are both convex and concave.
    b. There exist functions that are neither convex nor concave.
    c. Any function with a convex set as domain is convex.
    d. Any function with a convex set as domain is either convex or concave.
    e. Every convex function possesses a local minimum.
    f. No convex function possesses a local maximum.
    g. All local minima of convex functions are global minima.
    h. All convex sets are bounded.
    i. The intersection of two convex sets is a convex set.
    j. A hyperplane in $\mathbb{R}$ is a line
    k. A hyperplane in $\mathbb{R}^2$ is a line
5. Prove that $f(x) = x^2$ is a convex function.
6. Exercise 2.2 from BT

Session 4

1. Consider a polyhedron P defined by the constraints $y - 3x \leq 0, y + x \leq 8, y \geq 0$.
   a. Show graphically that $(x = 0, y = 0)$ is a vertex, using the definition of vertex.
   b. Prove that $(x = 2, y = 6)$ is a vertex, using the definition of basic feasible solution.
   c. Prove that $(x = 1, y = 1)$ is **not** a vertex, using the definition of extreme point.
   d. Prove that $(x = 0, y = 0)$ is a vertex, using the definition of vertex. Hint: Follow the proof that every Basic feasible solution is a vertex.
   e. Give parametric equations for all edges of P.
2. Are the following statements true or false?
   a. Every basic solution is a vertex.
   b. Every extreme point is a basic solution.
   c. Consider vertex $\boldsymbol{x} \in \mathbb{R}^n$, direction $\boldsymbol{d} \in \mathbb{R}^n$, and positive constant $\epsilon \in \mathbb{R}^+$. If $\boldsymbol{x} + \epsilon\boldsymbol{d}$ is feasible, then $\boldsymbol{x} - \epsilon\boldsymbol{d}$ is not feasible.
   d. Consider vertex $\boldsymbol{x} \in \mathbb{R}^n$, direction $\boldsymbol{d} \in \mathbb{R}^n$, and positive constant $\epsilon \in \mathbb{R}^+$. If $\boldsymbol{x} + \epsilon\boldsymbol{d}$ is not feasible, then $\boldsymbol{x} - \epsilon\boldsymbol{d}$ is feasible.
   e. Consider a polyhedron P with elements in $\mathbb{R}^n$. For all $\boldsymbol{d} \in \mathbb{R}^n$, $\max_{x \in P} \boldsymbol{d}\boldsymbol{x}'$ is attained at some vertex $\boldsymbol{x}^* \in P$.
   f. Consider a polyhedron P with elements in $\mathbb{R}^n$. For each vertex $\boldsymbol{x}^* \in P$, there exists some $\boldsymbol{d} \in \mathbb{R}^n$, such that $\max_{x \in P} \boldsymbol{d}\boldsymbol{x}'$ is attained at $\boldsymbol{x}^*$.
   g. Consider a polyhedron P with elements in $\mathbb{R}^n$. For all $\boldsymbol{d} \in \mathbb{R}^n$, if $\max_{x \in P} \boldsymbol{d}\boldsymbol{x}'$ is attained at some point $\boldsymbol{x}^* \in P$, then $\boldsymbol{x}^*$ is a vertex.
   h. Consider a set of linearly independent constraints $\{\boldsymbol{a}_1'\boldsymbol{x} = b_1, \ldots \boldsymbol{a}_n'\boldsymbol{x} = b_n\}$ that are active at some vertex $\boldsymbol{x}^*$. Then $\{\boldsymbol{a}_1', \boldsymbol{a}_2', \ldots \boldsymbol{a}_n'\}$ is a basis for $\mathbb{R}^n$.
   i. Consider a set of linearly independent constraints $\{\boldsymbol{a}_1'\boldsymbol{x} = b_1, \ldots \boldsymbol{a}_n'\boldsymbol{x} = b_n\}$ that are active at some vertex $\boldsymbol{x}^*$. Then $\{b_1, b_2, \cdots, b_n\}$ is a basis for $\mathbb{R}^n$.
   j. Every polyhedron contains at least one vertex.
   k. Every non-empty polyhedron contains at least one vertex.
3. Exercise 2.14 from BT. Hint: Graphically represent the problem in $\mathbb{R}^2$.

