Differential Equations - Module 4 - 2023

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1 Ordinary differential equations

This course is devoted to the study of Ordinary Differential Equations (ODE) and systems of ODEs. They are the first and easiest differential relations that one would like to study to understand natural phenomena and real-life modelling problems. However, despite the fact that they are a quite natural object their analysis could be very complicated and their study is the object of lively research fields.

1.1 Basic definitions and terminology

In this first section we introduce the basic definitions about ODEs and some useful terminologies. ODEs are defined quite informally in the following way.

Definition 1.1 (Ordinary Differential Equation (ODE))**.** An ordinary differential equation is an equation that relates the derivatives of a function $y(t)$ ($y'(t)$, $y''(t)$, $\dots$, $y^{(n)}(t)$) with the function $y(t)$ itself and its variable t, i.e.

$$\Phi(t, y(t), y'(t), y''(t), \dots, y^{(n)}(t)) = 0 \tag{1.1}$$

for a given function Φ of $\mathbb{R}^{n+1}$ variables.

A specific example could be

$$y'(t) = ty(t) + 2t \tag{1.2}$$

Note that for notational convenience the dependence of the function y and its derivative y' on t is often implicit. So the previous equation is often written as

$$y' = ty + 2t$$

Also there are various notations for the derivative of a function other than y'. For example $\frac{dy}{dt}$ or $\dot{y}$. For n-th order derivatives, a common alternative notation is $\frac{d^n y}{dt^n}$.

Definition 1.2 (Autonomous ODE)**.** We say that an ODE is **autonomous** is it does not depend explicitly on time, that is

$$\Phi(y(t), y'(t), y''(t), \dots, y^{(n)}(t)) = 0 \tag{1.3}$$

Definition 1.3 (Order of an ODE)**.** By **order** of an ordinary differential equation is meant the order of the highest derivative appearing in the equation.

For example (1.2) has order 1. It is useful consider ordinary differential equations in a specific form that is called normal form.

Definition 1.4 (Normal form of an ODE)**.** An ordinary differential equation is said to be in normal form if it can be written as

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

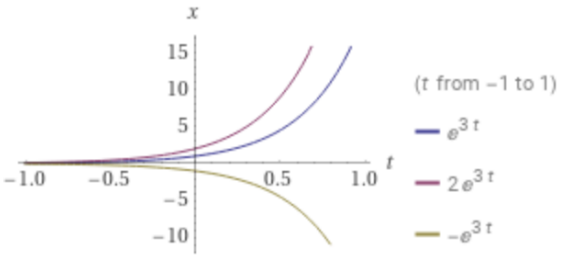
for an appropriate f, where n is the order of the ODE.

So in particular, (1.2) is in normal form, but $t^2y' = ty + 2t$ is not.
We are interested to solve ODEs, namely to find a function $y(t)$ that satisfies the ODE, so such that (1.1) holds. This is the meaning of solving an ODE. You are probably already familiar with basic solutions of ODEs. For example

$$y' = 3y$$

has solution $y(t) = Ce^{3t}$ for every $C \in \mathbb{R}$. In particular you have an infinite number of solutions for every choice of C. This concept is made precise in the following definition.

Definition 1.5 (General solution of an ODE / Solution curves)**.** Informally all solutions of and ODE (with the generic constant) is called **general solution**. You can plot the graph of such solutions for all $C \in \mathbb{R}$. These are called **solution curves**.



For first order ODE it is common to search solutions that also satisfy an initial condition, namely

$$y'(t) = f(t, y) \quad y(t_0) = y_0$$

This is called **initial value problem**. Typically a solution of an initial value problem is obtained by selecting one C from the general solution. So by imposing the constraint $y(t_0) = y_0$ the solution has one less degree of freedom, so it is completely determined. Obviously this is not a proof. There are suitable assumptions that allows to show that an initial value problem admits a solution at least close to (t_0, y_0) (and such solution is unique). We will discuss the issues of existence and uniqueness for solutions of ODEs later in the course.
Very often initial value problems have solutions, but only close to the point (t_0, y_0). Mathematically speaking this mean that there exists an interval $I \subset \mathbb{R}$ such that $t_0 \in I$ and

$$y'(t) = f(t, y) \quad \forall t \in I \qquad y(t_0) = y_0$$

In this case, we say that $y(t)$ is a solution of the initial value problem in I

Definition 1.6 (Interval of existence of a solution)**.** The **interval of existence** of a solution of an initial value problem is the largest interval $I \subset \mathbb{R}$ such that y is a solution of the initial value problem in I.

1.2 Explicit solutions of ODEs

ODEs are sometimes difficult to solve, where by solving we mean finding an explicit solution. However for some specific ODE, it is possible to obtain explicit solutions and there are methods to compute them. You have probably already seen some of them, but it is useful to go through them again. Note that being able to compute solutions of such ODEs is a necessary requirement for the exam of this course.

1.2.1 Separable equations

Separable equations are first order ODEs of the form

$$y' = g(t)h(y)$$

for some functions g and h. The name separable equations refers to the fact that you have the product of two functions, g and h, where g depend only on t and h depend only on y. For example the ODE

$$y' = t^2 y$$

is separable with $g(t) = t^2$ and $h(y) = y$. In this case, there exists a standard procedure to compute a solution of the ODE. The first step consists of "separating the variables", that is bringing the function the depends on y on the left side:

$$\frac{y'}{h(y)} = g(t)\,.$$

Then we integrate in t. So we get for $C \in \mathbb{R}$ that

$$\int \frac{y'(t)}{h(y(t))}\,dt = \int g(t) + C$$

We can now use substitution in integrals in the first integral to rewrite it as

$$\int \frac{1}{h(y)}\,dy = \int g(t)dt + C\,.$$

Now, by solving these integral and solve the resulting equation for $y(t)$ you get a solution for every $C \in \mathbb{R}$. Note that this method is not always providing an explicit solution. Indeed, it could happen that the integrals cannot be solved, or it is not possible to get an explicit dependence of $y(t)$ from t.

Remark 1.7. It is worth to notice that the method of separation of variables could fail when the solution that we find has points t such that $h(y(t)) = 0$. This is because we are dividing by $h(y(t))$ in the first place. In this case it often happens that through the method of separation of variables you cannot find solutions $y(t)$ such that $h(y(t)) = 0$. However, you can recover these solutions by directly verify that such $y(t)$ solve the ODE.

An example is Example 2.14 page 29.

$$y' = ty^2$$

Here $h(y) = y^2$ and $g(t)$ and if you solve the ODE you get

$$y(t) = \frac{-2}{t^2 + 2C} \tag{1.4}$$

for every $C \in \mathbb{R}$. However the function $y(t) = 0$ is a solution of the ODE, but cannot be recovered by (1.4) for any values of C (or informally when $C = \infty$).
If you are unsure about the division by zero, it is always worth to check directly if the solution that you get by separation of variables solves the ODE.
Note also that if the equation is simply

$$y'(t) = g(t)$$

then it is enough to integrate in time both sides to get

$$y(t) = \int g(t)\, dt + C$$

1.2.2 First order linear equations

A class of ODEs that is easy to solve are the **first order linear equations**.

Definition 1.8 (First order linear equations)**.** A first order linear ODE is and ODE of the form

$$y' = a(t)y + f(t) \tag{1.5}$$

for $a : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. We say that a first order ODE is **homogeneous** if $f = 0$, that is $y' = a(t)y$, otherwise we say that the ODE is **inhomogeneous**.

Why are they called linear? Because the map $y \mapsto a(t)y$ is linear in y for every t. For example $y' = a(t)y^2 + f(t)$ is non-linear.

How do you solve a first order linear ODE such as (1.5)? We start with the simple case of the homogeneous equation

Homogeneous ODE

Consider

$$y' = a(t)y$$

One can compute a general solution of such equation using the method of separation of variables. So

$$\frac{y'}{y} = a(t) \;\Leftrightarrow \log|y| = \int a(t)\, dt + C \Leftrightarrow |y|(t) = e^{\int a(t)\, dt + C} = e^C e^{\int a(t)\, dt}$$

So, noticing that $e^C \in \mathbb{R}_+$ for every $C \in \mathbb{R}$ we deduce that $|y|(t) = De^{\int a(t)\, dt}$ for every $D > 0$. Now we can remove the absolute value in $|y|(t)$ and allow the constant D to be both positive and negative. This leads to the expression of the solution for the homogeneous equation as

$$y(t) = \tilde{C}e^{\int a(t)\, dt} \quad \forall \tilde{C} \in \mathbb{R}$$

Remark 1.9. When you solve an exercise you can directly use that $y(t) = \tilde{C}e^{\int a(t)\,dt}$ is a solution of the homogeneous ODE $y' = a(t)y$, without doing the separation of variables again.

Inhomogeneous ODE

An inhomogeneous ODE cannot be solved with the method of separation of variables. Therefore other ways are needed. We present two methods that leads to the same formula for a solution of an inhomogeneous equation: **integrating factor method** and **variation of constants method (or variation of parameters method)**

- **Integrating factor method**

To apply this method we rewrite the inhomogeneous ODE as

$$y' - ay = f.$$

We want to find a function $u(t)$ such that

$$(uy)' = u(y' - ay) \tag{1.6}$$

Such $u(t)$ is called **integrating factor**. Suppose now that we are able to find the integrating factor. Then by integrating (1.6) we get

$$u(t)y(t) = \int u(t)(y'(t) - a(t)y(t))\,dt = \int u(t)f(t)\,dt + C \quad C \in \mathbb{R}$$

So dividing by $u(t)$ (supposing that the integration factor is different than zero for every t) we get

$$y(t) = \frac{\int u(t)f(t)\,dt}{u(t)} + \frac{C}{u(t)} \quad C \in \mathbb{R} \tag{1.7}$$

that is the solution of the inhomogeneous ODE. Clearly we still have to find an integrating factor $u(t)$. To do that we note that (1.6) can be rewritten as

$$u'y + uy' = uy' - auy \;\Leftrightarrow\; u'y = -auy$$

In particular this is true whenever $u' = -au$ (we just need to find one integrating factor, not all integrating factors). This equation can be easily solved by the method of separation of variables to get

$$u(t) = Ce^{-\int a(t)\,dt} \quad C \in \mathbb{R}\,.$$

Since we just need one integrating factor, so we can simply choose for example $C = 1$. Now that we have found the integrating factor we can substitute it into (1.7) (note also that the integrating factor is always positive) to get all the solutions of the inhomogeneous ODE as

$$y(t) = e^{\int a(t)\,dt}\int e^{-\int a(t)\,dt}f(t)\,dt + Ce^{\int a(t)\,dt} \quad C \in \mathbb{R}$$

Remark 1.10. This is a nice formula that allows you to compute the solution of every inhomogeneous **linear** ODE. At the exam you can use such formula directly, but if you do not remember it, you can derive it by e.g. the integrating factor method.

- **Variation of parameters method**

The second method that we see it is the so-called variation of parameters method. This is important because gives you a way to compute solutions of higher order inhomogeneous ODE (we will see that in the next section). So it is instructive to see it for first order ODE as well. We start again from the first order linear inhomogeneous ODE

$$y' = a(t)y + f(t)$$

First we compute the general solution of the associated homogeneous continuity equation $y' = a(t)y$. We have already done that and we know that the general solution is $y_{hom}(t) = Ce^{\int a(t)\,dt}$ for $C \in \mathbb{R}$. We now search for a solution of the inhomogeneous ODE of the form

$$y(t) = v(t)y_{hom}(t)$$

for a suitable $v(t)$. This procedure justifies the name *variation of parameters*, since we search solutions such that the constant that multiplies the homogeneous solution $y_{hom}(t)$ varies and this it is a function $v(t)$. We search for $v(t)$ imposing that $v(t)y_{hom}(t)$ has to satisfy the ODE. So we impose that

$$(vy_{hom})' = a(vy_{hom}) + f$$

and we expand the derivative to get that

$$v'y_{hom} + vy'_{hom} = avy_{hom} + f \;\; \Rightarrow v'y_{hom} = f$$

where we also use that $y'_{hom} = a(t)y_{hom}$ since y_{hom} is a solution of the homogeneous equation. In particular,

$$v' = \frac{f}{y_{hom}} = fe^{-\int a(t)\,dt}$$

So integrating in t we get to $v(t) = \int f(t)e^{-\int a(t)\,dt}\,dt + C$ for $C \in \mathbb{R}$ and therefore

$$y(t) = v(t)y_{hom}(t) = Ce^{\int a(t)\,dt} + e^{\int a(t)\,dt}\int f(t)e^{-\int a(t)\,dt}\,dt \qquad C \in \mathbb{R}$$

that is the same formula as in the integrating factor method.
If you look at the structure of the general solution of an inhomogeneous linear ODE you can see that the structure is really specific. It is the sum of two quantities: the first one is the general solution of the homogeneous equation and then there is an additional term that does not depend on a constant: $e^{\int a(t)\,dt}\int f(t)e^{-\int a(t)\,dt}\,dt$. You can verify that this is a solution of the inhomogeneous continuity equation. So one can deduce the following

Theorem 1.11. *All solutions of a linear inhomogeneous first order ODE are given by the sum of the general solution of the homogeneous ODE and a particular solution of the inhomogeneous ODE.*

1.2.3 Exact differential equations

This is a further class of ODEs that is possible to solve explicitly.
We consider a very general class of ODEs (that is much more general than linear ODE or separable ODEs). Denoting by $y(x)$ the solution depending on the variable x, we consider the ODE

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

for general functions P and Q. This class of ODEs can be written with the language of differential forms. In particular, by formally multiplying the ODE by dx we get

$$P(x,y)dx + Q(x,y)dy = 0$$

The left-hand-side $\omega = P(x,y)dx+Q(x,y)dy$ is called differential form. We will not enter into the details of such a theory and for this reason we intend the equation $P(x,y)dx+Q(x,y)dy = 0$ in a formal way. Here dx and dy can be viewed as the increment in the x and y axis, but formally they are just symbols.

Definition 1.12 (Exact differential form)**.** A differential form $\omega = P(x,y)dx + Q(x,y)dy$ is exact if there exists a continuously differentiable function F such that

$$\omega = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Why exact differential forms are important? Because if we divide $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$ by dx we get by the **chain rule** applied to the function F that

$$\omega = \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}\right)dx = \frac{d}{dx}F(x,y(x))dx$$

Therefore if we look at the solutions of $\omega = P(x,y)dx + Q(x,y)dy = 0$, then solutions are characterized by the property that $\frac{d}{dx}F(x,y(x)) = 0$ and thus

$$F(x,y(x)) = C \quad \text{for all } x$$

for arbitrary constants $C \in \mathbb{R}$. Let us see an example.

Example:

Consider the equation

$$4y^3\frac{dy}{dx} + 2x = 0$$

Actually we can solve it by separation of variables (try it!) as well. However we want to try the new method. So we first write the equivalent equation with the differential form

$$4y^3dy + 2xdx = 0$$

Unfortunately we do not know that the differential form $\omega = 4y^3dy + 2xdx$ is exact. For the time being we assume so and we try to solve the equation assuming that there exists F

such that $4y^3dy + 2xdx = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$. If this is true then $4y^3 = \frac{\partial F}{\partial y}$ and $2x = \frac{\partial F}{\partial x}$. From $4y^3 = \frac{\partial F}{\partial y}$ we get by integrating with respect to y that

$$y^4 = F(x, y) + C(x)$$

where $C(x)$ is a function depending only on x (note that we are integrating only respect the y variable). Now we use the relation $2x = \frac{\partial F}{\partial x}$ to get

$$2x = -\frac{\partial C(x)}{\partial x}$$

from which we deduce that $C(x) = -x^2 + D$ and thus $F(x, y) = y^4 + x^2$. In particular the differential form ω is exact for this specific F. This leads to say that all solutions of the starting ODE are the functions $y(x)$ such that

$$y(x)^4 + x^2 = C$$

QUESTION? Is it possible to know in advance if a differential form is exact? YES!

Theorem 1.13. *A differential form $\omega = P(x, y)dx + Q(x, y)dy$ is exact if and only if it is closed, that is*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Note that this theorem considers functions P, Q defined in all $\mathbb{R}^2$. This means that exactness and closedness hold for all (x, y) in $\mathbb{R}^2$. Is the same theorem true if we ask that exactness in a domain $\Omega \subset \mathbb{R}^2$ is equivalent to closedness in $\Omega \subset \mathbb{R}^2$? No! This is true only if the domain Ω does not have holes. These are called "simply connected" domains and they are the starting point on a much broader geometric theory that is outside the scope of this class.

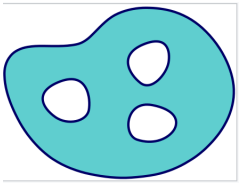


Figure 1: A domain that is not simply connected

It is good to know however that the theorem is true also if Ω is convex (Ω convex implies that Ω is simply-connected). So we can state

Theorem 1.14. *Given $\Omega \subset \mathbb{R}^2$ a convex set, a differential form $\omega = P(x, y)dx + Q(x, y)dy$ is exact in Ω if and only if it is closed in Ω, that is*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \textit{for all} \quad (x, y) \in \Omega$$

EXERCISE: In the book the proof of Theorem 6.20 is very (very!) imprecise. Why? How would you try to modify it?

Wrapping up: To solve the equation

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

first write the differential form $\omega = P(x,y)dx + Q(x,y)dy$ and verify that it is closed. Then, by Theorem 1.13, it is also exact. Therefore you can integrate separately in x and y as we saw in the example before to find $F(x,y)$. Solutions of the ODE are then solutions of the implicit equation

$$F(x, y(x)) = C$$

for every constant C.

1.2.4 Integrating factor for differential forms

Clearly if the differential form $\omega = Pdx + Qdy$ is not exact the previous method does not work. However, in some cases it is possible to adapt the ideas of the previous section to more general cases. The trick is to search for a function $\mu(x,y)$ such that there exists $F(x,y)$ with

$$\frac{\partial F}{\partial x} = \mu P \quad \frac{\partial F}{\partial y} = \mu Q$$

If such μ exists, then the differential form $\mu Pdx + \mu Qdy$ is exact. Such μ is called **integrating factor** for $Pdx + Qdy$. Why is this useful? Because since $\mu Pdx + \mu Qdy$ is exact, then

$$\mu Pdx + \mu Qdy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

if and only if $y(x)$ is a solution of

$$Pdx + Qdy = 0$$

provided that the integrating factor is different than zero everywhere. Therefore we can compute a general solution of $Pdx + Qdy = 0$ by equivalently computing solutions of $\mu Pdx + \mu Qdy = 0$ (that is exact) as in the previous section.
Clearly finding an integrating factor of a general differential form is a quite hard task, that can be done only in specific situation. On the book (Section 2.6) you can see how to compute solution of separable ODE and linear first order ODE by finding the integrating factor for the differential form. In this way you can get the same solution formula that we obtained in Section 1.2.1 and 1.2.2.

1.3 How to draw solutions of ODEs?

When studying ODEs it is an important skill to be able to draw and represent solutions. For a first order ODE this is very easy. Given a general solution $x(t)$ of an ODE in a normal form of the type

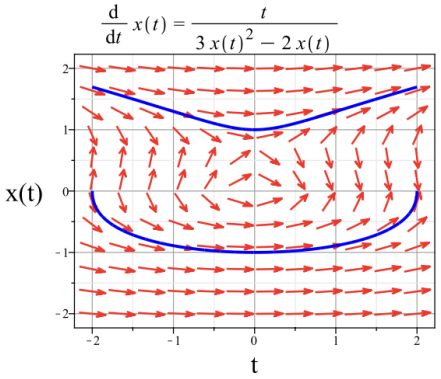
$$x'(t) = f(t, x(t))$$

we want to represent it in the plane (t,x). The first step is writing the so-called **direction field**. The direction field is represented as arrows in the plane (t,x) corresponding to

$(1, f(t,x))$. These arrows correspond to the tangent of the graph of a solution $x(t)$ in the point (t,x). This is because

$$(1, f(t,x)) = (1, x'(t))$$

that is the tangent to the graph of $x(t)$.



Clearly it is difficult to draw precisely the direction of the arrows. But you need to be able to find if they are increasing, decreasing, parallel to the t axis of parallel to the x axis. This depends on the sign of $f(t,x)$ in the point (t,x). Once you have the complete direction field, the solutions of the ODE are the curves that have the arrows of the direction field has tangent. In particular you can even select the solutions that satisfy a particular initial value. In this example the blue curves are the solution for the initial value problem with $x(0) = 1$ and $x(0) = -1$.

1.4 Existence and uniqueness of solutions

In the previous sections we saw how to compute explicit solutions of specific classes of ODEs. The question that we want to answer now is the following. Given an initial value problem for a first order ODE

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0$$

when can we ensure that a solution exists, at least in a small interval of existence $I \subset \mathbb{R}$. Can we ensure that the solution is unique in such interval? Can we also say that the interval of existence I is the all $\mathbb{R}$?

Very often it is not possible to find an explicit solution for an ODE, but at least it is possible to ensure that such a solution exists.

1.4.1 Picard–Lindelöf theorem

There are a lot of existence/uniqueness theorems for ODEs, dealing with different hypothesis and ensuring existence, uniqueness of both. Here we propose one of the most classical ones (the Picard–Lindelöf theorem). However, we suggest to keep in mind that this is not the only result... It is also fairly easy to get confused!

Theorem 1.15 (Picard–Lindelöf theorem)**.** *Consider the initial value problem*

$$y'(t) = f(t, y), \quad y(t_0) = y_0 \tag{1.8}$$

Suppose that f is continuous in the strip $S = [t_0, t_1] \times \mathbb{R}$. Suppose additionally that $y \mapsto f(t, y)$ is Lipschitz, uniformly in $[t_0, t_1]$. This means that there exists $C > 0$ such that

$$|f(t, y) - f(t, w)| \leq C|y - w|$$

for every $t \in [t_0, t_1]$ and $y, w \in \mathbb{R}$.
Then the solution of the initial value problem exists and is unique in an interval $[t_0, t_1]$.

We propose here a proof in a digestible form (this roughly follows the discussion in the book "Ordinary Differential Equations" by Wolfgang Walter, c.f. Section 6).

