

Instructor's Solutions Manual

JOHN POLKING

Rice University

DAVID ARNOLD

College of the Redwoods

Differential Equations

With Boundary
Value Problems

SECOND EDITION

POLKING

BOGGESS

ARNOLD



Upper Saddle River, NJ 07458

Editor-in-Chief: Sally Yagan
Supplement Editor: Jennifer Urban
Executive Managing Editor: Kathleen Schiaparelli
Assistant Managing Editor: Karen Bosch
Production Editor: Amanda Phillips
Supplement Cover Manager: Paul Gourhan
Supplement Cover Design: Joanne Alexandris
Manufacturing Buyer: Ilene Kahn
Manufacturing Manager: Alexis Heydt-Long



© 2006 Pearson Education, Inc.
Pearson Prentice Hall
Pearson Education, Inc.
Upper Saddle River, NJ 07458

All rights reserved. No part of this book may be reproduced in any form or by any means, without permission in writing from the publisher.

Pearson Prentice Hall™ is a trademark of Pearson Education, Inc.

The author and publisher of this book have used their best efforts in preparing this book. These efforts include the development, research, and testing of the theories and programs to determine their effectiveness. The author and publisher make no warranty of any kind, expressed or implied, with regard to these programs or the documentation contained in this book. The author and publisher shall not be liable in any event for incidental or consequential damages in connection with, or arising out of, the furnishing, performance, or use of these programs.

This work is protected by United States copyright laws and is provided solely for the use of instructors in teaching their courses and assessing student learning. Dissemination or sale of any part of this work (including on the World Wide Web) will destroy the integrity of the work and is not permitted. The work and materials from it should never be made available to students except by instructors using the accompanying text in their classes. All recipients of this work are expected to abide by these restrictions and to honor the intended pedagogical purposes and the needs of other instructors who rely on these materials.

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN 0-13-186235-9

Pearson Education Ltd., *London*
Pearson Education Australia Pty. Ltd., *Sydney*
Pearson Education Singapore, Pte. Ltd.
Pearson Education North Asia Ltd., *Hong Kong*
Pearson Education Canada, Inc., *Toronto*
Pearson Educación de México, S.A. de C.V.
Pearson Education—Japan, *Tokyo*
Pearson Education Malaysia, Pte. Ltd.

Contents

1	Introduction to Differential Equations	1
1.1	Differential Equation Models	1
1.2	The Derivative	2
1.3	Integration	5
2	First-Order Equations	12
2.1	Differential Equations and Solutions	12
2.2	Solutions to Separable Equations	21
2.3	Models of Motion	30
2.4	Linear Equations	32
2.5	Mixing Problems	46
2.6	Exact Differential Equations	52
2.7	Existence and Uniqueness of Solutions	56
2.8	Dependence of Solutions on Initial Conditions	64
2.9	Autonomous Equations and Stability	66
3	Modeling and Applications	79
3.1	Modeling Population Growth	79
3.2	Models and the Real World	85
3.3	Personal Finance	85
3.4	Electrical Circuits	91
4	Second-Order Equations	94
4.1	Definitions and Examples	94
4.2	Second-Order Equations and Systems	102
4.3	Linear, Homogeneous Equations with Constant Coefficients	119
4.4	Harmonic Motion	127
4.5	Inhomogeneous Equations; the Method of Undetermined Coefficients	149
4.6	Variation of Parameters	162
4.7	Forced Harmonic Motion	170

5	The Laplace Transform	194
5.1	The Definition of the Laplace Transform	194
5.2	Basic Properties of the Laplace Transform	204
5.3	The Inverse Laplace Transform	215
5.4	Using the Laplace Transform to Solve Differential Equations	225
5.5	Discontinuous Forcing Terms	245
5.6	The Delta Function	264
5.7	Convolutions	268
6	Numerical Methods	281
6.1	Euler's Method	281
6.2	Runge-Kutta Methods	301
6.3	Numerical Error Comparisons	323
6.4	Practical Use of Solvers	333
7	Matrix Algebra	343
7.1	Vectors and Matrices	343
7.2	Systems of Linear Equations with Two or Three Variables	350
7.3	Solving Systems of Equations	359
7.4	Homogeneous and Inhomogeneous Systems	371
7.5	Bases of a Subspace	380
7.6	Square Matrices	387
7.7	Determinants	393
8	An Introduction to Systems	404
8.1	Definitions and Examples	404
8.2	Geometric Interpretation of Solutions	410
8.3	Qualitative Analysis	420
8.4	Linear Systems	426
8.5	Properties of Linear Systems	433
9	Linear Systems with Constant Coefficients	447
9.1	Overview of the Technique	447
9.2	Planar Systems	460
9.3	Phase Plane Portraits	480
9.4	The Trace-Determinant Plane	489
9.5	Higher Dimensional Systems	505
9.6	The Exponential of a Matrix	530
9.7	Qualitative Analysis of Linear Systems	562

9.8	Higher-Order Linear Equations	571
9.9	Inhomogeneous Linear Systems	587

10 Nonlinear Systems 604

10.1	The Linearization of a Nonlinear System	604
10.2	Long-Term Behavior of Solutions	615
10.3	Invariant Sets and the Use of Nullclines	620
10.4	Long-Term Behavior of Solutions to Planar Systems	628
10.5	Conserved Quantities	637
10.6	Nonlinear Mechanics	642
10.7	The Method of Lyapunov	656
10.8	Predator–Prey Systems	664

11 Series Solutions to Differential Equations 668

11.1	Review of Power Series	668
11.2	Series Solutions Near Ordinary Points	677
11.3	Legendre's Equation	705
11.4	Types of Singular Points—Euler's Equation	718
11.5	Series Solutions Near Regular Singular Points	727
11.6	Series Solutions Near Regular Singular Points — the General Case	768
11.7	Bessel's Equation and Bessel Functions	813

12 Fourier Series 823

12.1	Computation of Fourier Series	823
12.2	Convergence of Fourier Series	846
12.3	Fourier Cosine and Sine Series	852
12.4	The Complex Form of a Fourier Series	882
12.5	The Discrete Fourier Transform and the FFT	889

13 Partial Differential Equations 893

13.1	Derivation of the Heat Equation	893
13.2	Separation of Variables for the Heat Equation	894
13.3	The Wave Equation	902
13.4	Laplace's Equation	907
13.5	Laplace's Equation on a Disk	911
13.6	Sturm Liouville Problems	915
13.7	Orthogonality and Generalized Fourier Series	918
13.8	Temperature in a Ball—Legendre Polynomials	923
13.9	The Heat and Wave Equations in Higher Dimension	926
13.10	Domains with Circular Symmetry—Bessel Functions	930

Chapter 1. Introduction

Section 1.1. Introduction to Differential Equations

1. Let $y(t)$ be the number of bacteria at time t . The rate of change of the number of bacteria is $y'(t)$. Since this rate of change is given to be proportional to $y(t)$, the resulting differential equation is $y'(t) = ky(t)$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the number of bacteria is growing).
2. Let $y(t)$ be the number of field mice at time t . The rate of change of the number of mice is $y'(t)$. Since this rate of change is given to be inversely proportional to the square root of $y(t)$, the resulting differential equation is $y'(t) = k/\sqrt{y(t)}$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the number of mice is growing).
3. Let $y(t)$ be the number of ferrets at time t . The rate of change of the number of ferrets is $y'(t)$. Since this rate of change is given to be proportional to the product of $y(t)$ and the difference between the maximum population and $y(t)$ (i.e. $100 - y(t)$), the resulting differential equation is $y'(t) = ky(t)(100 - y(t))$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the number of ferrets is growing provided $y(t) < 100$).
4. Let $y(t)$ be the quantity of radioactive substance at time t . The rate of change of the material is $y'(t)$. Since this rate of change (decay) is given to be proportional to $y(t)$, the resulting differential equation is $y'(t) = -ky(t)$. Note that k is a positive constant since $y'(t)$ must be negative (i.e. the quantity of radioactive material is decreasing).
5. Let $y(t)$ be the quantity of material at time t . The rate of change of the material is $y'(t)$. Since this rate of change (decay) is given to be inversely proportional to $y(t)$, the resulting differential equation is $y'(t) = -k/y(t)$. Note that k is a positive constant since $y'(t)$ must be negative (i.e. the quantity of material is decreasing).
6. Let $y(t)$ be the temperature of the potato at time t . The rate of change of the temperature is $y'(t)$. Since this rate of change is given to be proportional to the difference between the potato's temperature and that of the surrounding room (i.e. $y(t) - 65$), the resulting differential equation is $y'(t) = -k(y(t) - 65)$. Note that k is a positive constant since $y'(t)$ must be negative (i.e. the potato is cooling) and since $y(t) - 65 > 0$ (i.e. the potato is hotter than the surrounding room).
7. Let $y(t)$ be the temperature of the thermometer at time t . The rate of change of the temperature is $y'(t)$. Since this rate of change is given to be proportional to the difference between the thermometer's temperature and that of the surrounding room (i.e. $77 - y(t)$), the resulting differential equation is $y'(t) = k(77 - y(t))$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the thermometer is warming) and since $77 - y(t) > 0$ (i.e. the thermometer is cooler than the surrounding room).
8. Let $x(t)$ be the position (displacement) of the particle at time t . The force on the particle is given to be proportional to this displacement. Therefore, the force, F , is equal to $-kx(t)$ where k is a positive constant. The negative sign is present since the direction of F is opposite to that of $x(t)$. Newton's law states $F = ma$ where m is the mass of the object and $a = x''(t)$ is its acceleration. Therefore, $F = ma$ becomes $-kx(t) = mx''(t)$, which is the differential equation governing the motion of this particle.
9. Let $x(t)$ be the position (displacement) of the particle at time t . The force on the particle is given to be proportional to the square of the particle's velocity, i.e. $(x'(t))^2$. As a first guess, one might surmise that the force is given by $F = -k(x'(t))^2$, where k is a positive constant. However, closer inspection reveals that this will have the force pointing to

the left, regardless of whether the velocity is positive or negative. We can work around this difficulty by letting the force equal $F = -kx'(t)|x'(t)|$. The reader will recognize that the force is positive when $x'(t) < 0$, while the force is negative when $x'(t) > 0$, thus insuring that the force is always opposite the particle's motion. Newton's law states $F = ma$ where m is the mass of the object and $a = x''(t)$ is its acceleration. Therefore, $F = ma$ becomes $-k(x'(t))|x'(t)| = mx''(t)$, which is the differential equation governing the motion of this particle.

that we have written $x(t)|x(t)|$ instead of $x(t)^2$ since $-k/[x(t)|x(t)|]$ is negative when $x(t)$ is positive and $-k/[x(t)|x(t)|]$ is positive when $x(t)$ is negative. This agrees with the desired direction of F . Newton's law states $F = ma$ where m is the mass of the object and $a = x''(t)$ is its acceleration. Therefore, $F = ma$ becomes

$$\frac{-k}{x(t)|x(t)|} = mx''(t)$$

which is the differential equation governing the motion of this particle.

10. Let $x(t)$ be the position (displacement) of the particle at time t . The force on the particle is given to be inversely proportional to the square of this displacement. The direction of F is opposite to that of $x(t)$. Therefore, the force, F , is equal to $-k/[x(t)|x(t)|]$ where k is a positive constant. Note
11. Let $V(t)$ be the voltage drop across the inductor and $I(t)$ be the current at time t . The rate of change of the current is $I'(t)$. Since the voltage drop is proportional to the rate of change of I , we obtain the differential equation $V(t) = kI'(t)$, where k is a constant.

Section 1.2. The Derivative

1. $D_x(3x - 5) = 3D_x x - D_x 5$
 $= 3(1) - 0$
 $= 3$
2. $D_x(5x^2 - 4x - 8) = 5D_x x^2 - 4D_x x - D_x 8$
 $= 5(2x) - 4(1) - 0$
 $= 10x - 4$
3. $D_x(3 \sin 5x) = 3D_x \sin 5x$
 $= 3(\cos 5x)D_x(5x)$
 $= 15 \cos 5x$
4. $D_x(\cos 2\pi x) = (-\sin 2\pi x)D_x(2\pi x)$
 $= -2\pi \sin 2\pi x$
5. $D_x(e^{3x}) = e^{3x}D_x(3x)$
 $= 3e^{3x}$
6. $D_x(5e^{x^2}) = 5D_x e^{x^2}$
 $= 5e^{x^2}D_x(x^2)$
 $= 5e^{x^2}(2x)$
 $= 10xe^{x^2}$
7. $D_x \ln |5x| = \frac{1}{5x}D_x(5x)$
 $= \frac{1}{x}$
8. $D_x \ln(\cos 2x) = \frac{1}{\cos 2x}D_x \cos 2x$
 $= \frac{1}{\cos 2x}(-\sin 2x)D_x(2x)$
 $= -2 \tan 2x$
9. $D_x x \ln x = (D_x x) \ln x + x D_x \ln x$
 $= (1) \ln x + x \left(\frac{1}{x}\right)$
 $= 1 + \ln x$
10. $D_x e^x \sin \pi x = (D_x e^x) \sin \pi x + e^x D_x \sin \pi x$
 $= e^x \sin \pi x + e^x (-\cos \pi x)D_x(\pi x)$
 $= e^x (\sin \pi x - \pi \cos \pi x)$
11. $D_x \left(\frac{x^2}{\ln x}\right) = \frac{(D_x x^2) \ln x - x^2 D_x \ln x}{[\ln x]^2}$
 $= \frac{2x \ln x - x^2 \left(\frac{1}{x}\right)}{[\ln x]^2}$
 $= \frac{2x \ln x - x}{[\ln x]^2}$

$$\begin{aligned}
 12. \quad D_x \left(\frac{x \ln x}{\cos x} \right) &= \frac{D_x(x \ln x) \cos x - x \ln x D_x \cos x}{\cos^2 x} \\
 &= \frac{(1 + \ln x) \cos x - x \ln x (-\sin x)}{\cos^2 x} \\
 &= \frac{(1 + \ln x) \cos x + x \sin x \ln x}{\cos^2 x}
 \end{aligned}$$

13. If $L(x) = f(x_0) + f'(x_0)(x - x_0)$, then

$$\begin{aligned}
 R(x) &= f(x) - L(x) \\
 &= f(x) - f(x_0) - f'(x_0)(x - x_0).
 \end{aligned}$$

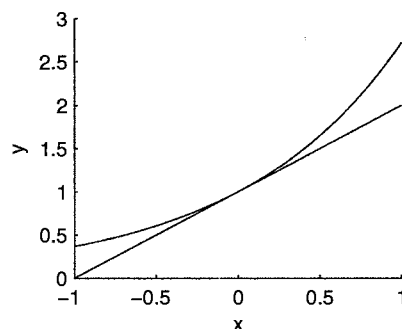
Thus,

$$\begin{aligned}
 \lim_{x \rightarrow x_0} \frac{R(x)}{x - x_0} &= \lim_{x \rightarrow x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right] \\
 &= f'(x_0) - f'(x_0) \\
 &= 0.
 \end{aligned}$$

14. Given that $f(x) = e^x$, the derivative is $f'(x) = e^x$. At $x_0 = 0$, $f'(0) = 1$. Thus, the linearization is

$$L(x) = f(0) + f'(0)(x - 0) = 1 + x.$$

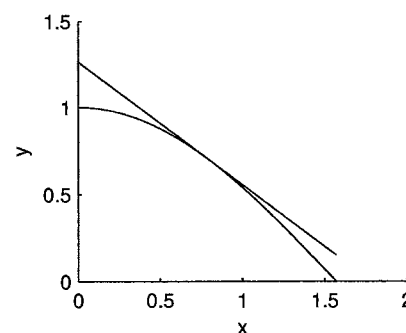
The graph of f , together with its linear approximation at $x_0 = 0$, is shown in the following figure.



15. Given that $f(x) = \cos x$, the derivative is $f'(x) = -\sin x$. At $x_0 = \pi/4$, $f'(\pi/4) = -\sqrt{2}/2$. Thus, the linearization is

$$\begin{aligned}
 L(x) &= f(\pi/4) + f'(\pi/4)(x - \pi/4) \\
 &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4} \right).
 \end{aligned}$$

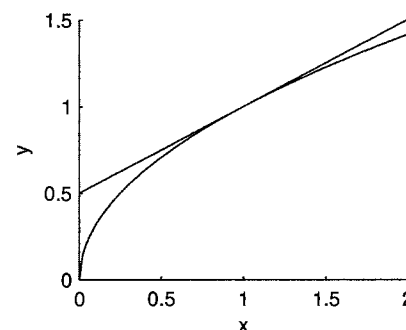
The graph of f , together with its linear approximation at $x_0 = \pi/4$, is shown in the following figure.



16. Given that $f(x) = \sqrt{x}$, the derivative is $f'(x) = 1/(2\sqrt{x})$. At $x_0 = 1$, $f'(1) = 1/2$. Thus, the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 1 + \frac{1}{2}(x - 1).$$

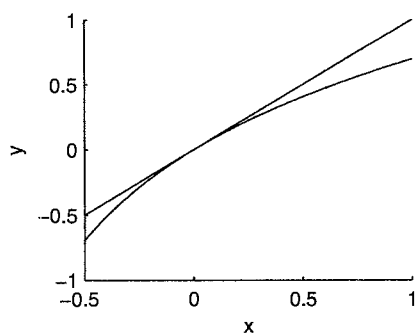
The graph of f , together with its linear approximation at $x_0 = 1$, is shown in the following figure.



17. Given that $f(x) = \ln(1 + x)$, the derivative is $f'(x) = 1/(1 + x)$. At $x_0 = 0$, $f'(0) = 1$. Thus, the linearization is

$$L(x) = f(0) + f'(0)(x - 0) = x.$$

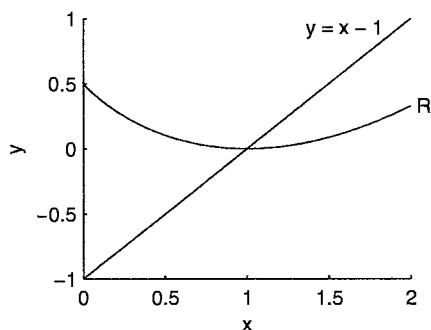
The graph of f , together with its linear approximation at $x_0 = 0$, is shown in the following figure.



18. Given that $f(x) = x^{3/2}$, the derivative is $f'(x) = (3/2)x^{1/2}$. At $x_0 = 1$, $f'(1) = 3/2$. Thus, the linearization is

$$L(x) = f(1) + f'(1)(x-1) = 1 + \frac{3}{2}(x-1).$$

The graph of $y = x - 1$, together with the graph of the remainder $R(x) = f(x) - L(x)$, is shown in the following figure.

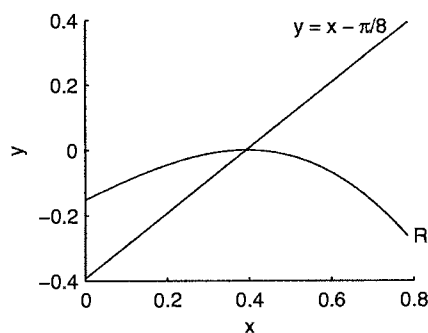


Note that both graphs approach zero as $x \rightarrow 1$, but the graph of R approaches zero at a more rapid rate.

19. Given that $f(x) = \sin 2x$, the derivative is $f'(x) = 2 \cos 2x$. At $x_0 = \pi/8$, $f'(\pi/8) = \sqrt{2}$. Thus, the linearization is

$$L(x) = f(\pi/8) + f'(\pi/8)(x - \pi/8) = \frac{\sqrt{2}}{2} + \sqrt{2}\left(x - \frac{\pi}{8}\right).$$

The graph of $y = x - \pi/8$, together with the graph of the remainder $R(x) = f(x) - L(x)$, is shown in the following figure.

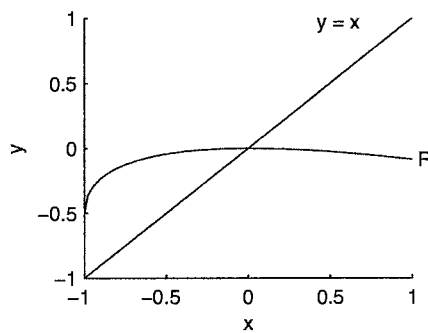


Note that both graphs approach zero as $x \rightarrow \pi/8$, but the graph of R approaches zero at a more rapid rate.

20. Given that $f(x) = \sqrt{x+1}$, the derivative is $f'(x) = 1/(2\sqrt{x+1})$. At $x_0 = 0$, $f'(0) = 1/2$. Thus, the linearization is

$$L(x) = f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x.$$

The graph of $y = x$, together with the graph of the remainder $R(x) = f(x) - L(x)$, is shown in the following figure.

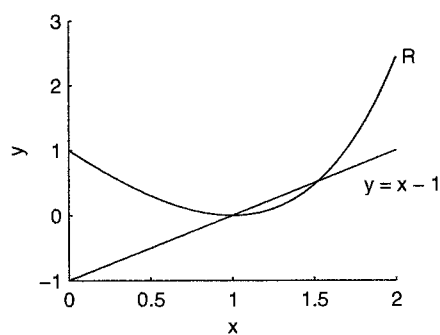


Note that both graphs approach zero as $x \rightarrow 0$, but the graph of R approaches zero at a more rapid rate.

21. Given that $f(x) = xe^{x-1}$, the derivative is $f'(x) = (x+1)e^{x-1}$. At $x_0 = 1$, $f'(1) = 2$. Thus, the linearization is

$$L(x) = f(1) + f'(1)(x-1) = 1 + 2(x-1).$$

The graph of $y = x - 1$, together with the graph of the remainder $R(x) = f(x) - L(x)$, is shown in the following figure.

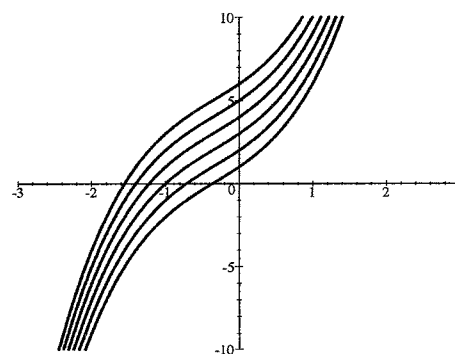
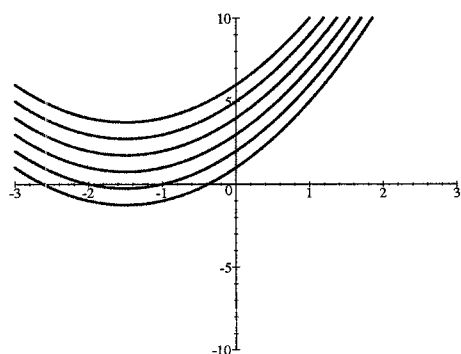


Note that both graphs approach zero as $x \rightarrow 1$, but the graph of R approaches zero at a more rapid rate.

