Homework assignment 2 – Mathematical Statistics

Because there is some R involved, students should hand in the answers via an assignment on canvas. Deadline is September 21 at 9 am.

For $\alpha > 0$, consider the p.d.f.

$$f(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}$$

(a) Compute the quantile function of the distribution with this p.d.f.

(b) For which values of α does the skewness exist? If it exists compute a closed-form expression depending on α. (Any expression only based on α works, no need to simplify it!)

Work with the dataset `Cholera.csv` and write your answers in an R script. You can include this script with the other files within one pdf file (you can check out the lecture notes how to include R code in a pdf).

(c) Select all the rows in the data that correspond to the year 1960.

(d) Make a histogram of all cholera numbers reported for 1980.

(e) Now take the logarithm of the cholera numbers for the year 1980 and make a histogram of those. Add the density of a normal density with mean and variance matching the mean and the variance of the data. Is the normal distribution a good fit?

Grading:	**a**	**b**	**c**	**d**	**e**	**Total**
	2	3	1	1	3	**10**

Solutions:

a) The cumulative distribution function (c.d.f.) is

$$F(x) = \begin{cases} 1 - \left(\frac{1}{x}\right)^{\alpha} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}$$

The quantile function, $Q(p)$, is found by setting $F(x) = p$ and solving for x. The solution is

$$Q(p) = \left(\frac{1}{1-p}\right)^{\frac{1}{\alpha}}.$$

b)

$$\text{Skewness}(\alpha) = E\left(\left(\frac{X-\mu}{\sigma}\right)^3\right) = \frac{E(X^3) - 3E(X^2)\mu + 3\mu^3 - \mu^3}{\sigma^3}.$$

For this expression to exist, the first three moments $E(X^k)$ with $k = 1,2,3$ need to exist. We see that $\int_1^\infty x^k x^{-\alpha-1}\,dx$ is finite if $k < \alpha$. In this case

$$E(X^k) = \int_1^\infty x^k x^{-\alpha-1}\,dx = \frac{1}{\alpha - k}.$$

This means that the skewness exists if and only if $\alpha > 3$. In this case, the variance is $\sigma^2 = \text{Var}(X) = E(X^2) - (EX)^2 = 1/(\alpha-2) - 1/(\alpha-1)^2$. Therefore, we obtain the expression

$$\text{Skewness}(\alpha) = \frac{1/(\alpha-3) - 3/((\alpha-2)(\alpha-1)) + 2/(\alpha-1)^3}{(1/(\alpha-2) - 1/(\alpha-1)^2)^{3/2}}.$$