Proof. The starting point is to consider an integral representation of the solution:

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau \tag{1.9}$$

for $t \in [t_0, t_1]$. Note that $y(t)$ defined in (1.9) is a solution of the initial value problem (1.11), just by taking the derivative in t and evaluating the function in $t = t_0$. Moreover a solution of the initial value problem can be written as in (1.9) by the fundamental theorem of calculus (here we are using the f is continuous). So it is true that

$$y(t) \text{ solves (1.11) if and only if (1.9) holds}$$

Denoting by $C([t_0, t_1])$ the space of continuous function on $[t_0, t_1]$ we define the operator $T : C([t_0, t_1]) \to C([t_0, t_1])$ as

$$T(y)(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau$$

Note that $T(y) \in C([t_0, t_1])$. Indeed, the function

$$t \mapsto y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau$$

is continuous in $[t_0, t_1]$. To show this take sequence $t_n \to t \in [t_0, t_1]$. Then

$$\begin{aligned} |T(y)(t_n) - T(y)(t)| &= \left| \int_{t_0}^{t_n} f(\tau, y(\tau))\, d\tau - \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau \right| \leq \int_{t_n}^{t} |f(\tau, y(\tau))|\, d\tau \\ &\leq \sup_{\tau \in [t_0, t_1]} |f(\tau, y(\tau))||t_n - t| \to 0 \end{aligned}$$

as $t_n \to t$. Note that $\sup_{\tau \in [t_0, t_1]} |f(\tau, y(\tau))| < \infty$. Indeed the map $t \mapsto f(t, y(t))$ is continuous in the compact interval $[t_0, t_1]$ due to the continuity of y. Therefore, since a continuous function defined on a compact interval is also bounded, it also holds that $\sup_{\tau \in [t_0, t_1]} |f(\tau, y(\tau))| < \infty$.

So we have constructed a functional $T : C([t_0, t_1]) \to C([t_0, t_1])$ with a very nice property:

$$T(y) = y \quad \text{if and only if} \quad y \text{ solves the initial value problem (1.11)}$$

Definition 1.16 (Fixed points)**.** Given a functional T, the points y such that $T(y) = y$ are called fixed points of T.

So we only have to show that the fixed points of a functional T exist and are unique. This is the classical statement for so-called **fixed point theorems** that are one of the backbones of functional analysis and theory of differential equations. We provide here a statement (without proof) we are going to use that is specific for our situation.

Theorem 1.17 (Banach fixed point theorem – A specific version)**.** *Given $T : C([t_0, t_1]) \to C([t_0, t_1])$, suppose that T is a contraction, that is,*

$$\sup_{t\in[t_0,t_1]} |T(y_1)(t) - T(y_2)(t)| < \sup_{t\in[t_0,t_1]} |y_1(t) - y_2(t)|$$

for every $y_1, y_2 \in C([t_0, t_1])$. Then T admits one fixed point and such fixed point is unique.

This is a specific version of a more general theorem

Theorem 1.18 (Banach fixed point theorem)**.** *Let X be a complete norm space. Given $T : X \to X$, suppose that T is a contraction, that is,*

$$\|T(y_1) - T(y_2)\|_X < \|y_1 - y_2\|$$

for every $y_1, y_2 \in X$. Then T admits one fixed point and such fixed point is unique.

Note that you can recover the specific version of the theorem for $X = C([t_0, t_1])$ with $\|y\| = \sup_{t\in[t_0,t_1]} |y(t)|$ that is complete normed space.

Also note that the contraction property is simply a Lipschitz property with a Lipschitz constant strictly less than one. So using Theorem 1.17 we can conclude the existence and uniqueness of the initial value problem provided we show that

$$T(y)(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau$$

is a contraction. So we estimate

$$\begin{aligned} |T(y_1)(t) - T(y_2)(t)| &= \left| \int_{t_0}^{t} f(\tau, y_1(\tau))\, d\tau - \int_{t_0}^{t} f(\tau, y_2(\tau))\, d\tau \right| \leq \int_{t_0}^{t} |f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))|\, d\tau \\ &\leq L \int_{t_0}^{t} |y_1(\tau) - y_2(\tau)|\, d\tau \leq L|t - t_0| \sup_{t\in[t_0,t_1]} |y_1(t) - y_2(t)| \end{aligned}$$

for every $y_1, y_2 \in C([t_0, t_1])$ and every $t \in [t_0, t_1]$. So, we would like to have $L|t - t_0| < 1$ for every $t \in [t_0, t_1]$. This is not true in general, but we can choose $\hat{t} \in [t_0, t_1]$ such that $L|t - t_0| < 1$ for every $t \in [t_0, \hat{t}]$. By doing that and repeating all these arguments in the interval $[t_0, \hat{t}]$ one gets existence and uniqueness of the initial value problem in the interval $[t_0, \hat{t}]$. The next step is to set up another initial value problem

$$y'(t) = f(t, y), \quad y(\hat{t}) = \hat{y} \tag{1.10}$$

where $\hat{y}$ is the solution computed in the previous step evaluated in $\hat{t}$. We can repeat the same argument for this initial value problem and find a unique solution in the interval $[\hat{t}, \bar{t}]$. Repeating this procedure we exhaust all the interval $[t_0, t_1]$ showing existence and uniqueness for the original value problem.

□

This theorem requires that the function f is Lipschitz in the second argument in all $\mathbb{R}$. It is possible to weaken this assumption by asking that f is Lipschitz in a rectangle $R = [t_0, t_1] \times [y_1, y_2]$ for $y_0 \in [y_1, y_2]$. In this case we have the following theorem

Theorem 1.19 (Picard–Lindelöf theorem – second version)**.** *Consider the initial value problem*

$$y'(t) = f(t, y), \quad y(t_0) = y_0 \tag{1.11}$$

Suppose that f is continuous in the rectangle $R = [t_0, t_1] \times [y_1, y_2]$ with $y_0 \in (y_1, y_2)$. Suppose additionally that $y \mapsto f(t, y)$ is Lipschitz in $[y_1, y_2]$, uniformly in $[t_0, t_1]$. This means that there exists $C > 0$ such that

$$|f(t, y) - f(t, w)| \leq C|y - w|$$

for every $t \in [t_0, t_1]$ and $y, w \in [y_1, y_2]$.
Then the solution of the initial value problem exists and is unique in an interval $[t_0, \hat{t}]$ with $\hat{t}$ possibly smaller than t_1.

We will not specify the proof for this version of the theorem. However, it can be obtained quite easily adapting the previous argument.
Let us conclude with some remarks about this result:

- It is possible to obtain existence and uniqueness of solutions in the interval $[t_1, t_2]$ with $t_0 \in (t_1, t_2)$ in case f is continuous in $[t_1, t_2] \times \mathbb{R}$ and f is Lipschitz in the second variable uniformly in $[t_1, t_2]$. The proof is exactly the same. One can even use the result that we showed, consider the reflection $t \mapsto -t$ and deduce the complete result
- The Lipschitz assumption on f can be replaced by f is differentiable in the second argument and

  $$\frac{\partial f}{\partial y}(t, y)$$

  is continuous in $[t_0, t_1] \times \mathbb{R}$.

1.5 Stability of first order ODEs

In this section we analyze the stability of first order ODEs. We focus on autonomous first order ODEs, that is

$$x'(t) = f(x(t)) \tag{1.12}$$

For this class of equation we have special solutions that are the equilibriums

Definition 1.20 (Equilibrium point)**.** An equilibrium point for (1.12) is x_0 such that $f(x_0) = 0$. It follows that is x_0 is an equilibrium point, then the function $x(t) = x_0$ for every t is a solution of (1.12).

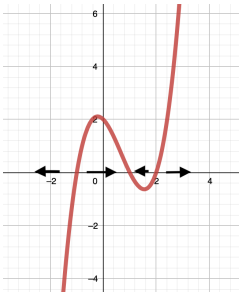
Equilibriums are very important object to understand the dynamic of an autonomous ODE. The reason is that it is possible to classify the dynamics around the equilibriums. Or in other words it is possible to say if the solution is being attracted to the equilibrium or repulsed. An useful graphical way to understand this dynamic is plotting the so called **phase line**. This is just the plot of the graph of f in the plane (x, f) and the equilibriums are precisely the intersection of the graph of f with the $x-$axis. Moreover, by computing the sign of f it is possible to understand if the solution is moving towards an equilibrium or away from an equilibrium. Let us understand that with an example (this is Example 9.6 in Polking).

An example

We consider the autonomous first order ODE

$$x' = (x^2 - 1)(x - 2)$$

We want to understand the equilibrium and write a phase line for such equation. To find the equilibrium we have to search the x such that $f(x) = (x^2 - 1)(x - 2) = 0$. If you solve this equation you get $x_1 = -1, x_2 = 1$ and $x_3 = 2$. So these are the only 3 equilibriums of the ODE. We now can draw the phase line:



Note that not only we can find the equilibriums, but we can also understand from the picture the behaviour of the function around the equilibrium. Indeed if we look around the equilibrium $x_1 = -1$, the function f is negative before x_1. This means that the derivative of the solution is negative and therefore the solution is decreasing. In particular, the solution is moving away from $x_1 = -1$. We can repeat this reasoning for every regions between equilibriums and get an intuition on how the dynamics work. We can represent the behaviour around equilibrium by using arrows (see the previous figure). The arrows are pointing in the direction that the solution is moving.

Can we formalize these considerations over the stability of equilibrium points and have a criteria to determine stability? First we need the definition of stability.

Definition 1.21 (Stability)**.** We say that an equilibrium point x_0 is **asymptotically stable** if there exists $\delta > 0$ such that every solution of the ODE, denoted by $x(t)$, such that $|x(0) - x_0| < \delta$ satisfies

$$\lim_{t \to +\infty} x(t) = x_0.$$

We say that an equilibrium point x_0 is **stable** if for every $\varepsilon > 0$ there exists δ such that for every solution of the ODE, denoted by $x(t)$, such that $|x(0) - x_0|$ it holds that

$$|x(t) - x_0| < \varepsilon \quad \forall t > 0$$

Clearly asymptotically stable implies stable. Finally we say that an equilibrium point is **unstable** if it is not stable.

We also have a criteria to determine if an equilibrium point of an autonomous ODE is asymptotically stable or unstable.

Theorem 1.22. *Let x_0 be an equilibrium point for the ODE $x' = f(x)$. Then*

- *If $f'(x_0) < 0$ then x_0 is asymptotically stable*
- *If $f'(x_0) > 0$, then x_0 is unstable*
- *If $f'(x_0) = 0$ then nothing can be said.*

This is the basic stability result we have. We will enter much more into the details of stability when we will deal with systems of ODEs. So this is basically just an appetizers. We can apply this theorem to the example $x' = (x^2-1)(x-2)$. So we have to compute the derivative of $f(x) = (x^2-1)(x-2)$ as $f'(x) = 2x(x-2) + x^2 - 1 = 3x^2 - 4x - 1$. So computed in -1 we have $f'(-1) = 6$ stating that -1 is unstable. $f'(1) = -2$, so that 1 is asymptotically stable. $f'(2) = 3$ so that 2 is unstable. Note that this result is consistent with the arrows drawn on the phase line diagram. You can try to write the direction field as well and represent the solutions to understand even better the dynamics (see Figure 6 Example 9.6 in Polking)

1.6 Second order ordinary differential equations

In the previous sections we focused on first order ODEs. These are the simple ones that often admits explicit solutions and a well developed existence and uniqueness theory. In this section we deal with the basics of second order differential equations. Perhaps you have already seen explicit solutions for them, but we repeat these concepts in this section.
A second order differential equation is an ODE of the form

$$y'' = f(t, y, y')$$

according to the definitions we have already given in the first section. There are a lot of physical phenomena that can be modelled as a second order ODE. This is the reason why they are so important. For example **Newton's second law** can be written as

$$F = ma$$

where F is the force applied to a body, m is the mass of the body and a is the resulting acceleration. By reminding that if the body was still and placed in the point 0, then its acceleration is simply $y''(t)$, then one can rewrite the second Newton law as

$$mf''(t) = F(t, y, y')$$

where the force is possibly depending on the position of the body, on its velocity and on time. A slightly more involved physical system is obtained when the force (typically the gravity force) is coupled with a spring. This example is called **vibrating spring** and it is an useful example on how second order ODEs can give insight over physical systems. We now derive the equation and then we will see how to solve it.

We consider a spring suspended from a beam and we attached at the lower end an object of mass m. The effect is that the gravity acceleration will apply a force equal to $F = mg$ where $g = 9.8m/s^2$. On the other hand the spring will apply an opposite force that is the restoring force of the spring that depends on how much the spring is deformed from the equilibrium position.

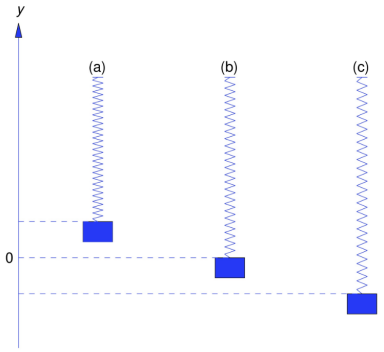


Figure 2: Figure from https://ximera.osu.edu/

The spring at (b) is at equilibrium, namely the gravity force compensate the restoring force of the spring. Note that we are supposing the the position of the object at equilibrium is for $y = 0$. Here our mass has velocity zero. If we move our mass we can describe the dynamics of the mass using the second Newton law:

$$my'' = R(y) + mg$$

where y'' is the acceleration, $R(y)$ is the restoring force applied by the spring and mg is the gravity. The restoring force of the spring, denoted by $R(y)$ depends on the position of the object. So in the configuration (a) the force will be pointing downwards, while in the configuration (c) the force will be pointing upwards. The experimental expression of $R(y)$ is given by the so-called **Hooke's law** and can be written as

$$R(y) = -ky$$

where $k > 0$ is the **spring constant**. Note that this formula cannot be "proven" but just follows from experimental observations. As expected, when $y > 0$, the force is negative, when $y < 0$, the force is positive and at $y = 0$ the force is zero and the system is at equilibrium. Therefore using Hooke's law we can write

$$my'' = -ky + mg$$

that is the second order ODEs describing the vibrating spring.
It is possible to relate g with k by looking at how much the spring is extended in the equilibrium configuration. This leads to the equivalent equation

$$my'' = -\frac{k}{m}y$$

For more details see Section 4.1 in Polking.
These examples leads us to search for way to solve second order ODEs and developed a theory for them. This is very difficult in general. This is the reason why we restrict our attention to **linear** second order ODEs.

1.6.1 Linear second order ODEs

A linear second order ODE is an ODE of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

for functions $p(t)$, $g(t), q(t)$. We remind that if $g = 0$, then the ODE is called **homogeneous**. Note also that for simplicity we already consider the equation in a normal form (that is the coefficient in front of y'' is 1). We start with an existence result.

Theorem 1.23 (**Existence and uniqueness** result for initial value problem of linear second order ODEs)**.** *Consider the following initial value problem:*

$$y'' + p(t)y' + q(t)y = g(t) \quad y(t_0) = y_0 \quad y'(t_0) = y_1$$

and suppose that $p(t)$, $g(t), q(t)$ are continuous in an interval (a, b) such that $t_0 \in (a, b)$. Then there exists a unique solution of the initial value problem in (a, b).

Note that this is radically different than Picard-Lindelof theorem. Indeed, Picard-Lindelof theorem prove existence and uniqueness for general first order ODE and not only linear. Moreover, Picard-Lindelof theorem requires only the initial condition on $y(t_0)$. This theorem instead, requires condition for $y(t_0)$ and $y'(t_0)$. This is clearly connected to the second order nature of the ODE.
So we know that a solution exists (at least locally around t_0). Can we say something more about the structure of the solutions?

Homogeneous second order linear ODEs

We start with general solutions of the homogeneous equation, that is

$$y'' + p(t)y' + q(t)y = 0 \tag{1.13}$$

and we note the following very simple fact.

Lemma 1.24. *A linear combination of solutions of (1.13) is again a solution of (1.13).*

Proof. The proof is straightforward. Let y_1 and y_2 solutions of (1.13) and consider their linear combination $y(t) = c_1y_1(t) + c_2y_2(t)$ for $c_1, c_2 \in \mathbb{R}$. Then

$$\begin{aligned} y'' + p(t)y' + q(t)y &= c_1y_1'' + c_2y_2'' + c_1p(t)y_1' + c_2p(t)y_2' + c_1q(t)y_1 + c_2q(t)y_2 \\ &= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) = 0 \end{aligned}$$

as we wanted to prove. □

This is typical property for all linear (and homogeneous) equations. Actually, much more is true! Not only linear combination of solutions are solutions, but also you can generate all solutions by linear combinations of linearly independent solutions. From now on you will realize how **linear algebra theory** (that you should know) is tightly linked with the theory of linear ODEs and linear systems of ODEs. There will be much more in the upcoming sections.
Applying the definition of **linearly independent vectors** (that you have seen in linear algebra) to the space of functions you should be able to deduce when two functions y_1 and y_2 are linearly independent, but let us rewrite it in this particular situation.

Definition 1.25 (Linearly independent functions)**.** Two functions y_1 and y_2 are **linearly independent** if neither is a constant multiple of the other. Otherwise they are **linearly dependent**.

Clearly the definition is very simple, but you can also generalize it for collections of more functions $(y_1, \dots y_n)$. Now we are ready to state the main result

Theorem 1.26. *Suppose that y_1, y_2 are linearly independent solutions of (1.13). Then the general solution of (1.13) is given by*

$$y = c_1 y_1 + c_2 y_2$$

for $c_1, c_2 \in \mathbb{R}$.

In particular this result is telling that once you have found two linearly independent solutions of (1.13), you can find all the solutions of (1.13) by considering all linear combinations. So two linearly independent solutions already characterized the solution set of the ODE. This justifies the following definition.

Definition 1.27 (Fundamental set of solutions)**.** Any two linearly independent solutions of (1.13) is called a **fundamental set of solutions** for (1.13).

Note also that once you have found the set of fundamental solutions y_1, y_2, the solution of an initial value problem of the type

$$y'' + p(t)y' + q(t)y = \quad y(t_0) = y_0 \quad y'(t_0) = y_1$$

can be obtained simply by considering the general solution $y = c_1 y_1 + c_2 y_2$ and then impose the initial values conditions

$$\begin{aligned} y(t_0) &= c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ y'(t_0) &= c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_1 \end{aligned}$$

This is a system of two equations in two unknowns c_1 and c_2 that can be easily solved.

Non-homogeneous second order linear ODEs

Can we say something about the structure of solutions of non-homogeneous second order linear ODEs? Actually a result similar to the one of first order ODEs holds.

Theorem 1.28. *Suppose that y_p is a particular solution of the inhomogeneous equation*

$$y'' + p(t)y' + q(t)y = g(t) \tag{1.14}$$

and y_1 and y_2 are the fundamental solutions of the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

Then the general solution of (1.14) is given by

$$y = y_p + c_1 y_1 + c_2 y_2$$

for $c_1, c_2 \in \mathbb{R}$. In particular this means that the general solution of a inhomogeneous linear second order ODE is given by the sum of a particular solution of the inhomogeneous equation and the general solution of the associated homogeneous equation.

In the next section we consider the case when $p(t)$ and $q(t)$ are constant and we give a method to solve this type of second order ODEs.

1.6.2 Linear homogeneous second order ODEs with constant coefficients

Are there simple situations when we can determined a set of fundamental set of solutions for a second order homogeneous ODE? Yes! In the case of constant coefficients the task is easy. We consider

$$y'' + py' + qy = 0 \tag{1.15}$$

for $p, q \in \mathbb{R}$. Note that constant coefficients refers to the fact that $q(t)$ and $p(t)$ in the definition of a general linear second order ODE are constant in t.
In order to get an intuition let us suppose that we are searching a solution that is of the form $y(t) = e^{\lambda t}$ for $\lambda \in \mathbb{R}$. This is somehow reasonable since its derivative is still an exponential, so, perhaps by choosing the correct λ we can find a solution. Let us plug $y(t) = e^{\lambda t}$ into (1.15). We get

$$0 = \lambda^2 e^{\lambda t} + \lambda y e^{\lambda t} + q e^{\lambda t} = e^{\lambda t}(\lambda^2 + p\lambda + q)$$

So, since $e^{\lambda t} \neq 0$ this is equivalent to $\lambda^2 + p\lambda + q = 0$. This tells us that if we set $y(t) = e^{\lambda t}$ for λ solution of

$$\lambda^2 + p\lambda + q = 0$$

then we have a solution

Definition 1.29 (Characteristic equation, characteristic polynom, characteristic root)**.** The equation $\lambda^2 + p\lambda + q = 0$ is called **characteristic equation** of (1.15). The polynomial $\lambda^2 + p\lambda + q$ is called **characteristic polynomial** and the solutions of $\lambda^2 + p\lambda + q = 0$ are called **characteristic roots**.