Session 5

1. For each of the polyhedra described by the following constraints, either find an equivalent description of the same polyhedron using linearly independent equalities, or conclude that it is empty.
   a. $x + 2y + 3z = 6$
      $x + y + 2z = 4$
      $2x + y + 4z = 3$
      $x - z = 2$
      $x, y, z \geq 0$
   b. $x + 2y + 3z = 6$
      $x + y + 2z = 4$
      $2x + y + 4z = 3$
      $x - z = 8$
      $x, y, z \geq 0$
2. Consider the following LP in general form. Rewrite it in standard form, and find one basic feasible solution.

   $\max x + y + z$
   s.t. $2x + y + z \leq 7$
   $x + z \leq 3$
   $x + 3y + 3z \leq 1$
   $x, y, z \geq 0$

3. Consider the three polyhedra described by the following sets of constraints (i,ii and iii)
   i. $x_1 + x_2 + 2x_3 + 2x_4 = 3$
      $2x_1 + x_2 + 3x_3 + x_4 = 5$
      $3x_1 + x_2 + 4x_3 + 4x_4 = 3$
      $x_1, x_2, x_3, x_4 \geq 0$
   ii. $x_1 + x_2 + 2x_3 + 2x_4 = 3$
      $2x_1 + x_2 + 2x_3 + x_4 = 5$
      $3x_1 + x_2 + 2x_3 + 4x_4 = 3$
      $x_1, x_2, x_3, x_4 \geq 0$
   iii. $x_1 + x_2 = 3$
      $2x_1 + x_2 + x_3 = 5$
      $3x_1 + x_2 + 4x_4 = 3$
      $x_1, x_2, x_3, x_4 \geq 0$
   a. Find a basic solution for polyhedron i, corresponding to basic variables (x_1, x_2, x_4).
   b. Find a basis for polyhedron iii, corresponding to the basic solution $(0\ 3\ 2\ 0)'$
   c. Find all basic solutions for all polyhedra. Feel free to use a computer for Gaussian elimination.
   d. Which of these solutions are feasible?
   e. Which of these solutions are degenerate?
   f. What is the maximum number of different basic solutions you can get in any standard form LP with 4 variables and 3 equality-constraints? For each of these LP's, comment on why it does not have that many different basic solutions.
   g. Construct a standard form LP with 4 variables, 3 equality-constraints and 4 different basic solutions.

4. Are the following statements true or false?
    a. Every LP in standard form with linearly dependent equality constraints is empty.
    b. Every LP in standard form with linearly independent equality constraints, is non-empty.
    c. Consider an LP in standard form with m linearly independent equality constraints and n variables.
        i. In any basic feasible solution, exactly $n - m$ variables are 0.
        ii. In any basic feasible solution, at least $n - m$ variables are 0.
        iii. In any basic solution, exactly $n - m$ variables are 0.
        iv. In any basic solution, at least $n - m$ variables are 0.
5. Exercise 2.8 from BT.
6. Exercise 2.9 from BT.
7. Exercise 2.10 from BT.
   Hint for 2.10a: Consider a graphical solution for the polyhedron $\boldsymbol{Ax} = \boldsymbol{b}$ (without nonnegativity constraints).
8. **Challenge:** Exercise 2.18 from BT.
9. **Challenge:** Exercise 2.6 from BT. Hint: Use that the polyhedron contains at least one vertex (Which you will prove in exercise 2.12).