Section 1.3. Integration

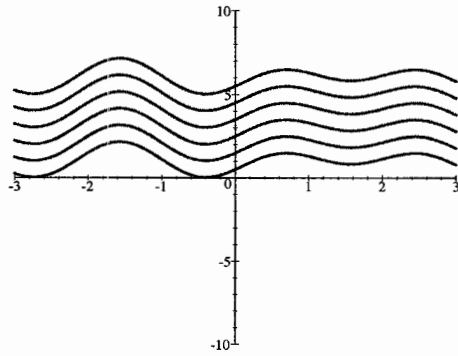
1. $y' = 2t + 3$. Integrate to obtain $y = t^2 + 3t + C$.

$$t^3 + t^2 + 3t + C.$$

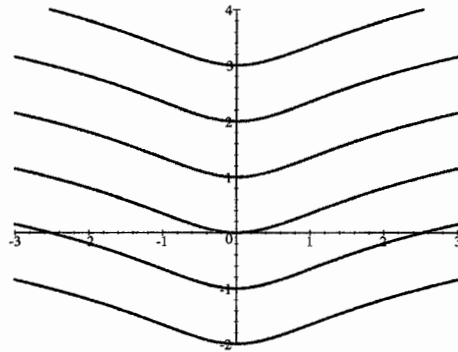


2. $y' = 3t^2 + 2t + 3$. Integrate to obtain $y =$ 3. $y' = \sin 2t + 2 \cos 3t$. Integrate to obtain $y =$

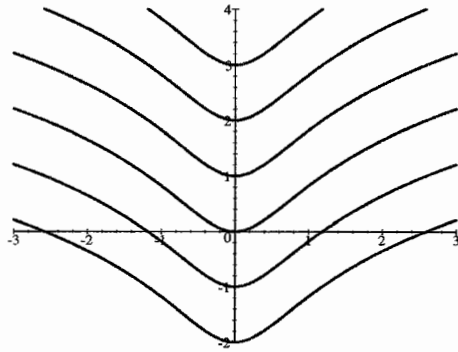
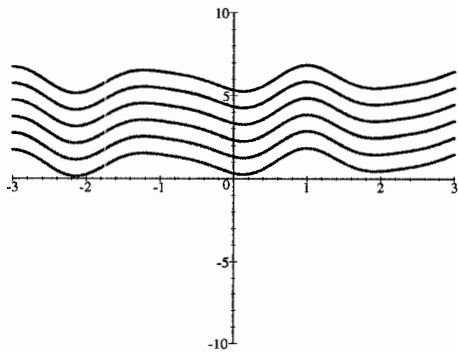
$$(-1/2) \cos 2t + (2/3) \sin 3t + C.$$



$$y = (1/2) \ln u + C = (1/2) \ln(1 + t^2) + C.$$



4. $y' = 2 \sin 3t - \cos 5t$. Integrate to obtain $y = (-2/3) \cos 3t - (1/5) \sin 5t + C$.



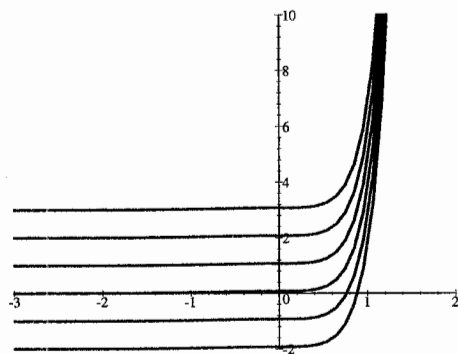
7. $y' = t^2 e^{3t}$. Integrate by parts with $u = t^2$ and $dv = e^{3t}$ to obtain

$$y = \frac{t^2 e^{3t}}{3} - \int \frac{e^{3t} 2t}{3} dt$$

Integrate by parts once more and obtain

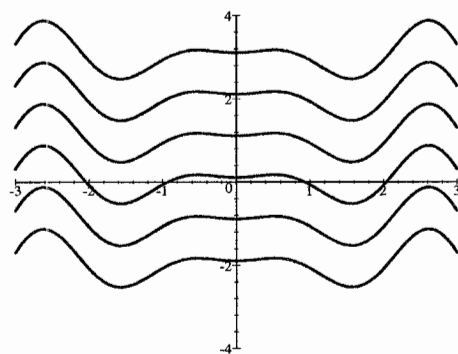
$$y = \frac{t^2 e^{3t}}{3} - \frac{2t e^{3t}}{9} + \frac{2e^{3t}}{27} + C$$

5. $y' = t/(1 + t^2)$. Use $u = 1 + t^2$, $du = 2t dt$ and get $dy = (1/2) du/u$. Integrate to obtain



8. $y' = t \cos 3t$. Integrate by parts with $u = t$, $dv = \cos 3t$ and obtain

$$y = \frac{t \sin 3t}{3} + \frac{\cos 3t}{9} + C$$



9. $y' = e^{-2\omega} \sin \omega$. Integrate by parts with $u = e^{-2\omega}$ and $dv = \sin \omega$ to obtain

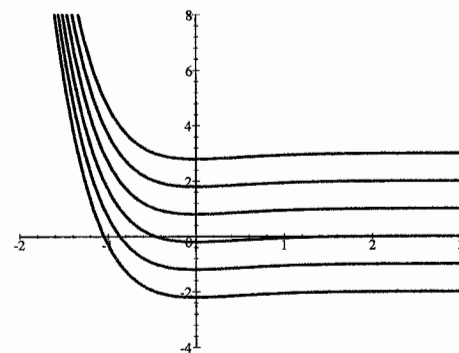
$$\int e^{-2\omega} \sin \omega d\omega = -e^{-2\omega} \cos \omega - 2 \int \cos \omega e^{-2\omega} d\omega.$$

Integrate by parts again with $u = e^{-2\omega}$ and $dv = \cos \omega$, to obtain

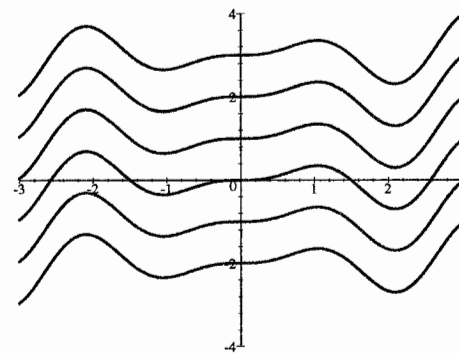
$$\begin{aligned} \int e^{-2\omega} \sin \omega d\omega &= -e^{-2\omega} \cos \omega - 2 \sin \omega e^{-2\omega} \\ &\quad - 4 \int \sin \omega e^{-2\omega} d\omega \end{aligned}$$

Then add the integral on the right to the integral on the left, which then becomes $5 \int \sin \omega e^{-2\omega} d\omega$; divide by the 5 and obtain the answer:

$$y = (-e^{-2\omega} \cos \omega - 2 \sin \omega e^{-2\omega}) / 5 + C$$



10. $y' = x \sin 3x$. Integrate by parts with $u = x$ and $dv = \sin 3x$ to obtain $y = (-x/3) \cos 3x + (1/9) \sin 3x + C$.

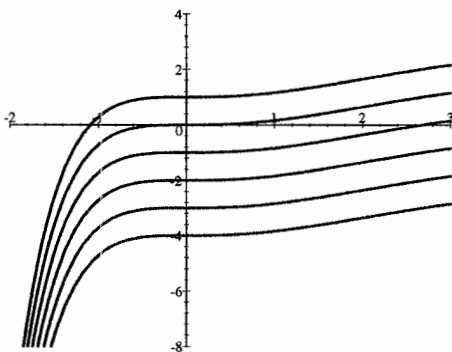


11. $x' = s^2 e^{-s}$. Integrate by parts with $u = s^2$ and $dv = e^{-s}$ and obtain

$$x = -s^2 e^{-s} + 2 \int e^{-s} s ds.$$

Integrate by parts again with $u = s$ and $dv = e^{-s}$ to obtain the answer:

$$x = -s^2 e^{-s} - 2s e^{-s} - 2e^{-s} + C.$$



12.

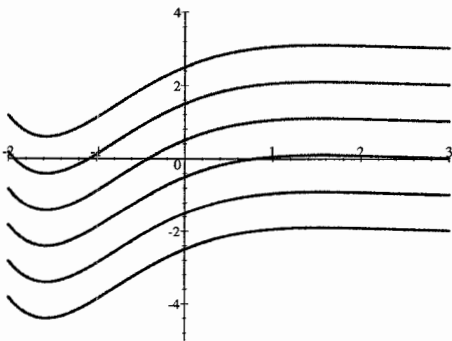
$$\int e^{-u} \cos u \, du = e^{-u} \sin u + \int \sin u e^{-u} \, du.$$

Integrate by parts again, with $U = e^{-u}$ and $dV = \sin u$ and obtain

$$\int e^{-u} \cos u \, du = e^{-u} \sin u - e^{-u} \cos u - \int \cos u e^{-u} \, du.$$

Add the integral on the right to the left side; then divide by 2 and obtain the answer:

$$y = (e^{-u} \sin u - e^{-u} \cos u) / 2 + C.$$

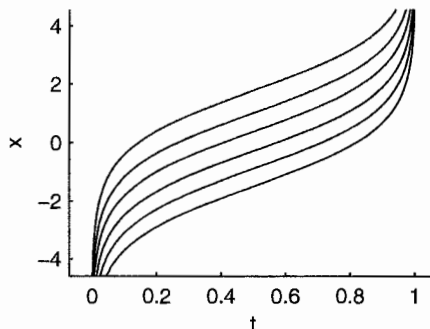


13. Use partial fractions to write

$$r' = \left[\frac{1}{u} + \frac{1}{1-u} \right].$$

Then integrate to obtain

$$r = \ln u - \ln(1-u) = \ln \left(\frac{u}{1-u} \right).$$

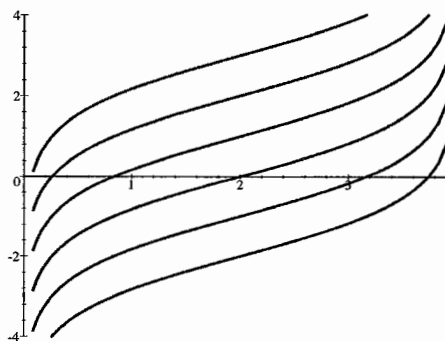


14. Use partial fractions to write

$$y' = \frac{3}{4} \left[\frac{1}{x} + \frac{1}{4-x} \right].$$

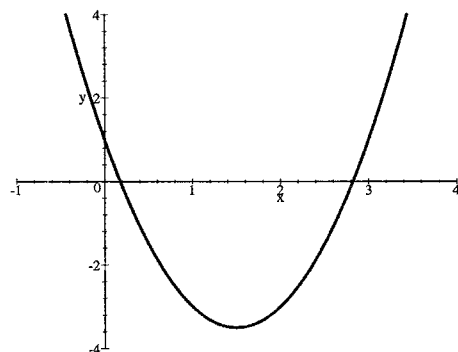
Then integrate to obtain

$$y = (3/4)(\ln x - \ln(4-x)) = \ln \left(\frac{x}{4-x} \right)^{3/4}.$$

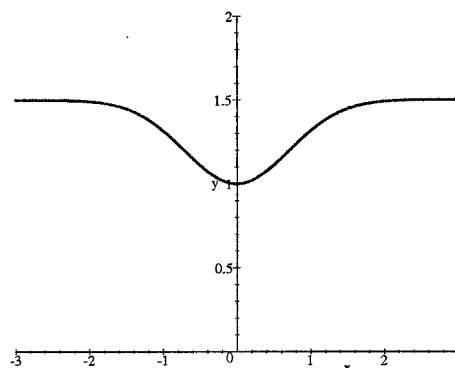


15. $y' = 4t - 6$. Integrate y' to obtain $y = 2t^2 - 6t + C$; the initial condition $y(0) = 1$ gives $1 = C$; so

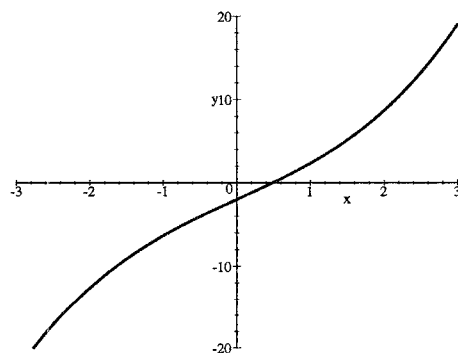
$$y(t) = 2t^2 - 6t + 1.$$



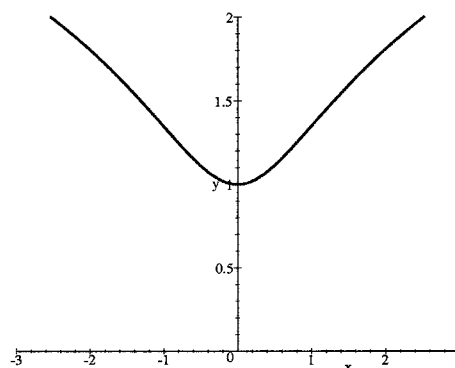
$$(-1/2)e^{-t^2} + (3/2).$$



16. $y' = x^2 + 4$. Integrate to obtain $y = x^3/3 + 4x + C$; the initial condition $y(0) = -2$ gives $-2 = C$; so $y(t) = x^3/3 + 4x - 2$.



18. $r'(t) = t/(1 + t^2)$. Integrate to obtain $r(t) = (1/2) \ln(1 + t^2) + C$; the initial condition, $r(0) = 1$ gives $1 = (1/2) \ln 1 + C$ or $C = 1$; so $r(t) = (1/2) \ln(1 + t^2) + 1$.



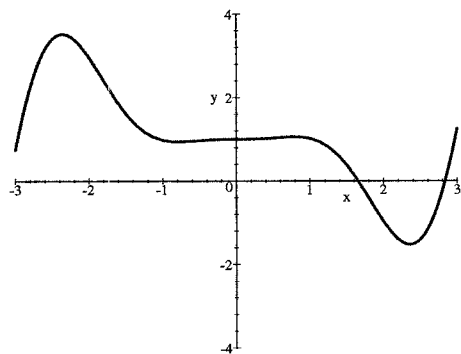
19. $s'(r) = r^2 \cos 2r$. Integrate by parts twice with dv being the trig - term ($\cos 2r$ and then $\sin 2r$ to obtain

$$s(r) = \frac{r^2 \sin 2r}{2} + \frac{r \cos 2r}{2} - \frac{\sin 2r}{4} + C.$$

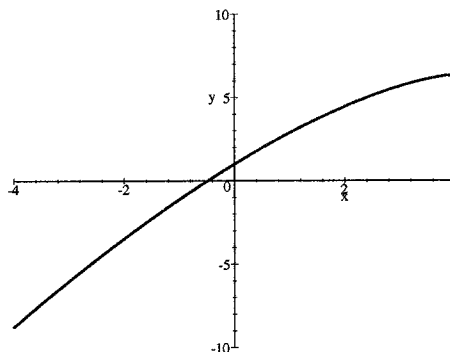
The initial condition, $s(0) = 1$ gives $1 = C$ so

$$s(r) = \frac{r^2 \sin 2r}{2} + \frac{r \cos 2r}{2} - \frac{\sin 2r}{4} + 1$$

17. $x'(t) = te^{-t^2}$. Integrate to obtain $x(t) = (-1/2)e^{-t^2} + C$; the initial condition $x(0) = 1$ gives $1 = (-1/2) + C$; so $C = 3/2$ and $x(t) =$



$C = 19/3$. So $x(t) = (-2/3)(4 - t)^{3/2} + 19/3$.

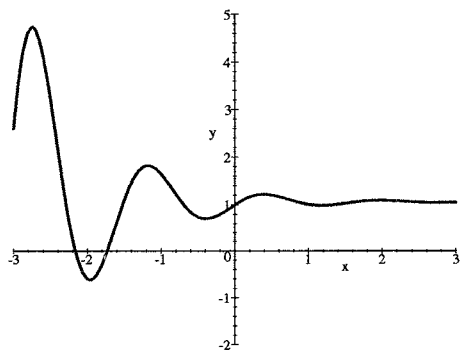


20. $P'(t) = e^{-t} \cos 4t$. Integrate by parts twice as done in the solution to Exercise 12 above to obtain

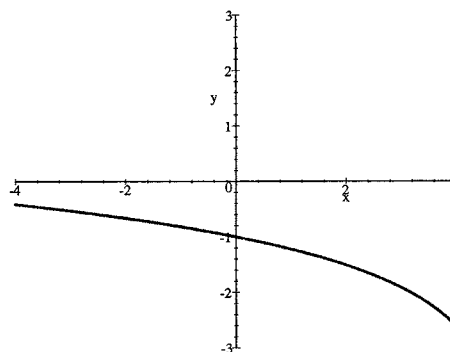
$$P(t) = \frac{4}{17} (e^{-t} \sin 4t - (1/4)e^{-t} \cos 4t) + C$$

The initial condition, $P(0) = 1$ gives $1 = -1/17 + C$ or $C = 18/17$; so

$$P(t) = \frac{4}{17} (e^{-t} \sin 4t - (1/4)e^{-t} \cos 4t) + \frac{18}{17}$$



22. $u'(x) = 1/(x - 5)$. Integrate to obtain $u(x) = \ln|x - 5| + C$. The initial condition, $u(0) = -1$ gives $-1 = \ln 5 + C$ or $C = -1 - \ln 5$; so $u(x) = \ln|x - 5| - 1 - \ln 5$.



23. $y'(t) = \frac{t+1}{t(t+4)}$. Partial fractions gives

$$y'(t) = \frac{1/4}{t} + \frac{3/4}{t+4}$$

Integrating, we obtain

$$y(t) = (1/4) \ln|t| + (3/4) \ln|t+4| + C$$

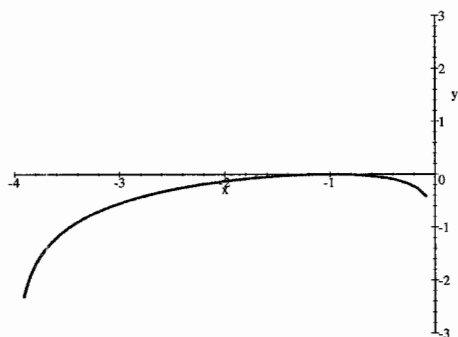
The initial condition, $y(-1) = 0$ gives

$$0 = (1/4) \ln 1 + (3/4) \ln 3 + C = (3/4) \ln 3 + C$$

or $C = -(3/4) \ln 3$. So

$$y(t) = (1/4) \ln|t| + (3/4) \ln|t+4| - (3/4) \ln 3$$

21. $x'(t) = \sqrt{4-t}$. Integrate to obtain $x(t) = (-2/3)(4-t)^{3/2} + C$. The initial condition, $x(0) = 1$ gives $1 = (-2/3)(4)^{3/2} + C = -16/3 + C$ or



24. $v'(r) = \frac{r^2}{r+1}$. By long division, we obtain

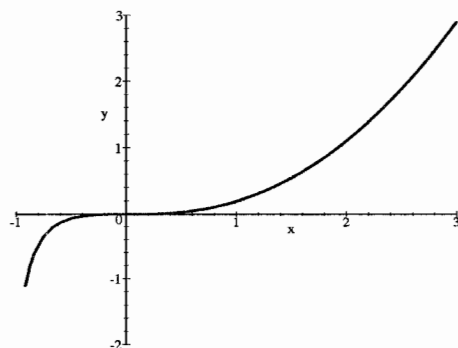
$$v'(r) = r - 1 + \frac{1}{r+1}$$

Integrating, we obtain

$$v(r) = \frac{r^2}{2} - r + \ln|r+1| + C$$

The initial condition, $v(0) = 0$ gives
 $0 = \ln 1 + C = C$; so

$$v(r) = \frac{r^2}{2} - r + \ln|r+1|$$



25. Let $s(t)$ be the height of the ball at time t seconds. If $g = -9.8$ is the gravitational constant, then $s''(t) = g$. Integrating we obtain, $s' = gt + v_0$, where v_0 is a constant. The initial condition $s'(0) = 50$ gives $v_0 = 50$; so $s'(t) = gt + 50$. The velocity at $t = 3$ seconds is $s'(3) = 3g + 50 = 20.6$ meters/second. Integrating s' , gives $s(t) = gt^2/2 + 50t + s_0$. The initial condition, $s(0) = 3$ gives $s_0 = 3$, so $s(t) = gt^2/2 + 50t + 3$. The height at $t = 3$ seconds is $s(3) = (9/2)g + 153 = 108.9$ meters.
26. Let $s(t)$ be the height of the ball at time t seconds. If $g = -9.8$ is the gravitational constant, then $s''(t) = g$. Integrating we obtain, $s' = gt + v_0$, where v_0 is a constant. The initial condition $s'(0) = 0$ (dropped from rest) gives $v_0 = 0$; so $s'(t) = gt$. The velocity at $t = 3$ is $s'(3) = 3g = -29.4$ meters/sec. Integrating s' , gives $s(t) = gt^2/2 + s_0$. The initial condition $s(0) = 200$ gives $s_0 = 200$, so $s(t) = gt^2/2 + 200$; the height at $t = 3$ seconds is $s(3) = (9/2)g + 200 = 155.9$ meters.
27. Let $s(t)$ be the height of the ball at time t seconds. If $g = -9.8$ is the gravitational constant, then $s''(t) = g$. Integrating we obtain, $s' = gt + v_0$, where v_0 is a constant. The initial condition $s'(0) = 120$ gives $v_0 = 120$, so $s'(t) = gt + 120$. The maximum height occurs when the velocity reaches zero, i.e. when $gt + 120 = 0$, or $t = -120/g = 12.24$ seconds. Integrating s' gives $s(t) = gt^2/2 + 120t + s_0$. The initial condition $s(0) = 6$ gives $s_0 = 6$, so $s(t) = gt^2/2 + 120t + 6$. When $t = 12.24$, the maximum height is $s(12.24) = 740.69$ meters.
28. Let $s(t)$ be the height of the ball at time t seconds. If $g = -9.8$ is the gravitational constant, then $s''(t) = g$. Integrating we obtain, $s' = gt + v_0$, where v_0 is a constant. The initial condition $s'(0) = -25$ gives $v_0 = -25$, so $s'(t) = gt - 25$. Integrating again gives $s(t) = gt^2/2 - 25t + s_0$. The initial condition $s(0) = 1000$ gives $s_0 = 1000$ and so $s(t) = gt^2/2 - 25t + 1000 = -4.9t^2 - 25t + 1000$. The ball hits the ground when $s(t) = 0$ which occurs at approximately $t = 11.96$ seconds.