We will see that the form of the solution of (1.15) depend on the solutions of the characteristic equation. In particular we will consider three cases:

- The characteristic roots are real and distinct
- The characteristic roots are complex
- The characteristic roots are real but equal

Real and distinct roots

If the characteristic roots are $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq \lambda_2$ then

$$y_1(t) = e^{\lambda_1 t} \quad y_2(t) = e^{\lambda_2 t}$$

are fundamental solution of (1.15). Note that since $\lambda_1 \neq \lambda_2$, then y_1 and y_2 are linearly independent. Therefore Theorem 1.26 applies and thus the general solution of (1.15) is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Complex roots

We now consider the case where the solutions of $\lambda^2 + p\lambda + q = 0$ are complex. Note that this implies that the two complex solutions are one the conjugate of the other one, that is $\lambda_1 = \bar{\lambda}_2$:

$$\lambda_1 = a + ib \quad \lambda_2 = a - ib$$

for some $a, b \in \mathbb{R}$. In this case the complex functions

$$z_1(t) = e^{(a+ib)t} \quad z_2(t) = e^{(a-ib)t}$$

are complex fundamental solutions of (1.15). We remind the Euler's formula for computing the complex exponentials:

Definition 1.30 (Euler's formula)**.** for $w_1, w_2 \in \mathbb{R}$ the Euler's formula is

$$e^{w_1+iw_2} = e^{w_1} e^{iw_2} = e^{w_1}(\cos(w_2) + i\sin(w_2))$$

Thanks to this formula we can rewrite z_1 and z_2 as

$$z_1(t) = e^{at}(\cos(bt) + i\sin(bt)) \tag{1.16}$$
$$z_2(t) = e^{at}(\cos(-bt) + i\sin(-bt)) = e^{at}(\cos(bt) - i\sin(bt)) \tag{1.17}$$

Note that z_1 and z_2 are not one multiple of the other one and so they are linearly independent. Therefore Theorem 1.26 applies and the general solution is

$$w(t) = d_1 e^{(a+ib)t} + d_2 e^{(a-ib)t}$$

where d_1 and d_2 are complex constants. These are however complex solutions of the ODEs. In a lot of situations this is not really preferrable. We would like to have real solutions of the ODE. Let us see how to get general real solutions of (1.15). Note that by (1.16) we have that

$$e^{at}\cos(bt) = \frac{1}{2}(z_1(t) + z_2(t))$$
$$e^{at}\sin(bt) = \frac{1}{2i}(z_1(t) - z_2(t))$$

Therefore thanks to Lemma 1.24 we have that

$$y_1(t) = e^{at}\cos(bt)$$
$$y_2(t) = e^{at}\sin(bt)$$

are solutions of (1.15) and are linearly independent. Therefore they form a fundamental set of solutions and thus a general (real) solution of (1.15) is given by

$$y(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$$

for $c_1, c_2 \in \mathbb{R}$. Let us see briefly an example.

Example

Find the solution of the initial value problem

$$y'' + 2y' + 2y = 0 \quad y(0) = 2 \quad y'(0) = 3$$

By Theorem 1.23 we know that there exists a unique solution (at least around 0) of the initial value problem. First we need to compute a general solution of $y'' + 2y' + 2y = 0$. The characteristic equation is

$$y^2 + 2y + 2 = 0$$

The solutions can be computed using standard formula for quadratic equation and this leads to complex solutions $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$. As we expected they are one conjugate with respect to the other one. We use directly the real fundamental solutions that are

$$y_1(t) = e^{-t}\cos(t) \quad y_2(t) = e^{-t}\sin(t)$$

so that a general solution is

$$y(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$$

for $c_1, c_2 \in \mathbb{R}$. In order to determine c_1 and c_2 we have to use the initial values $y(0) = 2$ and $y'(0) = 3$. So by imposing these initial value $y(0) = 2$ we get

$$2 = y(0) = c_1 \Rightarrow c_1 = 2$$

Then by computing the derivative y'

$$y'(t) = -c_1 e^{-t}\cos(t) - c_1 e^{-t}\sin(t) - c_2 e^{-t}\sin(t) + c_2 e^{-t}\cos(t)$$

Thus

$$3 = y'(0) = -c_1 + c_2 \Rightarrow c_2 = c_1 + 3 = 5$$

So the solution of the initial value problem is

$$y(t) = 2e^{-t}\cos(t) + 5e^{-t}\sin(t)$$

Repeated roots

Let us conclude with the case of repeated real roots (not that you cannot have repeated complex roots). In this case we have

$$\lambda^2 + p\lambda + q = (\lambda - \lambda_1)^2$$

for $\lambda_1 \in \mathbb{R}$ the only solution of $\lambda^2 + p\lambda + q = 0$. In particular we immediately see that

$$y_1(t) = e^{\lambda_1 t}$$

is a solution of (1.15). So, in order to find the general solution of (1.15) we need to find another solution of (1.15) that is not a multiple of $e^{\lambda_1 t}$. Actually is not difficult the check that the function

$$y_2(t) = te^{\lambda_1 t}$$

is a solution of (1.15) and it is not a multiple of y_1. This is summarized in the following proposition.

Proposition 1.31. *If the characteristic equation has only one real solution $\lambda_1 \in \mathbb{R}$, then a general solution of (1.15) is given by*

$$y(t) = c_1 e^{\lambda_1 t} + c_2 te^{\lambda_1 t}$$

for $c_1, c_2 \in \mathbb{R}$.

Then if you have an initial value problem you can compute the solution by imposing the initial value and determining c_1 and c_2.

1.6.3 Linear inhomogeneous ODEs with constant coefficients

In the previous section we saw how to compute solution of homonogeneous linear ODEs with constant coefficients. Now we face the case of inhomonogeoeus ODEs, namely equations such as

$$y''(t) + py'(t) + qy(t) = f(t)$$

for $p, q \in \mathbb{R}$ and $f = f(t)$ is a function.
According to Theorem 1.28 we know that the general solution can be constructed as the sum of a particular solution and the general solution of the associated homogeneous ODE. Therefore, we only need to find a particular solution of the inhomogeneous equation. Sometimes depending on the form of $f(t)$ there are easy way to compute a particular solution. We will not enter into too much details on that, since this case will be covered later for systems of ODEs. We just give an example

$$y'' - y' - 2y = 2e^{-2t} \tag{1.18}$$

We want to determine a specific solution for (1.18). When the **inhomogenenous term** $f(t) = 2e^{-2t}$ is **exponential**, then we can look at a particular solution of the form

$$y_p(t) = ae^{-2t}$$

for a to be determined. This choice for y_p is natural since derivatives are preserving the exponential. Therefore we just have to determine a by substituting y_p in the equation. If we substitute it we get

$$4ae^{-2t} + 2ae^{-2t} - 2ae^{-2t} = 2e^{-2t}$$

Therefore $4ae^{-2t} = 2e^{-2t}$ and thus $a = 1/2$. So our particular solution is

$$y_p(t) = \frac{1}{2}e^{-2t}$$

Similar considerations can be made when $f(t)$ is trigonometric or polynomial (see Section 4.5 in Polking).

2 Systems of first order ODEs

We now start a more challenging part of the course about the system of ODEs. They are simply a collection of ODEs that should be solved simoultaneously. We will focus our attention on **systems of first order ODEs**. Let us give a precise definition.

Definition 2.1 (System of first order ODEs)**.** A system of first order ODEs with n equations and n unknowns has the form

$$x_1'(t) = f_1(t, x_1, x_2, \dots, x_n) \tag{2.1}$$
$$x_2'(t) = f_2(t, x_1, x_2, \dots, x_n) \tag{2.2}$$
$$\dots \tag{2.3}$$
$$x_n'(t) = f_n(t, x_1, x_1, \dots, x_n) \tag{2.4}$$

A solution of (2.1) is a n-nuple of functions $(x_1(t), \dots, x_n(t))$ that satisfies all ODEs in (2.1).

Similarly to ODEs we say that a system is autonomous if all equations do not depend explicitly on time. Moreover, we can define an initial value problem as follows.

Definition 2.2 (Initial value problem for systems of ODEs)**.** An initial value problem for a system of ODEs is defined by adding the initial condition

$$(x_1(t_0), \dots, x_n(t_0)) = (x_0^1, \dots, x_0^n)$$

Note that in this course we will always have that the number of equations is equal to the number of unknowns. Moreover, we often (but not always) consider **planar systems**, namely when $n = 2$.

2.1 Reduction of higher-order ODEs to first order systems of ODEs

We now want to show that it is possible to write an higher order ODE as a system of first order ODEs. This is interesting because in many cases the theory of higher order ODEs can be deduced from the theory of first order systems of ODEs. Therefore in what follows we will focus more on the study of systems of first order ODEs.

Suppose that you have a third order ODE

$$x''' + xx'' = \cos t$$

Note that this is not linear (this method works for every ODE). We want to transform it in an equivalent first order system of ODEs with 3 unknowns. The trick now is to consider the following substitution of variables

$$\begin{aligned} u_1 &= x \\ u_2 &= x' \\ u_3 &= x'' \end{aligned}$$

The variables u_1, u_2, u_3 will be the three unknowns of our system. Now, which is the relation between u_1, u_2, u_3? Note that

$$u_1' = u_2 \quad u_2' = u_3$$

Moreover using the third order ODE we also have that

$$u_3' + u_1 u_3 = \cos t$$

So, the third order ODE is equivalent to the first order system

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ u_3' &= -u_1 u_3 + \cos t \end{aligned}$$

2.2 Real example of system of ODEs: predator-prey system

You will see plenty of examples of systems of ODEs in the project of this module. In these notes we present one to give you a flavour of the interesting questions that one can formulate about a system of ODEs. We thus consider the classical **predator-prey system**.
Suppose that in an ecosystem two species coexist and interact during time. In particular, one species is the "preys" and the other one is the "predators". As the name suggests, throughout time the predators are killing an amount of prey that depends on the number of predator present in the population. Moreover, both the prey and the predators reproduces in time. We want to design a system of ODEs that describe this interaction between preys and predators in time.
We denote by $F(t)$ the prey population and by $S(t)$ the predators population in time. Suppose that there is no interaction between prey and predators, then the change of population is determined only by the reproduction. In this case calling by r_F the reproductive rate of the preys and by r_S the reproduction rate of the predators we can modelled the evolution of the populations as

$$\begin{aligned} F'(t) &= r_F F(t) \\ S'(t) &= r_S S(t) \end{aligned}$$

This is a system of ODEs, but particularly simple. Indeed, there is no dependence of between F and S (species do not interact). Also the solution is particularly simple, since it is enough to compute the solution of each ODE separately.
We now want to include the interaction between the species. This affects the reproductive rate of the preys. Basically depending on the numbers of predators r_F is lower. We can therefore set

$$r_F = a - bS$$

for $a > 0$ and $b > 0$. The parameter a is the actual reproduction. Instead, the parameter b is the rate of "aggressiveness" of predators and acts negatively on r_F. Additionally the reproductive rate of the predators is dependent on the presence of the preys (no preys equals no food). Therefore we can set

$$r_S = -c + dF$$

Here $-c < 0$ since if there are no prey, then the predator population would decrease. Here $d > 0$ is the increase in the population determined by the presence of preys. Clearly this is not the only modelling solution. This leads to the system

$$\begin{aligned} F'(t) &= (a - bS(t))F(t) \\ S'(t) &= (-x + dF(t))S(t) \end{aligned}$$

that is the so-called Lotka-Volterra equations.

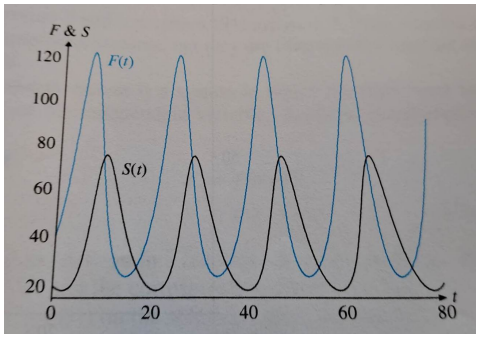
Relevant questions are: under which choices of the parameters a, b, c, d we have coexistence of the two species or the extinction of one of the two. This is not so easy to guess and it requires a computation. Clearly the parameters a, b, c, d are empirical, measured by observing the population. Using these measurement one can make intervention to avoid the extinction of the species.

2.3 How to represent solutions of a system of ODEs

The first question that we should as is how to represent the solution of a system of ODEs. There are several ways:

- **Plotting the components of the solution**

This is the analogous of plotting the solution curves as we have seen for standard ODEs. It consists in plotting the solution curves $F(t)$ and $S(t)$ in the same graph.



It is not easy to plot such solution curves. Indeed, typically one needs to compute the solution of the system $S(t)$ and $F(t)$ and then draw the curves. We would like to have a way to plot the solution without solving the system. Also, in this picture it does not make sense to draw the vector field as we did for ODEs.

- **The phase space and the phase plane**

The standard way to represent solutions and vector fields of **planar** systems of ODEs is the so-called phase plane. Let us consider a general system of equations

$$y_1' = f(t, y_1, y_2) \tag{2.5}$$
$$y_2' = g(t, y_1, y_2) \tag{2.6}$$

The phase plane is the graph of the general solution in the y_1y_2-plane. A solution of the system (2.5), denoted by $(y_1(t), y_2(t))$, can be represented as a curve $t \mapsto (y_1(t), y_2(t))$ in the y_1y_2-plane. Note that the variable t is not a variable on the graph, but it is implicit. A solution curve plotted on the phase plane is called **phase plane plot** or **solution curve**. Suppose now that the system is autonomous, that is there is not explicit dependence on the variable t. So it has the form

$$y_1' = f(y_1, y_2)$$
$$y_2' = g(y_1, y_2)$$

Do we have a way to draw solution curves without computing an explicit solution of the system? Yes! This is based on the following important remark

Remark 2.3. The vector in $(f(y_1, y_2), g(y_1, y_2)) \in \mathbb{R}^2$ is the tangent to a solution curve in the point (y_1, y_2) of the phase plane. This is because the derivative of the curve $(y_1(t), y_2(t))$ is

$$(y_1'(t), y_2'(t)) = (f(y_1(t), y_2(t)), g(y_1(t), y_2(t)))$$

and the derivative of a curve is a vector tangent to the curve.

Therefore, it is enough to draw the **vector field** $(f(y_1, y_2), g(y_1, y_2)) \in \mathbb{R}^2$ and the solution curves are determined by looking at the curves such that $(f(y_1, y_2), g(y_1, y_2))$ is the tangent.

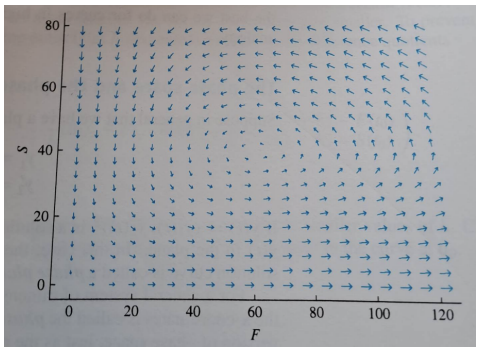
Definition 2.4. The phase plane consists in drawing the vector field in the plane y_1y_2. This allows you also to draw solution curves.

To give you an intuitive picture of the phase plane diagram let us look at this example. Consider a special case of the predator-prey system

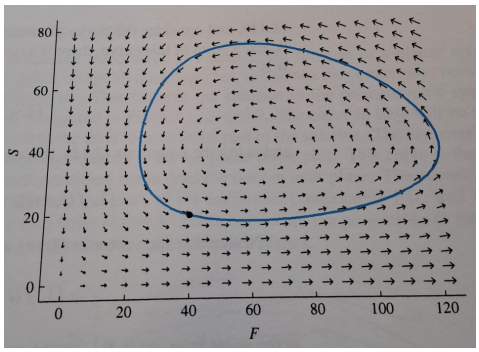
$$F' = (0.4 - 0.01S)F$$
$$S' = (-0.3 + 0.005F)S$$

restricted to the positive quadrant. We want $F > 0$ and $S > 0$ to have meaningful solutions. One can try to sketch the vector field $((0.4 - 0.01S)F, (-0.3 + 0.005F)S)$ in the FS plane. It would be possible to do this "by hand", by computing the vector field for specific values of F and S and interpolate. However, we will see that information about equilibriums of the

system and their stability information about the phase plane. In this case the phase plane is as follows



We can also try to draw a solution curve such that the vector field is the tangent in every point. The drawing should be as follows.



2.4 Analysis of systems of ODEs

We now start to look at qualitative properties of systems of ODEs. This will also help us to draw better **phase planes**. We will start with some result about existence and uniqueness of systems of ODEs

2.4.1 Existence and uniqueness of solutions

As for existence and uniqueness of systems of ODEs we give a result for completeness.

Theorem 2.5. *Consider an initial value problem for system of first order ODEs*

$$x_1'(t) = f_1(t, x_1, x_2, \ldots, x_n) \tag{2.7}$$
$$x_2'(t) = f_2(t, x_1, x_2, \ldots, x_n) \tag{2.8}$$
$$\ldots \tag{2.9}$$
$$x_n'(t) = f_n(t, x_1, x_1, \ldots, x_n) \tag{2.10}$$

with $x_i(t_0) = x_0^i$ *for every* $i = 1, \ldots, n$. *Given a rectangle* $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$ *with* $x_0^i \in (a_i, b_i)$ *for every* i. *Suppose that* f_i *are continuous in* $[t_0, t_1] \times R$ *for every* i *and*

$$\frac{\partial f_i}{\partial x_j}(t, x_1, \ldots, x_n)$$

exist for every $i, j = 1, \dots n$ and are continuous in $[t_0, t_1] \times R$. Then a solution of the initial value problem exists and it is unique in a small interval $[t_0, \hat{t}]$.

Note that this result is very similar to the one for standard ODEs. We are stating it with the assumption on the existence and continuity for the partial derivatives instead the Lipschitz condition. However, it will be also possible to give a Lipschitz condition on all the functions f_i.

2.4.2 Equilibrium points and nullclines

We are now ready to study the stability of solutions of system of ODEs. We start with the definition of **equilibrium** that is identical to the one for equilibriums of ODEs. At the moment we give this notion only for autonomous systems.

Definition 2.6 (Equilibrium point)**.** An equilibrium point for an autonomous system of ODEs is a point $(x_0^1, \dots, x_0^n)$ such that the vector field of the system is equal to zero. Namely

$$\begin{aligned} f_1(x_0^1, \dots, x_0^n) &= 0 \\ f_2(x_0^1, \dots, x_0^n) &= 0 \\ &\dots \\ f_n(x_0^1, \dots, x_0^n) &= 0 \end{aligned}$$

Note that we again have that the constant function $(x_1(t), \dots, x_n(t)) = (x_0^1, \dots, x_0^n)$ is a solution of the system. Such a solution is called **equilibrium solution**.
We also give the concept of **nullcline** for planar autonomous systems.

Definition 2.7 (Nullcline)**.** A nullcline of a planar system

$$\begin{aligned} x' &= f_1(x, y) = 0 \\ y' &= f_2(x, y) = 0 \end{aligned}$$

is the solution set of either $f_1(x, y) = 0$ or of $f_2(x, y) = 0$.

For example, suppose that we are still considering the predator-prey system

$$\begin{aligned} F' &= (a - bS)F \\ S' &= (-c + dF)S \end{aligned}$$

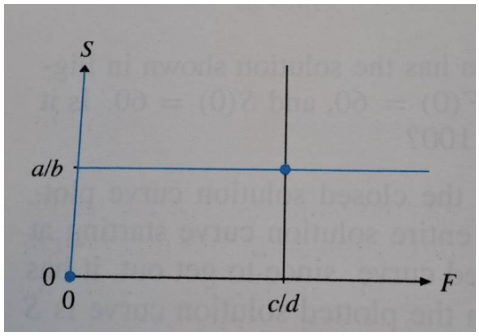
Then the F-nullcline is the solution set of $(a - bS)F = 0$. Therefore are all the points

$$\{(F, S) : F = 0 \text{ or } S = a/b\}$$

The S-nullcline is the solution set of $(-c + dF)S = 0$. Therefore are all the points

$$\{(F, S) : S = 0 \text{ or } S = c/d\}$$

We can represent them in the FS plane as follows.



The blue lines are the F-nullclines and the black lines are the S-nullclines.

Remark 2.8. Note that the equilibrium points of the system are precisely the intersections of the nullclines (the blue dots in the previous figure).

2.5 Linear systems of ODEs

2.5.1 Basic definitions

We first focus on the analysis of linear systems of ODEs.

Definition 2.9 (Linear system of ODEs)**.** A linear system of ODEs is a system of ODEs of the form

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + \ldots + a_{1n}(t)x_n(t) + f_1(t) \\ x_2'(t) &= a_{21}(t)x_1(t) + \ldots + a_{2n}(t)x_n(t) + f_2(t) \\ &\ldots \\ x_n'(t) &= a_{n1}(t)x_1(t) + \ldots + a_{nn}(t)x_n(t) + f_n(t) \end{aligned}$$

Here $x_i(t)$ are the unknowns and a_{ij} are called coefficients.

Similarly to ODEs we say that a system is **homogeneous** if $f_i = 0$ for every i. Otherwise it is **inhomogeneous**. Sometimes the f_i are called **forcing terms**.
When dealing with linear systems, using a matrix notation could be really useful. Suppose that we define the time dependent matrix $A(t)$ as

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}$$

By doing that we can define the linear system of ODEs
as

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \ldots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \ldots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \ldots \\ f_n(t) \end{pmatrix}$$

or in the compact form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

where bold letters denote vectors.
Note that for linear systems we immediately have existence and uniqueness of the initial value problem whenever A and f are continuous.

Theorem 2.10. *Suppose that A and f are continuous in an interval (a, b). The if $t_0 \in (a, b)$ then the initial value problem*

$$\boldsymbol{x}'(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{f}(t) \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

has a unique solution in (a, b).

2.5.2 Properties of homogeneous linear systems

From now one we will use the compact notation

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \tag{2.11}$$

where we intend each component to be vectors, namely

$$\mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \dots \\ x_n'(t) \end{pmatrix} \qquad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}$$

and $A(t)$ as before. We have the following easy result.

Lemma 2.11. *If $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ are solutions of (2.11), then any linear combination*

$$\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + \dots + c_k\boldsymbol{x}_k(t)$$

for $c_1, \dots, c_k \in \mathbb{R}$ is also a solution of (2.11).

Note that this property is completely analogous to the one for linear homogeneous second order ODEs. The proof of this fact is very easy and follows the proof for linear homogeneous second order ODEs.

We now want to follow the same path of second order homogeneous ODEs. In particular we would like to say that all the solution of the first order homogeneous system of ODEs can be constructed as linear combinations of special solutions. Again we need the notion of linear independence of a set of vector valued function $\mathbf{x}_1(t), \dots \mathbf{x}_k(t)$. Again you can check that this definition is equivalent to the standard one of linear algebra.

Definition 2.12 (Linearly independent functions)**.** We say that a set of vector valued functions $\mathbf{x}_1, \dots \mathbf{x}_k$ is linearly independent if

$$c_1\mathbf{x}_1(t) + \dots + c_k\mathbf{x}_k(t) = 0 \quad \forall t$$

implies that $c_1 = c_2 = \dots c_k = 0$.

