

Homework assignment 5 – Mathematical Statistics

Hand in your own solutions at the start of the tutorial on October 19.

A consumer organisation is assessing the performance of a call centre. One of the quality criteria is that at least 90% of the phone calls are answered within one minute, so at most 10% of the telephone calls should be answered after more than 1 minute. A survey is conducted by observing the waiting time for n calls. To test whether the proportion p of too late answers is significantly larger than 0.1 define X as the number of waiting times exceeding one minute.

- a) Let $p_1 > 0.1$. Show that the test that rejects $H_0 : p = 0.1$ against $H_1 : p = p_1$ for large values of X is most powerful and does not depend on p_1 .
- b) Using R, determine for $n = 25$ for the aforementioned test with level of significance $\alpha_0 = 0.05$:
 1. The rejection region,
 2. The probability of a type I error and
 3. The power of this test if in reality 15% of the calls are answered too late.

Hint: to compute quantiles and the c.d.f. of the binomial distribution, use the functions `qbinom()` and `pbinom()` in R.

- c) Using R, determine for $n = 900$ and for the test on $H_0 : p = 0.1$ against $H_1 : p > 0.1$ with $\alpha_0 = 0.05$:
 1. The rejection region,
 2. The probability of a type II error if in reality 15% of the calls are answered too late.
 3. The p-value if we observe 100 calls that are answered after 1 minute in the sample.
 4. Are the “verdicts” on the basis of the rejection region (1.) and the p-value (3.) the same?
- d) Apply a normal approximation to determine a sample size n , such that $\alpha_0 = 0.05$ and the power for $p = 0.15$ is at least 90%. (you can assume that n is sufficiently large for a normal approximation, no continuity correction is needed).

Grading:	a	b	c	d	Total
	2	1+1/2+1	1+1+1+1	1 1/2	10

Solutions:

- a) Neyman Pearson ratio is for any value $p_1 > 0.1$ (under $H_1 : p > 0.1$):

$$r(x) = \frac{P(X=x|p=0.1)}{P(X=x|p=p_1)} = \frac{\binom{n}{x} 0.1^x 0.9^{n-x}}{\binom{n}{x} p_1^x (1-p_1)^{n-x}} = \left(\frac{0.9}{1-p_1}\right)^n \left(\frac{1/9}{p_1/(1-p_1)}\right)^x = \left(\frac{0.9}{1-p_1}\right)^n \left(\frac{1-p_1}{9p_1}\right)^x$$

$\frac{1-p_1}{9p_1} < 1$, since $p_1 > 0.1$: the numerator $1-p_1 < 0.9$ and the denominator $9p_1 > 0.9$, which implies that $r(x)$ is a **decreasing function** of x . Hence: Since for every fixed value of p_1 the MP-test rejects for small values of $r(x)$, this is equivalent to rejecting for large values of X : for fixed p_1 the MP-test is:

“reject $H_0 : p = 0.1$ against $H_0 : p = p_1$ if $X \geq c$ ”, where c can be determined using $P(X \geq c | p = 0.1) = \alpha$: for given fixed α we will find the same value of c , no matter what p_1 is.

- b) 1. Right sided test (as was shown in a.): reject H_0 if $X \geq c$, where c is as small as possible but such that $P(X \geq c | p = 0.1) \leq \alpha_0 = 0.05$ or $P(X \leq c-1 | p = 0.1) \leq 0.95$.

Since $qbinom(0.95, 25, 0.1) = 5$, we find that: $c-1 = 5$, so $c = 6$

(One can achieve a slightly better test, using randomization, that is, if $c = 5$, then, we make a random selection of the null and alternative hypothesis).

2. $P(X \geq 6 | p = 0.1) = 1 - P(X \leq 5 | p = 0.1) = 1 - pbinom(5, 25, 0.1) = 0.033$

3. $P(X \geq 6 | p = 0.15) = 1 - P(X \leq 5 | p = 0.15) = 1 - pbinom(5, 25, 0.15) = 0.16$

- c) 1. Same as b.1. $qbinom(0.95, 900, 0.1) = 105$ and therefore $c = 106$.

2. The probability of a type II error for $p = 0.15$:

$$P(X < 106 | p = 0.15) = pbinom(105, 900, 0.15) = 0.0023$$

3. p-value for the observed $X = 100$:

$$P(X \geq 100 | p = 0.10) = 1 - pbinom(99, 900, 0.1) = 0.15$$

4. In 1. We will not reject H_0 , since $X = 100$ does not lie in the RR “ $X \geq 106$ ” In 3. we draw the same conclusion since the p-value 0.12 is not small enough: $0.12 > 0.05 = \alpha_0$.

- d) Now X is approximately $N(0.1n, 0.09n)$, so let us determine the rejection region, similarly to c.1: $P(X \leq c | p = 0.1) \approx \Phi\left(\frac{c-0.1n}{\sqrt{0.09n}}\right) \geq 95\%$, if $\frac{c-0.1n}{\sqrt{0.09n}} \geq 1.645$ So $c \geq 0.1n + 1.645 \cdot \sqrt{0.09n}$: we will continue with this approximated (possibly non-integer) boundary value of c .

The power of the test for $p = 0.15$ should be at least 90%:

$$P(X \geq c | p = 0.15) \approx 1 - \Phi\left(\frac{0.1n + 1.645 \cdot \sqrt{0.09n} - 0.15n}{\sqrt{0.15 \cdot 0.85n}}\right) = 1 - \Phi\left(1.645 \cdot \sqrt{\frac{0.09}{0.15 \cdot 0.85}} - \frac{0.05 \sqrt{n}}{\sqrt{0.15 \cdot 0.85}}\right) \geq 0.90$$

$$\text{Hence } 1.645 \cdot \sqrt{\frac{0.09}{0.15 \cdot 0.85}} - \frac{0.05 \sqrt{n}}{\sqrt{0.15 \cdot 0.85}} \leq -1.28, \text{ or } n \geq \frac{0.15 \cdot 0.85}{0.05^2} \cdot \left(1.28 + 1.645 \cdot \sqrt{\frac{0.09}{0.15 \cdot 0.85}}\right)^2 \approx 361.4.$$

n is (approximately) at least 362.