Chapter 2. First-Order Equations

Section 2.1. Differential Equations and Solutions

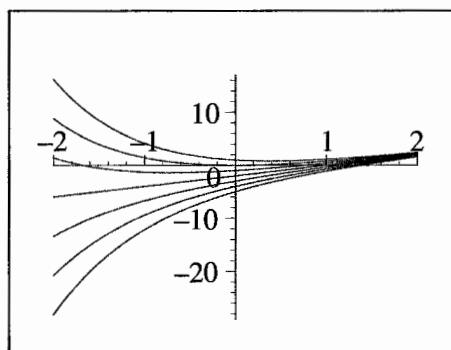
1. $\phi(t, y, y') = t^2 y' + (1+t)y = 0$ must be solved for y' . We get
4. $y'(t) + y(t) = (2 - Ce^{-t}) + (2t - 2 + Ce^{-t}) = 2t$

$$y' = -\frac{(1+t)y}{t^2}.$$

2. $\phi(t, y, y') = ty' - 2y - t^2$ must be solved for y' . We get

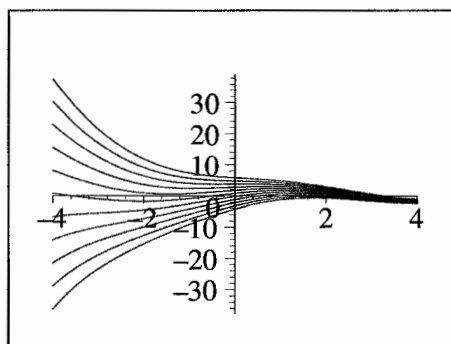
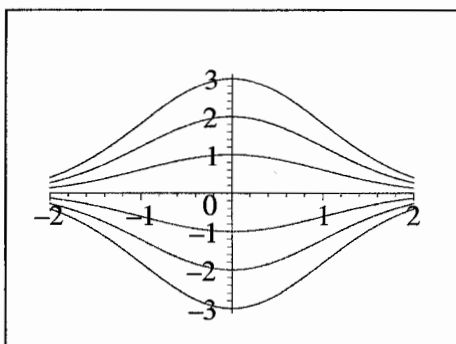
$$y' = \frac{2y + t^2}{t}.$$

3. $y'(t) = -Cte^{-(1/2)t^2}$ and $-ty(t) = -tCe^{-(1/2)t^2}$, so $y' = -ty$.



5. If $y(t) = (4/5) \cos t + (8/5) \sin t + Ce^{-(1/2)t}$, then

$$\begin{aligned} y(t)' + (1/2)y(t) &= [-(4/5) \sin t + (8/5) \cos t - (C/2)e^{-(1/2)t}] \\ &\quad + (1/2)[(4/5) \cos t + (8/5) \sin t + Ce^{-(1/2)t}] \\ &= 2 \cos t. \end{aligned}$$



6. If $y(t) = 4/(1 + Ce^{-4t})$, then

$$\begin{aligned} y' &= \frac{16C e^{(-4t)}}{(1 + C e^{(-4t)})^2} \\ y(4 - y) &= \frac{4}{1 + C e^{-4t}} \times \left[4 - \frac{4}{1 + C e^{-4t}} \right] \\ &= \frac{16(1 + C e^{-4t}) - 16}{(1 + C e^{(-4t)})^2} \\ &= \frac{16C e^{(-4t)}}{(1 + C e^{(-4t)})^2} \end{aligned}$$

7. For $y(t) = 0$, $y'(t) = 0$ and $y(t)(4 - y(t)) = 0(4 - 0) = 0$.

8. (a) If $t^2 + y^2 = C^2$, then

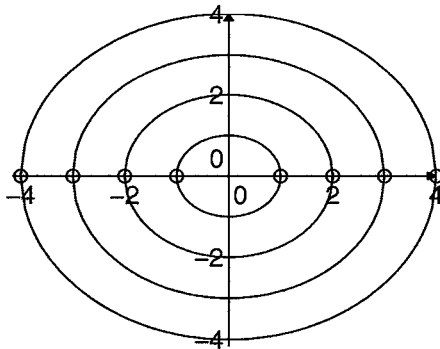
$$\begin{aligned} \frac{d}{dt}(t^2 + y^2) &= \frac{d}{dt}C^2 \\ 2t + 2yy' &= 0 \\ t + yy' &= 0 \end{aligned}$$

- (b) If $y(t) = \pm\sqrt{C^2 - t^2}$, then $y' = \mp t/\sqrt{C^2 - t^2}$, and

$$\begin{aligned} t + yy' &= t + [\pm\sqrt{C^2 - t^2}][\mp t/\sqrt{C^2 - t^2}] \\ &= t - t \\ &= 0. \end{aligned}$$

- (c) For $\sqrt{C^2 - t^2}$ to be defined we must have $t^2 \leq C^2$. In order that $y'(t) = \mp t/\sqrt{C^2 - t^2}$, we must restrict the domain further. Hence the interval of existence is $-C < t < C$.

- (d)



9. (a) If $t^2 - 4y^2 = c^2$, then

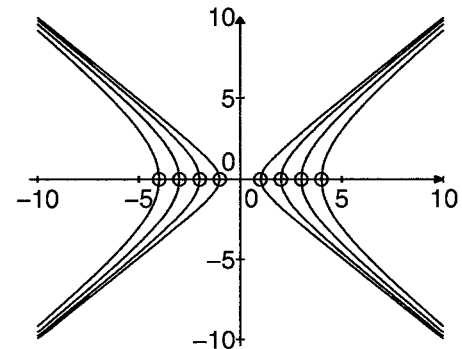
$$\begin{aligned} \frac{d}{dt}(t^2 - 4y^2) &= \frac{d}{dt}c^2 \\ 2t - 8yy' &= 0 \\ t - 4yy' &= 0. \end{aligned}$$

- (b) If $y(t) = \pm\sqrt{t^2 - C^2}/2$, then $y' = \pm t/(2\sqrt{t^2 - C^2})$, and

$$\begin{aligned} t - 4yy' &= t - 4[\pm\sqrt{t^2 - C^2}/2][\pm t/(2\sqrt{t^2 - C^2})] \\ &= t - t \\ &= 0. \end{aligned}$$

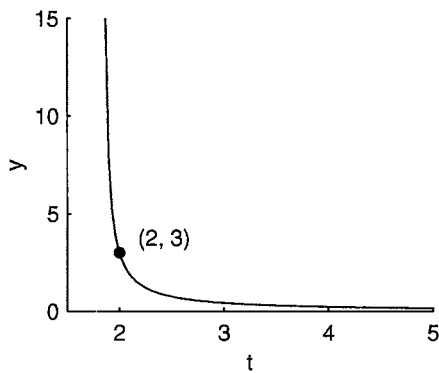
- (c) The interval of existence is either $-\infty < t < -C$ or $C < t < \infty$.

- (d)

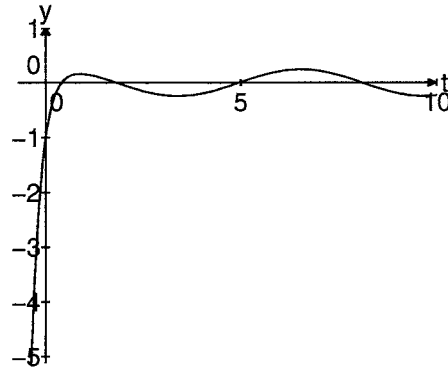


10. If $y(t) = 3/(6t - 11)$, then $y' = -3 \cdot 6/(6t - 11)^2 = -18/(6t - 11)^2$. On the other hand, $-2y^2 = -2[3/(6t - 11)]^2 = -18/(6t - 11)^2$, so we have a solution to the differential equation. Since $y(2) = -3/(12 - 11) = -3$, we have a solution to the initial value problem. The interval of existence is the interval containing 2 where $6t - 11 \neq 0$. This is the

interval $(11/6, \infty)$.

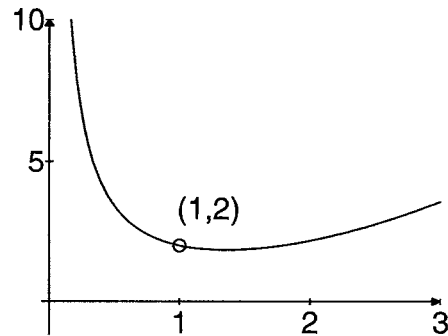
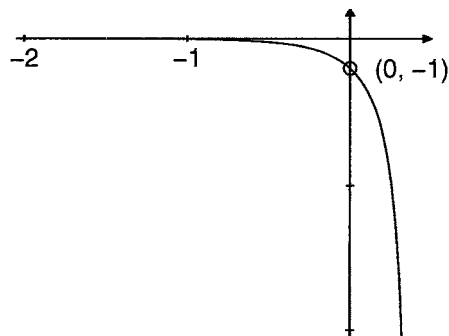


the interval $(-\infty, \infty)$.



11. See Exercise 6. The interval of existence is $(-\infty, \ln(5)/4)$.

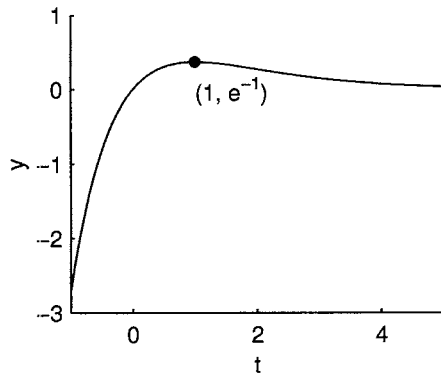
13. $y(t) = \frac{1}{3}t^2 + \frac{5}{3t}$. The interval of existence is $(0, \infty)$.



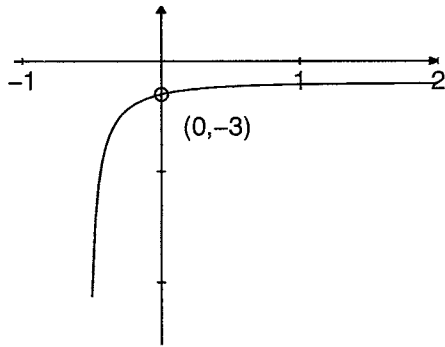
12. $y(t) = (4/17)\cos t + (1/17)\sin t - (21/17)e^{-4t}$ on

14. We need $e^{-1} = y(1) = e^{-1}(1 + C/(1)) = (1 + C)e^{-1}$. Hence $C = 0$, and our solution is $y(t) = te^{-t}$. This function is defined and differentiable on the whole real line. Hence the interval of existence

is the whole real line.



15. $y(t) = 2/(-1 + e^{-2t/3})$. The interval of existence is $(-\ln(3)/2, \infty)$.



16. The initial value problem is

$$y' = \sqrt{y}, \quad y(0) = 1.$$

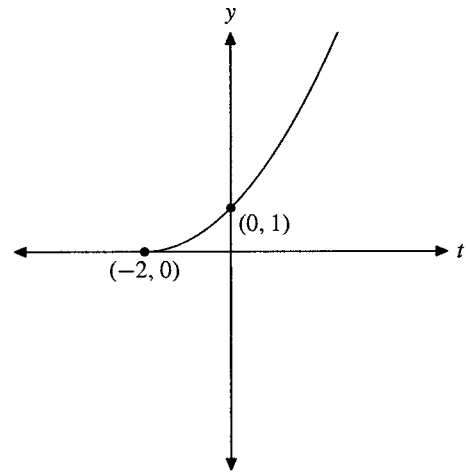
The first solution proposed by Maple, $y(t) = (1/4)(t+2)^2$, satisfies the initial condition, $y(0) = (1/4)(0+2)^2 = 1$. Next,

$$y'(t) = \frac{1}{2}(t+2),$$

and

$$\sqrt{y(t)} = \sqrt{\frac{1}{4}(t+2)^2} = \frac{1}{2}|t+2|.$$

But this equals $(1/2)(t+2)$ only if $t \geq -2$, as shown in the following figure.



The second solution proposed by Maple, $y(t) = (1/4)(t-2)^2$, satisfies the initial condition, as $y(0) = (1/4)(0-2)^2 = 1$. But

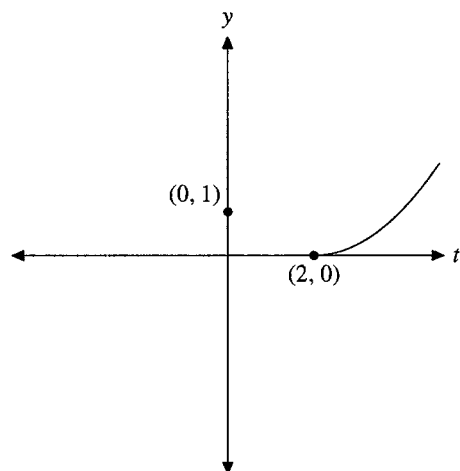
$$y'(t) = \frac{1}{2}(t-2),$$

and

$$\sqrt{y(t)} = \sqrt{\frac{1}{4}(t-2)^2} = \frac{1}{2}|t-2|.$$

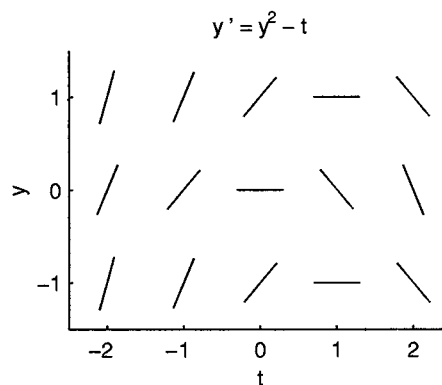
But this agrees with $(1/2)(t-2)$ only if $t \geq 2$, as

shown in the following figure.

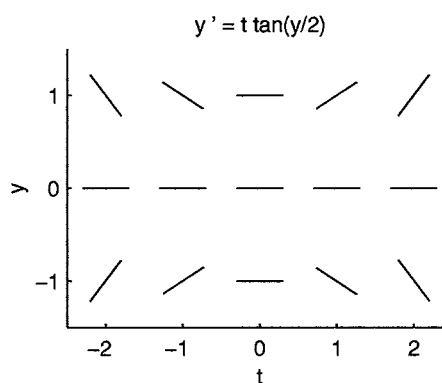


Note that this graph does *not* pass through $(0, 1)$. Hence, $y(t) = (1/4)(t - 2)^2$ is *not* a solution of the initial value problem.

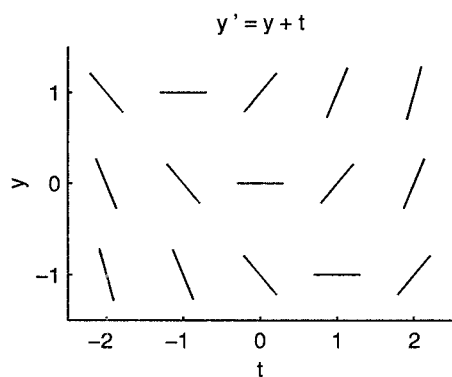
18.



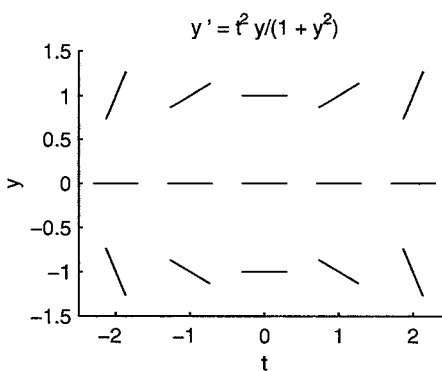
19.



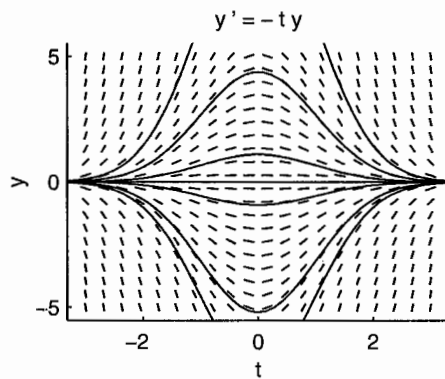
17.



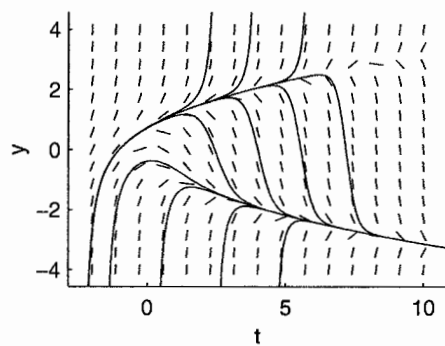
20.



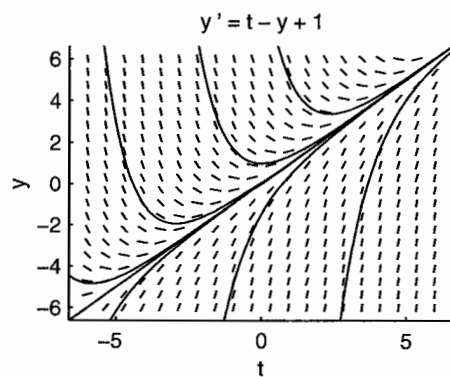
21.



22.



23.

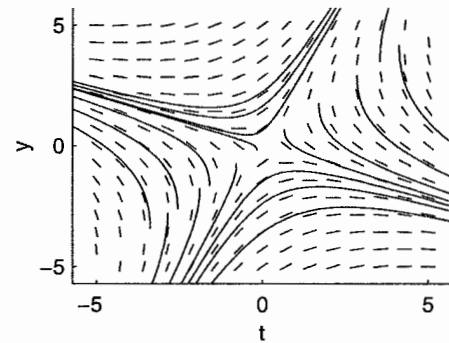


24. Note the difficulty experienced by the solver as it approaches the line $y = t$, where the denominator of

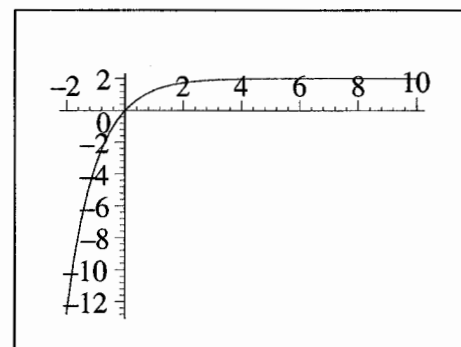
the right-hand side of

$$y' = \frac{y+t}{y-t}$$

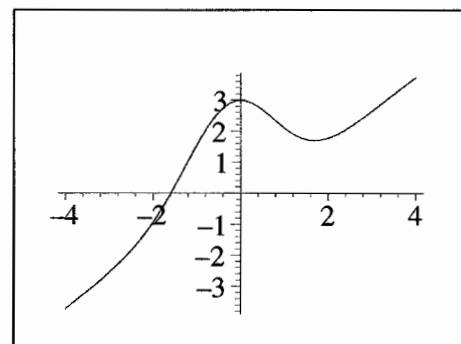
is undefined.



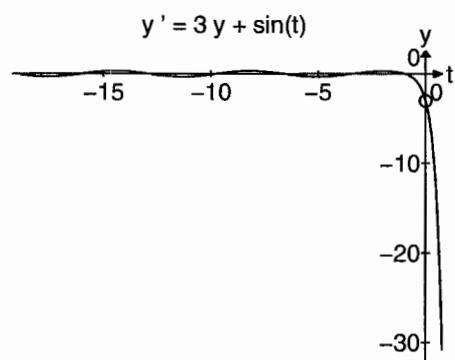
25.



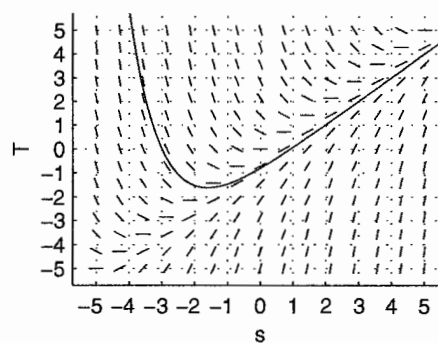
26.



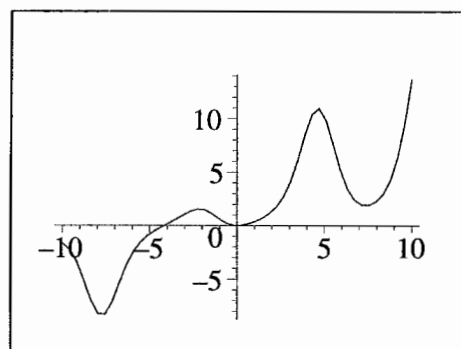
27.



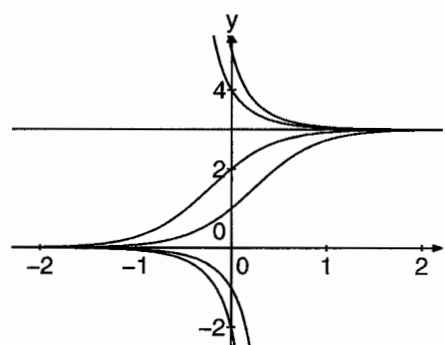
30.



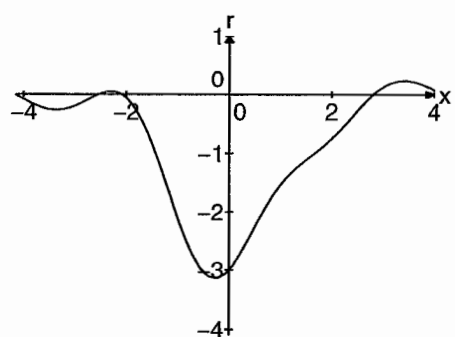
28.



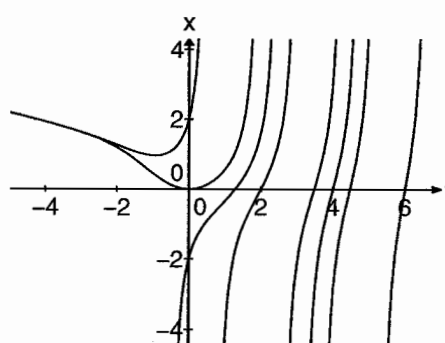
31.