Actually we have a very useful criteria for deciding when a set of solutions of a linear system is linearly independent. In a nutshell it says that it is enough to verify the linear independence on an arbitrary t. We make this precise in the next lemma

Lemma 2.13. *A set of k solutions $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ of the linear system (2.11) is linearly independent provided it exists a $\hat{t}$ such that the vectors $\boldsymbol{x}_1(\hat{t}), \dots, \boldsymbol{x}_k(\hat{t})$ are linearly independent.*

Viceversa, if a set of k solutions $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ of the linear system (2.11) is linearly independent, then for every $\hat{t}$ the vectors $\boldsymbol{x}_1(\hat{t}), \dots, \boldsymbol{x}_k(\hat{t})$ are linearly independent.

Proof. We prove the first statement. Suppose that $\mathbf{x}_1(\hat{t}), \dots \mathbf{x}_k(\hat{t})$ are linearly independent. Given the linear combination

$$c_1\mathbf{x}_1(t) + \dots + c_k\mathbf{x}_k(t) = 0 \quad \forall t$$

we can evaluate it in $\hat{t}$ to get

$$c_1\mathbf{x}_1(\hat{t}) + \dots + c_k\mathbf{x}_k(\hat{t}) = 0\,.$$

Thus $c_i = 0$ for all i since $\mathbf{x}_1(\hat{t}), \dots \mathbf{x}_k(\hat{t})$ are linearly independent.

Viceversa suppose by contradiction that there exists $\hat{t}$ such that $\mathbf{x}_1(\hat{t}), \dots \mathbf{x}_k(\hat{t})$ are linearly dependent. We want to prove that also $\mathbf{x}_1, \dots, \mathbf{x}_k$ are linearly dependent. Since $\mathbf{x}_1(\hat{t}), \dots \mathbf{x}_k(\hat{t})$ are linearly dependent, there exists $c_i \neq 0$ such that

$$c_1\mathbf{x}_1(\hat{t}) + \dots c_1\mathbf{x}_k(\hat{t}) = 0$$

Note that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \dots + c_k\mathbf{x}_k(t)$$

is also a solution of the system (2.11) by Lemma 2.11. However, also $\mathbf{z}(t) = 0$ for every t is a solution of the same system and $\mathbf{x}(\hat{t}) = 0$. So using the uniqueness for the initial value problem in Theorem 2.10 we obtain that $\mathbf{x}(t) = 0$ and thus $\mathbf{x}_1(t), \dots \mathbf{x}_k(t)$ are linearly independent.

□

So this criteria allows us to verify the linear independence of solutions of a system just by looking at a single t. This is important because of the next result that characterizes the general solution of a linear homogeneous system of ODEs

Theorem 2.14. *Suppose that $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ are linearly independent solutions of the n-dimensional system*

$$\boldsymbol{x}'(t) = A(t)\boldsymbol{x}(t) \tag{2.12}$$

Then any solution of (2.12) can be expressed as a linear combination of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$. In other words the general solution is

$$\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + \dots + \boldsymbol{x}_n(t)$$

for $c_1, \dots, c_n \in \mathbb{R}$.

Definition 2.15 (Fundamental set of solutions)**.** We name a collection of linearly independent solutions $\mathbf{x}_1, \dots, \mathbf{x}_n$ a **fundamental set of solutions**.

Following the same path of second order differential equations, the goal is now to construct (at least in certain situations) a **fundamental set of solutions** for (2.12).
We will do that for linear system of ODEs with constant coefficients. The study of linear systems of ODEs with constant coefficients will be the main subject of the next lectures.

2.6 Linear systems with constant coefficients

A linear system of ODEs with constant coefficients is a system of the form

$$\mathbf{y}' = A\mathbf{y} \tag{2.13}$$

where A is the matrix coefficients that does not depend on time.

Like in the ODE case we are tempted to look at exponential solutions of the system. So we can start to check when a function of the type

$$\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$$

is a solution of (2.13) where $\mathbf{v}$ is a vector. Or in other words, for which choice of $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ the function $e^{\lambda t}\mathbf{v}$ is a solution of (2.13). So we try to substitute it and we get that necessarily

$$\lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v}$$

therefore, dividing by $e^{\lambda t}$ we obtain that

$$\lambda\mathbf{v} = A\mathbf{v}$$

This means that λ needs to be an **eigenvalue** of A and $\mathbf{v}$ needs to be the **eigenvector** associated with the eigenvalue λ.

Therefore the first step to solve and study a system of first order ODEs with constant coefficients consists in finding the eigenvalues and eigenvectors of the matrix A. How can we compute the eigenvalues and eigenvectors of a matrix A?
We solve the equation

$$det(A - \lambda I) = 0$$

to find the eigenvalues and then for every eigenvalue we solve the system

$$A\mathbf{v} = \lambda\mathbf{v}$$

to find a vector $\mathbf{v} \neq 0$.

Definition 2.16. We remind that the the characteristic polynomial of a matrix A is defined as

$$p(\lambda) = (-1)^n det(A - \lambda I)$$

and the equation

$$p(\lambda) = 0$$

is called characteristic equation. The term $(-1)^n$ is added because we want the coefficient of the term λ^n of the polynom to be $+1$. This is just a convention. You can also write the characteristic polynomial without the $(-1)^n$. It is not a mistake.

In the next section we will try to solve a planar system of ODEs by using the methods of eigenvalues and eigenvectors

2.6.1 Planar systems with constant coefficients

We consider now planar systems, that is

$$\mathbf{y}' = A\mathbf{y} \tag{2.14}$$

where

$$\mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} \qquad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Thanks to the computation of the previous section we know that we are searching for solutions of the form $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$ where λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector. Since the matrix A is two dimensional we know that we have three possibilities

- The characteristic polynomial has two distinct real roots
- The characteristic polynomial has two distinct complex (and conjugate) roots
- The characteristic polynomial has one root with multiplicity two.

Let us examine the first case.

Two distinct real roots. This case is fairly straightforward. We have already seen that if there are two distinct real solutions λ_1 and λ_2, then

$$\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad \mathbf{y}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$$

are both solutions, where $\mathbf{v}_1$ is the eigenvector associated to the eigenvalue λ_1 and $\mathbf{v}_2$ is the eigenvector associated to the eigenvalue λ_2.
We just need to verify that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent. This is also true thanks to the following lemma

Lemma 2.17. *Suppose that λ_1 and λ_2 are eigenvalues of A and let $\boldsymbol{v}_1$ an eigenvector associated to λ_1 and $\boldsymbol{v}_2$ is en eigenvector associated to λ_2. Then $e^{\lambda_1 t}\boldsymbol{v}_1$ and $e^{\lambda_2 t}\boldsymbol{v}_2$ are linearly independent.*

Proof. We use the criteria of linear independence given in Lemma 2.13. So it is just enough to verify the linear independence in $t = 0$. Therefore it is enough to verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. This is as standard result in linear algebra: eigenvectors associated to distinct eigenvalues are linearly independent. □

In particular we obtain that a general solution of (2.14) is

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

Two distinct complex conjugate roots. In this case we have two complex and conjugate eigenvalues λ_1 and λ_2. Even if the eigenvalues are complex we can still compute the eigenvector $\mathbf{w}_1$ and $\mathbf{w}_2$ associated to λ_1 and λ_2 and they will be still complex.
Let us see an example. Suppose that we have the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$$

We compute the characteristic polynomial

$$p(\lambda) = det(A - \lambda I) = -\lambda(2-\lambda) + 2 = \lambda^2 - 2\lambda + 2$$

Eigenvalues are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. As expected one is the conjugate of the other ones. Now we compute the eigenvectors. We then have to solve $A\mathbf{v} = \lambda_1\mathbf{v}$ and $A\mathbf{v} = \lambda_2\mathbf{v}$. We start with the first one. The system is

$$\begin{aligned} v_2 &= (1+i)v_1 \\ -2v_1 + 2v_2 &= (1+i)v_2 \end{aligned}$$

As we expected we have an infinite amount of possibilities for v_1 and v_2. Indeed, we chose λ so that the matrix $A - \lambda I$ is singular. For our purposes it is enough to find one solution that is different from zero. So we can set $v_1 = 1$ and from the first equation we get $v_2 = 1 + i$. So the first eigenvector is $\mathbf{w}_1 = (1, 1+i)^T$. By doing the same for λ_2 we get that the second eigenvector is $\mathbf{w}_2 = (1, 1-i)^T$.

Remark 2.18. It is not by chance that the second eigenvector is the conjugate of the first one. This always happens. Indeed, let us consider the eigenvector equation

$$A\mathbf{w} = \lambda\mathbf{w}$$

By taking the conjugate on both sides

$$\overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}}$$

we now use that the conjugate is linear with respect to the complex multiplication. Thus, using that A is real we get

$$A\overline{\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}$$

So we conclude that $\overline{\mathbf{w}}$ is an eigenvector for $\overline{\lambda}$.

By putting together these considerations we arrive at the following result

Theorem 2.19. *Given A a 2×2 matrix with complex eigenvalues λ and $\overline{\lambda}$, suppose that $\boldsymbol{w}$ is an eigenvector associated with λ. Then the complex general solution of the planar system is*

$$\boldsymbol{y}(t) = d_1 e^{\lambda t}\boldsymbol{w} + d_2 e^{\overline{\lambda} t}\overline{\boldsymbol{w}}$$

for d_1, d_2 complex constants.

Similarly of what we did for second order ODEs, we would like to have a real general solution. We will see that the real and the imaginary part of a complex solution provide a real set of fundamental solution.
We have the following result that is actually general for $n \times n$ system

Proposition 2.20. *Let A a $n \times n$ matrix with real coefficient. Suppose that $\boldsymbol{z}(t) = \boldsymbol{x}(t) + i\boldsymbol{y}(t)$ is a complex solution of the system*

$$\boldsymbol{z}' = A\boldsymbol{z} \tag{2.15}$$

Then the conjugate $\overline{\boldsymbol{z}}$ is also a solution of (2.15) and the real and imaginary part $\boldsymbol{x}$ and $\boldsymbol{y}$ of $\boldsymbol{z}$ are also solutions of (2.15). Moreover, if $\boldsymbol{z}$ and $\overline{\boldsymbol{z}}$ are linearly independent, then also $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent.

Proof. The fact that $\overline{\mathbf{z}}$ is a solution of (2.15) follows from the following identities:

$$\overline{\mathbf{z}}' = \overline{\mathbf{z}'} = \overline{A\mathbf{z}} = \overline{A}\overline{\mathbf{z}} = A\overline{\mathbf{z}}$$

Now we can express the real part and the imaginary part $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{x} = \frac{1}{2}(\mathbf{z} + \overline{\mathbf{z}}) \qquad \mathbf{y} = \frac{1}{2i}(\mathbf{z} - \overline{\mathbf{z}})$$

Therefore $\mathbf{x}$ and $\mathbf{y}$ are linear combinations of solutions and therefore are solutions themselves. It remains to prove that if $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent, then also $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Suppose by contradiction that $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. So there is a constant c such that $\mathbf{y} = c\mathbf{x}$. This implies that $\mathbf{z} = (1 + ic)\mathbf{x}$ and $\overline{\mathbf{z}} = (1 - ic)\mathbf{x}$. Therefore

$$(1 - ic)\mathbf{z} - (1 + ic)\overline{\mathbf{z}} = 0$$

implying that $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly dependent and therefore reaching a contradiction. □

Now suppose that we have a 2×2 system of ODEs. If eigenvalues are complex we have shown that one solution is given by

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$$

where $\lambda = a + ib$ is an eigenvalue of A and $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ is the corresponding eigenvector. We want now to find the real and imaginary parts of $\mathbf{z}$.

$$\begin{aligned}\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{(a+ib)t}(\mathbf{v}_1 + i\mathbf{v}_2) = e^{at}(\cos(bt) + i\sin(bt))(\mathbf{v}_1 + i\mathbf{v}_2)\\ = e^{at}(\cos(bt)\mathbf{v}_1 - \sin(bt)\mathbf{v}_2) + ie^{at}(\sin(bt)\mathbf{v}_1 + \cos(bt)\mathbf{v}_2)\end{aligned}$$

So, thanks to Proposition 2.20 we have that a real fundamental set of solution is

$$\mathbf{y}_1(t) = e^{at}(\cos(bt)\mathbf{v}_1 - \sin(bt)\mathbf{v}_2) \qquad \mathbf{y}_2(t) = e^{at}(\sin(bt)\mathbf{v}_1 + \cos(bt)\mathbf{v}_2)$$

and therefore the general solution is

$$\mathbf{y}(t) = c_1e^{at}(\cos(bt)\mathbf{v}_1 - \sin(bt)\mathbf{v}_2) + c_2e^{at}(\sin(bt)\mathbf{v}_1 + \cos(bt)\mathbf{v}_2)$$

for $c_1, c_2 \in \mathbb{R}$.

A single eigenvalue with multiplicity two.
Suppose now that the eigenvalue λ has multiplicity 2.
We can compute a solution of the linear system by simply computing an eigenvector $\mathbf{v}$ associated to λ and set

$$y(t) = e^{\lambda t}\mathbf{v}$$

The question is: how do we compute a second solution that is linearly independent with $y(t)$?. Note that we can suppose that the eigenspace of λ, that is the space spanned by the eigenvectors associated to λ has dimension 1. Indeed, if the dimension were 2, this is a degenerate case that can happen only when the system is decoupled, that is

$$\begin{aligned} x_1' &= \lambda x_1 \\ x_2' &= \lambda x_2 \end{aligned}$$

that can be solved easily.

Similarly to the case of second order ODE we start by looking at a solution that is of the form

$$\mathbf{x}(t) = e^{\lambda_1 t}(\mathbf{v}_2 + t\mathbf{v}_1)$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ needs to be determined. We now plug $\mathbf{x}(t)$ into the system to find which conditions $\mathbf{v}_1$ and $\mathbf{v}_2$ need to satisfy.

$$\mathbf{x}'(t) = \lambda_1 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1) + e^{\lambda t}\mathbf{v}_1 = e^{\lambda t}((\lambda\mathbf{v}_2 + \mathbf{v}_1) + \lambda t\mathbf{v}_1)$$

This should be equal to $A\mathbf{x}(t) = e^{\lambda t}((A\mathbf{v}_2 + \mathbf{v}_1) + tA\mathbf{v}_1)$. In particular, pairing the terms that are multiplied by t we get that necessarily

$$\begin{aligned} A\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1 \\ A\mathbf{v}_1 &= \lambda\mathbf{v}_1 \end{aligned}$$

Note that the second condition is precisely satisfied when $\mathbf{v}_1$ is an eigenvector of λ. We have already found that. We just need to find $\mathbf{v}_2$ that satisfies $A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$.

Remark 2.21. For a linear algebra result, whenever you have an eigenvalue with multiplicity 2 it holds that

$$(A - \lambda I)^2\mathbf{w} = 0$$

for every $\mathbf{w}$.

Thanks to this remark we immediately obtain that $(A - \lambda I)\mathbf{w}$ is an eigenvector associated to λ and thus it is a multiple of $\mathbf{v}_1$ that is

$$(A - \lambda I)\mathbf{w} = a\mathbf{v}_1$$

Note also that if we choose $\mathbf{w}$ not a multiple of $\mathbf{v}_1$, then $\mathbf{w}$ is not an eigenvector and thus the constant a is not equal to zero. So if we choose

$$\mathbf{v}_2 = \frac{1}{a}\mathbf{w}$$

then we found $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfying the relations

$$\begin{aligned} A\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1 \\ A\mathbf{v}_1 &= \lambda\mathbf{v}_1 \end{aligned}$$

In conclusion we have the following theorem

Theorem 2.22. *Let λ an eigenvalue of A with multiplicity 2. Suppose that the eigenspace of λ has dimension 1. Take $\boldsymbol{v}_1$ an eigenvector and $\boldsymbol{v}_2$ such that $(A - \lambda I)\boldsymbol{v}_2 = \boldsymbol{v}_1$ (the way to find $\boldsymbol{v}_2$ is described in the previous paragraphs). Then*

$$\begin{aligned} \boldsymbol{x}_1(t) &= e^{\lambda t}\boldsymbol{v}_1 \\ \boldsymbol{x}_2(t) &= e^{\lambda t}(\boldsymbol{v}_2 + t\boldsymbol{v}_1) \end{aligned}$$

form a fundamental set of solutions for the system $\boldsymbol{x}' = A\boldsymbol{x}$.

2.7 Phase plane portraits and stability for linear systems

We now want to use all the notions that we have developed about linear systems of ODEs to draw phase plane portraits and understand stability.
We consider the planar linear homogeneous system ode ODEs (with constant coefficients) that is

$$\mathbf{y}' = A\mathbf{y} \tag{2.16}$$

where

$$\mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} \qquad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Remark 2.23. Note that is A is not singular the only equilibrium point of the system is $(0, 0)$. In what follows we will always consider matrix A that are not singular

We now want to understand the stability of the equilibrium $(0, 0)$. This will be dependent on the eigenvalues of the matrix A. Therefore we consider separate cases. Please, refer to Section 9.3 in Polking for more examples and figures.

2.7.1 Real distinct eigenvalues

First we consider the case of real and distinct eigenvalues λ_1 and λ_2. Due to the discussion of the previous section we know that the general solution is

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

for $c_1, c_2 \in \mathbb{R}$.

Note that if we look at the solutions

$$c_1 e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad c_2 e^{\lambda_2 t}\mathbf{v}_2$$

these solutions are called **exponential solutions** and they are half line directed by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Depending on the sign of λ_1 and λ_2 the solution will move towards zero or away from zero. Solutions that are moving towards zero are called stable, while solution that are moving away from zero are called unstable. If you think in this way then the stability of the equilibrium will be different if we have two positive eigenvalues, two negative eigenvalues, and one positive and one negative eigenvalues.

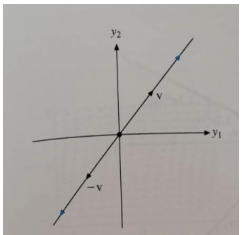


Figure 1. For a positive eigenvalue there are two unstable exponential solution curves.

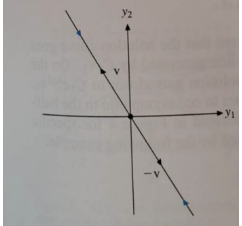


Figure 2. For a negative eigenvalue there are two stable exponential solution curves.

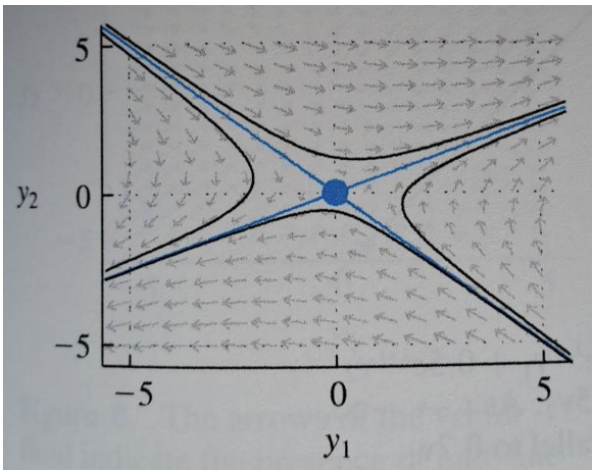
- **Saddle point**

Suppose that $\lambda_1 < 0$ and $\lambda_2 > 0$. Then the equilibrium $(0, 0)$ is called a **saddle point**. If we look at the exponential solutions

$$c_1 e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad c_2 e^{\lambda_2 t}\mathbf{v}_2$$

the first one is converging to the origin for $t \to +\infty$ (it is stable) while the second one is converging to $+\infty$ (or better its norm is converging to $+\infty$) when $t \to +\infty$ (it is diverging). In this case the dynamics around the equilibrium is hybrid. In some direction you will be stable, in some direction you will be unstable. This is the reason why the equilibrium is called saddle point.

How can we draw the vector field and the solution curves in the phase plane portrait? First we draw the half-lines corresponding to the eigenvalues $\mathbf{v}_1$ and $\mathbf{v}_2$ (these are also called **separatrices**). These lines corresponds to the direction of stability and instability. The behaviour of a typical solution curve is to approach zero following the direction of stability and then moving away from zero following the direction of instability.
For example this could be a phase portrait in the case of a saddle point



When you have to draw a phase portrait you need to compute the eigenvalues and the eigenvectors. Then draw the lines corresponding to the eigenvectors, draw the arrows in the direction of stability (or instability) and sketch one of a few solution curves.

- **Nodal sink**

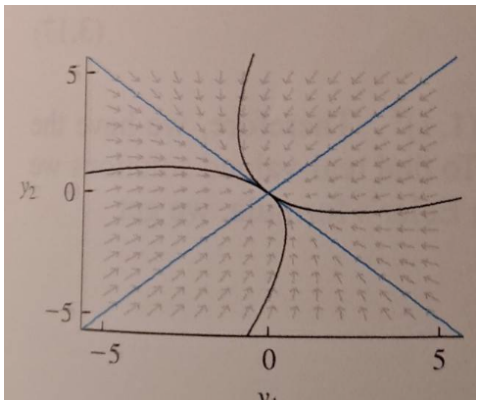
We consider the case when $\lambda_1 < \lambda_2 < 0$, so the eigenvalues are negative. In this case the exponential solutions

$$c_1 e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad c_2 e^{\lambda_2 t}\mathbf{v}_2$$

are all moving towards zero. So the origin is stable (also asymptotically stable). Moreover all the arrows are pointing towards the origin all solution curves end up in the origin. Moreover since on the eigenvector the arrows are pointing in the direction of the eigenvectors.
Note also that we can write

$$\mathbf{y}(t) = e^{\lambda_2 t}(c_1 e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + c_2\mathbf{v}_2)$$

So since $\lambda_1 - \lambda_2 < 0$ we get that $\mathbf{y}(t)$ is converge to zero, in the direction $c_1 e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + c_2\mathbf{v}_2$ that for $t \to +\infty$ is parallel to $\mathbf{v}_2$.



