Mathematical Statistics II	**Winter 2023**

Assignment 3

Handed Out: Dec 15 2023 | *Due: Dec 22 2023*

- Feel free to talk to other students in the class when doing this assignment. You should, however, write down your solution yourself.
- Only homeworks **submitted in the tutorial of week 6** are graded.

Task 1. Consider observing data X. Assume you have a parametric model, which depends on a parameter θ. The likelihood function is defined over values of $\theta \in \{-3, -2, 1, 0, 1, 2, 3, 4\}$. Likelihood and prior distribution values are given in the table below.

θ	-3	-2	-1	0	1	2	3	4
$L(\theta \mid X)$	0.5	2	1	3	1	3	2	0.5
$\pi(\theta)$	0.1	0.3	0.05	0.15	0.05	0.1	0.2	0.05

(a) Compute the posterior distribution over a valid range of θ.

(b) Compute the posterior mean.

Task 2. A researcher is trying to estimate the number of accidents per month within 100 feet of the Gervais Street/Assembly Street intersection in Columbia. She assumes a Poisson(λ) model for the number of accidents Y per month, so that the density function for Y given λ is

$$p(y|\lambda) = \frac{\lambda^y e^{-\lambda}}{y!}, \quad y = 0, 1, 2, \ldots, \ \lambda \geq 0$$

(a) She uses a standard exponential prior distribution for λ, i.e.,

$$\pi(\lambda) = \begin{cases} e^{-\lambda} & \lambda \geq 0 \\ 0 & \lambda < 0. \end{cases}$$

Derive the general form of a posterior distribution for λ given a random sample $y_1, \ldots, y_n$ for n months. Note that $X \sim \Gamma(\alpha, \beta)$ if the pdf is given by

$$f_{\alpha,\beta}(x) = \frac{x^{\alpha-1} e^{-\beta x} \beta^{\alpha}}{(\alpha - 1)!} = c \cdot x^{\alpha-1} \cdot e^{-\beta x}$$

for $\alpha > 0$. Then $EX = \alpha/\beta$.

(b) If she gathers the following accident counts from 15 randomly selected months

1 0 4 1 4 2 5 3 0 3 1 2 2 4 1

find the posterior mean and the 95% credible interval for λ using the standard exponential prior, along with these data.

Task 3. Consider the coin flipping problem, in which we treat the probability of heads θ as a random variable sampled from some prior distribution $\pi(\theta)$. We represent the ith coin flip by a random variable $X_i \in \{0, 1\}$, where $X_i = 1$ if the ith flip is a head. We assume that the X_i's are conditionally independent given θ.

(a) Suppose our prior distribution on θ is $\text{Beta}(h, t)$ for some $h, t > 0$ and that our sequence of flips $\mathcal{D}$ has n_h heads and n_t tails. Show that the posterior distribution of θ is $\text{Beta}(h + n_h, t + n_t)$.

(b) Give expressions of the MAP and the posterior mean estimate of θ. You may use the fact that a $\text{Beta}(a, b)$ distribution has mean $a/(a + \beta)$ and mode (i.e. the most likely value of the distribution) $(\alpha - 1)/(\alpha + \beta - 2)$.

(c) What happens to the MAP and the posterior mean estimate as the number of coin flips $n = n_h + n_t$ approaches infinity?

Grading:

	1		2		3			Total
	(a)	(b)	(a)	(b)	(a)	(b)	(c)	
Points	1	1	2	1	1	0.5	0.5	7