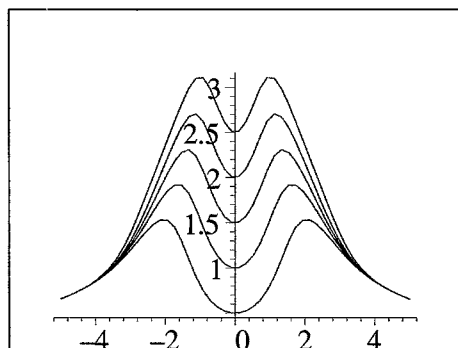
29.



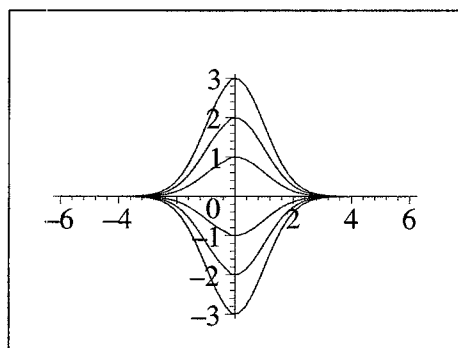
32.



33.



34.

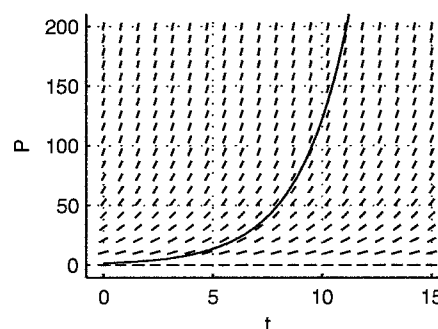


35. We must solve the initial value problem

$$\frac{dP}{dt} = 0.44P, \quad P(0) = 1.5.$$

Using our numerical solver, we input the equation and initial condition, arriving at the following solu-

tion curve.

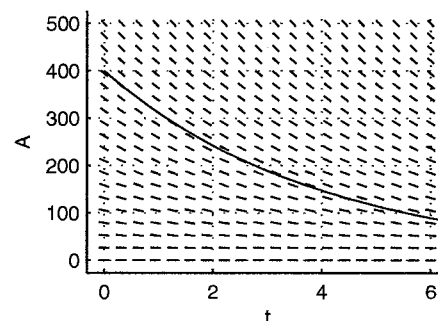


Using the solution curve, we estimate that $P(10) \approx 124$. Thus, there are approximately 124 mg of bacteria present after 10 days.

36. We must solve the initial value problem

$$\frac{dA}{dt} = -0.25A, \quad A(0) = 400.$$

Using our numerical solver, we input the equation and initial condition, arriving at the following solution curve.

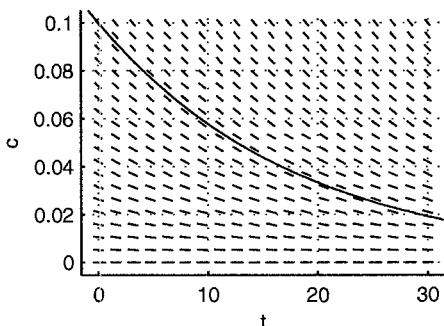


Use the solution curve to estimate $A(4) \approx 150$. Thus, there are approximately 150 mg of material remaining after 4 days.

37. We must solve the initial value problem

$$\frac{dc}{dt} = -0.055c, \quad c(0) = 0.10.$$

Using our numerical solver, we input the equation and initial condition, arriving at the following solution curve.



Use the solution curve to estimate that it takes a little more than 29 days for the concentration level to dip below 0.02.

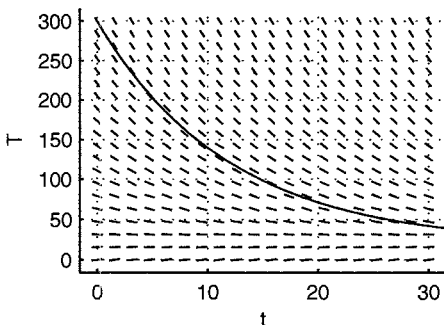
38. The rate at which the rod cools is proportional to the difference between the temperature of the rod and the surrounding air (20° Celsius). Thus,

$$\frac{dT}{dt} = -k(T - 20).$$

With $k = 0.085$ and an initial temperature of 300° Celsius, we must solve the initial value problem

$$\frac{dT}{dt} = -0.085(T - 20), \quad T(0) = 300,$$

where T is the temperature of the rod at t minutes. Note that since the initial temperature is larger than the surrounding air (20° Celsius), the minus sign insures that the model implies that the rod is cooling. Using our numerical solver, we input the equation and initial condition, arriving at the following solution curve.



Use the solution curve to estimate that it takes a little less than 15 minutes to cool to 100° Celsius.

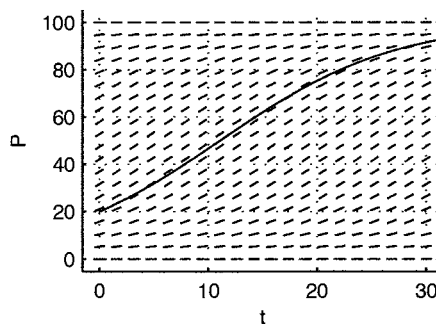
39. The rate at which the population is changing with respect to time is proportional to the product of the population and the number of critters less than the “carrying capacity” (100). Thus,

$$\frac{dP}{dt} = kP(100 - P).$$

With $k = 0.00125$ and an initial population of 20 critters, we must solve the initial value problem

$$\frac{dP}{dt} = 0.00125P(100 - P), \quad P(0) = 20.$$

Note that the right-hand side of this equation is positive if the number of critters is less than the carrying capacity (100). Thus, we have growth. Using our numerical solver, we input the equation and the initial condition, arriving at the following solution curve.



Use the solution curve to estimate that there are about 91 critters in the environment at the end of 30 days.

Section 2.2. Solutions to Separable Equations

1. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= xy \\ \frac{dy}{y} &= x dx \\ \ln |y| &= \frac{1}{2}x^2 + C \\ |y(x)| &= e^{x^2/2+C} \\ y(x) &= \pm e^C \cdot e^{x^2/2} \\ &= Ae^{x^2/2},\end{aligned}$$

Where the constant $A = \pm e^C$ is arbitrary.

2. Separate the variables and integrate.

$$\begin{aligned}x \frac{dy}{dx} &= 2y \\ \frac{1}{y} dy &= \frac{2}{x} dx \\ \ln |y| &= 2 \ln |x| + C \\ |y| &= e^{\ln x^2 + C} \\ y(x) &= \pm e^C x^2\end{aligned}$$

Letting $D = \pm e^C$, $y(x) = Dx^2$.

3. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= e^{x-y} \\ e^y dy &= e^x dx \\ e^y &= e^x + C \\ y(x) &= \ln(e^x + C)\end{aligned}$$

4. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= (1 + y^2)e^x \\ \frac{1}{1 + y^2} dy &= e^x dx \\ \tan^{-1} y &= e^x + C \\ y(x) &= \tan(e^x + C)\end{aligned}$$

5. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= y(x + 1) \\ \frac{1}{y} dy &= (x + 1) dx \\ \ln |y| &= \frac{1}{2}x^2 + x + C \\ |y| &= e^{(1/2)x^2 + x + C} \\ y(x) &= \pm e^C e^{(1/2)x^2 + x}\end{aligned}$$

Letting $D = \pm e^C$, $y = De^{(1/2)x^2 + x}$.

6. Separate the variables and integrate. *Note: Factor the right-hand side.*

$$\begin{aligned}\frac{dy}{dx} &= (e^x + 1)(y - 2) \\ \frac{1}{y - 2} dy &= (e^x + 1) dx \\ \ln |y - 2| &= e^x + x + C \\ |y - 2| &= e^{e^x + x + C} \\ y - 2 &= \pm e^C e^{e^x + x}\end{aligned}$$

Letting $D = \pm e^C$, $y(x) = De^{e^x + x} + 2$.

7. Separate the variables and integrate,

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{y + 2}, \\ (y + 2) dy &= x dx, \\ \frac{1}{2}y^2 + 2y &= \frac{1}{2}x^2 + C, \\ y^2 + 4y - (x^2 + D) &= 0,\end{aligned}$$

With D replacing $2C$ in the last step. We can use the quadratic formula to solve for y .

$$\begin{aligned}y(x) &= \frac{-4 \pm \sqrt{16 + 4(x^2 + D)}}{2} \\ y(x) &= -2 \pm \sqrt{x^2 + (D + 4)}\end{aligned}$$

If we replace $D + 4$ with another arbitrary constant E , then $y(x) = -2 \pm \sqrt{x^2 + E}$.

8. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= y \left(\frac{x}{x-1} \right) \\ \frac{1}{y} dy &= \left(1 + \frac{1}{x-1} \right) dx \\ \ln |y| &= x + \ln |x-1| + C \\ |y| &= e^{x+\ln|x-1|+C} \\ y(x) &= \pm e^C e^x e^{\ln|x-1|}\end{aligned}$$

Letting $D = \pm e^C$, $y(x) = De^x |x-1|$. It is important to note that this solution is not differentiable at $x = 1$ and further information (perhaps in the form of an initial condition) is needed to remove the absolute value and determine the interval of existence.

9. First a little algebra.

$$\begin{aligned}x^2 y' &= y \ln y - y' \\ (x^2 + 1)y' &= y \ln y\end{aligned}$$

Separate the variables and integrate.

$$\begin{aligned}\frac{1}{y \ln y} dy &= \frac{1}{x^2 + 1} dx \\ \frac{1}{u} du &= \frac{1}{x^2 + 1} dx,\end{aligned}$$

where $u = \ln y$ and $du = \frac{1}{y} dy$. Hence, $\ln |u| = \tan^{-1} x + C$. Solve for u :

$$\begin{aligned}|u| &= e^{\tan^{-1} x + C} \\ u &= \pm e^C e^{\tan^{-1} x}\end{aligned}$$

Let $D = \pm e^C$, replace u with $\ln y$, and solve for y .

$$\begin{aligned}\ln y &= D e^{\tan^{-1} x} \\ y(x) &= e^{D e^{\tan^{-1} x}}\end{aligned}$$

10.

$$\begin{aligned}x \frac{dy}{dx} &= y(1 + 2x^2) \\ \frac{dy}{y} &= \frac{1 + 2x^2}{x} dx = \left[\frac{1}{x} + 2x \right] dx \\ \ln |y| &= \ln |x| + x^2 + C \\ |y(x)| &= e^{\ln|x|+x^2+C} = e^C |x| e^{x^2} \\ y(x) &= A x e^{x^2}\end{aligned}$$

11.

$$\begin{aligned}(y^3 - 2) \frac{dy}{dx} &= x \\ (y^3 - 2) dy &= x dx \\ \frac{y^4}{4} - 2y &= \frac{x^2}{2} + C.\end{aligned}$$

The solution is given implicitly by the equation $y^4 - 8y - 2x^2 = A$, where we have set $A = 4C$.

12.

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x(y+1)}{x^2-1} \\ \frac{dy}{y+1} &= \frac{2x dx}{x^2-1} \\ \ln |y+1| &= \ln |x^2-1| + C \\ |y+1| &= e^{\ln|x^2-1|+C} = e^C |x^2-1| \\ y(x) &= A(x^2-1) - 1.\end{aligned}$$

13.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} \\ \frac{dy}{y} &= \frac{dx}{x} \\ \lambda |y| &= \ln |x| + C \\ |y(x)| &= e^{\ln|x|+C} = e^C |x| \\ y(x) &= Ax.\end{aligned}$$

The initial condition $y(1) = -2$ gives $A = -2$. The solution is $y(x) = -2x$. The solution is defined for all x , but the differential equation is not defined at $x = 0$ so the interval of existence is $(0, \infty)$.

14.

$$\begin{aligned}\frac{dy}{dt} &= -\frac{2t(1+y^2)}{y} \\ \frac{y dy}{1+y^2} &= -2t dt \\ \frac{1}{2} \ln(1+y^2) &= -t^2 + C \\ 1+y^2 &= e^{-2t^2+C} = e^C e^{-2t^2} \\ 1+y^2 &= A e^{-2t^2}\end{aligned}$$

With $y(0) = 1$, $1 + 1^2 = Ae^{-2(0)^2}$ and $A = 2$. Thus, 16.

$$1 + y^2 = 2e^{-2t^2}$$

$$y = \pm \sqrt{2e^{-2t^2} - 1}.$$

We must choose the branch that contains the initial condition $y(0) = 1$. Thus, $y = \sqrt{2e^{-2t^2} - 1}$. This solution is defined, provided that

$$2e^{-2t^2} - 1 > 0$$

$$e^{-2t^2} > \frac{1}{2}$$

$$-2t^2 > \ln \frac{1}{2}$$

$$t^2 < -2 \ln \frac{1}{2}$$

$$t^2 < \ln 4$$

$$|t| < \sqrt{\ln 4}.$$

Thus, the interval of existence is $(-\sqrt{\ln 4}, \sqrt{\ln 4})$.

15.

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$y dy = \sin x dx$$

$$\frac{1}{2} y^2 = -\cos x + C_1$$

$$y^2 = -2 \cos x + C \quad (C = 2C_1)$$

$$y(x) = \pm \sqrt{C - 2 \cos x}$$

Using the initial condition we notice that we need the plus sign, and $1 = y(\pi/2) = \sqrt{C}$. Thus $C = 1$ and the solution is

$$y(x) = \sqrt{1 - 2 \cos x}.$$

The interval of existence will be the interval containing $\pi/2$ where $2 \cos x < 1$. This is $\pi/3 < x < 5\pi/3$.

$$\frac{dy}{dx} = e^{x+y}$$

$$e^{-y} dy = e^x dx$$

$$-e^{-y} = e^x + C$$

$$e^{-y} = -e^x - C$$

$$-y = \ln(-e^x - C)$$

$$y = -\ln(-e^x - C)$$

With $y(0) = 1$,

$$1 = -\ln(-e^0 - C)$$

$$-1 - C = e^{-1}$$

$$C = -1 - e^{-1}.$$

Thus,

$$y = -\ln(-e^x + e^{-1} + 1).$$

This solution is defined provided that

$$-e^x + e^{-1} + 1 > 0$$

$$e^x < e^{-1} + 1$$

$$x < \ln(e^{-1} + 1).$$

Thus, the interval of existence is $(-\infty, \ln(e^{-1} + 1))$.

17.

$$\frac{dy}{dt} = 1 + y^2$$

$$\frac{dy}{1 + y^2} = dt$$

$$\tan^{-1}(y) = t + C$$

$$y(t) = \tan(t + C)$$

For the initial condition we have $1 = y(0) = \tan C$, so $C = \pi/4$ and the solution is $y(t) = \tan(t + \pi/4)$. Since the tangent is continuous on the interval $(-\pi/2, \pi/2)$, the solution $y(t) = \tan(t + \pi/4)$ is continuous on the interval $(-3\pi/4, \pi/4)$.

18.

$$\frac{dy}{dx} = \frac{x}{1 + 2y}$$

$$(1 + 2y) dy = x dx$$

$$y + y^2 = x^2/2 + C$$

This last equation can be written as $y^2 + y - (x^2/2 + C) = 0$. We solve for y using the quadratic formula

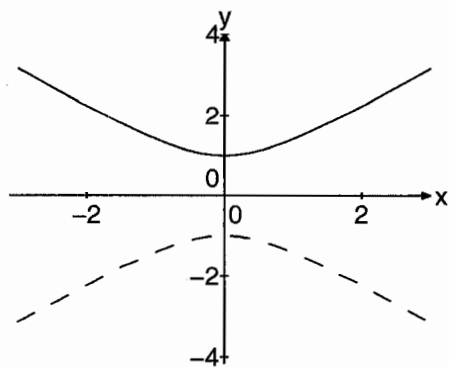
$$\begin{aligned} y(x) &= \left[-1 \pm \sqrt{1 + 4(x^2/2 + C_1)} \right] / 2 \\ &= \left[-1 \pm \sqrt{2x^2 + C} \right] / 2 \quad (C = 1 + 4C_1) \end{aligned}$$

For the initial condition $y(-1) = 0$ we need to take the plus sign in order to counter the -1 . Then the initial condition becomes $0 = [-1 + \sqrt{2 + C}]/2$, which means that $C = -1$. Thus the solution is

$$y(x) = \frac{-1 + \sqrt{2x^2 - 1}}{2}.$$

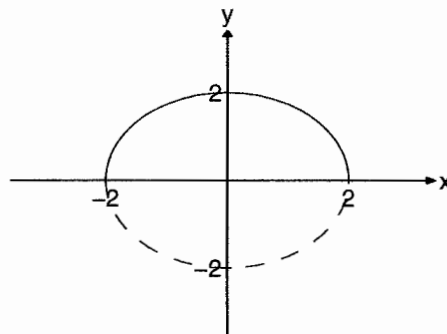
For the interval of existence we need the interval containing -1 where $2x^2 - 1 > 0$. This is $-\infty < x < -1/\sqrt{2}$.

19. With $y(0) = 1$, we get the solution $y(x) = \sqrt{1 + x^2}$, with interval of existence $(-\infty, \infty)$. This solution is plotted with the solid curve in the following figure. With $y(0) = -1$, we get the solution $y(x) = -\sqrt{1 + x^2}$, with interval of existence $(-\infty, \infty)$. This solution is plotted with the dashed curve in the following figure.



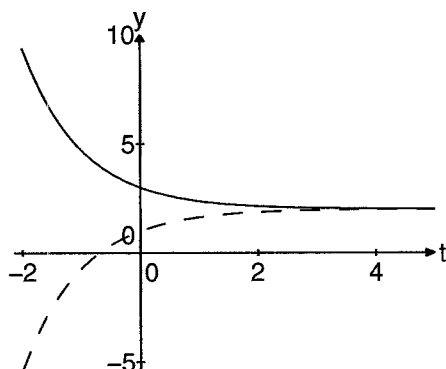
$$\begin{aligned} \frac{dy}{dx} &= -\frac{x}{y} \\ y dy &= -x dx \\ \frac{1}{2} y^2 &= -\frac{1}{2} x^2 + C \\ y^2 &= 2C - x^2 \\ y &= \pm \sqrt{2C - x^2} \end{aligned}$$

With $y(0) = 2$, we choose the positive branch and $2 = \sqrt{2C}$ leads to $C = 2$ and the solution $y = \sqrt{4 - x^2}$ with interval of existence $(-2, 2)$. This solution is plotted as a solid curve in the following figure. With $y(0) = -2$, we choose the negative branch and $-2 = -\sqrt{2C}$ leads to $C = 2$ and the solution $y = -\sqrt{4 - x^2}$ with interval of existence $(-2, 2)$. This solution is shown as a dashed curve in the figure.



21. With $y(0) = 3$ the solution is $y(t) = 2 + e^{-t}$, on $(-\infty, \infty)$. This solution is plotted in the next figure. With $y(0) = 1$ the solution is $y(t) = 2 - e^{-t}$, on $(-\infty, \infty)$. This solution is plot-

ted is the dashed curve in the next figure.



22.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y^2 + 1}{y} \\ \frac{y dy}{y^2 + 1} &= dx \\ \frac{1}{2} \ln(y^2 + 1) &= x + C \\ y^2 + 1 &= e^{2x+2C} = Ae^{2x} \quad (A = e^{2C}) \\ y^2 &= Ae^{2x} - 1 \\ y(x) &= \pm \sqrt{Ae^{2x} - 1}\end{aligned}$$

This is the general solution. The initial condition $y(1) = 2$ gives

$$\begin{aligned}2 &= +\sqrt{Ae^2 - 1} \\ 4 &= Ae^2 - 1 \\ A &= 5e^{-2}\end{aligned}$$

The particular solution is

$$y(x) = \sqrt{5e^{2x-2} - 1}.$$

The interval of existence requires that

$$\begin{aligned}5e^{2x-2} - 1 &> 0 \\ 2x - 2 &> \ln(1/5) \\ x &> 1 - \ln(5)/2 \approx 0.1953.\end{aligned}$$

Thus the interval of existence is $1 - \ln(5)/2 < x < \infty$.

23. We have $N(t) = N_0 e^{-\lambda t}$, and

$$\begin{aligned}N(t + T_{1/2}) &= N_0 e^{-\lambda(t+T_{1/2})} \\ &= N_0 e^{-\lambda t} \cdot e^{-\lambda T_{1/2}} \\ &= N(t) \cdot e^{-\lambda T_{1/2}} \\ &= N(t) \cdot \frac{1}{2}\end{aligned}$$

if $e^{-\lambda T_{1/2}} = 1/2$, or $T_{1/2} = \ln 2/\lambda$.

24. (a) $\lambda = \ln 2/T_{1/2} = 1.5507 \times 10^{-8}$.

(b) We have $N_0 = 1000$ and $N(t) = 100$. Hence $100 = 1000 \cdot e^{-\lambda t}$, or $t = \ln 10/\lambda = 1.4849 \times 10^8$ years.

25. We have $80 = N(4) = 100e^{-4\lambda}$. Hence $\lambda = \ln(100/80)/4 = 0.0558$. Then $T_{1/2} = \ln 2/\lambda = 12.4251$ hours.

26. Using $T_{1/2} = 6$ hours, we have $\lambda = \ln 2/T_{1/2} = 0.1155$. Then $N(9) = 10e^{-9\lambda} = 3.5355$ kg.

27. Using $T_{1/2} = 8.04$ days, we have $\lambda = \ln 2/T_{1/2} = 0.0862$. Then $N(20) = 500e^{-20\lambda} = 89.1537$ mg.

28. The decay constants are related to the half-lives by $\lambda_{210} = \ln(2)/2.42 = 0.2864$ and $\lambda_{211} = \ln(2)/15 = 0.0462$. The amount of ^{210}Rn is given by $x(t) = x_0 e^{-\lambda_{210}t}$ and of ^{211}Rn by $y(t) = y_0 e^{-\lambda_{211}t}$. The initial condition is that $y(0)/x(0) = y_0/x_0 = 0.2/0.8 = 1/4$, so $4x_0 = y_0$. We are looking for a time t when $0.8/0.2 = 4 = y(t)/x(t) = e^{t(\lambda_{210} - \lambda_{211})}/4$. Thus we need $e^{t(\lambda_{210} - \lambda_{211})} = 16$. From this we find that $t = 11.5$ hours.