- **Nodal source**

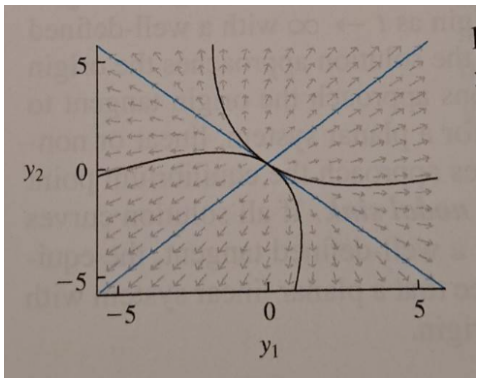
The opposite case is when the eigenvalues are both positive: $\lambda_2 > \lambda_1 > 0$. In this case the exponential solutions

$$c_1 e^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad c_2 e^{\lambda_2 t}\mathbf{v}_2$$

are converging away from zero.
Note also that we can write

$$\mathbf{y}(t) = e^{\lambda_2 t}(c_1 e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + c_2\mathbf{v}_2)$$

So since $\lambda_1 - \lambda_2 < 0$ we get that $\mathbf{y}(t)$ is converge to zero, in the direction $c_1 e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + c_2\mathbf{v}_2$ that for $t \to +\infty$ is parallel to $\mathbf{v}_2$.



We then pass to the case of complex eigenvalues

2.7.2 Complex eigenvalues eigenvalues

- Center

In this case both complex eigenvalues have zero real part (are only imaginary). So

$$\lambda_1 = i\beta \quad \lambda_2 = -i\beta$$

In this case the equilibrium is called **center**. Denoting by $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ the eigenvector associated to λ_1 we know that the general solution is thus

$$\mathbf{y}(t) = c_1(\cos(\beta t)\mathbf{v}_1 - \sin(\beta t)\mathbf{v}_2) + c_2(\sin(\beta t)\mathbf{v}_1 + \cos(\beta t)\mathbf{v}_2)$$

Note that solution $\mathbf{y}(t)$ is periodic. Therefore solution curves are always closed curves orbiting around the origin. Note that in order to determine better the ellipses it is enough to evaluate the vector field in some points (for example in the x-axis or y-axis). This will give you a better understanding on the shape of the ellipses

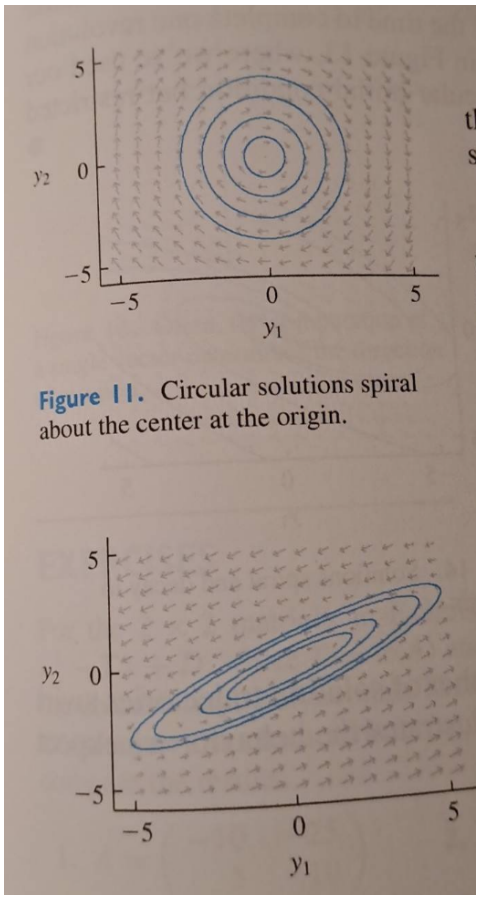


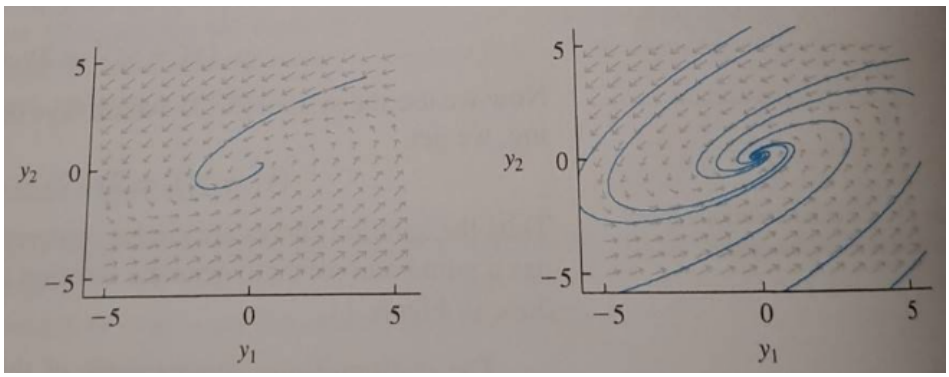
Figure 11. Circular solutions spiral about the center at the origin.

- Spiral Sink

This happens when the real part of the complex eigenvalues is negative. In this case solution is provided by

$$\mathbf{y}(t) = c_1 e^{\alpha t}(\cos(\beta t)\mathbf{v}_1 - \sin(\beta t)\mathbf{v}_2) + c_2 e^{\alpha t}(\sin(\beta t)\mathbf{v}_1 + \cos(\beta t)\mathbf{v}_2)$$

The solution is therefore not periodic anymore and it is converging to zero as $t \to +\infty$. Therefore solution curves behave as spirals spiralling towards the origin. Note that in order to determine better the spiral it is enough to evaluate the vector field in some points (for example in the x-axis or y-axis). This will give you information on the direction of the spiral.

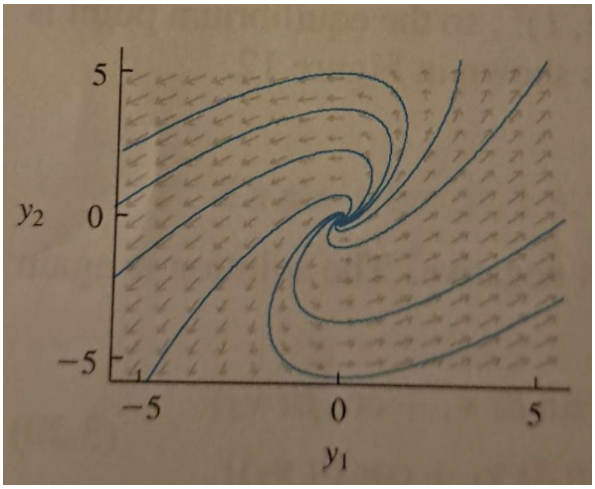


- Spiral Source

In this case the real part of the eigevalues is positive. The general solution is again

$$\mathbf{y}(t) = c_1 e^{\alpha t}(\cos(\beta t)\mathbf{v}_1 - \sin(\beta t)\mathbf{v}_2) + c_2 e^{\alpha t}(\sin(\beta t)\mathbf{v}_1 + \cos(\beta t)\mathbf{v}_2)$$

We have again spiral solution curves, but this time spiralling away from zero. Note that in order to determine better the spiral it is enough to evaluate the vector field in some points (for example in the x-axis or y-axis). This will give you better information on the direction of the spiral.



2.8 Higher dimensional systems of ODEs

In the previous section we learnt how to compute fundamental solutions of planar systems. We now look at the case of n-dimensional systems. Here the situation is much more involved since we can have real and complex eigenvalues with arbitrary multiplicity. Also it is worth to remark that we do not have a phase plane and drawing solution of a n-dimensional system (with $n \geq 3$) is extremely difficult.
Let us start with a reminder of linear algebra.

Remark 2.24. The **characteristic polynomial** $p(\lambda)$ of a matrix A can be written as

$$p(\lambda) = (\lambda - \lambda_1)^{q_1} \dots (\lambda - \lambda_k)^{q_k}$$

where λ_j are the k-distinct eigenvalues and q_j are their multiplicity. We define the **algebraic multiplicity** of λ_j as q_j. Moreover we define the **geometric multiplicity** of λ_j as d_j the dimension of the **eigenspace** of λ_j (that is the dimension of the space generated by the eigenvectors associated to λ_j). It holds that $d_j \leq q_j$, that is the geometric multiplicity is always less or equal than the algebraic multiplicity.

The goal is as usual to find a set of n-linearly independent solutions of the system. In many cases we already know how to do that.

- **Real distinct eigenvalues**

When λ_i are real and distinct we know that linearly independent solutions are

$$\mathbf{y}_j = e^{\lambda_j t}\mathbf{v}_j$$

where $\mathbf{v}_j$ is an eigenvector associated to λ_j.

- **Distinct complex eigenvalues**

We also know how to compute fundamental solution when λ_j are complex and distinct eigenvalues. Note that in this case we necessarily have pairs of conjugated eigenvalues. In this case linearly independent solutions are

$$\mathbf{y}_j^1(t) = e^{\alpha_j t}(\cos(\beta_j t)\mathbf{v}_j^1 - \sin(\beta_j t)\mathbf{v}_j^2)$$
$$\mathbf{y}_j^2(t) = e^{\alpha_j t}(\sin(\beta_j t)\mathbf{v}_j^1 + \cos(\beta_j t)\mathbf{v}_j^2)$$

where $\lambda_j = \alpha_j + i\beta_j$ is the complex eigenvalue and $\mathbf{v}_j = \mathbf{v}_j^1 + i\mathbf{v}_j^2$ is the complex eigenvector associated to λ_j. .

Remark 2.25. Note that when you have two complex eigenvalues, necessarily they have to be distinct (and conjugated). However, if you have more that two it could be that the they are identical. In this case the situation is more complicated.

- **Real repeated eigenvalues**

We now want to analyze the situation when the algebraic multiplicity of some eigenvalues is bigger than one. We have already seen that for planar systems.
Let us start with an example (this is example 6.23 in Polking). We want to find a fundamental set of solution of the system

$$\mathbf{x}' = A\mathbf{x} \quad A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -3 \end{pmatrix}$$

The first step is to compute the eigenvalues by solving $det(A - \lambda I) = 0$ (not that this is the determinant of a 3×3 matrix). One gets that the eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = -2 \ \text{ (with algebraic multiplicity two)}$$

For $\lambda_1 = -1$ we know how to compute a fundamental solution. We compute one eigenvector associated to λ_1 that is for example $\mathbf{v}_1 = (1, 1, -2)^T$ and we know that the first fundamental solution is simply

$$\mathbf{x}_1(t) = e^{\lambda_1}\mathbf{v}_1 = e^{-t}(1, 1, -2)^T$$

We now have to examine the eigenvalue with multiplicity two. Note that its algebraic multiplicity is two. Let us determine the **eigenspace**, that is the set of eigenvectors. We have to solve $det(A - \lambda I) = 0$. So we compute $A + 2I$ that is

$$A + 2I = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix}$$

and we solve the system $(A+2I)(v_1,v_2,v_3)^T=0$ for. If you do that you get the relation

$$v_2=0$$
$$v_1=-v_3$$

In particular the eigenspace is

$$\{(\alpha,0,-\alpha):\alpha\in\mathbb{R}\}$$

and therefore has dimension 1. We can pick one element, let say $\mathbf{v}_2=(1,0,-1)^T$ as a basis for this eigenspace. This also says that the geometric multiplicity of λ_2 is one. Also we can find one fundamental solution of the system simply considering

$$\mathbf{x}_2(t)=e^{\lambda_2}\mathbf{v}_2=e^{-2t}(1,0,-1)^T$$

Since we have a 3-dimensional system we have to find an additional fundamental solution from this eigenvalue. We follow a similar strategy to the planar systems. We have to find a vector $\mathbf{v}$ such that

$$(A-\lambda I)^2\mathbf{v}=0 \tag{2.17}$$

that is linearly independent to $\mathbf{v}_2$. Note that for the planar case we have that every vector $\mathbf{v}$ satisfies (2.17). For higher dimensional system this is not the case anymore. So one has really to find a solution of $(A-\lambda I)^2\mathbf{v}=0$. Calling $\mathbf{v}_3$ a solution of $(A-\lambda I)^2\mathbf{v}=0$ that is linearly independent with $\mathbf{v}_2$ the last fundamental solution is

$$\mathbf{x}_3(t)=e^{-2t}(\mathbf{v}_3+t(A+2I)\mathbf{v}_3)$$

Looking at our example we can compute

$$(A+2I)^2=\begin{pmatrix}0&5&0\\0&1&0\\0&-8&0\end{pmatrix}$$

So solving $(A-\lambda I)^2\mathbf{v}=0$ we get that all the solutions are

$$\{(\alpha,0,\beta):\alpha,\beta\in\mathbb{R}\}$$

so it has dimension 2. We expect that $\mathbf{v}_2$ belongs to this space and indeed $(1,0,-1)^T$ belongs to it. We need to choose another linearly independent vector. Let us say $(1,0,0)^T$. So the last fundamental solution is

$$\mathbf{x}_3(t)=e^{-2t}((1,0,0)^T+t(A+2I)(1,0,0)^T)$$

Remark 2.26. Solutions of $(A-\lambda I)^2\mathbf{v}=0$ are called **generalized eigenvectors**.

Remark 2.27. If the geometric multiplicity of the eigenvector is also 2, then it is easy. You can just take the two eigenvectors spanning the eigenspace $\mathbf{v}_1$, $\mathbf{v}_2$ and use them to construct the fundamental solution (without the use of generalized eigenvector) as

$$\mathbf{y}_1=e^{\lambda}\mathbf{v}_1\quad \mathbf{y}_2=e^{\lambda}\mathbf{v}_2$$

This is the same for planar systems. Basically if you are not able to find linearly independent eigenvectors in the eigenspace, then you have to resort to search for **generalized eigenvectors**. This is the reason why the case we analyze just before (when the geometric multiplicity is strictly lower than the algebraic multiplicity is called defective).

- What if we have eigenvalues with algebraic multiplicity greater than two?

Let us say that λ is an eigenvalues with algebraic multiplicity equal to q. In this case the procedure is as follows:
1) Find the smallest p such that the nullspace of $(A-\lambda I)^p$ has dimension q
2) Find a basis $\{\mathbf{v}_1, \dots \mathbf{v}_q\}$ for the nullspace of $(A-\lambda I)^p$
3) The q linearly independent solutions associated to the eigenvalue λ are

$$x_j = e^{\lambda t}(\mathbf{v}_j + t(A-\lambda I)\mathbf{v}_j + \ldots + \frac{t^{p-1}}{(p-1)!}(A-\lambda I)^{p-1}\mathbf{v}_j)$$

for $j = 1, \dots, q$. You can check that this is what we have done in the previous example, where q was precisely equal to 2.

Remark 2.28. Also in this case, solutions of $(A-\lambda I)^p\mathbf{v} = 0$ are called **generalized eigenvectors**.

- What about for repeated complex eigenvalues (note that the system has to be of dimension at least 4 to have this case)

Actually the procedure is similar and you can follow example 6.26 in Polking. We will not see the details of this case.

2.9 The exponential matrix

A classical way to solve linear systems of ODEs and initial value problems is to find the so called exponential matrix. When solving ODEs we found that solutions of linear homogeneous ODEs, $x' = ax$ can be computed simply as

$$x(t) = Ce^{at}$$

and the solution of the initial value problem for $x(0) = x_0$ was simply

$$x(t) = x_0 e^{at}$$

We now wonder if there exists an analogous of this formula for systems of ODEs. We would be tempted to try to define something like

$$\mathbf{x}(t) = e^{tA}\mathbf{v}$$

but obviously it is not clear what do we mean by exponential of a matrix. The next part is devoted to properly define the concept of exponential of a matrix and see how can we use it to solve systems of ODEs.

2.9.1 Definition and basic properties of the exponential matrix

To define the exponential matrix we have to remember how to write an exponential function as a series. You can recall that

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k!}a^k$$

for every $a \in \mathbb{R}$. We want to use a similar formula to define the exponential of the matrix as follows.

Definition 2.29 (Exponential of a matrix)**.** The **exponential of the matrix** A is defined as

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Note that here A^k is simply the matrix product $A \cdot A \cdot \ldots \cdot A$. Moreover the convergence of the series is never a problem since we are dividing by $k!$.

This is nice. Unfortunately, even for small matrices this is very difficult to compute. We now see one of the (few) examples when this is possible

Example:
We consider the diagonal matrix

$$A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

for $r_1, r_2 \neq 0$. We want to compute e^A. We simply use the definition and compute

$$A^2 = A \cdot A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

This computation can be repeated to determined A^k for every k:

$$A^k = \begin{pmatrix} r_1^k & 0 \\ 0 & r_2^k \end{pmatrix}$$

Therefore

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} r_1^k & 0 \\ 0 & r_2^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} r_1^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} r_2^k \end{pmatrix} = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}$$

Why is the exponential of a matrix useful to determine solutions of linear systems of ODEs. The key result is the following:

Theorem 2.30. *Let be given a $n \times n$ initial value problem for a linear system of ODEs*

$$\boldsymbol{x}' = A\boldsymbol{x} \quad \boldsymbol{x}(0) = \boldsymbol{v} \tag{2.18}$$

A solution of (2.18) can be computed as

$$\boldsymbol{x}(t) = e^{tA}\boldsymbol{v}$$

Note that this theorem says that the exponential matrix of A provides a complete description of all solutions of a linear system.

Proof. The proof is fairly straightforward. First let us observe that

$$\begin{aligned} \frac{d}{dt} e^{tA} &= \frac{d}{dt} \left(\sum_{k=0}^{\infty} \frac{1}{k!} (At)^k \right) = \sum_{k=1}^{\infty} \frac{kA}{k!} (At)^{k-1} = \sum_{k=1}^{\infty} \frac{kA}{k!} (At)^{k-1} \\ &= A \sum_{k=1}^{\infty} \frac{1}{(k-1)!} (At)^{k-1} = A \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k = Ae^{tA} \end{aligned}$$

Therefore

$$\mathbf{x}'(t) = Ae^{tA}\mathbf{v} = A\mathbf{x}(t)$$

meaning that $\mathbf{x}(t)$ solves the system of ODEs. Moreover it has the right initial value since

$$\mathbf{x}(0) = \mathbf{v}$$

□

Remark 2.31. Note that if the initial condition is not computed in zero, but it is $\mathbf{x}(t_0) = \mathbf{v}$ then the solution to the initial value problem is

$$\mathbf{x}(t) = e^{(t-t_0)A}\mathbf{v}$$

This can be verified using the fact that the exponential matrix satisfies the property

$$e^{A+B} = e^A e^B$$

- How can we construct an exponential matrix for a system?

As we said before computing an exponential matrix is hard in general. This is no surprise, since we found in the previous theorem that this is basically equivalent to finding a solution of a linear system of ODEs. However, it is possible to write the exponential matrix starting from a fundamental set of solutions of a system of ODEs.
We first give the notion of fundamental matrix of the system.

Definition 2.32 (Fundamental matrix)**.** The fundamental matrix of system $\mathbf{x}' = A\mathbf{x}$ is defined as the $n \times n$ matrix whose coloumns are the n fundamental solutions of the system. So given $\mathbf{x}_1, \dots \mathbf{x}_n$ a set of fundamental solutions for $\mathbf{x}' = A\mathbf{x}$ the fundamental matrix is

$$Y(t) = \begin{pmatrix} | & | & \dots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ | & | & \dots & | \end{pmatrix}$$

Theorem 2.33. *Let $\boldsymbol{x}_1, \dots \boldsymbol{x}_n$ a set of fundamental solution for a $n \times n$ system of ODEs*

$$\boldsymbol{x}' = A\boldsymbol{x}$$

Let $Y(t)$ the $n \times n$ fundamental matrix of the system. Then

$$e^{tA} = Y(t)Y(0)^{-1}$$

Let us see why

Proof. Define $E(t) = Y(t)Y(0)^{-1}$ and note that $E(0) = Id$ and

$$\frac{d}{dt}E(t) = \frac{d}{dt}Y(t)Y(0)^{-1} = AY(t)Y(0)^{-1} = AE(t)$$

but also

$$\frac{d}{dt}e^{tA} = Ae^{tA} \quad e^{0A} = Id$$

So, by the uniqueness of solution for linear systems of ODEs with constant coefficients one gets that $E(t) = e^{tA}$. □

2.10 Inhomogeneous linear systems of ODEs

We now switch our attention to inhomogeneous $n \times n$ linear systems of ODEs, namely systems of the form

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$$

for some $\mathbf{f}$ different than zero. Note that the results of this section hold also when the coefficients are time dependent. However, since we know how to compute solutions only for systems with constant coefficient, we will use them only applied to linear systems with constant coefficients.
Similarly to the case of ODEs we start by deriving results about the structure of solution of inhomogeneous systems of ODEs. We prove that a general solution is given by the sum of a general solution of the homogeneous system and a particular solution of the inhomogeneous system.

Proposition 2.34. *A general solution of the inhomogeneous liner system*

$$\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{f}(t) \tag{2.19}$$

is

$$\boldsymbol{y} = \boldsymbol{y}_p + c_1\boldsymbol{y}_1 + \dots c_n\boldsymbol{y}_n$$

where $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ is a fundamental set of solution for $\boldsymbol{y}' = A(t)\boldsymbol{y}$ and y_p is one particular solution of $\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{f}(t)$.

Remark 2.35. Note that $c_1\mathbf{y}_1 + \dots c_n\mathbf{y}_n$ is a general solution of the linear homogeneous system $\mathbf{y}' = A(t)\mathbf{y}$.

Since the proof is based on the linear structure of the system, it is very similar to the proof of the similar result for single ODEs.

Proof. Let $\mathbf{y}$ be an arbitrary solution of (2.19) and $\mathbf{y}_p$ a particular solution. Then $\mathbf{y} - \mathbf{y}_p$ solves the homogeneous system $\mathbf{y}' = A(t)\mathbf{y}$. Thus it can be written as

$$\mathbf{y} - \mathbf{y}_p = c_1\mathbf{y}_1 + \dots c_n\mathbf{y}_n$$

and the thesis follows. □

Now we want to find a general solution of the inhomogeneous systems of ODEs with constant coefficients. Let us remind that in case of a single ODEs we derived (by variations of parameters or by the integrating factor method) the formula

$$y(t) = Ce^{\int a(t)\,dt} + e^{\int a(t)\,dt} \int f(t)e^{-\int a(t)\,dt}\,dt \qquad C \in \mathbb{R}$$

for the general solution of the ODE

$$y' = a(t)y + f(t)$$

We want to generalize this formula in the case of systems of ODEs with constant coefficients. A key ingredient will be the fundamental matrix for the homogeneous system $\mathbf{x}' = A\mathbf{x}$ defined as

$$Y(t) = \begin{pmatrix} | & | & \dots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ | & | & \dots & | \end{pmatrix}$$

where $\mathbf{x}_1, \dots \mathbf{x}_n$ a set of fundamental solutions for $\mathbf{x}' = A\mathbf{x}$.