Session 6

1. Consider the following LP:

$$\begin{aligned} &\text{Min } 2x_1 + x_2 + 3x_3 + 2x_4 \\ s.t.\ & x_1 + x_2 + x_3 + 2x_4 + 3x_5 - x_6 = 2 \\ & x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 - 2x_6 = 2 \\ & x_1 + 3x_2 + 2x_3 + 4x_4 + 4x_5 - 3x_6 = 3 \end{aligned}$$

   a. Find the basic solution $\boldsymbol{x}$ corresponding to basic variables (x_1, x_2, x_3).
   b. Compute all basic directions $\boldsymbol{d_4}, \boldsymbol{d_5}, \boldsymbol{d_6}$ for $\boldsymbol{x}$ and basic variables (x_1, x_2, x_3).
   c. Compute all reduced costs for $\boldsymbol{x}$ and basic variables (x_1, x_2, x_3).
   d. Which of $\boldsymbol{d_4}, \boldsymbol{d_5}, \boldsymbol{d_6}$ are feasible directions?
   e. Which of the solutions $\boldsymbol{x} + \boldsymbol{d_4}, \boldsymbol{x} + \boldsymbol{d_5}, \boldsymbol{x} + \boldsymbol{d_6}$ are feasible? Comment on the difference between this question, and question d.
2. Are the following statements true or false?
   a. In any basic solution, all variables are positive.
   b. In any basic solution, all variables are non-negative.
   c. When moving in a basic feasible direction, the objective increases.
   d. When moving in a basic feasible direction, the objective decreases.
   e. Consider two bases B, B' corresponding to the same basic solution $\boldsymbol{x}$. The reduced costs corresponding to $\boldsymbol{x}$ and B are equal to the reduced costs corresponding to $\boldsymbol{x}$ and B'.
   f. Look up the definition of conical hull. Denote by $\{\boldsymbol{d_1}, \boldsymbol{d_2}, \dots, \boldsymbol{d_n}\}$ the set of all basic feasible directions.
      i. The set of all feasible directions is $span(\boldsymbol{d_1}, \boldsymbol{d_2}, \dots, \boldsymbol{d_n})$
      i. The set of all feasible directions is $cone(\boldsymbol{d_1}, \boldsymbol{d_2}, \dots, \boldsymbol{d_n})$
      ii. The set of all feasible directions is $conv(\boldsymbol{d_1}, \boldsymbol{d_2}, \dots, \boldsymbol{d_n})$
3. Exercise 3.2 from BT.
4. Exercise 3.3 from BT.

Session 7

1. For each of the following LP's with basic variables B and basic feasible solutions $\boldsymbol{x}$, either:
   i. state that theorem 3.1 does not imply anything on the optimality of $\boldsymbol{x}$,
   ii. or use theorem 3.1 to show that $\boldsymbol{x}$ is optimal,
   iii. or use theorem 3.1 to show that $\boldsymbol{x}$ is not optimal.

   If you conclude that $\boldsymbol{x}$ is not optimal, use the reduced costs to find a better solution.

   a. $\text{Min } x_1 + 2x_2 + 3x_3 + 12x_4 + 8x_5 + 12x_6$
      $s.t. x_1 + x_4 + x_5 + 3x_6 = 4$
      $x_2 + 2x_4 + x_5 + 2x_6 = 4$
      $x_3 + 3x_4 + x_5 + x_6 = 4$
      $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$
      $\boldsymbol{x} = (4\ 4\ 4\ 0\ 0\ 0)', B = (x_1, x_2, x_3)$
   b. $\text{Min } x_1 + 2x_2 + 3x_3 + 20x_4 + 8x_5 + 12x_6$
      $s.t. x_1 + x_4 + x_5 + 3x_6 = 4$
      $x_2 + 2x_4 + x_5 + 2x_6 = 4$
      $x_3 + 3x_4 + x_5 + x_6 = 4$
      $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$
      $\boldsymbol{x} = (4\ 4\ 4\ 0\ 0\ 0)', B = (x_1, x_2, x_3)$
   c. $\text{Min } 5x_1 + x_2 + 2x_3 + 8x_4 + 5x_5 + 5x_6$
      $s.t. x_1 + x_2 + x_4 + x_5 + 3x_6 = 4$
      $2x_1 + 2x_4 + x_5 + 2x_6 = 4$
      $2x_1 + x_3 + 3x_4 + x_5 + x_6 = 4$
      $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$
      $\boldsymbol{x} = (2\ 2\ 0\ 0\ 0\ 0)', B = (x_1, x_2, x_3)$
   d. $\text{Min } 5x_1 + x_2 + 2x_3 + 6x_4 + 5x_5 + 5x_6$
      $s.t. x_1 + x_2 + x_4 + x_5 + 3x_6 = 4$
      $2x_1 + 2x_4 + x_5 + 2x_6 = 4$
      $2x_1 + x_3 + 3x_4 + x_5 + x_6 = 4$
      $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$
      $\boldsymbol{x} = (2\ 2\ 0\ 0\ 0\ 0)', B = (x_1, x_2, x_3)$