29. (a) If $N = N_0 e^{-\lambda t}$, then substituting $T_\lambda = 1/\lambda$,

$$\begin{aligned}N &= N_0 e^{-\lambda T_\lambda} \\ &= N_0 e^{-\lambda(1/\lambda)} \\ &= N_0 e^{-1}.\end{aligned}$$

Therefore, after a period of one time constant $T_\lambda = 1/\lambda$, the material remaining is $N_0 e^{-1}$. Thus, the amount of radioactive substance has decreased to e^{-1} of its original value N_0 .

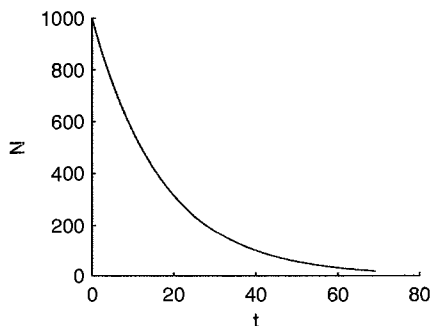
(b) If the half-life is 12 hours, then

$$\begin{aligned}\frac{1}{2}N_0 &= N_0 e^{-\lambda(12)} \\ e^{-12\lambda} &= \frac{1}{2} \\ -12\lambda &= \ln \frac{1}{2} \\ \lambda &= \frac{\ln \frac{1}{2}}{-12}.\end{aligned}$$

Hence, the time constant is

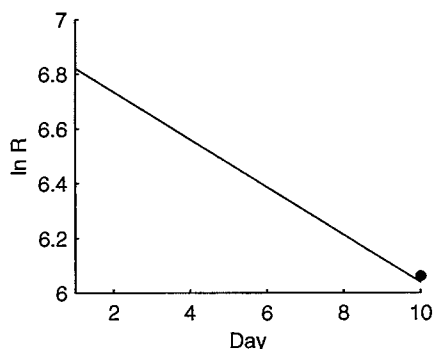
$$T_\lambda = \frac{1}{\lambda} = \frac{-12}{\ln \frac{1}{2}} \approx 17.3 \text{ hr.}$$

(c) If 1000 mg of the substance is present initially, then the amount of substance remaining as a function of time is given by $N = 1000e^{(t \ln(1/2))/12}$. The graph over four time periods ($[0, 4T_\lambda]$) follows.



30. The data is plotted on the following figure. The line is drawn using the slope found by linear regression. It has slope $-\lambda = -0.0869$. Hence the half-life is

$$T_{1/2} = \ln 2 / \lambda = 7.927 \text{ days.}$$



31. The half-life is related to the *decay constant* by

$$T_{1/2} = \frac{\ln 2}{\lambda_{226}}.$$

The *decay rate* is related to the number of atoms present by

$$R = \lambda_{226} N.$$

Substituting,

$$T_{1/2} = \frac{N \ln 2}{R}.$$

Calculate the number of atoms present in the 1 g sample.

$$\begin{aligned}N &= 1 \text{ g} \times \frac{1 \text{ mol}}{226 \text{ g}} \times \frac{6.02 \times 10^{23} \text{ atoms}}{\text{mole}} \\ &= 266 \times 10^{21} \text{ atoms.}\end{aligned}$$

Now,

$$T_{1/2} = \frac{(2.66 \times 10^{21} \text{ atoms})(\ln 2)}{3.7 \times 10^{10} \text{ atom/s}} = 4.99 \times 10^{10} \text{ s.}$$

In years, $T_{1/2} \approx 1582 \text{ yr.}$ The dedicated reader might check this result in the CRC Table.

32. (a) Because half of the existing ^{14}C decays every 5730 years, there will come a time when physical instruments can no longer measure the remaining ^{14}C . After about 10 half-lives (57300

years), the amount of original material remaining is

$$N_0 \left(\frac{1}{2}\right)^{10} \approx 0.00097N_0,$$

a very small amount.

(b) The decay constant is calculated with

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5568} \approx 0.0001245.$$

We can now write

$$N = N_0 e^{-0.0001245t}.$$

The ratio remaining is 0.617 of the current ratio, so

$$\begin{aligned} 0.617N_0 &= N_0 e^{-0.0001245t} \\ e^{-0.0001245t} &= 0.617 \\ -0.0001245t &= \ln 0.617 \\ t &= \frac{\ln 0.617}{-0.0001245} \end{aligned}$$

Thus, the charcoal is approximately 3879 years old.

33. Let $t = 0$ correspond to midnight. Thus, $T(0) = 31^\circ\text{C}$. Because the temperature of the surrounding medium is $A = 21^\circ\text{C}$, we can use $T = A + (T_0 - A)e^{-kt}$ and write

$$T = 21 + (31 - 21)e^{-kt} = 21 + 10e^{-kt}.$$

At $t = 1$, $T = 29^\circ\text{C}$, which can be used to calculate k .

$$\begin{aligned} 29 &= 21 + 10e^{-k(1)} \\ k &= -\ln(0.8) \\ k &\approx 0.2231 \end{aligned}$$

Thus, $T = 21 + 10e^{-0.2231t}$. To find the time of death, enter "normal" body temperature, $T = 37^\circ\text{C}$ and solve for t .

$$\begin{aligned} 37 &= 21 + 10e^{-0.2231t} \\ t &= \frac{\ln 1.6}{-0.2231} \\ t &\approx -2.1 \text{ hrs} \end{aligned}$$

Thus, the murder occurred at approximately 9:54 PM.

34. Let $y(t)$ be the temperature of the beer at time t minutes after being placed into the room. From Newton's law of cooling, we obtain

$$y'(t) = k(70 - y(t)) \quad y(0) = 40$$

Note k is positive since $70 > y(t)$ and $y'(t) > 0$ (the beer is warming up). This equation separates as

$$\frac{dy}{70 - y} = k dt$$

which has solution $y = 70 - Ce^{-kt}$. From the initial condition, $y(0) = 40$, $C = 30$. Using $y(10) = 48$, we obtain $48 = 70 - 30e^{-10k}$ or $k = (-1/10)\ln(11/15)$ or $k \approx .0310$. When $t = 25$, we obtain $y(25) = 70 - 30e^{-.598} \approx 56.18^\circ$.

35. The same differential equation and solution hold as in the previous problem:

$$y(t) = 70 - Ce^{-kt}$$

We let $t = 0$ correspond to when the beer was discovered, so $y(0) = 50$. This means $C = 20$. We also have $y(10) = 60$ or

$$60 = 70 - 20e^{-10k}$$

Therefore, $k = (-1/10)\ln(1/2) \approx .0693$. We want to find the time T when $y(T) = 40$, which gives the equation

$$70 - 20e^{-kT} = 40$$

Since we know k , we can solve this equation for T to obtain

$$T = (-1/k)\ln(3/2) \approx -5.85$$

or about 5.85 minutes before the beer was discovered on the counter.

36. $x' = [at + by + c]' = a + by' = a + bf(at + by + c) = a + f(x)$. For the equation $y' = (y + t)^2$ we use $x = t + y$. Then $x' = 1 + y' = 1 + (y + t)^2 = 1 + x^2$. Solving this separable equation in the usual way we get the general solution $x(t) = \tan(t + C)$. In terms of the unknown y , we get $y(t) = x(t) - t = \tan(t + C) - t$.

37. The tangent line at the point (x, y) is $\hat{y} - y = y'(x)(\hat{x} - x)$ (the variables for the tangent line have the hats). The $\hat{x}$ intercept is $\hat{x}_{\text{int}} = -y/y' + x$. Since (x, y) bisects the tangent line, we have $\hat{x}_{\text{int}} = 2x$. Therefore

$$2x = \frac{-y}{y'} + x$$

or

$$\frac{y'}{y} = \frac{-1}{x}$$

This separable differential equation is easily solved to obtain $y(x) = C/x$, where C is an arbitrary constant.

38. With the notation as in the previous problem, the equation of the normal line is

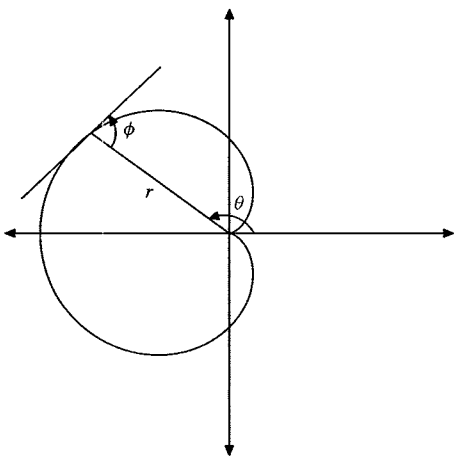
$$\hat{y} - y = \frac{-1}{y'(x)}(\hat{x} - x)$$

The $\hat{x}$ -intercept is found to be $\hat{x}_{\text{int}} = yy' + x$. Since $\hat{x}_{\text{int}} - x$ is given to be ± 1 , we obtain

$$yy' = \pm 1$$

with solution $y^2 = \pm 4x + C$, where C is an arbitrary constant.

39. Let ϕ be the angle from the radius to the tangent.



From geometry, $\tan \phi = r d\theta/dr$. Since $\theta = 2\phi$, we obtain

$$\frac{dr}{d\theta} = r \cot \phi = r \cot(\theta/2)$$

which can be separated as $dr/r = \cot(\theta/2)d\theta$. This can be solved for r as $r(\theta) = C \sin^2(\theta/2)$, where C is a constant.

40. The area under the curve $y = y(t)$ from 0 to x is

$$\int_0^x y(t) dt$$

which by assumption, equals $(1/4)xy(x)$ (one-fourth the area of the rectangle). Therefore

$$\int_0^x y(t) dt = (1/4)xy(x).$$

Differentiating this equation with respect to x and using the Fundamental Theorem of Calculus for the left side gives

$$y(x) = (1/4)(y(x) + xy'(x)).$$

This equation separates as

$$\frac{y'}{y} = \frac{3}{x}$$

which has the solution $y(x) = Cx^3$.

41. Center the football at the origin with equation

$$z^2 + x^2 + \frac{y^2}{4} = 4.$$

The top half of the football is the graph of the function

$$z = \sqrt{4 - x^2 - y^2/4}.$$

The (x, y) -components of the path of a rain drop form a curve in the (x, y) -plane which must always point in the direction of the gradient of the function z (the path of steepest descent). The gradient of z is given by

$$\nabla z = \frac{-xi - (y/4)j}{\sqrt{4 - x^2 - y^2/4}}$$

where i and j are the unit vectors parallel to the x and y -axis. Since the path traced by the drop, $y = y(x)$, must point in the direction of ∇z , we must have

$$\frac{dy}{dx} = \text{slope of the gradient} = \frac{z_y}{z_x} = \frac{y}{4x}.$$

This differential equation can be separated and solved as $x = Cy^4$. C can be solved from the initial position of the drop (x_0, y_0) to be $C = x_0/y_0^4$. The final answer is given by inserting $x = Cy^4$ into the expression for z :

$$(x, y, z) = (Cy^4, y, \sqrt{4 - C^2y^8 - y^2/4}),$$

(here, y is the independent variable).

42. Let $y(t)$ be the level of the water and let $V(t)$ be the volume of the water in the bowl at time t . On the one hand, we have $dV/dt =$ cross sectional area of the water level $\times dy/dt$. The cross sectional area of the water level is πx^2 , where x is the radius. Using the equation of the side of the bowl ($y = x^2$), we obtain

$$\frac{dV}{dt} = \pi y(t) \frac{dy}{dt}.$$

On the other hand, dV/dt is equal to $\pi a^2 v$ where v is the speed of the water exiting the bowl. From the hint, $v = \sqrt{2gy(t)}$. Thus we obtain the following differential equation:

$$\pi y(t) \frac{dy}{dt} = \frac{dV}{dt} = -\pi a^2 \sqrt{2gy(t)}.$$

This equation can be separated and solved for y as

$$y(t) = \left(C - \frac{3}{2} a^2 \sqrt{2g} t \right)^{2/3}$$

Since $y(0) = 1$, we obtain $C = 1$. Setting $y(t) = 0$, we obtain

$$t = \frac{2}{3a^2 \sqrt{2g}}$$

in units of seconds (here $g = 32$).

43. Let the unknown curve forming the outside of the bowl be given by $y = y(x)$ (the bowl is then formed by revolving this curve around the y -axis). We can also write this equation as $x = x(y)$ (reversing the roles of the independent and dependent variables). As in the analysis of the last problem, the rate of

change of volume, dV/dt is the cross sectional area multiplied by the rate of change in height, dy/dt . The cross sectional area is $\pi x^2 = \pi x(y)^2$. Thus

$$\frac{dV}{dt} = \pi x^2 \frac{dy}{dt}$$

From Torricelli's law:

$$\frac{dV}{dt} = -\pi a^2 v = -\pi a^2 \sqrt{2gy}$$

Since $dy/dt = C$ (a negative constant), we obtain

$$C\pi x^2 = -\pi a^2 \sqrt{2gy}$$

Solving for y , we obtain $y = Kx^4$ where K is a constant.

44. Following the hint, let θ be the polar angle and locate the destroyer at 4 miles along the positive x -axis. The destroyer wants to follow a path so that its arc length is always three times that of the sub. To account for the possibility that the sub heads straight along the positive x -axis, the destroyer should first head from $x = 4$ to $x = 1$ (the sub would move from $x = 0$ to $x = 1$ in this same time frame under this scenario). Now the destroyer must circle around the sub along a polar coordinate path $r = r(\theta)$. We have $r(0) = 1$. If the destroyer intersects the sub at $(\theta, r(\theta))$, then the sub will have traveled $r(\theta)$ and the destroyer would have traveled $\int_0^\theta \sqrt{r'(t)^2 + r(t)^2} d\theta$ (arc length along the curve $r = r(\theta)$). Since the speed of the destroyer is three times that of the sub, we obtain

$$3(r(\theta) - 1) = \int_0^\theta \sqrt{r'(t)^2 + r(t)^2} d\theta.$$

Differentiating this equation gives

$$3r' = \sqrt{r'^2 + r^2} \quad \text{or} \quad \frac{dr}{d\theta} = \frac{r(\theta)}{\sqrt{8}} \quad r(0) = 1$$

with solution $r(\theta) = e^{\theta/\sqrt{3}}$.

Section 2.3. Models of Motion

1. We need $gt = c/5$, or $t = c/5g = 612,240$ seconds. The distance traveled will be $s = gt^2/2 = 1.84 \times 10^{14}$ meters.
2. We need $0 = -9.8t^2/2 + 15t + 100$. The answer is 6.3 seconds.
3. The depth of the well satisfies $d = 4.9t^2$, where t is the amount of time it takes the stone to hit the water. It also satisfies $d = 340s$, where $s = 8 - t$ is the amount of time it takes for the noise of the splash to reach the ear. Thus we must solve the quadratic equation $4.9t^2 = 340(8 - t)$. The solution is $t = 7.2438$ sec. The depth is $d = 340(8 - t) = 257.1$ m.
4. In the first 60s the rocket rises to an elevation of $(100 - 9.8)t^2/2 = 162,360$ m and achieves a velocity of $v(60) = (100 - 9.8) \cdot 60 = 5412$ m/s. After that the velocity is $5412 - 9.8t$. This is zero at the highest point, reached when $t_1 = 552.2$ s. The altitude at that point is $162,360 + 5412t_1 - 9.8t_1^2/2 = 1.657 \times 10^6$ m. From there to the ground it takes t_2 s, where $4.9t_2^2 = 1.657 \times 10^6$, or $t_2 = 581.5$ s. The total trip takes $60 + 552.2 + 581.5 = 1193.7$ s.
5. The distance dropped in time t is $4.9t^2$. If T is the time taken for the first half of the trip, then $4.9(T+1)^2 = 2 \times 4.9T^2$, or $4.9(T^2 - 2T - 1) = 0$. Solving we find that $T = 1 + \sqrt{2} = 2.4142$ s. So the body fell $2 \times 4.9T^2 = 57.12$ m, and it took $T + 1 = 3.4142$ s.
6. (a) $v_0^2/2g$
(b) Both times are equal to v_0/g .
(c) v_0 .
7. The velocities must be changed to ft/s, so $v_0 = 60$ mi/h $= 60 \times 5280/3600 = 88$ ft/s, and $v = 30$ mi/h $= 44$ ft/s. Then $a = (v^2 - v_0^2)/2(x - x_0) = -5.8$ ft/s².
8. We have $v(t) = Ce^{-rt/m} - mg/r$. If $v(0) = 0$ then $C = mg/r$, and $v(t) = mg(e^{-rt/m} - 1)/r$. This is equal to $-mg/2r$ when $e^{-rt/m} = 1/2$. Thus the time

required is $t = m \ln(2)/r$. The distance traveled is

$$\begin{aligned} x &= \int_0^t v(s) ds \\ &= \frac{mg}{r} \int_0^t (e^{-rs/m} - 1) ds \\ &= \frac{mg}{r} \left[\frac{m}{r} (1 - e^{-rt/m}) - t \right] \\ &= \frac{mg}{r} \left[\frac{m}{2r} - \frac{m \ln 2}{r} \right] \\ &= \frac{m^2 g}{r^2} \left[\frac{1}{2} - \ln 2 \right] \end{aligned}$$

9. The resistance force has the form $R = -rv$. When $v = 0.2$, $R = -1$ so $r = 5$. The terminal velocity is $v_{\text{term}} = -mg/r = -0.196$ m/s.
10. (a) First, the terminal velocity gives us $20 = mg/r$, or $r = mg/20 = 70 \times 9.8/20 = 34.3$. Next, we have $v(t) = Ce^{-rt/m} - mg/r$. Since $v(0) = 0$, $C = mg/r$, and $v(t) = mg(e^{-rt/m} - 1)/r$. Integrating and setting $x(0) = 0$, we get

$$\begin{aligned} x &= \int_0^t v(s) ds \\ &= \frac{mg}{r} \int_0^t (e^{-rs/m} - 1) ds \\ &= \frac{mg}{r} \left[\frac{m}{r} (1 - e^{-rt/m}) - t \right] \end{aligned}$$

Hence $v(2) = -12.4938$ and $x(2) = -14.5025$.

- (b) The velocity is 80% of its terminal velocity when $1 - e^{-rt/m} = 0.8$. For the values of $m = 70$ and $r = 34.3$ this becomes $t = 3.2846$ s.
11. Without air resistance, $v_0 = \sqrt{2 \times 13.5g} = 16.2665$ m/s. With air resistance, v_0 is defined by

$$\int_{v_0}^0 \frac{v dv}{v + mg/r} = -\frac{r}{m} \int_{1.5}^{15} dy.$$

Hence,

$$\begin{aligned} &-v_0 + (mg/r) \ln(v_0 + mg/r) - (mg/r) \ln(mg/r) \\ &= -13.5 \frac{r}{m} \quad \text{or} \\ &-v_0 + 49 \ln(v_0 + 49) - 49 \ln(49) = -2.7 \end{aligned}$$

This is an implicit equation for v_0 . Solving on a calculator or a computer yields $v_0 = 18.1142\text{m/s}$.

12. The impact velocity v_i is defined by

$$\int_0^{v_i} \frac{v dv}{v + mg/r} = -\frac{r}{m} \int_{50}^0 dy$$

From which we get

$$\begin{aligned} v_i - (mg/r) \ln(v_i + mg/r) + (mg/r) \ln(mg/r) \\ = -50 \frac{r}{m}, \quad \text{or} \\ v_i - 19.6 \ln(v_i + 19.6) + 19.6 \ln(19.6) = -25 \end{aligned}$$

This is an implicit equation for v_i . Solving on a calculator or a computer yields $v_i = -17.3401\text{m/s}$.

13. Following the lead of Exercise 11, we find that

$$v dv = (-g + R(v)/m) dy = (-9.8 - 0.5v^2) dy$$

Hence if y_1 is the maximum height we have

$$\int_{230}^0 \frac{v dv}{v^2 + 19.6} = -0.5 \int_0^{y_1} dy.$$

Hence

$$\begin{aligned} y_1 &= \log(v^2 + 19.6) \Big|_0^{230} \\ &= 7.9010. \end{aligned}$$

14. (a) Follows from $a = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy}$.

(b)

$$\begin{aligned} v dv &= -\frac{GM}{(R+y)^2} dy \\ \int_{v_0}^v v dv &= -\int_0^y \frac{GM}{(R+s)^2} ds \\ \frac{1}{2}(v^2 - v_0^2) &= -GM \left(\frac{1}{R} - \frac{1}{R+y} \right) \\ v^2 &= v_0^2 - 2GM \left(\frac{1}{R} - \frac{1}{R+y} \right) \end{aligned}$$

- (c) If y is the maximum height, the corresponding velocity is $v = 0$, so from (3.16)

$$0 = v_0^2 - 2GM \left(\frac{1}{R} - \frac{1}{R+y} \right).$$

Solving for y we get the result.

- (d) If $v_0 < \sqrt{2GM/R}$, then $v_0^2 < 2GM/R$, and $2GM/R - v_0^2 > 0$. Hence by (c) the object has a finite maximum height and does not escape. However, when $v_0 = \sqrt{2GM/R}$, $2GM/R - v_0^2 = 0$, and there is no maximum height.

15. Let $x(t)$ be the distance from the mass to the center of the Earth. The force of gravity is kx (proportional to the distance from the center of the Earth). Since the force of gravity at the surface (when $x = R$) is $-mg$, we must have $k = -mg/R$. Newton's law becomes

$$m \frac{d^2x}{dt^2} = \frac{-mgx}{R}$$

Using the reduction of order technique as given in the hint, we obtain

$$v \frac{dv}{dx} = \frac{-gx}{R}$$

which can be separated with a solution given by $v = \sqrt{C - gx^2/R}$. The constant C can be evaluated from the initial condition, $v(x = R) = 0$, to be $C = gR$. When $x = 0$ (the center of the Earth), we obtain $v = \sqrt{C} = \sqrt{gR}$ or approximately 4.93 miles per second.

16. We will use $GM = gR^2$. Once more we use

$$\begin{aligned} v dv &= -\frac{GM}{(R+y)^2} dy \\ \int_0^v s ds &= -GM \int_a^0 \frac{dy}{(R+y)^2} \\ v^2 &= 2GM \left[\frac{1}{R} - \frac{1}{R+a} \right] \\ &= \frac{2agR}{a+R} \end{aligned}$$

17. The force acting on the chain is the force of gravity applied to the piece of the chain that hangs off

the table. This force is $mgx(t)$ where m is the mass density of the chain. Newton's law gives

$$mx''(t) = mgx(t)$$

Using the hint, $x''(t) = dv/dt = v(dv/dx)$ and so this equation becomes

$$v \frac{dv}{dx} = gx$$

Separating this equation and integrating gives $v^2 = gx^2 + C$. Since $v = 0$ when $x = 2$ (initial velocity is zero), we obtain $C = -4g$. Therefore

$$v = \sqrt{g(x^2 - 4)}.$$

Since $dx/dt = v = \sqrt{g(x^2 - 4)}$, we can separate this equation and integrate to obtain

$$\ln(x + \sqrt{x^2 - 4}) = \sqrt{g}t + K$$

where K is a constant. From the initial condition that $x = 2$ when $t = 0$, we obtain $K = \ln 2$. Inserting $x = 10$ and solving for t , we obtain

$$t = \frac{1}{\sqrt{g}} \ln \left(\frac{10 + \sqrt{96}}{2} \right) \approx .405 \text{ seconds}$$

18. $v = -g - (k/m)v$, v = velocity. Velocity on impact is 58.86 meters/per second downward and 90.9 seconds until he hits the ground.

