Homework assignment 1 – Mathematical Statistics 2023
Hand in your own solutions at the start of the tutorial on September 14.
Consider a continuous random variable X with the following probability density function:

$$f(x) = \begin{cases} k(2x - x^2) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where k is a normalization constant.

a) Determine the value of k such that $f(x)$ is a valid probability density function.

b) Compute the cumulative distribution function (c.d.f.) of X.

c) Determine the median of X.

d) Calculate $P(0.5 \le X \le 1.5)$.

e) Show that X and $Y = 2 - X$ are identically distributed.

Let Z be another random variable defined as $Z = X^2$.

f) Determine the probability density function of Z.

g) Calculate the expected value $E(Z)$.

h) Compute the variance of Z.

Grading:

	a	b	c	d	e	f	g	h	Total
Points	1	1	1	1	1	1	2	2	**10**

Solutions for Homework Assignment 1 – Mathematical Statistics 2023
For a continuous random variable X with the given probability density function:

$$f(x) = \begin{cases} k(2x - x^2) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) To find k:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Computing within the interval [0,2]:

$$\int_0^2 k(2x - x^2)\,dx = 1$$

$$k\left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = 1$$
$$k\left(4 - \frac{8}{3}\right) = 1$$
$$k = \frac{3}{4}$$

b) The cumulative distribution function (c.d.f.) of X:

$$F(x) = \int_0^x \frac{3}{4}(2t - t^2)\,dt$$
$$F(x) = \frac{3}{4}\left(t^2 - \frac{1}{3}t^3\right)\Big|_0^x$$
$$F(x) = \frac{3}{4}\left(x^2 - \frac{1}{3}x^3\right)$$

Therefore,

$$F(x) = \begin{cases} 1 & \text{for } x \geq 2 \\ \frac{3}{4}\left(x^2 - \frac{1}{3}x^3\right) & \text{for } x \in [0,2] \\ 0 & \text{for } x \leq 0\,. \end{cases}$$

c) The median is the point x such that $F(x) = 0.5$. Noticing that $F(0) = 0$, $F(2) = 1$ and the fact that F is non-decreasing, the problem boils down to solving the equation:

$$\frac{3}{4}\left(x^2 - \frac{1}{3}x^3\right) = 0.5$$

This equation is solved for $x = 1$. The p.d.f. is positive around $x = 1$. This means that F is strictly monotone increasing around $x = 1$. Thus, $x = 1$ is the unique solution and the median is 1.

d) To calculate $P(0.5 \leq X \leq 1.5)$:

$$P(0.5 \leq X \leq 1.5) = F(1.5) - F(0.5) = 0.84375 - 0.15625 = 0.6875\,.$$

e) We can relate the c.d.f. F_Y of Y to the c.d.f. F_X of X via

$$F_Y(y) = P(Y \leq y) = P(2 - X \leq y) = P(X \geq 2 - y) = 1 - F_X(2 - y).$$

Considering the three cases $y \geq 2$, $0 < y < 2$ and $y \leq 0$, we find using that

$$1 - \frac{3}{4}\left((2-y)^2 - \frac{1}{3}(2-y)^3\right) = 1 - \left(3 - 3y + \frac{3}{4}y^2 - 2 + 3y - \frac{3}{2}y^2 + \frac{1}{4}y^3\right)$$
$$= \frac{3}{4}\left(y^2 - \frac{1}{3}y^3\right).$$

This means that X and Y have the same c.d.f. Since the c.d.f. uniquely defines the distribution, X and Y have the same distribution.

For $Z = X^2$:

f) For positive values of x

$$F_Z(x) = \mathbb{P}(Z \leq x) = \mathbb{P}(X^2 \leq x) = \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}) = F_X(\sqrt{x}) - F_X(-\sqrt{x})\,.$$

Due to $x > 0$, $F_X(-\sqrt{x}) = 0$. Taking the derivative w.r.t to x on both sides, we obtain

$$f_Z(x) = \frac{1}{2\sqrt{x}} f_X(\sqrt{x})\,.$$

Plugging $\sqrt{x}$ in the expression of f_X:

$$f_Z(x) = \begin{cases} \frac{3}{4} - \frac{3}{8}\sqrt{x} & \text{for } x \in [0,4], \\ 0 & \text{else }. \end{cases}$$

Alternatively, one can use the formula for the transformation of random variables:

$$f_Z(y) = \begin{cases} f_X(g^{-1}(y))\,|\det(J_{g^{-1}}(y))| & \text{if } y \text{ is in the range of } g \\ 0 \quad \text{else}\,. \end{cases}$$

Here, $g(y) = y^2$ and g is a diffeomorphism on $\mathbb{R}^+$ with inverse $g^{-1}(y) = \sqrt{y}$.

g) By definition, the expected value $E(Z)$ is:

$$E(Z) = \int_{-\infty}^{\infty} z f_Z(z)\,dz\,.$$

Therefore,

$$E(Z) = \int_{-\infty}^{\infty} x f_Z(x)\,dx = \int_{-\infty}^{\infty} x\left(\frac{3}{4} - \frac{3}{8}\sqrt{x}\right)\mathbf{1}_{[0,4]}\,dx = \frac{3}{4}\int_0^4 x\,dx - \frac{3}{8}\int_0^4 x^{3/2}\,dx$$
$$= \frac{3}{8}x^2\Big|_0^4 - \frac{3}{4\cdot 5}x^{5/2}\Big|_0^4 = 6 - \frac{24}{5} = \frac{6}{5}\,.$$

h) By definition, the variance of Z is:

$$\mathrm{Var}(Z) = E(Z^2) - [E(Z)]^2$$

Since we have already computed the expected value $E[Z]$, we only need to compute the second moment $E[Z^2]$:

$$E[Z^2] = \int_{-\infty}^{\infty} x^2 f_Z(x)\, dx = \int_{-\infty}^{\infty} x^2 \left(\frac{3}{4} - \frac{3}{8}\sqrt{x}\right) \mathbf{1}_{[0,4]}\, dx = \frac{3}{4}\int_0^4 x^2\, dx - \frac{3}{8}\int_0^4 x^{5/2}\, dx$$

$$= \frac{1}{4}x^3\Big|_0^4 - \frac{3}{8}\cdot\frac{2}{7}x^{7/2}\Big|_0^4 = 16 - \frac{3}{8}\cdot\frac{2}{7}\cdot 2\cdot 4^3 = \frac{16}{7}\,.$$

Thus, the variance $Var(Z) = \frac{16}{7} - \frac{36}{25} \approx 0.845$.