Mathematical Statistics II	**Winter 2023**

Assignment 4

Handed Out: Jan 12 2024 | *Due: Jan 18 2024*

- Feel free to talk to other students in the class when doing this assignment. You should, however, write down your solution yourself.
- Only homeworks **submitted in the first tutorial 2024** are graded.
- Note that this homework sheet is optional, in the end we will count the best 3 out of 4 homework assignments.

Task 1. Suppose that $\{W_t\}$ and $\{Z_t\}$ are independent and identically distributed (iid) sequences, with $P(W_t = 0) = P(W_t = 1) = 0.5$ and $P(Z_t = -1) = P(Z_t = 1) = 0.5$. Define the time series model

$$X_t = W_t(1 - W_{t-1})Z_t.$$

Show that $\{X_t\}$ is white noise but not iid noise.

Solution. First we show that $\{X_t\}$ is white noise. For that we need to compute its mean and autocovariance. For the mean we note that

$$\mathbf{E}(Z_t) = (-1) \cdot 0.5 + 1 \cdot 0.5 = 0$$

and therefore

$$\mathbf{E}X_t = \mathbf{E}(W_t(1 - W_{t-1})Z_t) = \mathbf{E}(W_t)(1 - \mathbf{E}(W_{t-1}))\mathbf{E}(Z_t) = 0,$$

where the last equality holds as the random variables are independent. For the autocovariance, we have

$$\begin{aligned} \gamma(s,t) &= \mathbf{E}(W_s(1 - W_{s-1})Z_s W_t(1 - W_{t-1})Z_t) \\ &= \mathbf{E}(W_s(1 - W_{s-1})W_t(1 - W_{t-1})) \cdot \mathbf{E}Z_s Z_t \end{aligned}$$

If $s \neq t$, then $\mathbf{E}(Z_s Z_t) = \mathbf{E}(Z_s) \cdot \mathbf{E}(Z_t) = 0$. Therefore $\{X_t\}$ is uncorrelated. If $s = t$, then

$$\mathbf{E}Z_s Z_t = \mathbf{E}Z_t^2 = (-1)^2 \cdot 0.5 + 1^2 \cdot 0.5 = 1$$

and therefore

$$\gamma(t,t) = \mathbf{E}\left(W_t^2(1 - W_{t-1})^2\right) = \mathbf{E}(W_t^2) \cdot (1 - \mathbf{E}(W_{t-1}^2)) = \frac{1}{4},$$

where we use that

$$\mathbf{E}(W_t^2) = 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5.$$

Therefore, $\{X_t\}$ has constant variance and is therefore white noise.

To show that $\{X_t\}$ is not iid note that $X_{t-1} = 0$ implies that $W_{t-1} = 1$, which implies that $X_t = 0$. Therefore

$$P(X_{t-1} = 1, X_t = 1) = 0.$$

As

$$\begin{aligned} P(X_{t-1} = 1) = P(W_t(1 - W_{t-2})Z_{t-1} = 1) &= P(W_t = 1)P((1 - W_{t-2}) = 1)P(Z_{t-1} = 1) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \end{aligned}$$

and

$$P(X_{t-1} = -1) = \frac{1}{8}$$

we have

$$P(X_{t-1} = 1)P(X_t = 1) = 1/64 \neq 0 = P(X_{t-1} = 1, X_t = 1),$$

X_t and X_{t-1} are not independent.

Task 2. Decide in the following, if the time series is a stationary process. If so, give the mean and the autocovariance function. Here, $\{W_t\}$ is i.i.d. $\mathcal{N}(0, 1)$

(a) $X_t = W_t - W_{t-3}$

(b) $X_t = W_3$

(c) $X_t = t + W_3$

(d) $X_t = W_t^2$

(e) $X_t = W_t W_{t-2}$.

Hint: You might use that $\mathbf{E}(W_t^4) = 3$.

Solution.