19. Let x be the height of the parachuter and let v be his velocity. The resistance force is proportional to v and to e^{-ax} . Hence it is given by $R(x, v) = -ke^{ax}v$, where k is a positive constant. Newton's second law gives us $mx'' = -mg - ke^{ax}x'$, or $mx'' + ke^{ax}x' + mg = 0$.

Section 2.4. Linear Equations

1. Compare $y' = -y + 2$ with $y' = a(t)y + f(t)$ and note that $a(t) = -1$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t) dt} = e^{\int 1 dt} = e^t.$$

Multiply both sides of our equation by this integrating factor and check that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} e^t(y' + y) &= 2e^t \\ (e^t y)' &= 2e^t \end{aligned}$$

Integrate and solve for y .

$$\begin{aligned} e^t y &= 2e^t + C \\ y(t) &= 2 + Ce^{-t} \end{aligned}$$

2. We have $a(t) = 3$, so $u(t) = e^{-3t}$. Multiplying we see that the equation becomes

$$e^{-3t}y' - 3e^{-3t}y = 5e^{-3t}.$$

We verify that the left-hand side is the derivative of $e^{-3t}y$, so when we integrate we get

$$e^{-3t}y(t) = -\frac{5}{3}e^{-3t} + C.$$

Solving for y , we get

$$y(t) = -\frac{5}{3} + Ce^{3t}.$$

3. Compare $y' + (2/x)y = (\cos x)/x^2$ with $y' = a(x)y + f(x)$ and note that $a(x) = -2/x$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x) dx} = e^{\int 2/x dx} = e^{2 \ln |x|} = |x|^2 = x^2.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting

equation is the derivative of a product.

$$\begin{aligned}x^2 \left(y' + \frac{2}{x} y \right) &= \cos x \\x^2 y' + 2xy &= \cos x \\(x^2 y)' &= \cos x\end{aligned}$$

Integrate and solve for x .

$$\begin{aligned}x^2 y &= \sin x + C \\y(x) &= \frac{\sin x + C}{x^2}\end{aligned}$$

4. We have $a(t) = -2t$, so $u(t) = e^{t^2}$. Multiplying by u , the equation becomes

$$e^{t^2} y' + 2te^{t^2} y = 5te^{t^2}.$$

We verify that the left-hand side is the derivative of $e^{t^2} y$, so when we integrate we get

$$e^{t^2} y(t) = \frac{5}{2} e^{t^2} + C.$$

Solving for y we get the general solution

$$y(t) = \frac{5}{2} + Ce^{-t^2}.$$

5. Compare $x' - 2x/(t+1) = (t+1)^2$ with $x' = a(t)x + f(t)$ and note that $a(t) = -2/(t+1)$. Consequently, an integrating factor is found with

$$\begin{aligned}u(t) &= e^{\int -a(t) dt} = e^{\int -2/(t+1) dt} \\&= e^{-2 \ln |t+1|} = |t+1|^{-2} = (t+1)^{-2}.\end{aligned}$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned}(t+1)^{-2} \left(x' - \frac{2}{t+1} x \right) &= 1 \\((t+1)^{-2} x)' &= 1\end{aligned}$$

Integrate and solve for x .

$$\begin{aligned}(t+1)^{-2} x &= t + C \\x(t) &= t(t+1)^2 + C(t+1)^2\end{aligned}$$

6. If we write the equations as $x' = (4/t)x + t^3$, we see that $a(t) = 4/t$. Thus the integrating factor is

$$u(t) = e^{-\int (4/t) dt} = e^{-4 \ln t} = t^{-4}.$$

Multiplying by u , the equation becomes

$$t^{-4} x' - 4t^{-5} x = t^{-1}.$$

After verifying that the left-hand side is the derivative of $t^{-4}x$, we can integrate and get

$$t^{-4} x(t) = \ln t + C.$$

Hence the general solution is

$$x(t) = t^4 \ln t + Ct^4.$$

7. Divide both sides by $1+x$ and solve for y' .

$$y' = -\frac{1}{1+x} y + \frac{\cos x}{1+x}$$

Compare this result with $y' = a(x)y + f(x)$ and note that $a(x) = -1/(1+x)$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x) dx} = e^{\int 1/(1+x) dx} = e^{\ln |1+x|} = |1+x|.$$

If $1+x > 0$, then $|1+x| = 1+x$. If $1+x < 0$, then $|1+x| = -(1+x)$. In either case, if we multiply both sides of our equation by either integrating factor, we arrive at

$$(1+x)y' + y = \cos x.$$

Check that the left-hand side of this result is the derivative of a product, integrate, and solve for y .

$$\begin{aligned}((1+x)y)' &= \cos x \\(1+x)y &= \sin x + C \\y(x) &= \frac{\sin x + C}{1+x}\end{aligned}$$

8. Divide by $1+x^3$ to put the equation into normal form

$$y' = \frac{3x^2}{1+x^3} y + x^2.$$

We see that $a(x) = 3x^2/(1+x^3)$. Hence the integrating factor is

$$u(x) = e^{-\int 3x^2/(1+x^3) dx} = e^{-\ln(1+x^3)} = \frac{1}{1+x^3}.$$

Multiplying by this we get

$$\frac{1}{1+x^3} y' - \frac{3x^2}{(1+x^3)^2} y = \frac{x^2}{1+x^3}.$$

We first verify that the left-hand side is the derivative of $(1+x^3)^{-1}y$. Then we integrate, getting

$$\frac{1}{1+x^3} y(x) = \frac{1}{3} \ln(1+x^3) + C.$$

Solving for y , we get

$$y(x) = \frac{1}{3(1+x^3)\ln(1+x^3)} + C(1+x^3).$$

9. Divide both side of this equation by L and solve for di/dt .

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{E}{L}$$

Compare this with $i' = a(t)i + f(t)$ and note that $a(t) = -R/L$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t) dt} = e^{\int R/L dt} = e^{Rt/L}$$

Multiply both sides of our equation by this integrating factor and note that the resulting left-hand side is the derivative of a product.

$$e^{Rt/L} \left(\frac{di}{dt} + \frac{R}{L}i \right) = \frac{E}{L} e^{Rt/L}$$

$$(e^{Rt/L} i)' = \frac{E}{L} e^{Rt/L}$$

Integrate and solve for i .

$$e^{Rt/L} i = \frac{E}{R} e^{Rt/L} + C$$

$$i(t) = \frac{E}{R} + C e^{-Rt/L}$$

10. Compare $y' = my + c_1 e^{mx}$ with $y' = a(x)y + f(x)$ and note that $a(x) = m$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x) dx} = e^{\int -m dx} = e^{-mx}.$$

Multiply both sides of the differential equation by the integrating factor and check that the resulting left-hand side is the derivative of a product.

$$y' - my = c_1 e^{mx}$$

$$e^{-mx}(y' - m e^{-mx}) = c_1$$

$$(e^{-mx} y)' = c_1$$

Integrate and solve for y .

$$e^{-mx} y = c_1 x + c_2$$

$$y = (c_1 x + c_2) e^{mx}$$

11. Compare $y' = \cos x - y \sec x$ with $y' = a(x)y + f(x)$ and note that $a(x) = -\sec x$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x) dx} = e^{\int \sec x dx}$$

$$= e^{\ln |\sec x + \tan x|} = |\sec x + \tan x|.$$

If $\sec x + \tan x > 0$, then $|\sec x + \tan x| = \sec x + \tan x$. If $\sec x + \tan x < 0$, then $|\sec x + \tan x| = -(\sec x + \tan x)$. In either case, when we multiply both sides of the differential equation by this integrating factor, we arrive at

$$(\sec x + \tan x)(y' + y \sec x) = \cos x(\sec x + \tan x),$$

or

$$(\sec x + \tan x)y' + (\sec^2 x + \sec x \tan x)y = 1 + \sin x$$

Again, check that the left-hand side of this equation is the derivative of a product, then integrate and solve for y .

$$((\sec x + \tan x)y)' = 1 + \sin x$$

$$(\sec x + \tan x)y = x - \cos x + C$$

$$y = \frac{x - \cos x + C}{\sec x + \tan x}$$

12. Compare $x' - (n/t)x = e^t t^n$ with $x' = a(t)x + f(t)$ and note that $a(t) = n/t$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t) dt} = e^{\int -n/t dt} = e^{-n \ln |t|} = |t|^{-n}.$$

Depending on the sign of t and whether n is even or odd, $|t|^{-n}$ either equals t^{-n} or $-t^{-n}$. In either case, when we multiply our equation by either of these integrating factors, we arrive at

$$t^{-n}x' - nt^{-n-1}x = e^t.$$

Note that the left-hand side of this result is the derivative of a product, integrate, and solve for x .

$$\begin{aligned}(t^{-n}x)' &= e^t \\ t^{-n}x &= e^t + C \\ x &= t^n e^t + Ct^n\end{aligned}$$

13. (a) Compare $y' + y \cos x = \cos x$ with $y' = a(x)y + f(x)$ and note that $a(x) = -\cos x$. Consequently, an integrating factor is found with

$$u(x) = e^{-\int a(x) dx} = e^{\int \cos x dx} = e^{\sin x}.$$

Multiply both sides of the differential equation by the integrating factor and check that the resulting left-hand side is the derivative of a product.

$$\begin{aligned}e^{\sin x}(y' + y \cos x) &= e^{\sin x} \cos x \\ (e^{\sin x} y)' &= e^{\sin x} \cos x\end{aligned}$$

Integrate and solve for y .

$$\begin{aligned}e^{\sin x} y &= e^{\sin x} + C \\ y(x) &= 1 + Ce^{-\sin x}\end{aligned}$$

- (b) Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= \cos x(1 - y) \\ \frac{dy}{1 - y} &= \cos x dx \\ -\ln|1 - y| &= \sin x + C.\end{aligned}$$

Take the exponential of each side.

$$\begin{aligned}|1 - y| &= e^{-\sin x - C} \\ 1 - y &= \pm e^{-C} e^{-\sin x}\end{aligned}$$

If we let $A = \pm e^{-C}$, then

$$y(x) = 1 - Ae^{-\sin x},$$

where A is any real number, except zero. However, when we separated the variables above by dividing by $y - 1$, this was a valid operation only if $y \neq 1$. This hints at another solution. Note that $y = 1$ easily checks in the original equation. Consequently,

$$y(x) = 1 - Ae^{-\sin x},$$

where A is any real number. Note that this will produce the same solutions as $y = 1 + Ce^{-\sin x}$, C any real number, the solution found in part (a).

14. Compare $y' = y + 2xe^{2x}$ with $y' = a(x)y + f(x)$ and note that $a(x) = 1$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x) dx} = e^{\int -1 dx} = e^{-x}.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned}e^{-x}y' - e^{-x}y &= 2xe^x \\ (e^{-x}y)' &= 2xe^x\end{aligned}$$

Integration by parts yields

$$\int 2xe^x dx = 2xe^x - \int 2e^x = 2xe^x - 2e^x + C.$$

Consequently,

$$\begin{aligned}e^{-x}y &= 2xe^x - 2e^x + C \\ y(x) &= 2xe^{2x} - 2e^{2x} + ce^x\end{aligned}$$

The initial condition provides

$$3 = y(0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^0 = -2 + C.$$

Consequently, $C = 5$ and $y(x) = 2xe^{2x} - 2e^{2x} + 5e^x$.

15. Solve for y' .

$$y' = -\frac{3x}{x^2 + 1}y + \frac{6x}{x^2 + 1}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = -3x/(x^2 + 1)$. Consequently, an integrating factor is found with

$$\begin{aligned} u(x) &= e^{\int -a(x) dx} = e^{\int 3x/(x^2+1) dx} \\ &= e^{(3/2)\ln(x^2+1)} = (x^2 + 1)^{3/2}. \end{aligned}$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (x^2 + 1)^{3/2} y' + 3x(x^2 + 1)^{1/2} y &= 6x(x^2 + 1)^{1/2} \\ ((x^2 + 1)^{3/2} y)' &= 6x(x^2 + 1)^{1/2} \end{aligned}$$

Integrate and solve for y .

$$\begin{aligned} (x^2 + 1)^{3/2} y &= 2(x^2 + 1)^{3/2} + C \\ y &= 2 + C(x^2 + 1)^{-3/2} \end{aligned}$$

The initial condition gives

$$-1 = y(0) = 2 + C(0^2 + 1)^{-3/2} = 2 + C.$$

Therefore, $C = -3$ and $y(x) = 2 - 3(x^2 + 1)^{-3/2}$.

16. Solve for y' .

$$y' = -\frac{4t}{1+t^2} y + \frac{1}{1+t^2}$$

Compare this with $y' = a(t)y + f(t)$ and note that $a(t) = -4t/(1+t^2)$. Consequently, an integrating factor is found with

$$\begin{aligned} u(t) &= e^{\int -a(t) dt} = e^{\int 4t/(1+t^2) dt} \\ &= e^{2\ln|1+t^2|} = (1+t^2)^2. \end{aligned}$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (1+t^2)^2 y' + 4t(1+t^2) y &= \frac{1}{1+t^2} \\ ((1+t^2)^2 y)' &= \frac{1}{1+t^2} \\ (1+t^2)^2 y &= \tan^{-1} t + C \end{aligned}$$

The initial condition $y(1) = 0$ gives

$$(1+1^2)^2(0) = \tan^{-1} 1 + C.$$

Consequently, $C = -\pi/4$ and

$$y(t) = \frac{\tan^{-1} t - \frac{\pi}{4}}{(1+t^2)^2}.$$

17. Compare $x' + x \cos t = (1/2) \sin 2t$ with $x' = a(t)x + f(t)$ and note that $a(t) = -\cos t$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t) dt} = e^{\int \cos t dt} = e^{\sin t}.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} e^{\sin t} x' + e^{\sin t} (\cos t) x &= \frac{1}{2} e^{\sin t} \sin 2t \\ (e^{\sin t} x)' &= \frac{1}{2} e^{\sin t} \sin 2t \end{aligned}$$

Use $\sin 2t = 2 \sin t \cos t$.

$$(e^{\sin t} x)' = e^{\sin t} \sin t \cos t$$

Let $u = \sin t$ and $dv = e^{\sin t} \cos t dt$. Then,

$$\begin{aligned} \int e^{\sin t} \cos t \sin t dt &= \int u dv \\ &= uv - \int v du \\ &= (\sin t) e^{\sin t} - \int e^{\sin t} \cos t dt \\ &= (\sin t) e^{\sin t} - e^{\sin t} + C \end{aligned}$$

Therefore,

$$\begin{aligned} e^{\sin t} x &= e^{\sin t} \sin t - e^{\sin t} + C \\ x(t) &= \sin t - 1 + C e^{-\sin t} \end{aligned}$$

The initial condition gives

$$1 = x(0) = \sin(0) - 1 + C e^{-\sin(0)} = -1 + C.$$

Consequently, $C = 2$ and $x(t) = \sin t - 1 + 2e^{-\sin t}$.

18. Solve
- $xy' + 2y = \sin x$
- for
- y'
- .

$$y' = -\frac{2}{x}y + \frac{\sin x}{x}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = -2/x$ and $f(x) = (\sin x)/x$. It is important to note that neither a nor f is continuous at $x = 0$, a fact that will heavily influence our interval of existence.

An integrating factor is found with

$$u(x) = e^{\int -a(x) dx} = e^{\int 2/x dx} = e^{2 \ln |x|} = |x|^2 = x^2.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} x^2 y' + 2xy &= x \sin x \\ (x^2 y)' &= x \sin x \end{aligned}$$

Integration by parts yields

$$\begin{aligned} \int x \sin x dx &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + C. \end{aligned}$$

Consequently,

$$\begin{aligned} x^2 y &= -x \cos x + \sin x + C, \\ y &= -\frac{1}{x} \cos x + \frac{1}{x^2} \sin x + \frac{C}{x^2}. \end{aligned}$$

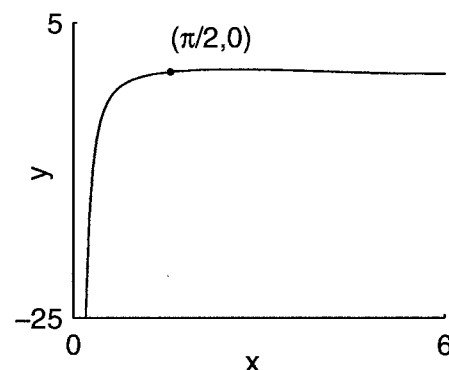
The initial condition provides

$$0 = y(\pi/2) = \frac{4}{\pi^2} + \frac{4C}{\pi^2}.$$

Consequently, $C = -1$ and $y = -(1/x) \cos x + (1/x^2) \sin x - 1/x^2$.

We cannot extend any interval to include $x = 0$, as our solution is undefined there. The initial condition $y(\pi/2) = 0$ forces the solution through a point with $x = \pi/2$, a fact which causes us to select $(0, +\infty)$ as the interval of existence. The solution curve is shown in the following figure. Note how it drops to

negative infinity as x approaches zero from the right.



19. Solve for
- y'
- .

$$y' = \frac{1}{2x+3}y + (2x+3)^{-1/2}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = 1/(2x+3)$ and $f(x) = (2x+3)^{-1/2}$. It is important to note that a is continuous everywhere except $x = -3/2$, but f is continuous only on $(-3/2, +\infty)$, facts that will heavily influence our interval of existence.

An integrating factor is found with

$$\begin{aligned} u(x) &= e^{\int -a(x) dx} = e^{\int -1/(2x+3) dx} \\ &= e^{-(1/2) \ln |2x+3|} = |2x+3|^{-1/2}. \end{aligned}$$

However, we will assume that $x > -3/2$ (a domain where both a and f are defined), so $u(x) = (2x+3)^{-1/2}$. Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (2x+3)^{-1/2} y' - (2x+3)^{-3/2} y &= (2x+3)^{-1} \\ ((2x+3)^{-1/2} y)' &= (2x+3)^{-1} \end{aligned}$$

Integrate and solve for y .

$$(2x+3)^{-1/2} y = \frac{1}{2} \ln(2x+3) + C,$$

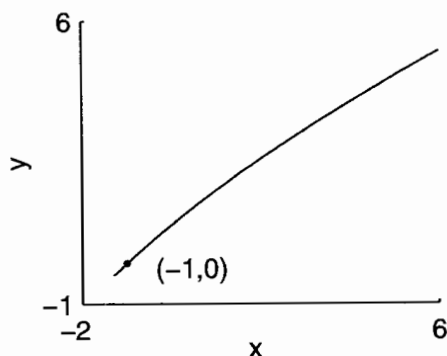
or

$$y = \frac{1}{2} (2x+3)^{1/2} \ln(2x+3) + C(2x+3)^{1/2}$$

The initial condition provides

$$0 = y(-1) = C.$$

Consequently, $y = (1/2)(2x+3)^{1/2} \ln(2x+3)$. The interval of existence is $(-3/2, +\infty)$ and the solution curve is shown in the following figure.



20. Compare $y' = \cos x - y \sec x$ with $y' = a(x)y + f(x)$ and note that $a(x) = -\sec x$ and $f(x) = \cos x$. Although f is continuous everywhere, a has discontinuities at $x = \pi/2 + k\pi$, k an integer.

An integrating factor is found with

$$\begin{aligned} u(x) &= e^{\int -a(x) dx} = e^{\int \sec x dx} \\ &= e^{\ln |\sec x + \tan x|} = |\sec x + \tan x|. \end{aligned}$$

If $\sec x + \tan x > 0$, the $|\sec x + \tan x| = \sec x + \tan x$. If $\sec x + \tan x < 0$, the $|\sec x + \tan x| = -(\sec x + \tan x)$. Multiplying our equation by either integrating factor produces the same result.

$$\begin{aligned} (\sec x + \tan x)y' + (\sec x \tan x + \sec^2 x)y \\ = 1 + \sin x, \end{aligned}$$

From which follows:

$$\begin{aligned} ((\sec x + \tan x)y)' &= 1 + \sin x \\ (\sec x + \tan x)y &= x - \cos x + C \end{aligned}$$

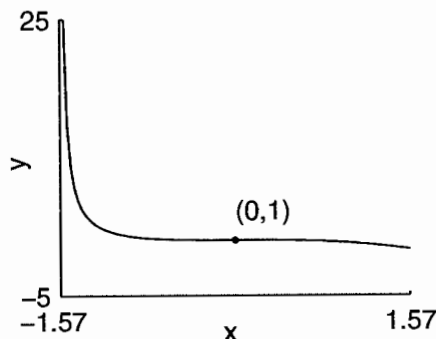
Use the initial condition, $y(0) = 1$.

$$(\sec 0 + \tan 0)(1) = 0 - \cos(0) + C$$

Consequently, $C = 2$ and

$$y = \frac{x - \cos x + 2}{\sec x + \tan x}.$$

The initial condition forces the graph to pass through $(0, 1)$, but $a(x)$ has nearby discontinuities at $x = -\pi/2$ and $x = \pi/2$. Consequently, the interval of existence is maximally extended to $(-\pi/2, \pi/2)$, as shown in the following figure.



