Mathematical Statistics II	**Winter 2023**

Assignment 2

Handed Out: Dec 1 2023 *Due: Dec 8 2023*

- Feel free to talk to other students in the class when doing this assignment. You should, however, write down your solution yourself.
- Only homeworks **submitted in the tutorial of week 4** are graded.

Task 1. A charter bus line has 48-passenger buses and 38-passenger buses. With X and Y denoting the number of miles traveled per day for the 48-passenger and the 38-passenger buses, respectively, the bus company is interested in testing the equality of the two distributions

$$H_0 : \quad F(z) = G(z).$$

The company observed the following data on a random sample of $n_1 = 10$ buses holding 48 passengers and $n_2 = 11$ buses holding 38 passengers.

$$X : 104\ 252\ 300\ 308\ 315\ 323\ 331\ 396\ 414\ 452$$
$$Y : 184\ 196\ 197\ 248\ 260\ 279\ 355\ 386\ 393\ 432\ 450.$$

What conclusion can you make about the equality of the two distribution functions? Run a Wald-Wolfowitz runs test. Note that $n_1, n_2 \geq 10$.

Solution. The combined order sample, with the x values in blue and the y values in red, looks like this:

13.00 15.50 16.75 17.25 17.50 19.00 19.25 19.75
20.50 20.75 21.50 22.00 22.50 22.75 23.50 24.75

Counting we see that there are 9 runs. We should reject the null hypothesis if the number of runs is smaller than expected. Using the normal approximation of R, with $n_1 = 10$ and $n_2 = 11$, we have

$$\mu_R = \frac{2n_1n_2}{n_1 + n_2} + 1 = \frac{2 \cdot 10 \cdot 11}{10 + 11} + 1 = 11.476$$

and

$$\begin{aligned} Var(R) &= \frac{2n_1 2n_2(2n_1n_2 - n_1 - n_2)}{(n_1 + n_2)^2 - n_1 - n_2)}(n_1 + n_2)^2(n_1 + n_2 - 1) \\ &= \frac{2 \cdot 10 \cdot 11(2 \cdot 10 \cdot 11 - 10 - 11}{(10 + 11)^2(10 + 11 - 1)} = 4.9637. \end{aligned}$$

Now we observe $r = 9$ runs. Therefore, the approximate p-value is

$$P \approx P\left[Z \leq \frac{9 - 11.476}{\sqrt{4.9637}}\right] = P[Z \leq -1.11] = 0.1135.$$

We fail to reject the null hypothesis at the level 0.05, because the p-value is greater than 0.05.

Task 2. [**Rock-Paper-Scissors**] Recall that in the Rock-Paper-Scissors game, rock beats scissors which beats paper which beats rock. We collect the choices made by 119 players on the first turn of the game. A player gains an advantage in playing this game if there is evidence that the choices made on the first turn are not equally distributed among the three options. Use a goodness-of-fit test to see if there is evidence that any of the proportions are different from 1/3.

Option Selected	Frequency
Rock	66
Paper	39
Scissors	14
Total	119

Solution. If we let p_r, p_p and p_s represent the proportions of rock, paper and scissor choices, the hypotheses are

$$\begin{aligned} H_0 &: \quad p_r = p_p = p_s = 1/3 \\ H_1 &: \quad \text{Some } p_i \neq 1/3. \end{aligned}$$

The expected count is $119(1/3) = 39.7$ for each cell. The chi-square statistic is

$$\begin{aligned} \chi^2 &= \frac{(66-39.7)^2}{39.7} + \frac{(39-39.7)^2}{39.7} + \frac{(14-39.7)^2}{39.7} \\ &= 17.4 + 0.01 + 16.6 \\ &= 34.01. \end{aligned}$$

The test statistic, $\chi^2 = 34.01$, lies very far in the tail of a chi-square distribution with 2 degrees of freedom, so the p-value is very close to zero. This gives a strong evidence that the choices made on the first turn of a rock-paper-scissors game are not equally likely. Comparing the expected counts to the observed counts it appears that "rock" is used more often and "scissors" are less frequent then expected. Unless your opponent also looked at this study, it might be smart to start with paper.

Task 3. The table below shows the hours of relief provided by two anagelsic drugs in 12 patients suffering from arthritis. Is there any evidence that one drug provides longer relief than the other? Run a Wilcoxon signed-rank test. Note that $n \geq 10$.

Case	Drug A	Drug B
1	2.0	3.5
2	3.6	5.7
3	2.6	2.9
4	2.6	2.4
5	7.3	9.9
6	3.4	3.3
7	14.9	16.7
8	6.6	6.0
9	2.3	3.8
10	2.0	4.0
11	6.8	9.1
12	8.5	20.9

Solution. For m denoting the median of the difference between Drug A and Drug B, our testing problem is the following

$$\begin{aligned} H_0 : &\quad m = 0 \\ H_1 : &\quad m \neq 0. \end{aligned}$$

Our actual differences (Drug B - Drug A) are

$$+1.5, +2.1, +0.3, -0.2, +2.6, -0.1, +1.8, -0.6, +1.5, +2.0, +2.3, +12.4,$$

ordering and ranking them leads to

Diff.	0.1	0.2	0.3	0.6	1.5	1.5	1.8	2.0	2.1	2.3	2.6	12.4
Rank R_i	1	2	3	4	5.5	5.5	7	8	9	10	11	12
Z_i	0	0	1	0	1	1	1	1	1	1	1	1

where $Z_i = \mathbf{1}(\text{Drug B} - \text{Drug A} \geq 0)$. Note that the difference 1.5 occured twice, therefore the corresponding rank is $\frac{5+6}{2} = 5.5$. Then the corresponding test statistic is

$$W = \sum_{i=1}^{12} Z_i R_i = 3 + 5.5 + 5.5 + 7 + 8 + 9 + 10 + 11 + 12 = 71.$$

We can use a normal approximation, i.e., that

$$W' = \frac{W - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} \sim \mathcal{N}(0,1).$$

As $\frac{n(n+1)}{4} = \frac{12 \cdot 13}{4} = 39$ and $\sqrt{\frac{n(n+1)(2n+1)}{24}} = \sqrt{\frac{13 \cdot 14 \cdot 25}{24}} = \sqrt{162.5}$, we have

$$W' = \frac{71 - 39}{\sqrt{162,5}} = 2.510.$$

This gives a two-sided p-value of $p = 2 \cdot 0.006 = 0.012$. There is a strong evidence that Drug B provides more relief than Drug A.

Grading:

Task	1	2	3	Total
Points	2	2	3	7