(a) $X_t = W_t - W_{t-3}$ is a stationary process: $\mathbf{E}X_t = \mathbf{E}W_t - \mathbf{E}W_{t-3} = 0$ and

$$\begin{aligned} \gamma(t + \tau, t) = \mathbf{E}X_{t+\tau}X_t &= \mathbf{E}((W_{t+\tau} - W_{t+\tau-3})(W_t - W_{t-3})) \\ &= \mathbf{E}(W_{t+\tau}W_t) - \mathbf{E}(W_{t+\tau-3}W_t) - \mathbf{E}(W_{t+\tau}W_{t-3}) + \mathbf{E}(W_{t+\tau-3}W_{t-3}) \\ &= \mathbf{1}_{[t+\tau=t]} - \mathbf{1}_{[t+\tau-3=t]} - \mathbf{1}_{[t+\tau=t-3]} + \mathbf{1}_{[t+\tau-3=t-3]} \\ &= 2 \cdot \mathbf{1}_{[\tau=0]} - \mathbf{1}_{[|\tau|=3]}, \end{aligned}$$

which is not depending on t.

(b) $X_t = W_3$ is a stationary process because $\mathbf{E}X_t = \mathbf{E}W_3 = 0$ and

$$\gamma(t+\tau, t) = \mathbf{E}X_{t+\tau}X_t = \mathbf{E}W_3^2 = 1.$$

(c) $X_t = W_3 + t$ is not a stationary process as the mean is not constant: $\mathbf{E}X_t = t$.

(d) $X_t = W_t^2$ is a stationary process: $\mathbf{E}X_t = \mathbf{E}W_t^2 = 1$ and

$$\mathbf{E}X_{t+\tau}X_t = \mathbf{E}W_{t+\tau}^2 W_t^2 = \begin{cases} 3, & \text{if } \tau = 0 \\ 1, & \text{if } \tau \neq 0. \end{cases}$$

(e) $X_t = W_t W_{t-2}$ is a stationary process: $\mathbf{E}X_t = \mathbf{E}W_t \mathbf{E}W_{t-2} = 0$ and

$$\gamma(t+\tau, t) = \mathbf{E}W_{t+\tau}W_{t+\tau-2}W_tW_{t-2} = \mathbf{1}_{\tau=0}$$

Task 3. For a moving average process of the form

$$X_t = W_{t-1} + 2W_t + W_{t+1},$$

where W_t are independent with zero mean and variance σ_w^2, determine the autocovariance and the autocorrelation function as a function of lag τ.

Solution. If $X_t = W_{t-1} + 2W_t + W_{t+1}$, then $\mathbf{E}X_t = 0$. As further $\mathbf{E}W_sW_t = 0$ for $s \neq t$,

$$\begin{aligned}
\gamma(t,t) &= \mathbf{E}X_t^2 = \mathbf{E}(W_{t-1} + 2W_t + W_{t+1})^2 = \mathbf{E}W_{t-1}^2 + 4\mathbf{E}W_t + \mathbf{E}W_{t+1}^2 = 6\sigma_w^2 \\
\gamma(t,t+1) &= \mathbf{E}(W_{t-1} + 2W_t + W_{t+1})(W_t + 2W_{t+1} + W_{t+2}) = 2\mathbf{E}W_t^2 + 2\mathbf{E}W_{t+1}^2 = 4\sigma_w^2 \\
\gamma(t,t+2) &= \mathbf{E}(W_{t-1} + 2W_t + W_{t+1})(W_{t+1} + 2W_{t+2} + W_{t+3}) = \mathbf{E}W_{t+1}^2 = \sigma_w^2.
\end{aligned}$$

and $\gamma(t, t+\tau) = 0$ for $\tau \geq 3$. By symmetrie, $\gamma(t, t-\tau) = \gamma(t, t+\tau)$. For the autocorrelation function, we saw above that $\gamma(t,t) = 6\sigma_w^2$ for all t. Therefore,

$$\rho(\tau) = \frac{\gamma(t, t+\tau)}{\gamma(t,t)}$$

and so $\rho(0) = 1$, $\rho(1) = \frac{2}{3}$, $\rho(2) = \frac{1}{6}$ and $\rho(\tau) = 0$ for $\tau \geq 3$.

Grading:

Task	1	2					3	Total
		(a)	(b)	(c)	(d)	(e)		
Points	2	1	0.5	0.5	0.5	0.5	2	7