Theorem 2.36. *The general solution of the inhomogeneous linear system with constant coefficients $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t)$ can be computed as*

$$\boldsymbol{y}(t) = Y(t)\boldsymbol{c} + Y(t)\left(\int Y(t)^{-1}\boldsymbol{f}(t)\,dt\right)$$

where $\boldsymbol{c} = (c_1, \dots c_n)^T$.

We will not provide a proof of this fact. However note that the first part $Y(t)\mathbf{c}$ is precisely the general solution of the homogeneous equation. A similar formula holds for the initial value problem.

Theorem 2.37. *The solution of the initial value problem*

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t) \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

can be computed as

$$\boldsymbol{y}(t) = Y(t)\left(Y(t_0)^{-1}\boldsymbol{y}_0 + \int_{t_0}^{t} Y(s)^{-1}\boldsymbol{f}(s)\,ds\right)$$

2.11 Qualitative analysis of linear systems

By describing the fundamental solution of a n dimensional linear system of ODEs we can also get information about its stability. We saw that if an eigenvalue λ is real, then a fundamental solution is

$$\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$$

where $\mathbf{v}$ is the eigenvector associated to λ. On the other hand if the eigenvalue $\lambda = a + ib$ is complex, then

$$\mathbf{y}(t) = e^{\lambda t}\mathbf{v} = e^{at}e^{ibt}\mathbf{v} = e^{at}(\cos(bt) + i\sin(bt))\mathbf{v}$$

where $\mathbf{v}$ is the complex eigenvector associated to λ. Finally in case of repeated eigenvalues a fundamental solution has the form

$$\mathbf{y}(t) = e^{\lambda t}(\mathbf{v} + t(A - \lambda I)\mathbf{v} + \ldots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v})$$

where $\mathbf{v}$ is a generalized eigenvector.

What appears clear in all cases is that the asymptotic behaviour for $t \to +\infty$ depends on λ (if the eigenvalue is real) and on the real part of λ (if λ is complex).
Therefore we deduce the following theorem

Theorem 2.38. *Given A a $n \times n$ matrix.*

- *If the real part of every eigenvalues of A is negative then every solution of the system $\boldsymbol{x}' = A\boldsymbol{x}$ tends to the origin for $t \to +\infty$.*
- *If A has at least one eigenvalue with positive real part then there are solutions of the system $\boldsymbol{x}' = A\boldsymbol{x}$ starting arbitrarily close to the origin that get arbitrarily large for $t \to +\infty$.*

In particular the sign of the real part of the eigenvalues determine the stability of the system. The definition of being stable, unstable or asymptotically stable is the same as for one dimensional ODEs. Let us recall it.

Definition 2.39 (Stability)**.** We say that an equilibrium point x_0 is **asymptotically stable** if there exists $\delta > 0$ such that every solution of the system of ODE, denoted by $\mathbf{x}(t)$, such that $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ satisfies

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}_0.$$

We say that an equilibrium point $\mathbf{x}_0$ is **stable** if for every $\varepsilon > 0$ there exists δ such that for every solution of the ODE, denoted by $\mathbf{x}(t)$, such that $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ it holds that

$$|\mathbf{x}(t) - \mathbf{x}_0| < \varepsilon \quad \forall t > 0$$

Clearly asymptotically stable implies stable. Finally we say that an equilibrium point is **unstable** if it is not stable.

In particular Theorem 2.38 can be rephrased as

Theorem 2.40. *For a linear system of ODEs with constant coefficients the origin is asymptotically stable if the real part of every eigenvalue of A is negative. On the other hand, if there exists an eigenvalue with a positive real part, then the origin is unstable.*

3 Non linear systems of ODEs

We now start a new section that is about non-linear system of ODEs. While for linear systems we had a quite complete way to find solutions and study the dynamics, for non-linear systems of ODEs the description of the dynamics is much harder. We will see that the first basic way to get information on the stability of the equilibrium will be the linearization of the system.

Non linear systems simply refers to general n dimensional systems

$$x_1'(t) = f_1(t, x_1, x_2, \dots, x_n) \tag{3.1}$$
$$x_2'(t) = f_2(t, x_1, x_2, \dots, x_n) \tag{3.2}$$
$$\dots \tag{3.3}$$
$$x_n'(t) = f_n(t, x_1, x_1, \dots, x_n) \tag{3.4}$$

We will only study the stability for autonomous non-linear systems, namely without an explicit dependence of t:

$$x_1'(t) = f_1(x_1, x_2, \dots, x_n) \tag{3.5}$$
$$x_2'(t) = f_2(x_1, x_2, \dots, x_n) \tag{3.6}$$
$$\dots \tag{3.7}$$
$$x_n'(t) = f_n(x_1, x_1, \dots, x_n) \tag{3.8}$$

We now want to understand what linearization of a non-linear system means and how such techniques allows for a stability analysis around the equilibriums.

3.1 The intuition behind linearization in 1D

To clarify the idea let us start with one-dimensional autonomous systems, that simply means autonomous ODEs

$$y' = f(y)$$

Suppose that y_0 is an equilibrium, that is $f(y_0) = 0$. Then if f is differentiable we can use Taylor formula around y_0 to write

$$f(y_0 + u) = f(y_0) + f'(y_0)u + R(u) = f'(y_0)u + R(u)$$

where $R(u)$ is the classical remainder in Taylor expansion and thus

$$\lim_{u \to 0} \frac{R(u)}{u} = 0.$$

Since $R(u)$ is negligible compared to $f'(y_0)$ we can imagine to neglect it and for u small enough consider the approximation

$$f(y_0 + u) \approx f'(y_0)u.$$

This is simply the linearization of the autonomous ODE $y' = f(y)$ around the equilibrium y_0. Now suppose that we make a change of variable in $y' = f(y)$ to move the equilibrium y_0 in the origin by setting $y - y_0 = u$. In this way using the linearization we get

$$u' = f(u + y_0) \approx f'(y_0)u$$

that we can simply solve as

$$u(t) = Ce^{f'(y_0)t}$$

We can then make the following observation:

- If $f'(y_0) > 0$ then $u(t) \to +\infty$ (or changing back the variables $y(t) \to y_0$)
- If $f'(y_0) < 0$ then $u(t) \to 0$ (or changing back the variables, $y(t) \to y_0$)

In particular we recover the result about stability of ODEs and we discovered that the dynamics of the ODEs is determined by the linearization around the equilibrium.
The goal is now to do the same for systems and understand what does it mean to linearize them.

3.2 Linearization of non-linear systems of ODEs

We start with the case of planar (automonous) systems

$$x' = f(x, y)$$
$$y' = g(x, y)$$

and again we want to look at a linearization around an equilibrium (x_0, y_0), so such that $f(x_0, y_0) = g(x_0, y_0) = 0$.
We can repeat the same linearization procedure as in the one-dimensional case. Suppose that f and g are differentiable. Then we can apply Taylor expansion (the 2-dimensional version) around (x_0, y_0) to f and to g to get

$$f(x_0 + u, y_0 + v) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_1(u, v)$$

and

$$g(x_0 + u, y_0 + v) = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_2(u, v)$$

where R_1 and R_2 are the remainder and are small when $\|(u, v)\| = \sqrt{u^2 + v^2}$ is small. This means that

$$\lim_{\|(u,v)\|\to 0} \frac{R_i(u, v)}{\|(u, v)\|} = 0 \quad \forall i = 1, 2$$

In particular, since $f(x_0, y_0) = g(x_0, y_0) = 0$ we can assume that

$$f(x_0 + u, y_0 + v) \approx \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v$$

and

$$g(x_0 + u, y_0 + v) \approx \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v$$

Again by rescaling the ODE to place (x_0, y_0) in the origin by the traslation $(u - x_0, v - y_0)$ we get the linearized system

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v$$
$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v$$

Note that this is a linear system $(u', v')^T = J(x_0, y_0)(u, v)^T$ where the matrix $J(x_0, y_0)$ is defined as

$$J(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

Remark 3.1. Note that $J(x_0, y_0)$ is the Jacobian matrix of the vector valued function $F = (f, g)^T$ evaluated in the point $J(x_0, y_0)$.

Definition 3.2 (Linearization of a system of ODEs)**.** To summarize we call linearization of a (planar) system of ODEs around the equilibrium (x_0, y_0) the linear system

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v$$
$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v$$

3.3 Characterization of equilibriums of planar non-linear systems through their linearization

We now want to be able to characterize equilibriums in non-linear planar systems by looking at the linearization in a specific equilibrium.

Let us have a look back to the characterization of equilibriums for planar linear systems. We saw

- Saddle point
- Nodal sink
- Nodal source
- Center
- Spiral sink
- Spiral source
- Repeated eigenvalues
- At least one eigenvalue is zero (matrix is singular)

We did not see the last two types in details. Between these equilibrium types we call the following ones **generic**:

- Saddle point
- Nodal sink
- Nodal source
- Spiral sink
- Spiral source

while all the other ones **non-generic**. Note that among the ones that we analyze the only non-generic is the center (and it is also the most important one).
The following result tell us the equilibrium type of a non-linear system if the linearized one has a generic equilbrium.

Theorem 3.3. *Consider the planar system*

$$x' = f(x, y)$$
$$y' = g(x, y)$$

with f and g continuously differentiable. Let (x_0, y_0) be an equilibrium point. Suppose that the linearization in (x_0, y_0) defined as in Definition 3.2 has a generic equilibrium, then the equilibrium point of the non-linear system has the same type.

Remark 3.4. Note that in particular, this gives use information about the stability of equilibrium points when the linearization has a generic equilibrium.

Theorem 3.5. *Given a autonomous non-linear system of ODEs let A be its linearization in the equilibrium (x_0, y_0). Then*

- *If the real part of every eigenvalue of A is negative (nodal sink, spiral sink) then the equilibrium is asymptotically stable*
- *If there exists an eigenvalue with a positive real part (saddle point, nodal source, spiral source) then the equilibrium is unstable.*

Let us see an example several examples

An example:
We consider a model for interacting species (similar to the predator-prey system).

$$x' = (1 - x - y)x$$
$$y' = (4 - 7x - 3y)y$$

where we can think $r_x = (1 - x - y)$ and $r_y = (4 - 7 - 3y)$ to be the reproductive rate of the first and second species respectively. Note that not only the species are competing (the reproductive rate decrease if the other population is larger), but also the rate of a population decreases if the number of individuals in that population is higher. This means that there is a limited amount of resources available for both population. When resources are limited the growth follow a logistic model, while if the reproductive rate is simply $r = ax$ for $a > 0$ the model is called Malthusian.
Apart from this discussion, let us try to understand the equilibrium points in this case. To find the equilibrium we need to find all x_0, y_0 such that

$$(1 - x - y)x = 0$$
$$(4 - 7x - 3y)y = 0$$

In particular looking at all cases the four equilibrium points are

$$(0, 0) \quad (0, 4/3) \quad (1, 0) \quad (1/4, 3/4)$$

We just analyze the most interesting equlibrium, that is the one where both population are positive, that is $(1/4, 3/4)$. Here we need to compute the linearization of the system. This

mean we have to compute the Jacobian matrix of $F(x,y) = ((1-x-y)x, (4-7x-3y)y)^T$ in $(1/4, 3/4)$. First it is worth to compute the Jacobian matrix in a generic point (x,y):

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-2x-y & -x \\ -7y & 4-7x-6y \end{pmatrix}$$

So in the point $(1/4, 3/4)$ we have

$$J(1/4, 3/4) = \begin{pmatrix} -1/4 & -1/4 \\ -21/4 & -9/4 \end{pmatrix}$$

We can compute the eigenvalues as

$$p(\lambda) = (-1/4 - \lambda)(-9/4 - \lambda) - 21/16 = 0$$

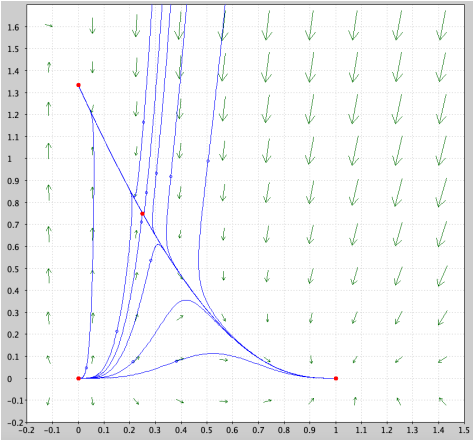
that is

$$\lambda^2 + \frac{5}{2}\lambda - \frac{3}{4} = 0$$

So

$$\lambda_{1/2} = -\frac{5}{4} \pm \sqrt{25/16 + 3/4} = -\frac{5}{4} \pm \sqrt{37/16} = \frac{-5 \pm \sqrt{37}}{4}$$

Therefore the two eigenvalues have different sign. In particular, since a saddle in a generic equilibrium for linear system, using Theorem 3.3 we can say that the equilibrium $(1/4, 3/4)$ is a saddle also for the non-linear system and using Theorem 3.5 we also conclude that it is unstable.
We will see in the next section how to draw a good phase plane portrait. For the moment it is enough to know that the equilibrium $(1/4, 3/4)$ is a saddle, so we expect a behaviour as follows



Clearly, compared to the linearized system we have more equilibrium points and the dynamic far from $(1/4, 3/4)$ could be radically different, but at least close to $(1/4, 3/4)$ we know that we have a saddle point behaviour.

3.4 Linearization and stability for higher-dimensional system

For n-dimensional autonomous systems of the form

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) \tag{3.9}$$

where $\mathbf{f} = (f_1, \dots, f_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function the linearization procedure is completely analogous to the 2 dimensional case.
Let us say that $\mathbf{y}_0 = (y_0^1, \dots, y_0^n)^T$ is an equilibrium, that is $\mathbf{f}(\mathbf{y}_0) = 0$. We want to linearize the system (3.9) around this equilibrium. In this case the Taylor expansion can be written in the compact form

$$\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = \mathbf{f}(\mathbf{y}_0) + J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u})$$

where $\mathbf{u} = (u_1, \dots, u_n)^T$ is a n dimensional perturbation, $J(\mathbf{y}_0)$ is the Jacobian matrix of $\mathbf{f}$ computed in the point $\mathbf{y}_0$ that is

$$J(\mathbf{y}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\mathbf{y}_0) & \frac{\partial f_1}{\partial y_2}(\mathbf{y}_0) & \dots & \frac{\partial f_1}{\partial y_n}(\mathbf{y}_0) \\ \frac{\partial f_2}{\partial y_1}(\mathbf{y}_0) & \frac{\partial f_2}{\partial y_2}(\mathbf{y}_0) & \dots & \frac{\partial f_2}{\partial y_n}(\mathbf{y}_0) \\ \dots & \dots & & \\ \frac{\partial f_n}{\partial y_1}(\mathbf{y}_0) & \frac{\partial f_n}{\partial y_2}(\mathbf{y}_0) & \dots & \frac{\partial f_n}{\partial y_n}(\mathbf{y}_0) \end{pmatrix}$$

and $\mathbf{R}(\mathbf{u})$ is the remainder such that

$$\lim_{\|\mathbf{u}\| \to 0} \frac{\mathbf{R}(\mathbf{u})}{\mathbf{u}} = 0$$

Following the same approach to the 2 dimensional case we can neglect the remainder and define the linearization in the equilibrium $\mathbf{y}_0$ as

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$$

Again the linearization is providing information on the stability of the equilibrium as the next theorem states.

Theorem 3.6. *Given $\boldsymbol{y}_0$ an equilibrium of the system (3.9), let $J(\boldsymbol{y}_0)$ be the Jacobian of $\boldsymbol{f}$ evaluated at $\boldsymbol{y}_0$. Then*

- *If the real part of all eigenvalues of $J(\boldsymbol{y}_0)$ is negative, then $\boldsymbol{y}_0$ is asymptotically stable*
- *If $J(\boldsymbol{y}_0)$ has at least an eigenvalue with positive real part, then $\boldsymbol{y}_0$ is unstable.*

3.5 Invariant sets and the use of nullclines

Up to now we are able to study the stability of equilibrium points of non-linear systems, thanks to the linearization. We now try to go one step further and get more information for regions that are non necessarily close to the equilibrium. A useful notion is the one of invariant sets. These are specific regions in $\mathbb{R}^n$ where the solution of an n dimensional system is forced to remain for all times. We now give the precise definition.

Definition 3.7 (Positively and negatively invariant set)**.** Given an autonomous n-dimensional system

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) \tag{3.10}$$

We say that $S \subset \mathbb{R}^n$ is a **positively invariant set** for (3.10) if every solution of (3.10) such that $\mathbf{y}(0) \in S$ is such that

$$\mathbf{y}(t) \in S \quad \forall t \geq 0$$

We say that $S \subset \mathbb{R}^n$ is a **negatively invariant set** for (3.10) if every solution of (3.10) such that $\mathbf{y}(0) \in S$ is such that

$$\mathbf{y}(t) \in S \quad \forall t \leq 0$$

Remark 3.8. Let us see some easy example of invariant set. The set $S = \{\mathbf{y}_0\}$ where $\mathbf{y}_0$ is an equilibrium of (3.10) is positively invariant (not necessarily negative invariant).

Remark 3.9. If the uniqueness of the initial value problems holds in all $\mathbb{R}^n$ (note that this is ensure for example if the $\mathbf{f}$ is differentiable with continuous derivative in all $\mathbb{R}^n$) then all solution curves are positively and negative invariant.

From now on we will assume that the uniqueness of initial values problems holds in all $\mathbb{R}^n$. This will allow a cleaner and easier analysis of invariant sets.

3.5.1 How to use nullclines to understand better solutions of non-linear systems?

Let us remind that given a planar system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

the x-nullcline is the set of points (x, y) such that $f(x, y) = 0$ and the y-nullcline is the set of points (x, y) such that $g(x, y) = 0$
We start with an important observation about the vector field on the nullclines.

On the x-nullclines the vector field of the system is always of the form $(0, a)$ for $a \in \mathbb{R}$. Therefore the arrows are vertical. On the other hand on the y-nullclines the vector field of the system is always of the form $(a, 0)$ for $a \in \mathbb{R}$. Therefore the arrows are horizontal. The direction of the arrows on the nullcline can be computed directly by evaluating the vector field on the nullclines. Typically the direction could change when we reach the equilibrium on a nullcline. Let us see an example

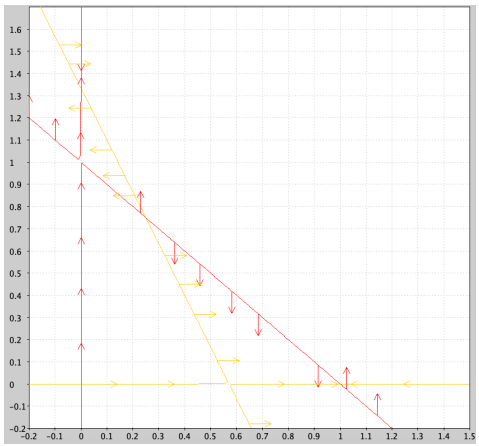
Consider again the competing species example

$$\begin{aligned} x' &= (1 - x - y)x \\ y' &= (4 - 7x - 3y)y \end{aligned}$$

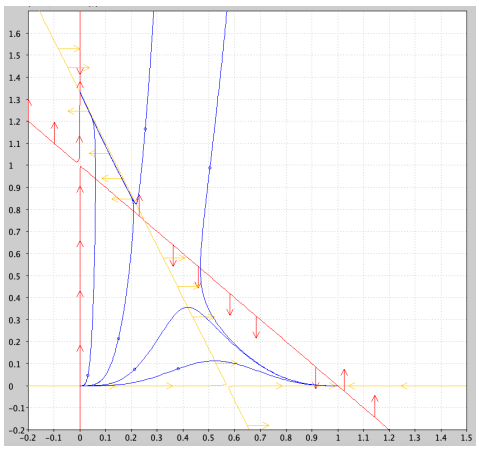
and compute the nullclines. The x-nullcline is $x = 0$ and $x + y = 1$. The y-nullcline is $y = 0$ and $7x + 3y = 4$. Moreover the equilibriums are

$$(0, 0) \quad (0, 4/3) \quad (1, 0) \quad (1/4, 3/4)$$

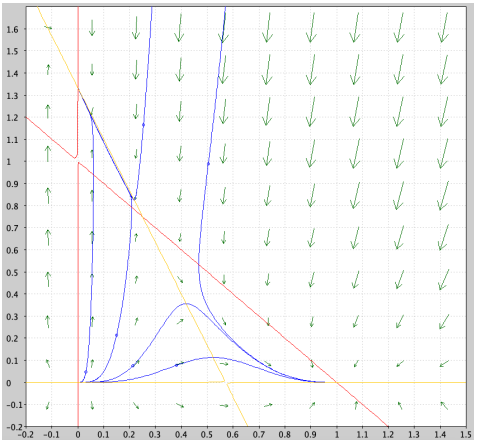
and we already found that the equilbrium $(1/4, 3/4)$ is a saddle point. Using the linearization and the characterization of the equilibrium it is also easy to see that $(0, 0)$ is a nodal source. Moreover, $(0, 4/3)$ and $(1, 0)$ are nodal sinks. We can try to draw the phase portrait now.



Already we know a lot of things. Solutions curves will escape from the unstable equilibrium $(0, 0)$ and cross the nullclines in the direction given by the arrows ending up in the two nodal sinks $(0, 4/3)$, $(1, 0)$. Following these indications we can already guess the solution curves as follows.



and consequently the vector field of the phase plane portrait



So this is the recipe for drawing a good phase plane portrait in case of non-linear system

- Find equilibrium points, characterized them and determined their stability.
- Find and draw the nullclines
- Draw the vector field on the nullclines (pay attention to the direction)
- Draw vector field and solution curves following the information given by the stability of the equilibrium points and the nullclines.