21. Solve for x .

$$x' = -\frac{1}{1+t}x + \frac{\cos t}{1+t}$$

Compare this result with $x' = a(t)x + f(t)$ and note that $a(t) = -1/(1+t)$ and $f(t) = \cos t/(1+t)$, neither of which are continuous at $t = -1$. An integrating factor is found with

$$u(t) = e^{\int -a(t) dt} = e^{\int 1/(t+1) dt} = e^{\ln |1+t|} = |1+t|.$$

However, the initial condition dictates that our solution pass through the point $(-\pi/2, 0)$. Because of the discontinuity at $t = -1$, our solution must remain to the left of $t = -1$. Consequently, with $t < -1$, $u(t) = -(1+t)$. However, multiplying our equation by $u(t)$ produces

$$\begin{aligned} (1+t)x' + x &= \cos t, \\ ((1+t)x)' &= \cos t, \\ (1+t)x &= \sin t + C. \end{aligned}$$

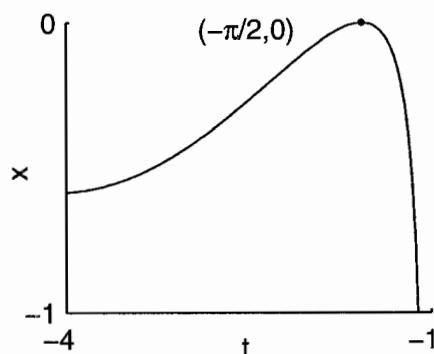
Use the initial condition.

$$\left(1 - \frac{\pi}{2}\right)(0) = \sin\left(-\frac{\pi}{2}\right) + C$$

Consequently, $C = 1$ and

$$x = \frac{1 + \sin t}{1 + t}.$$

The interval of existence is maximally extended to $(-\infty, -1)$, as shown in the following figure.



22. Let $z = x^{1-n}$. Then

$$\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} = (1-n)x^{-n} \frac{dx}{dt}.$$

This motivates multiplying our equation by $(1-n)x^{-n}$ to produce

$$(1-n)x^{-n} \frac{dx}{dt} = (1-n)a(t)x^{1-n} + (1-n)f(t).$$

Replacing $(1-n)x^{-n}dx/dt$ with dz/dt and x^{1-n} with z produces the desired result.

$$\frac{dz}{dt} = (1-n)a(t)z + (1-n)f(t)$$

23. In this case $n = 2$, so we set $z = y^{-1}$. Then

$$\begin{aligned} \frac{dz}{dx} &= \frac{dz}{dy} \frac{dy}{dx} \\ &= -y^{-2} \left[xy^2 - \frac{y}{x} \right] \\ &= -x + \frac{1}{xy} \\ &= -x + \frac{z}{x}. \end{aligned}$$

This is a linear equation for z . The integrating factor is $1/x$, so we have

$$\begin{aligned} \frac{1}{x} \left[\frac{dz}{dx} - \frac{z}{x} \right] &= \frac{1}{x}(-x) \\ \left[\frac{z}{x} \right]' &= -1 \\ \frac{z}{x} &= -x + C \\ z(x) &= x(C - x) \end{aligned}$$

Since $z = 1/y$, our solution is $y(x) = \frac{1}{x(C-x)}$.

24. In this case $n = 2$, so we set $z = y^{-1}$. Then

$$\begin{aligned} \frac{dz}{dx} &= \frac{dz}{dy} \frac{dy}{dx} \\ &= y^{-2} [y^2 - y] \\ &= -1 + \frac{1}{y} \\ &= -1 + z \end{aligned}$$

This is a linear equation for z . The integrating factor is e^{-x} , so we have

$$\begin{aligned} e^{-x} \left[\frac{dz}{dx} - z \right] &= -e^{-x} \\ [e^{-x}z]' &= -e^{-x} \\ e^{-x}z &= e^{-x} + C \\ z(x) &= 1 + Ce^x. \end{aligned}$$

Since $z = 1/y$ our solution is $y(x) = \frac{1}{1 + Ce^x}$.

25. Solve for y' .

$$y' = -\frac{1}{x}y + x^3y^3$$

Compare this with $y' = a(x)y + f(x)y^n$ and note that this has the form of Bernoulli's equation with $n = 3$. Let $z = y^{1-3} = y^{-2}$. Then

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = -2y^{-3} \frac{dy}{dx}.$$

Multiply the equation by $-2y^{-3}$.

$$-2y^{-3} \frac{dy}{dx} = \frac{2}{x} y^{-2} - 2x^3$$

Replace $-2y^{-3}(dy/dx)$ with dz/dx and y^{-2} with z .

$$\frac{dz}{dx} = \frac{2}{x}z - 2x^3$$

This equation is linear with integrating factor

$$u(x) = e^{\int -a(x) dx} = e^{\int -2/x dx} = e^{-2 \ln |x|} = x^{-2}.$$

Multiply by the integrating factor and integrate.

$$\begin{aligned} x^{-2}z' - 2x^{-3}z &= -2x \\ (x^{-2}z)' &= -2x \\ x^{-2}z &= -x^2 + C \\ z &= -x^4 + Cx^2 \end{aligned}$$

Replace z with y^{-2} and solve for y .

$$\begin{aligned} y^{-2} &= -x^4 + Cx^2 \\ y &= \pm 1/\sqrt{Cx^2 - x^4} \end{aligned}$$

26. Compare this with $P' = a(t)P + f(t)P^n$ and note that this has the form of Bernoulli's equation with $n = 2$. Let $z = P^{1-2} = P^{-1}$. Then

$$\frac{dz}{dt} = \frac{dz}{dP} \frac{dP}{dt} = -P^{-2} \frac{dP}{dt}.$$

Multiply the equation by $-P^{-2}$.

$$-P^{-2} \frac{dP}{dt} = -aP^{-1} + b$$

Replace $-P^{-2}(dP/dt)$ with dz/dt and P^{-1} with z .

$$\frac{dz}{dt} = -az + b$$

This equation is linear with integrating factor

$$u(t) = e^{\int -a(t) dt} = e^{\int a dt} = e^{at}.$$

Multiply by the integrating factor and integrate.

$$\begin{aligned} e^{at} \frac{dz}{dt} + ae^{at}z &= be^{at} \\ \frac{d}{dt}(e^{at}z) &= be^{at} \\ e^{at}z &= \frac{b}{a}e^{at} + C \\ z &= \frac{b}{a} + Ce^{-at} \end{aligned}$$

Replace z with P^{-1} and solve for y .

$$\begin{aligned} P^{-1} &= \frac{b}{a} + Ce^{-at} \\ P &= \frac{1}{b/a + Ce^{-at}} \end{aligned}$$

27. (a) Since $y = y_1 + z$, $y^2 = y_1^2 + 2y_1z + z^2$. Hence

$$\begin{aligned} z' &= y' - y_1' \\ &= -[\psi y^2 + \phi y + \chi] + [\psi y_1^2 + \phi y_1 + \chi] \\ &= \psi[y_1^2 - y^2] + \phi[y_1 - y] \\ &= -\psi[2y_1z + z^2] - \phi z \\ &= -(2y_1\psi + \phi)z - \psi z^2 \end{aligned}$$

- (b) Since $y_1 = 1/t$ is a solution, we set $z = y + 1/t$. Then $y = z - 1/t$, and $y^2 = z^2 - 2z/t + 1/t^2$, so

$$\begin{aligned} z' &= y' - \frac{1}{t^2} \\ &= -\frac{1}{t^2} - \frac{y}{t} + y^2 - \frac{1}{t^2} \\ &= -\frac{3z}{t} + z^2. \end{aligned}$$

This is a Bernoulli equation with $n = 2$. Thus we set $w = 1/z$. Differentiating, we get

$$\begin{aligned} w' &= -\frac{1}{t^2}z' \\ &= -\frac{1}{t^2} \left[-\frac{3z}{t} + z^2 \right] \\ &= \frac{3}{tz} - 1 \\ &= \frac{3w}{t} - 1. \end{aligned}$$

This is a linear equation, and t^{-3} is an integrating factor.

$$\begin{aligned} t^{-3} \left[w' - \frac{3w}{t} \right] &= -t^{-3} \\ [t^{-3}w] &= -t^{-3} \\ t^{-3}w(t) &= \frac{1}{2}t^{-2} + C \\ w(t) &= \frac{1}{2}t + Ct^3 \end{aligned}$$

Now it is a matter of unravelling the changes of variable. First

$$z(t) = \frac{1}{w(t)} = \frac{2}{t + 2Ct^3} = \frac{2}{t + Bt^3},$$

Where we have set $B = 2C$. Then

$$y(t) = z(t) - \frac{1}{t} = \frac{2}{t + Bt^3} - \frac{1}{t}.$$

This looks a little better if we use partial fractions to write

$$\begin{aligned} y(t) &= \frac{2}{t + Bt^3} - \frac{1}{t} \\ &= \frac{2}{t} - \frac{2Bt}{1 + Bt^2} - \frac{1}{t} \\ &= \frac{1}{t} - \frac{2Bt}{1 + Bt^2}. \end{aligned}$$

28. The model is $N' = kN(1000 - N)$, where the proportionality constant k is yet to be determined. Since we know that when $N = 100$, the rate of infection is 90/day, we have $k \cdot 100 \cdot 900 = 90$, we find that $k = 1 \times 10^{-3}$. Hence the model equation is $N' = N - N^2/1000$. This is a Bernoulli equation, with $n=2$. Accordingly we set $x = 1/N$. Then

$$\begin{aligned} x' &= -N'/N^2 \\ &= -1/N + 10^{-3} \\ &= -x + 10^{-3}. \end{aligned}$$

Solving this linear equation, we get $x(t) = 10^{-3}[1 + Ce^{-t}]$. Hence $N(t) = 1/x(t) = 1000/[1 + Ce^{-t}]$. At $t = 0$, $N = 20 = 1/[1 + C]$. Hence $C = 49$, and the solution is $N(t) = 1000/[1 + 49e^{-t}]$.

We have $N(t) = 0.9 \times 1000 = 900$ when $t = 6.089$ days.

29. Newton's law of cooling says the rate of change of temperature is equal to k times the difference between the current temperature and the ambient temperature. In this case the ambient temperature is decreasing from 0°C , and at a constant rate of 1°C per hour. Hence the model equation is $T' = -k(T + t)$, where we are taking $t = 0$ to be midnight. This is a linear equation. The solution is

$T(t) = -t + 1/k + Ce^{-kt}$. Since $T(0) = 31$, the constant evaluates to $C = 31 - 1/k$. The solution is $T(t) = -t + 1/k + (31 - 1/k)e^{-kt}$.

We need to compute the time t_0 when $T(t_0) = 37$, using $k \approx 0.2231$ from Exercise 35 of Section 2. This is a nonlinear equation, but using a calculator or a computer we can find that $t_0 \approx -0.8022$. Since $t = 0$ corresponds to midnight, this means that the time of death is approximately 11:12 PM.

30. The homogeneous equation, $y' = -3y$ has solution $y_h(t) = e^{-3t}$. We look for a particular solution in the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - 3vy_h \\ &= v'y_h - 3y_p, \end{aligned}$$

and $y_p' = -3y_p + 4$, we have $v' = 4/y_h = 4e^{3t}$. Integrating we see that $v(t) = 4e^{3t}/3$, and

$$y_p(t) = v(t)y_h(t) = \frac{4}{3}e^{3t} \cdot e^{-3t} = \frac{4}{3}.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = \frac{4}{3} + Ce^{-3t}.$$

31. The homogeneous equation, $y' = -2y$ has solution $y_h(t) = e^{-2t}$. We look for a particular solution in the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - 2vy_h \\ &= v'y_h - 2y_p, \end{aligned}$$

and $y_p' = -2y_p + 5$, we have $v' = 5/y_h = 5e^{2t}$. Integrating we see that $v(t) = 5e^{2t}/2$, and

$$y_p(t) = v(t)y_h(t) = \frac{5}{2}e^{2t} \cdot e^{-2t} = \frac{5}{2}.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = \frac{5}{2} + Ce^{-2t}.$$

32. The homogeneous equation, $y' = -(2/x)y$ has solution $y_h(x) = x^{-2}$. We look for a particular solution in the form $y_p(x) = v(x)y_h(x)$, where v is an unknown function. Since

$$\begin{aligned} y'_p &= v'y_h + vy'_h \\ &= v'y_h - 2y_p/x, \end{aligned}$$

and $y'_p = -(2/x)y_p + 8x$, we have $v' = 8x/y_h = 8x^3$. Hence $v(x) = 2x^4$, and

$$y_p(x) = v(x)y_h(x) = 2x^2.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = 2x^2 + Cx^{-2}.$$

33. The homogeneous equation, $y' = -y/t$, has solution $y_h(t) = 1/t$. We look for a particular solution in the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y'_p &= v'y_h + vy'_h \\ &= v'y_h - 2vy_h \\ &= v'y_h - y_p/t, \end{aligned}$$

and $y'_p = -y_p/t + 4t$, we have $v' = 4t/y_h = 4t^2$. Integrating we get $v(t) = 4t^3/3$, and

$$y_p(t) = v(t)y_h(t) = \frac{4}{3}t^2.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = \frac{4}{3}t^2 + C/t.$$

34. The homogeneous equation, $x' = -2x$ has solution $x_h(t) = e^{-2t}$. We look for a particular solution in the form $x_p(t) = v(t)x_h(t)$, where v is an unknown function. Since

$$\begin{aligned} x'_p &= v'x_h + vx'_h \\ &= v'x_h - 2vx_h \\ &= v'x_h - 2x_p, \end{aligned}$$

and $x'_p = -2x_p + t$, we have $v' = t/x_h = te^{2t}$. Integrating, we get $v(t) = (t/2 - 1/4)e^{2t}$, and

$$x_p(t) = v(t)x_h(t) = \frac{1}{4}[2t - 1].$$

$$x(t) = x_p(t) + Cx_h(t) = \frac{1}{4}[2t - 1] + Ce^{-2t}.$$

35. The homogeneous equation, $y' = -2xy$ has solution $y_h(x) = e^{-x^2}$. We look for a particular solution in the form $y_p(x) = v(x)y_h(x)$, where v is an unknown function. Since

$$\begin{aligned} y'_p &= v'y_h + vy'_h \\ &= v'y_h - 2xy_p, \end{aligned}$$

and $y'_p = -2xy_p + 4x$, we have $v' = 4x/y_h = 4xe^{x^2}$. Hence $v(x) = 2e^{x^2}$, and

$$y_p(x) = v(x)y_h(x) = 2.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = 2 + Ce^{x^2}.$$

36. The homogeneous equation, $y' = 3y$ has solution $y_h(t) = e^{3t}$. We look for a particular solution in the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y'_p &= v'y_h + vy'_h \\ &= v'y_h + 3vy_h \\ &= v'y_h + 3y_p, \end{aligned}$$

and $y'_p = 3y_p + 4$, we have $v' = 4/y_h = 4e^{-3t}$. Integrating we see that $v(t) = -4e^{-3t}/3$, and

$$y_p(t) = v(t)y_h(t) = -\frac{4}{3}e^{-3t} \cdot e^{3t} = -\frac{4}{3}.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = -\frac{4}{3} + Ce^{3t}.$$

Since $y(0) = 2$, we must have $2 = -4/3 + C$, or $C = 10/3$. Thus the solution is

$$y(t) = (-4 + 10e^{3t})/3.$$

37. The homogeneous equation $y' = -y/2$ has solution $y_h(t) = e^{-t/2}$. We look for a particular solution of the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned}y_p' &= v'y_h + vy_h' \\&= v'y_h - vy_h/2 \\&= v'y_h - y_p/2,\end{aligned}$$

and $y_p' = -y_p/2 + t$, we have $v' = t/y_h(t) = te^{t/2}$. Integrating we find that $v(t) = (2t - 4)e^{t/2}$, and $y_p(t) = v(t)y_h(t) = (2t - 4)$. The general solution is $y(t) = y_p(t) + Cy_h(t) = (2t - 4) + Ce^{-t/2}$. From $y(0) = 1$ we compute that $C = 5$, so the solution is

$$y(t) = (2t - 4) + 5e^{-t/2}.$$

38. The homogeneous equation $y' = -y$ has solution $y_h(t) = e^{-t}$. We look for a particular solution of the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned}y_p' &= v'y_h + vy_h' \\&= v'y_h - vy_h \\&= v'y_h - y_p,\end{aligned}$$

and $y_p' = -y_p + e^t$, we see that $v' = e^t/y_h = e^{2t}$. Integrating we get $v(t) = e^{2t}/2$, and $y_p(t) = e^t/2$. The general solution is $y(t) = y_p(t) + Cy_h(t) = e^t/2 + Ce^{-t}$. From $y(0) = 1$, we compute that $C = 1/2$, so the solution is

$$y(t) = \frac{1}{2}(e^t + e^{-t}).$$

39. The homogeneous equation $y' = -2xy$ has solution $y_h(x) = e^{-x^2}$. We look for a particular solution of the form $y_p(x) = v(x)y_h(x)$, where v is an unknown function. Since

$$\begin{aligned}y_p' &= v'y_h + vy_h' \\&= v'y_h - 2xy_h \\&= v'y_h - 2xy_p,\end{aligned}$$

and $y_p' = -2xy_p + 2x^3$, we see that $v' = 2x^3e^{x^2}$.

Integrating we get $v(x) = (x^2 - 1)e^{x^2}$, and $y_p(x) = x^2 - 1$. The general solution is $y(x) = y_p(x) +$

$Cy_h(x) = x^2 - 1 + Ce^{-x^2}$. Since $y(0) = -1$, we have $C = 0$, and the solution is

$$y(x) = x^2 - 1.$$

40. The homogeneous equation $x' = (2/t^2)x$ has solution $x_h(t) = e^{-2/t}$. We look for a particular solution of the form $x_p(t) = v(t)x_h(t)$, where v is an unknown function. Since

$$\begin{aligned}x_p' &= v'x_h + vx_h' \\&= v'x_h + 2vx_h/t^2 \\&= v'x_h + 2x_p/t^2,\end{aligned}$$

and $x_p' = 2x_p/t^2 + 1/t^2$, we have $v' = 1/(t^2x_h) = e^{2/t}/t^2$. Integrating we find that $v(t) = -e^{2/t}/2$, and $x_p(t) = -1/2$. The general solution is $x(t) = x_p(t) + Cx_h(t) = -1/2 + Ce^{-2/t}$. Since $x(1) = 0$, we find that $C = e^2/2$, and the solution is

$$x(t) = \frac{1}{2}(-1 + e^{2(1-1/t)}).$$

41. The homogeneous equation $x' = -4tx/(1+t^2)$ has solution $x_h(t) = (1+t^2)^{-2}$. We look for a particular solution of the form $x_p(t) = v(t)x_h(t)$, where v is an unknown function. Since

$$\begin{aligned}x_p' &= v'x_h + vx_h' \\&= v'x_h - 4tx_p/(1+t^2),\end{aligned}$$

and $x_p' = -4tx_p/(1+t^2) + t/(1+t^2)$, so

$$v' = \frac{t}{1+t^2} \cdot \frac{1}{x_h(t)} = t(1+t^2).$$

Integrating, we get $v(t) = t^2/2 + t^4/4$. Thus

$$x_p(t) = v(t)x_h(t) = \frac{2t^2 + t^4}{4(1+t^2)^2}.$$

The general solution is

$$x(t) = x_p(t) + Cx_h(t) = \frac{4C + 2t^2 + t^4}{4(1+t^2)^2}.$$

The initial condition $x(0) = 1$ implies that $C = 1$, so the solution is

$$y(t) = \frac{4 + 2t^2 + t^4}{4(1+t^2)^2}.$$

42. (a) The equation $T' + kT = 0$ is separable, with solution $T_h = Ce^{-kt}$, C an arbitrary constant.
- (b) The equation $T' = -k(T - A)$ is autonomous. We seek a constant solution (see Section 2.9) that makes the right side equal to zero. Hence, $T_p = A$ is a particular solution of the inhomogeneous equation.
- (c) The general solution is $T = T_h + T_p = Ce^{-kt} + A$, with C an arbitrary constant.
- (d) Again, the solution of the homogeneous equation $T' + kT = 0$ is $T_h = Ce^{-kt}$, with C an arbitrary constant. The inhomogeneous equation $T' = -k(T - A) + H$ is also autonomous (the right side is independent of t). We seek a constant solution by setting the right side equal to zero.

$$\begin{aligned} -k(T_p - A) &= H \\ T_p - A &= \frac{H}{k} \\ T_p &= A + \frac{H}{k} \end{aligned}$$

Hence, the general solution is given by the equation

$$T = T_h + T_p = Ce^{-kt} + \left(A + \frac{H}{k}\right).$$

43. (a) The solution of the homogeneous equation $T' + kT = 0$ is $T_h = Fe^{-kt}$, with F an arbitrary constant.
- (b) We guess that $T_p = C \cos \omega t + D \sin \omega t$ is a particular solution. Substituting T_p and $T_p' = -C\omega \sin \omega t + D\omega \cos \omega t$ in the left side of $T' + kT = kA \sin \omega t$, then gathering coefficients of $\cos \omega t$ and $\sin \omega t$, we obtain
- $$T_p' + kT_p = (-\omega C + kD) \sin \omega t + (kC + \omega D) \cos \omega t. \quad (4.1)$$
- Comparing this with the right side of $T_p' + kT_p = kA \sin \omega t$, we see that
- $$-\omega C + kD = kA \quad \text{and} \quad kC + \omega D = 0.$$
- (c) Solving these equations simultaneously (for example, multiply the first equation by k , the second by ω , then add the equations to eliminate

C) provides

$$C = -\frac{\omega k A}{k^2 + \omega^2} \quad \text{and} \quad D = \frac{k^2 A}{k^2 + \omega^2}.$$

Substituting these results in $T_p = C \cos \omega t + D \sin \omega t$ provides the particular solution

$$T_p = -\frac{\omega k A}{k^2 + \omega^2} \cos \omega t + \frac{k^2 A}{k^2 + \omega^2} \sin \omega t.$$

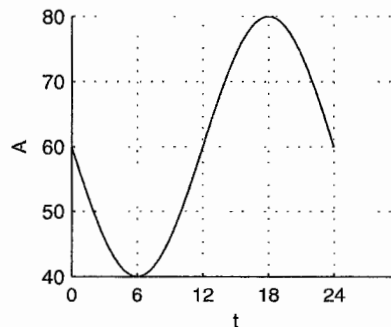
Hence, the general solution is

$$\begin{aligned} T &= T_h + T_p \\ &= Fe^{-kt} + \frac{kA}{k^2 + \omega^2} [k \sin \omega t - \omega \cos \omega t]. \end{aligned}$$

44. (a) If the period of the ambient temperature is 24 hours, then the computation

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24} = \frac{\pi}{12}$$

gives the angular frequency. Because the sinusoid has a maximum of 80° F and a minimum of 40° F, the amplitude will be half of the difference, or 20. A sketch of the ambient temperature follows.



