Mathematical Statistics II	**Winter 2023**
Assignment 1	
Handed Out: Nov 17 2023	*Due: Nov 24 2023*

- Feel free to talk to other students in the class when doing this assignment. You should, however, write down your solution yourself.
- Only homeworks **submitted in the tutorial of week 2** are graded.

Task 1: Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}^T = (2, -3, 1)$ and $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$.

(a) Find the distribution of $3X_1 - 2X_2 + X_3$.

(b) Relabel the variables if necessary and find a 2×1 vector $\mathbf{a}$ such that X_2 and $X_2 - \mathbf{a} \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ are independent.
Hint: Two vectors $\mathbf{Y}$ and $\mathbf{Z}$ are uncorrelated if $Cov(\mathbf{Y}, \mathbf{Z}) = 0$.

Solution:

(a) Let $\mathbf{a} = (3, -2, 1)^T$, then $\mathbf{a}\boldsymbol{X} = 3X_1 - 2X_2 + X_3$. Therefore,

$$\mathbf{a}^T \boldsymbol{X} \sim \mathcal{N}(\mathbf{a}^T \boldsymbol{\mu}, \mathbf{a}^T \Sigma \mathbf{a}),$$

where

$$\mathbf{a}^T \boldsymbol{\mu} = (3, -2, 1) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 13$$

and

$$\mathbf{a}^T \boldsymbol{\Sigma} \mathbf{a} = (3, -2, 1) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 9.$$

The distribution of $3X_1 - 2X_2 + X_3$ is $\mathcal{N}(13, 9)$.

(b) Let $\mathbf{a} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}^T$, then $Y = X_2 - \mathbf{a}^T \begin{pmatrix} X_1 \\ X_3 \end{pmatrix} = -aX_1 + X_2 - a_2 X_3$.

Now, let $A = \begin{pmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{pmatrix}$, then $A\boldsymbol{X} = \begin{pmatrix} X_2 \\ Y \end{pmatrix} \sim \mathcal{N}(A\boldsymbol{\mu}, A\Sigma A^T)$, where

$$\begin{aligned} A\Sigma A^T &= \begin{pmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & -a_1 \\ 1 & 1 \\ 0 & -a_2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -a_1 - 2a_2 + 3 \\ -a_1 - 2a_2 + 3 & a_1^2 - 2a_1 - 4a_2 + 2a_1 a_2 + 2a_2^2 + 3 \end{pmatrix}. \end{aligned}$$

Since we want to have X_2 and Y independent, this implies that $-a_1 - 2a_2 + 3 = 0$. So we have

$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \text{ for } c \in \mathbb{R}.$$

Task 2: Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu}^T = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{pmatrix}$. Which of the following random variables are independent? Explain.

(a) X_1 and X_2

(b) X_1 and X_3

(c) X_2 and X_3

(d) (X_1, X_3) and X_2

(e) X_1 and $X_1 + 3X_2 - 2X_3$

Solution:

(a) $\sigma_{12} = \sigma_{21} = 0$, X_1 and X_2 are independent.

(b) $\sigma_{13} = \sigma_{31} = -1$, X_1 and X_3 are not independent.

(c) $\sigma_{23} = \sigma_{32} = 0$, X_2 and X_3 are independent.

(d) We rearrange the covariance matrix and partition it. The new covariance matrix is the following

$$\Sigma^* = \left(\begin{array}{cc|c} 4 & -1 & 0 \\ -1 & 2 & 0 \\ \hline 0 & 0 & 5 \end{array}\right)$$

It is clear that $\begin{pmatrix} X_1 & X_3 \end{pmatrix}$ and X_2 are independent.

(e) Let $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{pmatrix}$, then $A\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_1 + 3X_2 - 2X_3 \end{pmatrix}$ and $A\boldsymbol{X} \sim \mathcal{N}(A\boldsymbol{\mu}, A\Sigma A^T)$, where

$$A\Sigma A^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 61 \end{pmatrix}.$$

It is clear that X_1 and $X_1 + 3X_2 - X_3$ are not independent.

Task 3: Suppose that

$$Y_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \epsilon_i, \quad i = 1, \ldots, n,$$

where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$. Notice that there is no intercept. Suppose that

$$\sum_i X_{1i} X_{2i} = 0.$$

Show that the least squares estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ from the multiple regression are the same as if we were to fit separate, simple regressions on X_1 and X_2.
Solution: Consider the multiple regression model

$$Y_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \epsilon_i, \quad i = 1, \ldots, n, .$$

To find the least squares estimator, we need to minimize

$$L(\beta_1, \beta_2) = \sum_{i=1}^{n} \left(Y_i - \beta_1 X_{1i} - \beta_2 X_{2i}\right)^2 .$$

Denote by $r_i = Y_i - \beta_1 X_{1i} - \beta_2 X_{2i}$ the residuals. To minimize L, we need to solve the following system of linear equations

$$\frac{\partial L}{\partial \beta_1} = -2 \sum_{i=1}^{n} X_{1i} r_i = 0$$
$$\frac{\partial L}{\partial \beta_2} = -2 \sum_{i=1}^{n} X_{2i} r_i = 0.$$

Now consider the simple regression models:

$$Y_i = \alpha_1 X_{1i} + \eta_i$$

and

$$Y_i = \alpha_2 X_{2i} + \xi_i.$$

The least squares estimators for the simple regression models are obtained by minimizing

$$L_1(\alpha_1) = \sum_{i=1}^{n} (Y_i - \alpha_1 X_{1i})^2 := \sum_{i=1}^{n} r_{1i}$$

and

$$L_2(\alpha_2) = \sum_{i=1}^{n} (Y_i - \alpha_2 X_{2i})^2 := \sum_{i=1}^{n} r_{2i},$$

respectively. The estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are then obtained by solving

$$\frac{\partial L}{\partial \alpha_1} = -2\sum_{i=1}^{n} X_{1i} r_{1i} = 0$$

and

$$\frac{\partial L}{\partial \alpha_2} = -2\sum_{i=1}^{n} X_{2i} r_{2i} = 0.$$

As $\sum_i X_{1i} X_{2i} = 0$, we have

$$-2\sum_{i=1}^{n} X_{1i} r_i = -2\sum_{i=1}^{n} X_{1i} Y_i - \beta_1 X_{1i}^2 - \beta_2 X_{1i} X_{2i} = -2\sum_{i=1}^{n} X_{1i}(Y_i - \beta_1 X_{1i}) = -2\sum_{i=1}^{n} X_{1i} r_{1i} = 0$$

and analogously

$$-2\sum_{i=1}^{n} X_{2i} r_i = -2\sum_{i=1}^{n} X_{2i} r_{2i} = 0.$$

Therefore, it follows that $\hat{\alpha}_1 = \hat{\beta}_1$ and $\hat{\alpha}_2 = \hat{\beta}_2$.

Grading:

	1		2					3	Total
	(a)	(b)	(a)	(b)	(c)	(d)	(e)		
Points	1	1	0.5	0.5	0.5	1	0.5	2	10