Solutions exercises of Chapter 8

Exercise 1 (see section 8.2)

Define the random vector $X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, then X has a multivariate normal distribution with expectation $\mu \times e$,
with e being the n-dimensonal (column-)vector with all elements being equal to 1,
and variance-covariance matrix equal to $\sigma^2 I_n$, shortly $X \sim N(\mu \times e\, , \sigma^2 I_n\,)$.

Consider an orthonormal basis $u_1, u_2, \ldots, u_n$ for the space of n-dimensional vectors with the first basis vector specified as $u_1 = \begin{pmatrix} 1/\sqrt{n} \\ 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{pmatrix}$.
Note that $u_1^T X, u_2^T X, \ldots, u_n^T X$ are the coefficients of X with respect to the basis $u_1, u_2, \ldots, u_n$.
So $X = (u_1^T X) \times u_1 + (u_2^T X) \times u_2 + \cdots + (u_n^T X) \times u_n$ and note

$$(u_1^T X) \times u_1 = \left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) \times \begin{pmatrix} 1/\sqrt{n} \\ 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{pmatrix} = \bar{X} \times e\ .$$

Note $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2}$, we rewrite its numerator

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \|X - \bar{X} \times e\|^2$$
$$= \left\|(u_1^T X) \times u_1 + (u_2^T X) \times u_2 + \cdots + (u_n^T X) \times u_n - \bar{X} \times e\right\|^2$$
$$= \left\|(u_2^T X) \times u_2 + (u_3^T X) \times u_3 + \cdots + (u_n^T X) \times u_n\right\|^2 = \sum_{i=2}^{n}\left(u_i^T X\right)^2$$

The random vector $W = \begin{pmatrix} u_2^T X \\ u_3^T X \\ \vdots \\ u_n^T X \end{pmatrix} = AX$, where $A = \begin{pmatrix} u_2^T \\ u_3^T \\ \vdots \\ u_n^T \end{pmatrix}$ has a multivariate distribution
with expectation $E(W) = A\, E(X) = A\, \mu \times e = \mu \times Ae = 0$
(note $e = \sqrt{n} \times u_1$ and $Au_1 = 0$ (vector))
and variance-covariance matrix $A\, \sigma^2 I_n A^T = \sigma^2 A A^T = \sigma^2 I_{n-1}$.
From the zero covariances we conclude that $u_2^T X, u_3^T X, \ldots, u_n^T X$ are independent.
Furthermore we conclude $u_i^T X \sim N(0, \sigma^2)$, $\frac{u_i^T X}{\sigma} \sim N(0,1)$ and $\frac{(n-1)S^2}{\sigma^2} = \sum_{i=2}^{n}\left(\frac{u_i^T X}{\sigma}\right)^2$ has the χ^2 distribution with $df = n-1$ degrees of freedom.

Exercise 2

In this exercise we only use the general property

(1) $E(Y^2) = (E(Y))^2 + var(Y)$

and the following general equality:

(2) $\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n} X_i^2 \ - \ n(\bar{X})^2$.

We have to show $E(S^2) = \sigma^2$.
Further elaboration:

$E(S^2) = E\left[\frac{\sum_{i=1}^n (X_i-\bar{X})^2}{n-1}\right] = \frac{1}{n-1}E[\sum_{i=1}^n (X_i-\bar{X})^2]$

$= \frac{1}{n-1}E[\sum_{i=1}^n X_i^2 \;-\; n(\bar{X})^2]$ because of (2)

$= \frac{1}{n-1}[\sum_{i=1}^n E(X_i^2) - n\,E(\bar{X}^2)]$

$= \frac{1}{n-1}\left[\sum_{i=1}^n (\mu^2+\sigma^2) - n\left(\mu^2+\frac{\sigma^2}{n}\right)\right]$ because of (1), note that $E(\bar{X}) = \mu$ and $var(\bar{X}) = \frac{\sigma^2}{n}$

$= \frac{1}{n-1}[n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2]$

$= \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2$

Exercise 3

a. $X_i \sim N(0,\sigma^2) \Rightarrow \frac{X_i}{\sigma} \sim N(0,1) \Rightarrow \frac{nS_{\mu=0}^2}{\sigma^2} = \sum_{i=1}^n \frac{X_i^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{X_i}{\sigma}\right)^2 = \sum_{i=1}^n Z_i^2$ has a χ_n^2-distribution, using the definition of the Chi-square distribution, since $Z_1 = \frac{X_1}{\sigma}, \ldots, Z_n = \frac{X_n}{\sigma}$ are independent and all $N(0,1)$.

b. The χ_n^2-distribution has expectation n and variance $2n$:

From $E\left(\frac{nS_{\mu=0}^2}{\sigma^2}\right) = n$ it follows that $E(S_{\mu=0}^2) = \sigma^2$ and

from $var\left(\frac{nS_{\mu=0}^2}{\sigma^2}\right) = 2n$ that $\left(\frac{n}{\sigma^2}\right)^2 var(S_{\mu=0}^2) = 2n$ or $var(S_{\mu=0}^2) = \frac{2\sigma^4}{n}$.

