Homework assignment 3 – Mathematical Statistics

Hand in your own solutions at the start of the tutorial on September 28.

(a) Consider independent $X_1, \dots, X_n \sim \text{Unif}(0, \theta)$ for unknown parameter $\theta > 0$. Show that $T(X_1, \dots, X_n) = \max_{i=1,\dots,n} X_i$ is a sufficient statistic.

(b) Show that the p.d.f. of $T(X_1, \dots, X_n)$ is given by

$$x \mapsto n\theta^{-n} x^{n-1} \mathbf{1}(0 \le x \le \theta).$$

Hint: Compute first the c.d.f.

(c) Let us use now $\widehat{\theta} = \max_{i=1,\dots,n} X_i$ as an estimator for θ (for this reason we rename T into $\widehat{\theta}$). Compute the bias, the variance and the MSE (mean squared error) of $\widehat{\theta}$.

(d) Based on the analysis in part (c), propose now an unbiased estimator. Compute the MSE of this unbiased estimator. Is it smaller than MSE($\widehat{\theta}$)?

Grading:	**a**	**b**	**c**	**d**	**Total**
	2	2	3	3	**10**

Solutions:

a) The joint density is

$$\begin{aligned}\prod_{i=1}^{n} \theta^{-1} \mathbf{1}(0 \le x_i \le \theta) &= \theta^{-n} \mathbf{1}\Big(0 \le x_i \le \theta, \text{ for all } i\Big) \\ &= \theta^{-n} \mathbf{1}\Big(0 \le \min_i x_i \le \max_i x_i \le \theta\Big) \\ &= \theta^{-n} \mathbf{1}\Big(0 \le \min_i x_i\Big) \mathbf{1}\Big(\max_i x_i \le \theta\Big).\end{aligned}$$

That $\max_i X_i$ is a sufficient statistic follows now from the factorization theorem applied with $h(x_1, \dots, x_n) = \mathbf{1}\Big(0 \le \min_i x_i\Big)$ and $g_\theta(T(x_1, \dots, x_n)) = \theta^{-n} \mathbf{1}\Big(\max_i x_i \le \theta\Big)$.

b) We first compute the c.d.f.

$$F_T(x) = P(T \le x) = P(\max_{i=1,\dots,n} X_i \le x) = P(X_1 \le x, \dots, X_n \le x) = \prod_{i=1}^{n} P(X_i \le x) = \prod_{i=1}^{n} F_{X_i}(x).$$

Since $X_1, \dots, X_n$ are uniform on $[0,\theta]$ the density of an individual observation is $\theta^{-1}\mathbf{1}(0 \le x \le \theta)$ and the c.d.f. of an individual observation is $F_{X_i}(x) = P(X_i \le x) = \int_0^x \theta^{-1}\mathbf{1}(0 \le u \le \theta)\,du$. Therefore $F_{X_i}(x) = 0$ for $x < 0$, $F_{X_i}(x) = x/\theta$ for $0 \le x \le \theta$ and $F_{X_i}(x) = 1$ for $x > \theta$.

Thus, $F_T(x) = 0$ for $x < 0$, $F_T(x) = (x/\theta)^n$ for $0 \le x \le \theta$ and $F_T(x) = 1$ for $x > \theta$. The p.d.f. is the derivative of the c.d.f. Therefore,

$$f_T(x) = F_T(x)' = n\theta^{-n}x^{n-1}\mathbf{1}(0 \le x \le \theta).$$

c) Thanks to the previous part, for any $k > 0$,

$$E[(\widehat{\theta})^k] = \int x^k n\theta^{-n}x^{n-1}\mathbf{1}(0 \le x \le \theta)\,dx = \frac{n}{n+k}\theta^{-n}\theta^{n+k} = \frac{n}{n+k}\theta^k.$$

This means the bias is

$$\mathrm{Bias}(\widehat{\theta}) = E[\widehat{\theta}] - \theta = \frac{n}{n+1}\theta - \theta = -\frac{1}{n+1}\theta. \tag{1}$$

The variance is

$$\mathrm{Var}(\widehat{\theta}) = E[\widehat{\theta}^2] - (E[\widehat{\theta}])^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{(n+2)(n+1)^2}\theta^2.$$

Bias-variance decomposition yields for the MSE

$$\mathrm{MSE}(\widehat{\theta}) = \mathrm{Bias}(\widehat{\theta})^2 + \mathrm{Var}(\widehat{\theta}) = \left(\left(-\frac{1}{n+1}\right)^2 + \frac{n}{(n+2)(n+1)^2}\right)\theta^2 = \frac{2}{(n+2)(n+1)}\theta^2.$$

d) Equation 1 shows that one has to look at the estimator $\widetilde{\theta} := (n+1)\widehat{\theta}/n$. Then

$$\mathrm{Bias}(\widetilde{\theta}) = E[\widetilde{\theta}] - \theta = \frac{n+1}{n}E[\widehat{\theta}] - \theta = 0.$$

The variance of this estimator is

$$\mathrm{Var}(\widetilde{\theta}) = \left(\frac{n+1}{n}\right)^2 \mathrm{Var}(\widehat{\theta}) = \left(\frac{n+1}{n}\right)^2 \frac{n}{(n+2)(n+1)^2}\theta^2 = \frac{1}{n(n+2)}\theta^2.$$

Bias-variance decomposition yields then

$$\mathrm{MSE}(\widetilde{\theta}) = \mathrm{Bias}(\widetilde{\theta})^2 + \mathrm{Var}(\widetilde{\theta}) = \frac{1}{n(n+2)}\theta^2.$$

Since $1/n \le 2/(n+1)$, the MSE of the estimator $\widetilde{\theta}$ is smaller.