Can we find other invariant set for this system? Yes!

Remark 3.10. The two triangles determined by the intersection of the nullclines are positively invariant. Indeed, because of the direction of the arrows, once a solution curve has entered in a triangle it is force to remain inside the triangle for all subsequent t.

3.6 Long-term behaviour of solutions to planar systems

For non-linear planar system we were just able to know something about the stability of equilibrium points (locally around them), the behaviour of the solutions on the nullclines and, sometimes, find invariant regions. In this section we show that something more is possible. First we need the definition of **forward limit set** of a solution.

Definition 3.11 ((Forward) limit set of a solution)**.** Consider a solution $\mathbf{y}$ of the initial value problem for a planar non-linear system

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0$$

We define the **forward limit set** (or ω-limit set) of the solution of the initial value problem $\mathbf{y}$ as the set of points $\mathbf{x} \in \mathbb{R}^2$ for which there exists an increasing sequence of times $t_1 < t_2 < \ldots < t_n$ such that

$$\lim_{n\to+\infty} t_n = \infty \quad \textbf{and} \quad \lim_{n\to+\infty} \mathbf{y}(t_n) = \mathbf{x}$$

We denote the forward limit set of $\mathbf{y}$ with the symbol $\omega(\mathbf{y}_0)$.

The forward limit set is basically the set of points in $\mathbb{R}^2$ that will be reached "at infinity" by a solution of

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) \quad \mathbf{y}(0) = \mathbf{y}_0$$

Remark 3.12. If a solution curve passing through $\mathbf{y}_0$ is approaching an asymptotically stable equilibrium, then $\omega(\mathbf{y}_0)$ is precisely that equilibrium. However $\omega(\mathbf{y}_0)$ could be more complicated. If you have a center, then the solution curves around the center are closed and the solution is periodic. Therefore $\omega(\mathbf{y}_0)$ for y_0 close to the equilibrium are precisely this closed curves. Such periodic and closed solution curves are often called **periodic orbit**.

Definition 3.13 (Informal)**.** If a set is the forward limit set of all the curves starting in the vicinity of that set, then the set is called attractor. You can think about asymptotically stable equilibrium points or closed solution curves, such that if you start close to them, then you get attracted to them.

The following result provides more information about forward limit sets

Proposition 3.14. *The following facts hold:*

- *If a solution curve starting in $\boldsymbol{y}_0$ remains in a bounded set, then $\omega(\boldsymbol{y}_0)$ is non-empty.*
- *A forward limit set is positively and negatively invariant*

Proof. (Informal) The first fact simply follows by compactness of sequences in $\mathbb{R}^2$. By that it is meant that a bounded sequence always admit a converging subsequence (this is also called Weierstrass Theorem). The second fact is a consequence of the fact that if a forward limit set is not invariant there should be a sequence of time such that you can leave the set, but this contradicts the definition of forward limit set.

□

3.6.1 Limit cycles

In this section we want to understand better the case when you have a periodic solution curves. In this case it can happen that solution curves that start close to such periodic solution will collapse toward the curve of get repelled away. Let us try to understand that with an example.

Remark 3.15. Polar coordinates are useful to understand such type of dynamics, because allows to understand better periodic behaviour of curves. Let us remind that a polar coordinate transformation is defined as

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

implying that

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}$$

Moreover, by chain rule we can compute the derivative of r and θ in terms of x and y as follows. From the relation $r^2 = x^2 + y^2$ we get

$$2rr' = \frac{d}{dt}r^2 = \frac{d}{dt}(x^2 + y^2) = 2xx' + 2yy'$$

and from $\tan(\theta) = \frac{y}{x}$ we get

$$\frac{1}{\cos(\theta)^2}\theta' = \frac{d}{dt}\tan(\theta) = \frac{xy' - yx'}{x^2}$$

There will be an exercise in the next tutorial to practise this change of variables! Repeating these computation is useful.

Consider the non-linear system

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2) \end{aligned}$$

We apply the change to polar coordinates. This will give us a system in r and θ. Using the previous relations

$$2rr' = 2(xx' + yy') = 2(x^2 + y^2)(1 - x^2 - y^2) = 2r^2(1 - r^2)$$

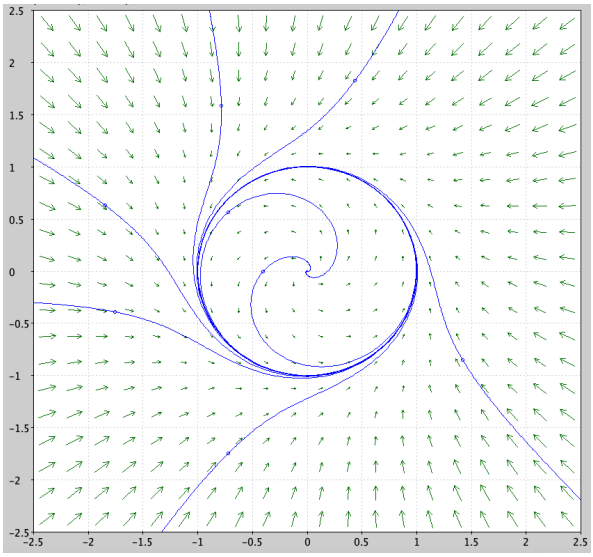
and

$$\frac{1}{\cos(\theta)^2}\theta' = \frac{x^2 + y^2}{x^2}$$

So simplyfying we get for $r \neq 0$.

$$\begin{aligned} r' &= r(1 - r^2) \\ \theta' &= 1 \end{aligned}$$

This substitution tells us that solution curves rotate around the origin with constant speed equal to 1 and the distance to the origin is governed by the equation $r' = r(1 - r^2)$. We can even see that if we start with $r < 1$, then $r' > 0$ and thus we get far from the origin. If we start with $r = 1$, then $r' = 0$, so we circle around the origin on a circle of radius 1. In particular the circle is a solution curve. If $r > 1$, then $r' < 0$ and thus we want to go closer to the origin. In particular we deduce that solution curves that start inside the circle of radius 1 will spiral outwards and get attracted to the circle of radius 1. On the other hand, solution curves that start outside the circle will spiral inwards and get attracted by the circle of radius 1.
In this case the forward limit set $\omega(\mathbf{y}_0)$ for every $\mathbf{y}_0 \neq (0,0)^T$ is precisely the circle of radius 1. Also $\omega((0,0)^T)$ is precisely the origin.



Definition 3.16 (Limit cycle)**.** A closed solution curve toward which other solution curves spiral is called a limit cycle. If solution curves spiral towards the cycle for $t \to +\infty$, then the cycle is an attractor it is called **attracting limit cycle**.

If the solution curves spiral away from the limit cycle for $t \to +\infty$ then the limit cycle is called **repelling limit cycle**.

Note that for the previous example we have an attracting limit cycle.
QUESTION: Is it possible to prove the existence of closed solution curves (so periodic solutions)? For the example seen just before, we found one (the unit circle) using polar coordinates. Also this is the case for linear system when you have a center. This is the goal of Poincare' Bendixson Theorem.

Theorem 3.17 (Poincare'-Bendixson)**.** *Consider a non-linear planar system*

$$x' = f(x, y)$$
$$y' = g(x, y)$$

such that f, h have continuous derivative. Suppose that R is a closed, bounded, positively invariant set for this system. If R does not contain equilibrium points, then there exists a closed solution curve (or periodic orbit) in R.

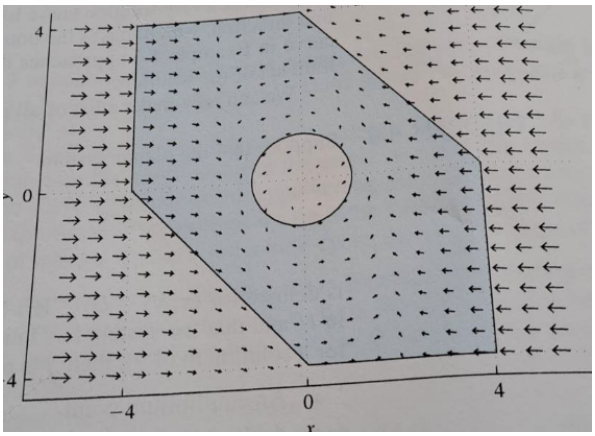
We will not see the prove, since it is quite hard and it requires at least one full class. However, we want to use this theorem in practice to find closed solution curves for a planar non-linear system.

Example: The van der Pol system

This is an example to see how the application of Poincare' Bendixson works. Consider the non-linear planar system

$$x' = 2x - y - x^3$$
$$y' = x$$

Clearly the only equilibrium point is the origin. We want to prove the existence of a closed curve solution (also called periodic orbit) by applying Poincare'-Bendixson Theorem. So we need to find a positively invariant set R that does not contain the origin. In general it is not easy to guess the correct region R to try out. A good rule of thumb is to try to represent the vector field of the system and try a guess on R. Basically one needs that the arrows on the boundary of R are pointing inside R. Following this heuristic rule an attempt is the region R drawn in the following figure. Note that R is closed and bounded. The small circle has radius 1.



Note that all arrows on the boundary of R are pointing inside R and the circle is cut-off to exclude the only equilibrium point. Our goal is now to verify that R is indeed positively invariant. This a procedure that you should keep in mind. What we want to do is to verify that in all points of the boundary of R the vector field is pointing inwards. So, denoting by $(f,g)^T = (2x - y - x^3, x)^T$ the vector field of the system and by ν the outward normal vector to the boundary of R we want that

$$\nu \cdot (f,g)^T \leq 0 \quad \text{in all points of the boundary of } R$$

We have to consider each piece of R. Let us start with the upper segment

$$\{(x,y)^T : y = 4, -4 \leq x \leq 0\}$$

the outer normal is $\nu = (0,1)^T$ in all points of the segment. Therefore on this segment we have

$$\nu \cdot (f,g)^T = (0,1)^T \cdot (2x - 4 - x^3, x)^T = x \leq 0$$

for all x with $-4 \leq x \leq 0$. OK!

Let us also verify the same on the slanted portion of the boundary. This means, the points

$$\{(x,y)^T : 0 \leq x \leq 4, y = 4 - x\}.$$

The outward normal is $\nu = (1,1)^T$ for every point of this edge. We compute the scalar product

$$\nu \cdot (f,g)^T = (1,1)^T \cdot (2x - y - x^3, x)^T = 3x - y - x^3 = 4x - x^3 - 4$$

To compute the sign we can take the derivative that is $-3x^2+4$. The derivative is zero for $x=\sqrt{4/3}$ where $\nu\cdot(f,g)^T$ has a maximum. In $x=\sqrt{4/3}$ the value of $\nu\cdot(f,g)^T$ is

$$\frac{8}{\sqrt{3}}-\frac{8}{(\sqrt{3})^3}-4=\frac{8}{\sqrt{3}}-\frac{8}{3\sqrt{3}}-4=$$
$$=\frac{24-8-12\sqrt{3}}{3\sqrt{3}}\leq\frac{24-8-12\frac{3}{2}}{3\sqrt{3}}=\frac{-2}{3\sqrt{3}}<0$$

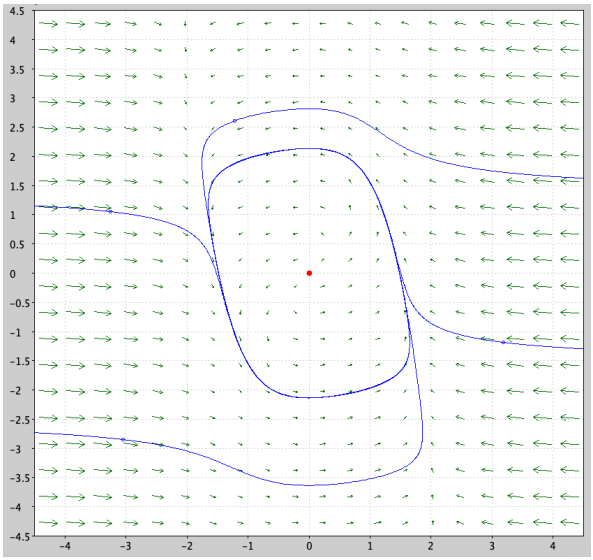
and therefore is negative. You can verify that the same computation holds for all the other edges of the hexagon.

It remains to prove that the vector field is pointing inward the region R on the circle around the origin. The outer normal to the circle is $(x,y)^T$ (important to know!). So the outer normal to R is $\nu=(-x,-y)^T$. So

$$\nu\cdot(f,g)^T=(-x,-y)^T\cdot(2x-y-x^3,x)^T=-2x^2+xy+x^4-xy$$
$$=-2x^2+x^4=x^2(-2+x^2)\leq 0$$

since $|x|$ is always less or equal than 1. Pay attention to the direction of the normal (are you taking the outer or the inner normal?). Alternatively you can also use polar coordinates (see example 4.12 in Polking for this alternative method).

We are done! We proved that R is positively invariant and it is closed and bounded. Moreover it does not contain equilibrium points. Therefore by Poincare'-Bendixson there exists a closed solution curve inside R.



3.7 Conserved quantities

We now see an instrument that sometimes could be useful to understand better the dynamic in a planar system. These are called **conserved quantities** and they are functions that can be proven to be constant through time for solutions of the system.

Consider a planar non-linear system

$$x'=f(x,y)$$
$$y'=g(x,y)$$

We can try to calculate the variation of y with respect to x in the following way. Using the **chain rule** and the **inverse function theorem** we have

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} = \frac{g(x,y)}{f(x,y)}$$

Already by solving this one dimensional ODE we could be able to find the graph of $y(x)$ in the phase space. Let us see an example and try to understand what we can say more.

Vibrating spring example

Consider the second order ODE obtain by modelling the the vibrating spring:

$$my'' + ky = 0$$

where m is the mass and k is the spring constant. We can transform this (linear) ODE in a (linear) system:

$$\begin{aligned} y' &= v \\ v' &= -\frac{k}{m}y \end{aligned}$$

Now we can compute

$$\frac{dv}{dy} = \frac{-ky}{mv}$$

and note that in this case we have a solvable ODE with the method of separation of variables. This solution gives

$$\frac{1}{2}mv(t)^2 + \frac{1}{2}ky(t)^2 = C$$

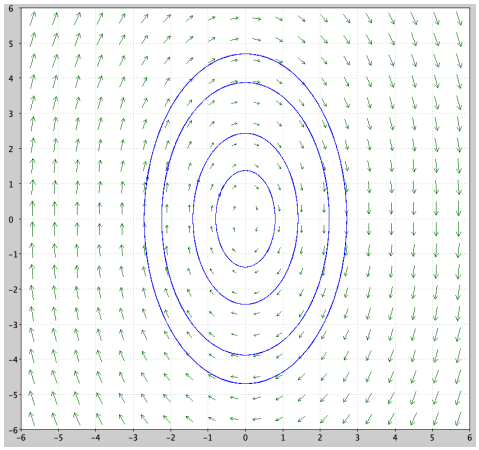
for every $C \in \mathbb{R}$. This says that the quantity $\frac{1}{2}mv(t)^2 + \frac{1}{2}ky(t)^2$ is conserved in time for solutions of the system. Can we give a physics interpretation of this fact? Yes, in case of the vibrating spring the variable y represent its positioning on the y axis, while $y' = v$ is its velocity. Therefore the quantities

$$\frac{1}{2}mv(t)^2 \quad \text{and} \quad \frac{1}{2}ky(t)^2$$

are the **kinetic energy** of the system and the **potential energy** stored in the spring (respectively). Therefore the conserved quantity is precisely the total energy of the system. You can also observe the classical situation where potential energy gets transformed into kinetic energy and viceversa.
Note that from this conserved quantity we can also get information on the phase plane portrait (this is a linear system, so clearly you should be able to draw it independently). Calling $E(y,v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$ we can represent the curves

$$\{(y,v) : E(y,v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = C\}$$

in the phase plane (y, v) for some values of C, that are ellipses of different eccentricity dependent on the parameters m and k. These curves are called **level set curves** of the energy E.



We know that since on solutions $(y(t), v(t))$ of the system we have that $\frac{1}{2}mv(t)^2+\frac{1}{2}ky(t)^2 = C$, then it can be seen that the total energy of solutions of the system remain constant in time. This means that in the phase plane, the solution curves should be necessarily contained in the level set curves of E.

3.8 Conservative systems

The considerations of the previous section can be generalized for a large class of planar systems, denoted by **conservative systems**. A conservative system is a planar system of the form

$$\begin{aligned} y' &= v \\ v' &= f(y) \end{aligned}$$

Remark 3.18. Note that these systems derive from second order ODEs of the form $y'' = f(y)$. Also, in general there systems are non-linear (since f can be non-linear). So our linear analysis results are not helpful here.

We now try to obtain a conserved quantity from these systems. Following the strategy of the previous section we find that

$$\frac{dv}{dy} = \frac{f(y)}{v}$$

so that using separation of variables

$$\frac{1}{2}v^2 = \int f(y)\, dy + C$$

for $C \in \mathbb{R}$. We thus see that the form of the conserved quantity structure is very similar to the spring example. Indeed, here $K(v) = \frac{1}{2}v^2$ is the **kinetic energy**, while $U(v) = -\int f(y)\, dy$ is the **potential energy**. So the **total energy** that is conserved in time is

$$E(y, v) = K(v) + U(y) = \frac{1}{2}v^2 - \int f(y)\, dy$$

This is the reason why these systems are called **conservative**. We can always find an conserved quantity that is the total energy and can be split into kinetic and potential energy.

3.8.1 Drawing phase plane portrait for conservative systems

We have found that conservative systems always have a conserved energy of the form

$$E(y,v) = K(v) + U(y) = \frac{1}{2}v^2 - \int f(y)\,dy$$

How can we use that to understand the phase portrait? First note that the equilibrium points of a conservatives systems are of the form

$$(y,0)^T \quad \text{such that} \quad f(y) = 0$$

So, since $U(y) = -\int f(y)\,dy$. The equilibrium points are of the form $(y,0)^T$ such that $U'(y) = 0$ and thus are completely determined by U. Moreover, the equilibrium points correspond to the stationary points of U.

Similarly to the case of the vibrating spring we can determine the solution curves in the phase plane by looking at the level sets of $E(y,v)$. So we look for $E \in \mathbb{R}$ at the curves $E(y,v) = E$ that can be written as

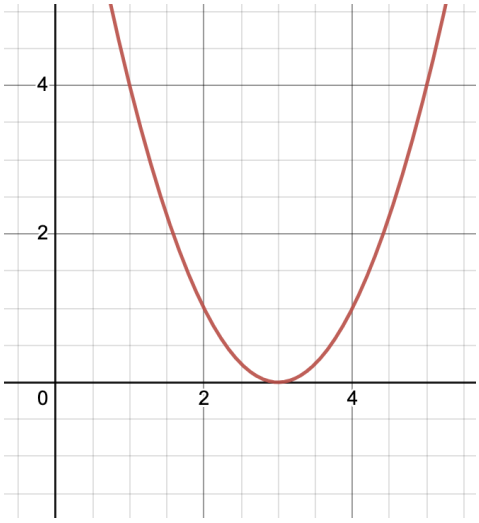
$$\frac{1}{2}v^2 - \int f(y)\,dy = E$$

We can even solve explicitly in v to obtain that

$$v = \pm\sqrt{2(E - U(y))}$$

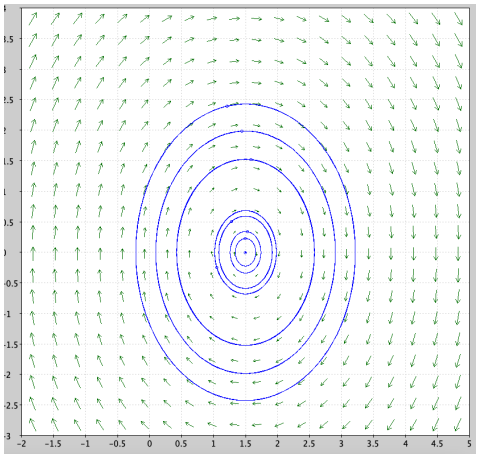
So, the solution curves have two branches: the positive one $\sqrt{2(E-U(y))}$ and the negative one $\sqrt{2(E-U(y))}$ and thus the solution curves are symmetric with respect to the y-axis.

For example let us say that the potential energy $U(y)$ has the shape of a parabola with a minimum in $y_0 = 3$. This means in particular that the only equilibrium point is in $y_0 = 3$.



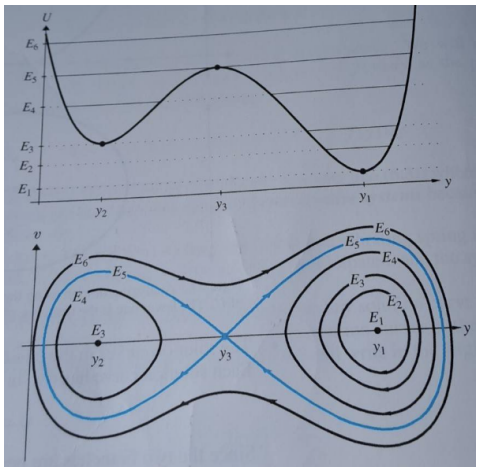
Then the solutions curves are of the form $\pm\sqrt{2(E-U(y))}$ and thus are closed curve solution curves. So we also see that the equilibrium point is stable.

Remark 3.19. In this situation we say that we have one potential well. You can think about that, as a point where we have the minimum of potential energy. So in this case a ball placed in the well will oscillate around y_0 (the minimum of the potential energy). Since there is no dissipation, then potential and kinetic energy will exchange them themselves. This help understand that the dynamic is periodic and the ball is constrained to be in the potential well.



The direction of the arrows can be found or computing it for specific point. Or noticing that the since $y' = v$, then on the positive v the first component of the vector field is positive. For v negative the first component of the vector field is negative.

What happens for more complicated potentials? Suppose that we have more wells. Or in other words more local minimums of our $U(y)$. In the case sketched below $U(y)$ has three stationary point, so the system has three equilibrium points $(y_1, 0)^T, (y_2, 0)^T, (y_3, 0)^T$.