2. Are the following statements true or false?
   a. Denote by $\bar{\boldsymbol{c}}$ the reduced costs corresponding to a solution $\boldsymbol{x}$. If $\boldsymbol{x}$ is optimal and degenerate, then $\bar{\boldsymbol{c}} < \mathbf{0}$.
   b. Suppose there exists a vertex for which all basic feasible directions have at least one negative component. Then the feasible region is bounded.
   c. Suppose there exists a vertex for which all basic feasible directions have only non-negative components. Then no optimum exists.
   d. Denote by $\bar{\boldsymbol{c}}$ the reduced costs corresponding to a solution $\boldsymbol{x}$. If $\boldsymbol{x}$ is optimal and degenerate, then $\overline{\boldsymbol{c}_i} < \mathbf{0}$ for some i.
3. Exercise 3.6 from BT.
4. Exercise 3.7 from BT. In this exercise $\boldsymbol{c}$ denotes the cost vector of the original LP, and A denotes the matrix of equality constraints such that $\boldsymbol{Ax} = \boldsymbol{b}$ in the original LP.

Session 8

1. Consider the following LP

   $\text{Max}\, x + 3y$

   $s.t.\, x \leq 4$

   $y \leq 3$

   $y - x \leq 2$

   $x, y \geq 0$

   a. Graphically represent the feasible region of the LP.
   b. Rewrite the LP in standard form.
   c. Solve the LP using the simplex method, starting at the vertex where $x = 0, y = 0$ (and slack variables have the right values). If you can choose between multiple feasible directions with negative reduced cost, consider the following tie-breaking rules for choosing a column to enter the basis:
      i. Greatest (most negative) reduced cost.
      ii. Largest distance until next vertex
      iii. Smallest cost at next vertex.

      Which of these methods leads to the smallest number of steps in this case?
   d. Construct LP's where each of the other rules results in the smallest number of steps. (You can describe your answers graphically; you don't need to do simplex again.)

2. Consider the following LP:

   $\text{Max}\, x + 3y$

   $s.t.\, y - x \leq 2$

   $2y - x \geq -4$

   $x, y \geq 0$

   a. Graphically represent the feasible region of the LP.
   b. Rewrite the LP in standard form.
   c. Solve the LP using the simplex method, starting at the vertex where $x = 0, y = 0$.
   d. Use the final vector of basic directions and reduced costs to find a solution where the original LP has an objective value of 100.

3. Are the following statements true or false?
   a. Consider a basic feasible direction $\boldsymbol{d_i}$ of a vertex $\boldsymbol{x}$. There exists an $\epsilon > 0$ such that $\boldsymbol{x} + \epsilon \boldsymbol{d_i}$ is a feasible solution.
   b. Consider a basic feasible direction $\boldsymbol{d_i}$ of a vertex $\boldsymbol{x}$. There exists an $\epsilon > 0$ such that $\boldsymbol{x} + \epsilon \boldsymbol{d_i}$ is a vertex.
   c. Consider a basic feasible direction $\boldsymbol{d_i}$ of a vertex $\boldsymbol{x}$. Either there exists an $\epsilon > 0$ such that $\boldsymbol{x} + \epsilon \boldsymbol{d_i}$ is a vertex, or for all ϵ, $\boldsymbol{x} + \epsilon \boldsymbol{d_i}$ is a feasible solution.
   d. The simplex method always finds an optimal solution.
   e. In any non-degenerate LP, the simplex method finds an optimal solution if it exists.
   f. In any non-degenerate LP, if an optimal solution exists, then there exists a vertex at which it is attained.
   g. In any non-degenerate LP, if the optimum is attained at $\boldsymbol{x}$, then $\boldsymbol{x}$ is a vertex.