Hence $MSE(S_{\mu=0}^2) = var(S_{\mu=0}^2) = \frac{2\sigma^4}{n} \rightarrow 0$ if $n \rightarrow \infty$: $S_{\mu=0}^2$ is a consistent estimator of σ^2.

c. $f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x_i}{2\sigma^2}}$, then $L(\sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x_i^2}{2\sigma^2}} = (2\pi)^{-\frac{1}{2}n}(\sigma^2)^{-\frac{1}{2}n}e^{-\frac{\sum x_i^2}{2\sigma^2}}$ $(\sigma^2 > 0)$

The loglikelihood function $\ln L(\sigma^2) = -\frac{1}{2}n \cdot \ln(2\pi) - \frac{1}{2}n \cdot \ln(\sigma^2) - \frac{\sum x_i^2}{2\sigma^2}$ has derivative

$$\frac{d}{d(\sigma^2)}\ln L(\sigma^2) = -\frac{n}{2\sigma^2} + \frac{\sum x_i^2}{2(\sigma^2)^2} = -\frac{n}{2(\sigma^2)^2}\cdot\left(\sigma^2 - \frac{1}{n}\sum x_i^2\right) = 0 \text{ if } \sigma^2 = \frac{1}{n}\sum x_i^2 = \hat{\sigma}^2$$

Since the derivative is positive if $0 < \sigma^2 < \hat{\sigma}^2$ and negative if $\sigma^2 > \hat{\sigma}^2$, we found that $\hat{\sigma}^2 = \frac{1}{n}\sum X_i^2 = S_{\mu=0}^2$ is the mle of σ^2.

d. Reject $H_0: \sigma^2 \leq 10$ in favour of $H_1: \sigma^2 > 10$ if $S_{\mu=0}^2 \geq c$, such that

$P(S_{\mu=0}^2 \geq c | \sigma^2 = 10) = P\left(\frac{15S_{\mu=0}^2}{10} \geq \frac{15c}{10} \middle| \sigma^2 = 10\right) = \alpha = 5\%$, if $\frac{15c}{10} = 25.00$. Hence $c = 16.67$.

e. The likelihood ratio is defined as follows:

$$\Lambda(x_1,\ldots,x_n) = \frac{\sup_{0<\sigma^2\leq 10} L(\sigma^2)}{\sup_{\sigma^2>0} L(\sigma^2)}$$

From c. we can conclude that a maximum exists for both numerator and denominator:
If $0 < \hat{\sigma}^2 \leq 10$, then numerator and denominator have the same maximum $\sigma^2 = \hat{\sigma}^2$, so $\Lambda = 1$.
And if $\hat{\sigma}^2 > 10$ the numerator attains its maximum at bound $\sigma^2 = 10$ and the denominator at $\hat{\sigma}^2 = \frac{1}{n}\sum x_i^2$

$$\Lambda(x_1,\ldots,x_n) = \frac{L(10)}{L(\hat{\sigma}^2)} = \frac{(2\pi)^{-\frac{1}{2}n}(10)^{-\frac{1}{2}n}e^{-\frac{\sum x_i^2}{2\cdot 10}}}{(2\pi)^{-\frac{1}{2}n}\left(\frac{1}{n}\sum x_i^2\right)^{-\frac{1}{2}n}e^{-\frac{\sum x_i^2}{2\cdot\frac{1}{n}\sum x_i^2}}} = (10)^{-\frac{1}{2}n}\left(\frac{1}{n}\sum x_i^2\right)^{\frac{1}{2}n}e^{-\frac{\sum x_i^2}{20}+\frac{1}{2}n}$$

The function $g(y) = y^{\frac{1}{2}n}e^{-\frac{ny}{20}}$ attains its maximum at $y = \frac{1}{n}\sum x_i^2 = 10$ and decreases on $(10,\infty)$.

Furthermore $\Lambda(x_1,\ldots,x_n)$ does not decrease on values of $y = \frac{1}{n}\sum x_i^2 \leq 10$, since $\Lambda = 1$ in that case.

Hence the likelihood ratio test that rejects H_0 for small values of $\Lambda(X_1, \dots, X_n)$ is equivalent to the test that rejects H_0 for large values of $S^2_{\mu=0} = \frac{1}{n}\sum X_i^2$. (The critical value c follows from $n = 15$ and $\alpha = 5\%$.)