Moreover we have two wells and two local minimums. The dynamics is such that if we start close to y_1 or y_2 (the local minimums of the potential), then we stay close to the local minimum following a periodic solution. On the other hand, if we are way enough there is enough total energy to cross the stationary point y_3. The solution is still periodic, but it encloses both equilibrium. In particular the equilibrium point y_3 is a saddle.

3.9 Hamiltonian systems

We now analyze a special type of system that has a conserved quantity and it is a generalization of the conservative systems that we saw in the previous section. This type of system is called **Hamiltonian system**. They are defined as non-linear planar systems of the form

$$\begin{aligned} x' &= \frac{\partial H}{\partial y} \\ y' &= -\frac{\partial H}{\partial x} \end{aligned}$$

Here the function H is continuously twice differentiable. H is called **Hamiltonian**.

Remark 3.20. Note that conservative systems are Hamiltonian with the choice $H(y, v) = \frac{1}{2}v^2 + U(y)$.

Why Hamiltonian systems are interesting? The following lemma is the key reason to that.

Lemma 3.21. *The Hamiltonian H is a conserved quantity for the corresponding Hamiltonian system.*

Proof. It is enough to differentiate using the chain rule

$$\frac{d}{dt}H(x(t), y(t)) = \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}y' = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} = 0$$

□

The next theorem tells us a criteria to understand when a planar non-linear system is Hamiltonian.

Theorem 3.22. *Consider the planar system*

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{3.11}$$

- *If* (3.11) *is Hamiltonian, then*

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$$

- *Viceversa if*

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$$

  in a rectangle, then the system is Hamiltonian in a rectangle.

Proof. We verify only the first bullet point (that is the easy part of this result).
Note that if the system is Hamiltonian, then

$$\frac{\partial f}{\partial x} = \frac{\partial^2 H}{\partial x \partial y}$$

and

$$\frac{\partial g}{\partial x} = -\frac{\partial^2 H}{\partial y \partial x}$$

Since H is continuously twice differentiable the mixed second order derivatives are equal. So we verify that $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$. Note that it is important to require that H is twice continuously differentiable. Indeed, only twice differentiability is not enough to ensure the equality between the mixed derivatives. The second part of this result is more involved. However I suggest you to compare this result with the equivalence between closed and exact differential forms.

□

How can we use this result? Let us see that with an example.

Example

Suppose that we have the system

$$\begin{aligned} x' &= y \\ y' &= x(x-1) \end{aligned}$$

In principle the system is conservative, so we know how to solve that. However, forgetting about that, we want to find an Hamiltonian H. First defining $f(x,y) = y$ and $g(x,y) = x(x-1)$ note that

$$\frac{\partial f}{\partial x} = 0 = -\frac{\partial g}{\partial v}$$

by Theorem 3.22 we can infer that the system is Hamiltonian. To find the Hamiltonian H we start from the observation that

$$y = f(x,y) = \frac{\partial H}{\partial y}$$

So, integrating in y we get that

$$\frac{y^2}{2} = H(x,y) + \varphi(x) \tag{3.12}$$

for an unknown function $\varphi(x)$. Now using that $x(x-1) = g(x,y) = -\frac{\partial H}{\partial x}$ in (3.12) we we obtain that

$$\varphi'(x) = x(x-1)$$

so that $\varphi(x) = \frac{x^3}{3} - \frac{x^2}{2} + C$. So plugging the value of φ in (3.12) (and choosing $C = 1$) we finally conclude that

$$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^3}{3}$$

3.10 Lyapunov method

We conclude the analysis of non-linear systems with a method designed to study the stability of equilibrium points for which the linearization is not providing information (so in case that the linearization has a non-generic equilibrium).
Consider an autonomous non-linear planar system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{3.13}$$

We have already seen the concept of deriving in time a function $V(x,y)$ along a solution curve $(x(t),y(t))$. By the chain rule we have

$$\frac{d}{dt}V(x(t),y(t)) = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{\partial V}{\partial x}f(x,y) + \frac{\partial V}{\partial y}g(x,y)$$

We denote such quantity with the following notation

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}f(x,y) + \frac{\partial V}{\partial y}g(x,y)$$

Definition 3.23. We say that V is increasing along the solution curve through the point (x,y) if $\dot{V}(x,y) > 0$ and it is decreasing if $\dot{V}(x,y) < 0$. Note that denoting the vector field of the system as $F(x,y) = (f(x,y), g(x,y))^T$ we can also write

$$\dot{V}(x,y) = \nabla V(x,y) \cdot F(x,y)$$

These definitions can be easily given for an n dimensional autonomous system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

In this case given a function V defined in $\mathbb{R}^n$ we can compute

$$\frac{d}{dt}V(\mathbf{x}(t)) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$$

and we can similarly define

$$\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$$

The goal of Lyapunov method is to find a suitable function V with good properties that is able to characterize the stability of an equilibrium of a system of ODEs.
We now consider a special class of functions V that have minimum in a point $\mathbf{x}_0$ and are strictly positive everywhere else. They are called positive definite functions. Let us see the precise definition.

Definition 3.24. Let $\mathbf{x}_0 \in S$. A function V defined in S is said to be positive definite in S with minimum at $\mathbf{x}_0$ if

- $V(\mathbf{x}_0) = 0$
- $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in S$ such that $\mathbf{x} \neq \mathbf{x}_0$.

Moreover, we say that S is semidefinite if in the second condition we replace the strict inequality with $V(\mathbf{x}) \geq 0$. Similarly if the inequalities are reversed we talk about negative definite V and negative semidefinite V.

Now suppose that we want to prove that an equilibrium $\mathbf{x}_0$ for the system is asymptotically stable. We search for a function V that is positive definite with minimum in $\mathbf{x}_0$. Moreover, we require the additional property that $\dot{V}$ is negative definite. This means that solution curves, for every point is S are going "downhill" in the direction of $\mathbf{x}_0$. These heuristic observations are formalized in the following theorem, due to Lyapunov.

Theorem 3.25 (Lyapunov)**.** *Suppose that the n dimensional autonomous system*

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$$

has an equilibrium point at $\boldsymbol{x}_0$. Suppose additionally that there exists a continuously differentiable function V defined in a neighborhood U of $\boldsymbol{x}_0$ that is positive definite with minimum in $\boldsymbol{x}_0$. Then

- *If $\dot{V}$ is negative semidefinite in U, then $\boldsymbol{x}_0$ is a stable equilibrium point*
- *If $\dot{V}$ is negative definite in U, then $\boldsymbol{x}_0$ is an asymptotically stable equilbrium.*

Remark 3.26. This theorem is useful when the linearization does not give information about the type and the stability of the equilibrium. You can recall that this happens when the origin is a non-generic equilibrium for the linearized system.

Remark 3.27. Note also that this result is non-local in nature, in the sense that if we find U and V with the properties of the theorem. Then the stability properties of the theorem extend to all equilibrium points in U. This was not true for the linearization technique (there we had to linearize separately in each equilibrium points).

Let us see an example. Consider the system

$$\begin{aligned} x' &= y + \alpha x(x^2 + y^2) \\ y' &= -x + \alpha y(x^2 + y^2) \end{aligned}$$

for some $\alpha < 0$. We want to prove that the origin is asymptotically stable by using Lyapunov theorem. We have to find the right V. We thus try with

$$V(x, y) = x^2 + y^2$$

that is positive definite with minimum in the origin. Then we compute

$$\begin{aligned} \dot{V}(x, y) &= \nabla V \cdot (y + \alpha x(x^2 + y^2), -x + \alpha y(x^2 + y^2))^T \\ &= 2xy + 2\alpha x^2(x^2 + y^2) - 2xy + 2\alpha y^2(x^2 + y^2) = 2\alpha(x^2 + y^2)^2 \end{aligned}$$

So this is negative definite and thus the origin is asymptotically stable.

We can also consider a second example that revolves around the vibrating spring system when we also have a damping term

$$\begin{aligned} y' &= v \\ v' &= -\frac{k}{m}y - \frac{\mu}{m}v \end{aligned}$$

with $\mu > 0$. Here the velocity decreases by a factor of $\frac{\mu}{m}v$. Still we can see that the total energy

$$E(y, v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$$

can be used as the function V in the Lyapunov theorem. Indeed it is positive definite with minimum in the origin and

$$\dot{E} = \nabla E \cdot (y', v')^T = kyv + mv\left(-\frac{k}{m}y - \frac{\mu}{m}v\right) = -\mu v^2$$

that is negative semidefinite (it is zero for all points $(y, 0)$ not only for the origin). So thanks for Lyapunov theorem the origin is a stable equilibrium.

We conclude this section with a second result in the same style. This is called Lasalle invariance principle.

Theorem 3.28 (Lasalle invariance principle)**.** *Suppose that a n dimensional system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is defined in U and there exists a V continuously differentiable in U and such that*

$$\dot{V}(\boldsymbol{x}(t)) \leq 0 \quad \forall t \geq 0$$

Then the forward limit set ω of a solution of the system is such that

$$\omega \subset \{x \in U : \dot{V}(x) = 0\}$$

Now we can come back at the example about the damped vibrating spring.

$$\begin{aligned} y' &= v \\ v' &= -\frac{k}{m}y - \frac{\mu}{m}v \end{aligned}$$

We found that E was such that $\dot{E} \leq 0$. In particular

$$\dot{E}(\mathbf{x}(t)) \leq 0 \quad \forall t \geq 0$$

So $\omega \subset \{(y, 0) : y \in \mathbb{R}\}$. Since the forward limit set is the union of solution curves (is a fact that we did not see) and the only solution curve contained in $\{(y, 0) : y \in \mathbb{R}\}$ is the origin, then this means that ω is precisely the origin. So every solution of the system is attracted to the origin. Thus the origin is asymptotically stable.

4 What are Partial Differential Equations? A brief intuitive introduction

Partial differential equations are equations that involves partial derivative of functions. In this lecture we will see what is the role of ODEs in the resolution of PDE. Clearly this is not a course on PDE. You will see much more about PDE in the next years.

4.1 The heat equation

In this lecture we will only analyze the heat equation. This PDE regulates the propagation of heat in a medium and it is one of the most studied and popular PDE. It is written as

$$\frac{\partial u}{\partial t}(x,t) - k\frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

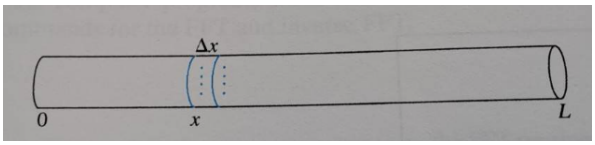
where k is a positive constant that is called **thermal diffusivity** of the material and measures how well the heat propagates through the material. Here $u(x,t)$ in the unknown and note that this is a function of space x and time t. $u(x,t)$ represent the temperature of the material in the point x at time t.

4.1.1 Derivation of the heat equation

In this section we describe how the heat equation can be derived from physical considerations. Let $u(x,t)$ be the temperature of a body in a location x and time t and $Q(t)$ be the heat contained it at a given time t. (you can think the heat to be the energy in the all body and the temperature the effect that you measure). It is reasonable that if the temperature is constant, then changing the volume implies a corresponding change in the heat (in a bigger volume we need more heat to reach the given temperature) as follows

$$\Delta Q = c\rho u \Delta V \tag{4.1}$$

where ρ is the density of the material (that we suppose to be constant in space), c is a constant called **specific heat** that tells you how much the temperature of a body is affected by the heat applied to it. Now consider a thin rod of length L. This is the body in which we want to study the evolution of temperature.



Consider also a small cross-section of the rod between x and $x + \Delta x$ and let S be the cross-sectional area of the rod. Then the volume of the small cross section is $S\Delta x$. Using the formula (4.1) we have

$$\Delta Q = c\rho u S \Delta x$$

Therefore writing $\frac{dQ}{dx} = \frac{\Delta Q}{\Delta x} = c\rho u S$, we obtain that the amount of heat at time t in the portion (a,b) is

$$Q(t) = S\int_a^b c\rho u(x,t)\,dx$$

Therefore by differentiating $Q(t)$ in time we obtain

$$\frac{dQ}{dt} = S\frac{d}{dt}\int_a^b c\rho u(x,t)\,dx = S\int_a^b c\rho\frac{\partial u}{\partial t}(x,t)\,dx \tag{4.2}$$

Now we also observe that the change of heat in the region (a,b) in time is given by

$$\frac{dQ}{dt} = \text{heat flow through a} - \text{heat flow through b}$$

Moreover we make the assumption that **the heat flow trough the point a is determined by a difference in the temperature of the rod before a and after a** (the same for the point b). So we can write

$$\begin{aligned}\frac{dQ}{dt} &= \text{heat flow through a} - \text{heat flow through b} = SC\frac{\partial u}{\partial x}(b,t) - SC\frac{\partial u}{\partial x}(a,t)\\ &= S\int_a^b \frac{\partial}{\partial x}(C\frac{\partial u}{\partial x})\,dx = CS\int_a^b \frac{\partial^2 u}{\partial x^2}\,dx\end{aligned} \tag{4.3}$$

where C is another positive constant called **thermal conductivity**. So, comparing the two equations (4.2) and (4.3) we get

$$\int_a^b c\rho\frac{\partial u}{\partial t}(x,t) - C\frac{\partial^2 u}{\partial x^2}\,dx = 0$$

Since this equation is true for every a, b this implies that

$$c\rho\frac{\partial u}{\partial t}(x,t) - C\frac{\partial^2 u}{\partial x^2} = 0$$

and dividing by $c\rho$ and setting $k = \frac{C}{c\rho}$ the equation becomes

$$\frac{\partial u}{\partial t}(x,t) - k\frac{\partial^2 u}{\partial x^2} = 0$$

Here k is called **thermal diffusivity** of the material and this constant is typically measured for all the materials. This PDE is called heat equation.

4.2 Initial and boundary conditions

Typically we want to understand the evolution of the temperature through time. Therefore we usually set the distribution of temperature at the initial time $t = 0$. This consists in setting the solution of the heat equation $u(x,t)$ in the point $t = 0$ as

$$u(x,0) = f(x) \qquad \text{for } 0 \le x \le L$$

However, this is typically not enough to have a well-defined problem. We need also to impose conditions at the beginning and at the end of the rod for every L. For example we could want to set the temperature constant at the boundary of the rod by setting

$$u(0,t) = T_0 \qquad u(L,t) = T_L \qquad \forall t$$

This type of boundary conditions are called **Dirichlet boundary conditions**. The other possible boundary condition model the case when there is no flow of temperature at the boundary of the rod. So we impose that the derivative (in space) at the boundary of the rod is zero:

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \frac{\partial u}{\partial x}(L,t) = 0 \qquad \forall t$$

These boundary conditions are called **Neumann boundary conditions**.

4.2.1 Initial/boundary value problem

Therefore an example of initial/boundary value problem is

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - k\frac{\partial^2 u}{\partial x^2} &= 0 \\ u(0,t) = T_0 \qquad u(L,t) = T_L \qquad & \forall t \\ u(x,0) = f(x) \qquad \text{for } 0 \le x \le L & \end{aligned}$$

The goal is to find u that solves the heat equation and satisfies both the initial and boundary value conditions.

4.2.2 Solution for heat equation with homogeneous boundary conditions

In this section we analyze the case

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - k\frac{\partial^2 u}{\partial x^2} &= 0 \\ u(0,t) = 0 \qquad u(L,t) = 0 \qquad & \forall t \\ u(x,0) = f(x) \qquad \text{for } 0 \le x \le L & \end{aligned}$$

that is when the temperature at the boundary of the rod is zero. In this case we will derive an explicit solution of the heat equation.

The method of separation of variables

We will use a technique called separation of variables, thatm even if has the same name as the one for solving ODEs, it is quite different.
The idea is to search a solution of the heat equation that is of the form

$$u(x,t) = X(x)T(t)$$

This mean that we search a solution that can be written as the product of two functions: X, that depends only on x and T that depends only on t.
We now plug $u(x,t) = X(x)T(t)$ into the heat equation and check which equations X and T should satisfy. By plugging $u(x,t) = X(x)T(t)$ into $\frac{\partial u}{\partial t}(x,t) - k\frac{\partial^2 u}{\partial x^2} = 0$ we get

$$X(x)T'(t) = kX''(x)T(t)$$

and diving by $T(t)$ and $X(x)$ we get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

So, since the left side is a function of t and the right side is a function of x, necessarily they have to be equal to the same constant. We thus have that there exists λ such that

$$\frac{T'(t)}{kT(t)} = -\lambda \quad \frac{X''(x)}{X(x)} = -\lambda$$

Therefore we obtain the following two ODEs

$$T' + \lambda kT = 0 \qquad X'' + \lambda X = 0$$

that we would like to solve to find T an X. The solution of the first one is immediate and equal to

$$T(t) = Ce^{-\lambda kt}$$

for $C \in \mathbb{R}$.

4.3 Sturm-Lioville method

We now switch our attention to the second one

$$X'' + \lambda X = 0$$

and additionally we want to satisfy the boundary condition $X(0) = X(L) = 0$. This means that we look for solutions of the problem

$$X'' + \lambda X = 0 \quad X(0) = X(L) = 0 \tag{4.4}$$

Note that this is not a typical initial value problem for a second order ODE, since we are imposing the value of X at two points. This problem is called **Sturm-Liouville problem** or **two point boundary value problem**.
Moreover, we note that the constant λ is still undetermined. The goal of Sturm-Lioville problem is to find a λ such that there exists a solution of (4.4) that is not constantly equal to zero.

Definition 4.1. A number λ is called an **eigenvalue** for the Sturm-Liouvillle problem if there exists a non-zero solution X that solves (4.4). If λ is an eigenvalue, then any function that satisfies (4.4) is called an **eigenfunction**.

We first exclude the case that there exist eigenvalues λ for the Sturm-Liouville problem such that $\lambda \leq 0$.

Lemma 4.2. *Given the Sturm-Lioville problem*

$$X'' + \lambda X = 0 \quad X(0) = X(L) = 0 \tag{4.5}$$

the eigenvalues λ are such that $\lambda \geq 0$.

Proof. First consider the case $\lambda < 0$, then we can write $\lambda = -r^2$ for some $r > 0$. Therefore the equation $X'' - r^2 X = 0$ has now as general solution (remember that the characteristic polynomial is $\lambda^2 - r^2 = 0$)

$$X(x) = c_1 e^{rx} + c_2 e^{-rx}$$

Since we want to satisfy the boundary condition, we also need that

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{rL} + c_2 e^{-rL} &= 0 \end{aligned}$$

Therefore substituting the first one in the second one we obtain

$$c_1 e^{rL} - c_1 e^{-rL} = c_1(e^{rL} - e^{-rL})$$

that is zero only if $c_1 = 0$ or $L = 0$. If $c_1 = 0$ then also $c_2 = 0$, so $X(x)$ would be a zero solution. On the other hand $L = 0$ in non-phyisical, since we are starting with a rod of positive length. Therefore we cannot have $\lambda < 0$.
Suppose now that $\lambda = 0$. Then we have the equation $X'' = 0$ that is has solution $X(x) = ax+b$ for some a and b. This is therefore the equation of a line and imposing the boundary condition the only line such that $X(0) = X(L) = 0$ is the constant line equal to zero. □

We can thus assume that $\lambda > 0$ and thus we set $\lambda = \omega^2$ for $\omega > 0$. Now the characteristic polynomial has complex solutions $i\omega$ and $-i\omega$. Therefore the general solution of the second order ODE is

$$X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

We impose again the boundary conditions $X(0) = X(L) = 0$ ad we get

$$\begin{aligned} c_1 &= 0 \\ c_2 \sin(\omega L) &= 0 \end{aligned}$$

and therefore the solutions are

$$\omega L = n\pi$$

for $n \in \mathbb{Z} \setminus \{0\}$ (ω is non-zero). In particular the eigenvalue is $\lambda = \omega^2 = \frac{n^2\pi^2}{L^2}$ and one eigenfunction (for example when $c_2 = 1$) is

$$X(x) = \sin(\frac{n\pi x}{L})$$

Therefore all eigenvalues and a corresponding eigenfunction of the Sturm-Lioville problem are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad X_n(x) = \sin(\frac{n\pi x}{L})$$

for $n \in \mathbb{N} \setminus \{0\}$.

We can now come back to the solution of the heat equation. We already found $T(t)$ that is (choosing for example $C = 1$)

$$T(t) = e^{-\frac{n^2\pi^2}{L^2}kt}$$

Therefore

$$u_n(x,t) = X(x)T(t) = e^{-\frac{n^2\pi^2}{L^2}kt}\sin(\frac{n\pi x}{L})$$

are for every $n \in \mathbb{N} \setminus \{0\}$ solutions of the heat equation such that $u(0,t) = u(L,0) = 0$.

How to impose the initial value $f(x)$?

We are at a good point, but up to know our solutions of the heat equation do not satisfy the initial value $u(x,0) = f(x)$. We now see how it is possible to impose it.
First let us note that linear combinations of solutions of the heat equations are again solutions of the heat equation, that is if we consider

$$u(x,t) = \sum_{n=1}^{N} b_n u_n(x,t) = \sum_{n=1}^{N} b_n e^{-\frac{n^2\pi^2}{L^2}kt}\sin(\frac{n\pi x}{L})$$

$u(x,t)$ is a solution of the heat equation and also $u(0,t) = u(L,0) = 0$. We can also consider infinite sums. This procedure is just formal at this point, since one needs to make sure that the series is converging uniformly and thus the coefficients b_n should decay fast enough. If this is the case, then we will consider $u(x,t)$ of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{L^2}kt}\sin(\frac{n\pi x}{L}) \tag{4.6}$$

for $b_n \in \mathbb{R}$. The only thing that remain to do is to choose b_n for $n \in \mathbb{N} \setminus \{0\}$ such that $u(x,0) = f(x)$ for every $0 \leq x \leq L$. Therefore we impose that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \qquad 0 \leq x \leq L \tag{4.7}$$

Can we always find such b_n?. Yes! Indeed, we can see that (4.7) is the Fourier sine expansion of $f(x)$ and therefore its coefficients b_n can be computed as

$$b_n = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi x}{L})\,dx$$

So choosing such b_n a plug them into (4.6) we get that

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{L^2}kt}\sin(\frac{n\pi x}{L})$$

will give a solution of the initial/boundary value problem for the heat equation.