Session 9

1. Let $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 7 \\ 3 & -4 & 4 \end{bmatrix}$ and $\boldsymbol{A}^{-1} = \begin{bmatrix} 24 & -4 & -5 \\ 13 & -2 & -3 \\ -5 & 1 & 1 \end{bmatrix}$.

   Use this information to find $\boldsymbol{B}^{-1}$ for $\boldsymbol{B} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 7 \\ 1 & -4 & 4 \end{bmatrix}$.
2. Solve exercise 2c of session 8, using the full tableau implementation of the simplex method.
3. Consider the following LP:

   $\text{Min}\, x_1 + 3x_2 + 6x_3 + 4x_4 + 2x_5 + 5x_6$

   $s.t.\, x_1 + x_2 + 3x_3 + x_4 + 2x_5 - 2x_6 = 6$

   $x_1 + 2x_2 + 2x_3 - x_4 + x_5 + x_6 = 1$

   $x_1 + x_2 + 2x_3 + x_5 - x_6 = 3$

   $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

   a. Find the vertex corresponding to basic variables (x_1, x_3, x_4).
   b. Construct the initial tableau starting from this vertex.
   c. Solve the LP using the full tableau implementation of the simplex method.
4. Exercise 3.21 from BT.
5. Are the following statements true or false?
   a. For any LP, the full tableau implementation of the simplex method is approximately 1.5 times as fast as the naïve implementation.
   b. There exist LP's for which the full tableau implementation of the simplex method is 100 times as fast as the naïve implementation.
   c. For any LP, the full tableau implementation of the simplex method visits approximately 67% of vertices visited in the naïve implementation.
   d. There exist LP's for which the full tableau implementation of the simplex method visits only 1% of vertices visited in the naïve implementation.
   e. The full tableau implementation keeps track of the reduced costs at all times.
   f. The full tableau implementation never requires the direct computation of the inverse of a matrix.
6. Exercise 3.18 from BT.

Session 10

1. Order the following vectors lexicographically
    i. [1 2 3 4 5]
    ii. [0 1 2 3 4]
    iii. [100 0 0 0 0]
    iv. [1 2 3 4 4]
    v. [-3 8 -5 3 3]
2. Solve example 3.6 from BT, using the full tableau implementation of the simplex method with lexicographic pivoting rule instead.
3. Exercise 3.19 from BT.
4. Are the following statements true or false?
    a. If $\boldsymbol{a} >_L \boldsymbol{b}$ and $\boldsymbol{b} >_L \boldsymbol{c}$, then $\boldsymbol{a} >_L \boldsymbol{c}$.
    b. If $\boldsymbol{a} >_L \boldsymbol{b}$ then $\boldsymbol{a} + \boldsymbol{c} >_L \boldsymbol{b} + \boldsymbol{c}$.
    c. If $\boldsymbol{a} >_L \boldsymbol{b}$ then $\boldsymbol{A}\boldsymbol{a} >_L \boldsymbol{A}\boldsymbol{b}$, for any invertible matrix $\boldsymbol{A}$.
    d. In any LP, the simplex method with lexicographic pivoting rule finds an optimal solution if it exists.
    e. In any LP, if an optimal solution exists, then there exists a vertex at which it is attained.
    f. In any LP, if the optimum is attained at $\boldsymbol{x}$, then $\boldsymbol{x}$ is a vertex.
    g. In each iteration of the simplex method with lexicographic pivoting rule, the objective strictly decreases.
5. **Challenge:** Exercise 2.12 from BT. Hint: Use the extension to theorem 3.3 by defining a suitable objective function.
6. **Challenge:** Exercise 3.23 from BT. Hint: Consider a copy of this LP, except it does not contain the n-th variable.

Session 11

1. Consider the following LP:
   $\text{Min}\, x_1 + 3x_2 + 6x_3 + 4x_4$
   $s.t.\, x_1 + x_2 + 3x_3 + x_4 = 6$
   $x_1 + 2x_2 + 2x_3 - x_4 = 1$
   $x_1 + x_2 + 2x_3 = 3$
   $x_1, x_2, x_3, x_4 \geq 0$
   Use the first phase of the two phase simplex method to find a basic feasible solution and an initial tableau.
2. Exercise 3.20 from BT.
3. Are the following statements true or false?
   a. If the phase 1 LP is infeasible, then the phase 2 LP is infeasible.
   b. If the phase 1 LP is feasible, then the phase 2 LP is feasible.
   c. If we use a cost of $\sum_{i-1}^{n} iy_i$ instead of $\sum_{i-1}^{n} y_i$ in phase 1 (see page 116, step 2), then the two-phase simplex algorithm still works.
   d. Given enough time, the two-phase simplex algorithm with a lexicographic pivoting rule will find the optimum of any standard form LP, if it exists.
   e. Theoretically, the simplex algorithm is the fastest algorithm to solve large LP's, known at this time.
4. Exercise 3.22 from BT.

Session 13

1. Exercise 4.1 of BT
2. Consider the following (primal) LP:

   $\text{Min}\, x_1 + 3x_2 + 2x_3$
   $s.t.\ \ x_1 + 2x_2 \leq 1$
   $x_1 + 3x_3 \leq 3$
   $x_1, x_2, x_3 \geq 0$

   a. Construct its dual LP.
   b. Find a standard form LP that is equivalent to the primal LP, and construct its dual.
   c. Show that the duals constructed in a. and b. are equivalent.
   d. Consider any LP P of the form

      $\text{Min}\, \boldsymbol{c}'\boldsymbol{x}$
      $s.t.\ \ \boldsymbol{A_1 x} \leq \boldsymbol{b_1}$
      $\boldsymbol{A_2 x} = \boldsymbol{b_2}$
      $\boldsymbol{A_3 x} \geq \boldsymbol{b_3}$
      $\boldsymbol{x} \geq \boldsymbol{0}$

      , where $\boldsymbol{A_1}, \boldsymbol{A_2}, \boldsymbol{A_3}$ are matrices, and $\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3}$ are vectors. Denote its dual by D. Denote by P' a standard form LP equivalent to P, and denote its dual by D'. Prove that D and D' are equivalent.
3. Consider the following standard form (primal) LP and note that its rows are linearly dependent:

   $\text{Min}\, x_1 + x_2 + 2x_3$
   $s.t.\ \ x_1 - x_2 + 2x_3 = 1$
   $x_1 + x_3 = 2$
   $2x_1 + 2x_3 = 4$
   $x_1, x_2, x_3 \geq 0$

   a. Construct its dual LP.
   b. Find a standard form LP that is equivalent to the primal LP but has linearly independent rows, and construct its dual.
   c. Show that the duals constructed in a. and b. are equivalent.
   d. Consider any LP of the form

      $\text{Min}\, \boldsymbol{c}'\boldsymbol{x}$
      $s.t.\ \ \boldsymbol{Ax} = \boldsymbol{b}$
      $\boldsymbol{x} \geq \boldsymbol{0}$

      , where the rows of $\boldsymbol{A}$ are linearly dependent. For some j there exist $\lambda_1 \dots \lambda_j$ for which $\boldsymbol{a_j} = \sum_{i=1}^{j-1} \lambda_j \boldsymbol{a_j}$, and also $b_j = \sum_{i=1}^{j-1} \lambda_j b_j$. Denote its dual by D. Denote by P' the standard form LP where row j is omitted from P, and denote its dual by D'. Prove that D and D' are equivalent.
4. Consider the following standard form (primal) LP and note that its rows are linearly dependent:

   $\text{Min}\, x_1 + x_2 + 2x_3$
   $s.t.\ \ x_1 - x_2 + 2x_3 = 1$
   $x_1 + x_3 = 2$
   $2x_1 + 2x_3 = 3$
   $x_1, x_2, x_3 \geq 0$

   a. Construct its dual LP.
   b. Show that the dual is unbounded.

c. Consider any LP of the form

$\text{Min}\ \boldsymbol{c}'\boldsymbol{x}$

$s.t.\ \ \boldsymbol{Ax} = \boldsymbol{b}$

$\quad\ \ \boldsymbol{x} \geq \boldsymbol{0}$

, where the rows of $\boldsymbol{A}$ are linearly dependent. For some j there exist $\lambda_1 \dots \lambda_j$ for which $\boldsymbol{a_j} = \sum_{i=1}^{j-1} \lambda_j \boldsymbol{a_j}$, but $b_j \neq \sum_{i=1}^{j-1} \lambda_j b_j$. Prove that its dual is either infeasible or unbounded.

5. Are the following statements true or false?
    a. The dual contains a $\leq$ constraint for each $\geq$ constraint in the primal.
    b. The dual contains a free variable for each equality constraint in the primal.
    c. The dual contains an equality constraint for each free variable in the primal.
    d. The dual of the dual of a primal is the primal.
    e. The dual of a standard form LP is a standard form LP.

Session 14

1. Are the following statements true or false?
    a. If the primal has an optimal solution, then the dual has an optimal solution.
    b. If the primal is unbounded, then the dual is unbounded.
    c. If the primal is unbounded, then the dual is infeasible.
    d. If the primal is infeasible, then the dual is unbounded.
    e. If the dual is unbounded, then the primal is infeasible.
2. Exercise 4.5 from BT
3. Exercise 4.3 from BT

Session 15

1. Consider the following (primal) LP with optimum $x = 0, y = \frac{9}{4}, z = \frac{1}{4}$:

   $\min x + 2y + 3z$

   $s.t.\ \ x + y - z \geq 2$

   $-2x + y + 3z \geq 3$

   $x, y, z \geq 0$

   Construct its dual, and find the optimum to the dual, using complementary slackness. Use weak duality to check that your solution is indeed optimal.

2. Consider the following (primal) LP with optimum $x = 2, y = 2$:

   $\max 3x + 2y$

   $s.t.\ \ x + 2y \leq 6$

   $2x + y \leq 6$

   $x + y \leq 4$

   $2x \leq 5$

   $x, y \geq 0$

   a. Construct the dual to this LP.
   b. Use complementary slackness to find a dual variable that is 0 in the dual optimum.
   c. Since the primal optimum is degenerate, there are 3 possible systems of linear equalities that lead to the dual optimum. Construct and solve each system. Which system(s) correspond(s) to (an) optimal solution(s) to the dual?
   d. Use the simplex method on the primal to find a basis with non-negative reduced costs. Use this basis to construct a system of linear equalities that lead to the dual optimum.

3. Consider a linear program P in standard form with optimum $\boldsymbol{x}$ and denote its dual by D with optimum $\boldsymbol{p}$. Are the following statements true or false?

   a. For each nonzero component of $\boldsymbol{x}$ the corresponding dual constraint is active.
   b. For each zero component of $\boldsymbol{x}$ the corresponding dual constraint is not active.
   c. For each active constraint of P, the corresponding dual variable is non-zero.
   d. For each nonzero component of $\boldsymbol{p}$ the corresponding primal constraint is active.
   e. For each zero component of $\boldsymbol{p}$ the corresponding primal constraint is not active.
   f. For each non-active constraint of D, the corresponding primal variable is zero.
   g. For each active constraint if D, the corresponding primal variable is non-zero.

Session 16

Consider the oil refinery problem from exercises 1.16 and 3.21 in BT. Construct the dual for the problem in 3.21a, and use complementary slackness to compute shadow prices. How do the shadow prices relate to the answer found for 3.21c?