Note the minimum at 6 am, then the maximum at 6 pm. What we have is an upside-down sine, with angular frequency $\pi/12$, that is shifted upward 60° F. Thus, the equation for the ambient temperature must be

$$A = 60 - 20 \sin \frac{\pi t}{12}.$$

Therefore, the model, adjusted for this ambient temperature, becomes

$$\frac{dT}{dt} = -\frac{1}{2} \left(T - 60 + 20 \sin \frac{\pi t}{12} \right). \quad (4.2)$$

- (b) The homogeneous equation $T' + (1/2)T = 0$ has solution $T_h = Ce^{-t/2}$, where C is an arbitrary constant. Now, consider the inhomogeneous equation

$$T' + \frac{1}{2}T = 30 - 10 \sin \frac{\pi t}{12}. \quad (4.3)$$

Note that the right hand side consists of a constant and a sinusoid. Let's try a particular solution having the form

$$T_p = D + E \cos \frac{\pi t}{12} + F \sin \frac{\pi t}{12}. \quad (4.4)$$

Substitute this guess and its derivative into the left hand side of equation (4.4) and collect coefficients to get

$$\begin{aligned} T' + \frac{1}{2}T &= \frac{1}{2}D + \left(-\frac{\pi}{12}E + \frac{1}{2}F \right) \sin \frac{\pi t}{12} \\ &\quad + \left(\frac{1}{2}E + \frac{\pi}{12}F \right) \cos \frac{\pi t}{12} \end{aligned} \quad (4.5)$$

Comparing this with the right-hand side of equation (4.3), we see that

$$\begin{aligned} \frac{1}{2}D &= 30, \\ -\frac{\pi}{12}E + \frac{1}{2}F &= -10, \quad \text{and} \\ \frac{1}{2}E + \frac{\pi}{12}F &= 0. \end{aligned}$$

Clearly, $D = 60$, and solving the remaining two equations simultaneously, we obtain

$$E = \frac{120\pi}{36 + \pi^2} \quad \text{and} \quad F = \frac{-720}{36 + \pi^2}.$$

These values of D , E , and F , when inserted into equation (4.4), provide the particular solution

$$T_p = 60 + \frac{120\pi}{36 + \pi^2} \cos \frac{\pi t}{12} - \frac{720}{36 + \pi^2} \sin \frac{\pi t}{12}.$$

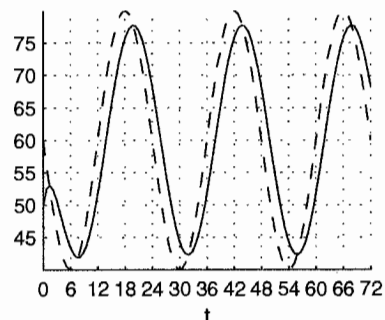
Thus, the general solution is $T = T_h + T_p$, or

$$\begin{aligned} T_p &= Ce^{-t/2} + 60 \\ &\quad + \frac{120}{36 + \pi^2} \left[\pi \cos \frac{\pi t}{12} - 6 \sin \frac{\pi t}{12} \right]. \end{aligned} \quad (4.6)$$

Now, when the initial condition $T(0) = 50$ is substituted into equation (4.6), we obtain $C = -10 - 120\pi/(36 + \pi^2)$. Thus, the general solution becomes

$$\begin{aligned} T &= -\left(10 + \frac{120\pi}{36 + \pi^2} \right) e^{-t/2} + 60 \\ &\quad + \frac{120}{36 + \pi^2} \left[\pi \cos \frac{\pi t}{12} - 6 \sin \frac{\pi t}{12} \right]. \end{aligned} \quad (4.7)$$

- (c) The plot of the ambient temperature is shown as a dashed curve in the following figure. The temperature T inside the cabin is shown as a solid curve.



Note that the transient part of the solution dies out quickly. Indeed, because of the factor of $e^{-t/2}$, the time constant (See Section 2.2, Exercise ??) is $T_c = 2$ hr. Thus, in about four time constants, or 8 hours, this part of the temperature solution is negligible. Finally, note how the temperature in the cabin reacts to and trails the ambient temperature outside, which makes sense.

Section 2.5. Mixing Problems

1. (a) Let $S(t)$ denote the amount of sugar in the tank, measured in pounds. The rate in is $3 \text{ gal/min} \times 0.2 \text{ lb/gal} = 0.6 \text{ lb/min}$. The rate out is $3 \text{ gal/min} \times S/100 \text{ lb/gal} = 3S/100 \text{ lb/min}$. Hence

$$\begin{aligned}\frac{dS}{dt} &= \text{rate in} - \text{rate out} \\ &= 0.6 - 3S/100\end{aligned}$$

This linear equation can be solved using the integrating factor $u(t) = e^{3t/100}$ to get the general solution $S(t) = 20 + Ce^{-3t/100}$. Since $S(0) = 0$, the constant $C = -20$ and the solution is $S(t) = 20(1 - e^{-3t/100})$.
 $S(20) = 10(1 - e^{-0.6}) \approx 9.038 \text{ lb}$.

- (b) $S(t) = 15$ when $e^{-3t/100} = 1 - 15/20 = 1/4$. Taking logarithms this translates to $t = (100 \ln 4)/3 \approx 46.2098 \text{ min}$.
 (c) As $t \rightarrow \infty$ $S(t) \rightarrow 20$.
2. (a) Let $x(t)$ represent the number of pounds of sugar in the tank at time t . The rate in is 0, and the rate out is $2 \text{ gal/min} \cdot x/50 \text{ lb/gal} = x/25 \text{ lb/min}$. Hence the model equation is $x' = -x/25$. The general solution is $x(t) = Ae^{-t/25}$. The initial condition implies that $A = x(0) = 50 \text{ gal} \times 2 \text{ lb/gal} = 100 \text{ lb}$. Hence the solution is $x(t) = 100e^{-t/25}$. After 10 minutes we have $x(10) = 67.032 \text{ lb}$ of sugar in the tank.
 (b) We have to find t such that $x(t) = 100e^{-t/25} = 20$. This comes to $t = 25 \ln 5 \approx 40.2359 \text{ min}$.
 (c) $x(t) = 100e^{-t/25} \rightarrow 0$ as $t \rightarrow \infty$.

3. (a) Let $x(t)$ represent the number of pounds of salt in the tank at time t . The rate at which the salt in the tank is changing with respect to time is equal to the rate at which salt enters the tank minus the rate at which salt leaves the tank, i.e.,

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}.$$

In order that the units match in this equation, dx/dt , the rate In, and the rate Out must each be measured in pounds per minute (lb/min).

Solution enters the tank at 5 gal/min, but the concentration of this solution is $1/4 \text{ lb/gal}$. Consequently,

$$\text{rate in} = 5 \text{ gal/min} \times \frac{1}{4} \text{ lb/gal} = \frac{5}{4} \text{ lb/min}.$$

Solution leaves the tank at 5 gal/min, but at what concentration? Assuming perfect mixing, the concentration of salt in the solution is found by dividing the amount of salt by the volume of solution, $c(t) = x(t)/100$. Consequently,

$$\text{rate out} = 5 \text{ gal/min} \times \frac{x(t)}{100} \text{ lb/gal} = \frac{1}{20}x(t) \text{ lb/min}.$$

As there are 2 lb of salt present in the solution initially, $x(0) = 2$ and

$$\frac{dx}{dt} = \frac{5}{4} - \frac{1}{20}x, \quad x(0) = 2.$$

Multiply by the integrating factor, $e^{(1/20)t}$, and integrate.

$$\begin{aligned}(e^{(1/20)t}x)' &= \frac{5}{4}e^{(1/20)t} \\ e^{(1/20)t}x &= 25e^{(1/20)t} + C \\ x &= 25 + Ce^{-(1/20)t}\end{aligned}$$

The initial condition $x(0) = 2$ gives $C = -23$ and

$$x(t) = 25 - 23e^{-(1/20)t}.$$

Thus, the concentration at time t is given by

$$c(t) = \frac{x(t)}{100} = \frac{25 - 23e^{-(1/20)t}}{100},$$

and the eventual concentration can be found by taking the limit as $t \rightarrow +\infty$.

$$\lim_{t \rightarrow +\infty} \frac{25 - 23e^{-(1/20)t}}{100} = \frac{1}{4} \text{ lb/gal}$$

Note that this answer is quite reasonable as the concentration of solution entering the tank is also $1/4 \text{ lb/gal}$.

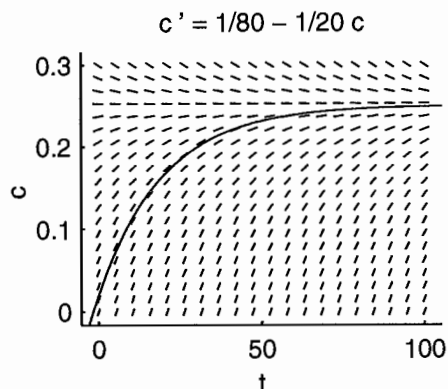
- (b) We found it convenient to manipulate our original differential equation before using our solver. The key idea is simple: we want to sketch the concentration $c(t)$, not the salt content $x(t)$. However,

$$c(t) = \frac{x(t)}{100} \quad \text{or} \quad x(t) = 100c(t).$$

Consequently, $x'(t) = 100c'(t)$. Substituting these into our balance equation gives

$$\begin{aligned} x' &= \frac{5}{4} - \frac{1}{20}x, \\ 100c' &= \frac{5}{4} - \frac{1}{20}(100c), \\ c' &= \frac{1}{80} - \frac{1}{20}c, \end{aligned}$$

with $c(0) = x(0)/100 = 2/100 = 0.02$. The numerical solution of this ODE is presented in the following figure. Note how the concentration approaches 0.25 lb/gal.



4. Let $x(t)$ represent the amount of salt in the solution at time t . Let r represent the rate (gal/min) that water enters (and leaves) the tank. Consequently, the rate at which salt enters the tank is 0 gal/min, but the

$$\text{rate out} = r \text{ gal/min} \times \frac{x(t)}{500} \text{ lb/gal} = \frac{r}{500}x(t) \text{ lb/min}.$$

Thus,

$$\begin{aligned} \frac{dx}{dt} &= \text{rate in} - \text{rate out}, \\ \frac{dx}{dt} &= -\frac{r}{500}x. \end{aligned}$$

Let $c(t)$ represent the concentration at time t . Thus, $c(t) = x(t)/500$, or $500c(t) = x(t)$ and $500c'(t) = x'(t)$. Substitute these into the rate equation to produce

$$\begin{aligned} 500c' &= -\frac{r}{500}(500c), \\ c' &= -\frac{r}{500}c. \end{aligned}$$

This equation is separable, with solution $c = Ae^{-(r/500)t}$. Use the initial concentration, $c(0) = .05$ lb/gal, to produce

$$c = 0.05e^{-(r/500)t}.$$

The concentration must reach 1% in one hour (60 min), so $c(60) = 0.01$ and

$$0.01 = 0.05e^{-(r/500)(60)},$$

$$\frac{1}{5} = e^{-(3/25)r},$$

$$r = \frac{25}{3} \ln 5,$$

$$r \approx 13.4 \text{ gal/min}.$$

5. The volume is increasing at the rate of 2 gal/min, so the volume at time t is $V(t) = 20 + 2t$. The tank is full when $V(t) = 50$, or when $t = 15$ min. If $x(t)$ is the amount of salt in the tank at time t , then the concentration is $x(t)/V(t)$. The rate in is 4 gal/min $\cdot$ 0.5 lb/gal = 2 lb/min. The rate out is 2 gal/min $\cdot$ x/V lb/gal. Hence the model equation is

$$x' = 2 - 2x/V = 2 - \frac{x}{10 + t}.$$

This linear equation can be solved using the integrating factor $u(t) = 10 + t$, giving the general solution $x(t) = 10 + t + C/(10 + t)$. The initial condition $x(0) = 0$ enables us to compute that $C = -100$, so the solution is $x(t) = 10 + t - 100/(10 + t)$. At $t = 15$, when the tank is full, we have $x(15) = 21$ lb.

6. The volume in the tank is decreasing at 1 gal/min, so the volume is $V(t) = 100 - t$. There is no sugar coming in, and the rate out is 3 gal/min $\times$ $S(t)/V(t)$ lb/gal. Hence the differential equation is

$$\frac{dS}{dt} = \frac{-3S}{100 - t}.$$

This equation is linear and homogeneous. It can be solved by separating variables. The general solution is $S(t) + A(100 - t)^3$. Since $S(0) = 100 \times 0.05 = 5$, we see that $A = 5 \times 10^{-6}$, and the solution is $S(t) = 5 \times 10^{-6} \times (100 - t)^3$. When $V(t) = 100 - t = 50$ gal,

$$S(t) = 5 \times 10^{-6} \times 50^3 = 0.625 \text{ lb.}$$

7. (a) The volume of liquid in the tank is increasing by 2 gal/min. Hence the volume is $V(t) = 100 + 2t$ gal. Let $x(t)$ be the amount of pollutant in the tank, measured in lbs. The rate in during this initial period is $6 \text{ gal/min} \cdot 0.5 \text{ lb/gal} = 3 \text{ lb/gal}$. The rate out is $8 \text{ gal/min} \cdot x/V = 4x/(50 + t)$. Hence the model equation is

$$x' = 3 - 4x/(50 + t).$$

This linear equation can be solved using the integrating factor $u(t) = (50 + t)^4$. The general solution is $x(t) = 3(50 + t)/5 + C(50 + t)^{-4}$. The initial condition $x(0) = 0$ allows us to compute the constant to be $C = -1.875 \times 10^8$. Hence the solution is

$$x(t) = \frac{3t}{5} + 30 - \frac{1.875 \times 10^8}{(50 + t)^4}.$$

After 10 minutes the tank contains $x(10) = 21.5324$ lb of salt.

- (b) Now the volume is decreasing at the rate of 4 gal/min from the initial volume of 120 gal. Hence if we start with $t = 0$ at the 10 minute mark, the volume is $V(t) = 120 - 4t$ gal. Now the rate in is 0, and the rate out is $8 \text{ gal/min} \cdot x/V = 2x/(30 - t)$. Hence the model equation is

$$x' = -\frac{2x}{30 - t}.$$

This homogeneous linear equation can be solved by separating variables to find the general solution $x(t) = A(30 - t)^2$. At $t = 0$ we have $x(0) = 21.5342$, from which we find that $A = 21.5342/900$, and the solution is

$$x(t) = \frac{21.5342}{900} (30 - t)^2.$$

We are asked to find when this is one-half of 21.5342. This happens when $(30 - t)^2 = 450$ or at $t = 8.7868$ min.

8. Let $x(t)$ represent the amount of drug in the organ at time t . The rate at which the drug enters the organ is

$$\text{rate in} = a \text{ cm}^3/\text{s} \times \kappa \text{ g/cm}^3 = a\kappa \text{ g/s.}$$

The rate at which the drug leaves the organ equals the rate at which fluid leaves the organ, multiplied by the concentration of the drug in the fluid at that time. Hence,

$$\text{rate out} = b \text{ cm}^3/\text{s} \times \frac{x(t)}{V_0 + rt} \text{ g/cm}^3 = \frac{b}{V_0 + rt} x(t) \text{ g/s.}$$

Consequently,

$$\frac{dx}{dt} = a\kappa - \frac{b}{V_0 + rt} x.$$

The integrating factor is

$$u(t) = e^{\int b/(V_0 + rt) dt} = e^{(b/r) \ln(V_0 + rt)} = (V_0 + rt)^{b/r}.$$

Multiply by the integrating factor and integrate.

$$\begin{aligned} ((V_0 + rt)^{b/r} x)' &= a\kappa (V_0 + rt)^{b/r} \\ (V_0 + rt)^{b/r} x &= \frac{a\kappa}{r(b/r + 1)} (V_0 + rt)^{b/r+1} + L \\ x &= \frac{a\kappa}{b + r} (V_0 + rt) + L(V_0 + rt)^{-b/r} \end{aligned}$$

No drug in the system initially gives $x(0) = 0$ and $L = -a\kappa V_0^{b/r+1}/(b + r)$. Consequently,

$$\begin{aligned} x &= \frac{a\kappa}{b + r} (V_0 + rt) - \frac{a\kappa V_0^{b/r+1}}{b + r} (V_0 + rt)^{-b/r}, \\ x &= \frac{a\kappa}{b + r} (V_0 + rt) \left[1 - V_0^{b/r+1} (V_0 + rt)^{-b/r-1} \right], \\ x &= \frac{a\kappa}{b + r} (V_0 + rt) \left[1 - \left(\frac{V_0}{V_0 + rt} \right)^{b/r+1} \right]. \end{aligned}$$

The concentration is found by dividing $x(t)$ by $V(t) = V_0 + rt$. Consequently,

$$c(t) = \frac{a\kappa}{b + r} \left[1 - \left(\frac{V_0}{V_0 + rt} \right)^{(b+r)/r} \right].$$

9. (a) The rate at which pollutant enters the lake is

$$\text{rate in} = p \text{ km}^3/\text{yr}.$$

The rate at which the pollutant leaves the lake is found by multiplying the flow rate by the concentration of pollutant in the lake.

$$\begin{aligned}\text{rate out} &= (r + p) \text{ km}^3/\text{yr} \times \frac{x(t)}{V} \text{ km}^3/\text{km}^3 \\ &= \frac{r + p}{V} x(t) \text{ km}^3/\text{yr}\end{aligned}$$

Consequently,

$$\frac{dx}{dt} = p - \frac{r + p}{V} x.$$

But $c(t) = x(t)/V$, so $Vc'(t) = x'(t)$ and

$$\begin{aligned}Vc' &= p - \frac{r + p}{V}(Vc), \\ c' + \frac{r + p}{V}c &= \frac{p}{V}.\end{aligned}$$

- (b) With $r = 50$ and $p = 2$, the equation becomes

$$\begin{aligned}c' &= \frac{2}{100} - \frac{50 + 2}{100}c, \\ c' &= 0.02 - 0.52c.\end{aligned}$$

This is linear and solved in the usual manner.

$$\begin{aligned}(e^{0.52t}c)' &= 0.02e^{0.52t} \\ e^{0.52t}c &= \frac{0.02}{0.52}e^{0.52t} + K \\ c &= \frac{1}{26} + Ke^{-0.52t}\end{aligned}$$

The initial concentration is zero, so $c(0) = 0$ produces $K = -1/26$ and

$$c = \frac{1}{26} - \frac{1}{26}e^{-0.52t}.$$

The question asks when the concentration reaches 2%, or when $c(t) = 0.02$. Thus,

$$\begin{aligned}0.02 &= \frac{1}{26}(1 - e^{-0.52t}), \\ e^{-0.52t} &= 0.48, \\ t &= -\frac{\ln 0.48}{0.52}, \\ t &\approx 1.41 \text{ years}.\end{aligned}$$

10. Because the factory stops putting pollutant in the lake, $p = 0$ and $c' + ((r + p)/V)c = p/V$ becomes

$$c' + \frac{50}{100}c = 0.$$

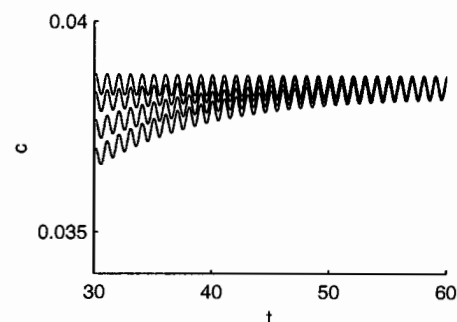
Note that we carried $r = 50$ from Exercise 9. This equation is separable, with general solution $c = Ke^{-(1/2)t}$. The initial concentration is 3.5%, so $c(0) = 0.035$ produces $K = 0.035$ and

$$c(t) = 0.035e^{-(1/2)t}.$$

The question asks for the time required to lower the concentration to 2%. That is, when does $c(t) = 0.02$?

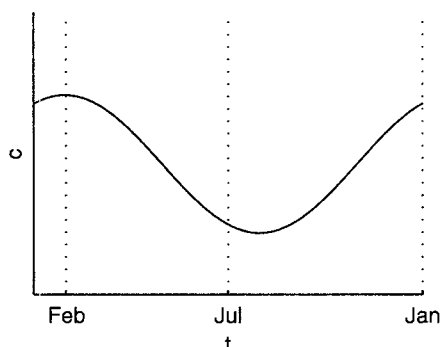
$$\begin{aligned}0.02 &= 0.035e^{-(1/2)t} \\ -\frac{1}{2}t &= \ln \frac{0.02}{0.035} \\ t &= 2 \ln \frac{0.035}{0.02} \\ t &\approx 1.1 \text{ years}\end{aligned}$$

11. (a) The concentrations are plotted in the following figure. In steady-state the concentration varies periodically.



- (b) The following figure shows one year of the oscillation, and indicates that the maximum concentration occurs early in February. This is four months after the time of the minimum flow.

Thus there is a shift of phase between the cause and the effect.



12. For Tank A we have a constant volume of 100 gal. Let $x(t)$ denote the amount of salt in Tank A. The rate into Tank A is 0, and the rate out is $5 \text{ gal/s} \times x/100 \text{ lb/gal} = x/20 \text{ lb/s}$. Hence the model equation is

$$x' = -\frac{x}{20}.$$

The solution with initial value $x(0) = 20$ is $x(t) = 20e^{-t/20}$.

The volume of solution in Tank B is increasing at 2.5 gal/s . Hence the volume at time t is $200 + 2.5t$. Let $y(t)$ denote the amount of salt in Tank B. Then the rate into Tank B is the same as the rate out of Tank A, $x/20$. The rate out of Tank B is $2.5 \text{ gal/s} \times y/(200 + 2.5t) \text{ lb/gal} = y/(80 + t) \text{ lb/s}$. Hence the model equation is

$$y' = \frac{x}{20} - \frac{y}{80 + t} = e^{-t/20} - \frac{y}{80 + t}.$$

This linear equation can be solved using the integrating factor $u(t) = 80 + t$. The general solution is

$$y(t) = \frac{C}{80 + t} - 20e^{-t/20} - \frac{400}{80 + t}e^{-t/20}.$$

Since $y(0) = 40$, we can compute that $C = 65 \times 80 = 5200$. Hence the solution is

$$y(t) = \frac{5200}{80 + t} - 20e^{-t/20} - \frac{400}{80 + t}e^{-t/20}.$$

Tank B will contain 250 gal when $t = 20$. At this point we have $y(20) = 43.1709 \text{ lb}$.

13. (a) Let $x(t)$ be the amount of pollutant (measured in km^3) in Lake Happy Times. The rate in for Lake Happy Times is $2 \text{ km}^3/\text{yr}$. The rate out is $52 \text{ km}^3/\text{yr} \times x/100 = 0.52x \text{ km}^3/\text{yr}$. Hence the model equation is

$$x' = 2 - 0.52x.$$

This linear equation can be solved using the integrating factor $u(t) = e^{0.52t}$. With the initial condition $x(0) = 0$ we find the solution $x(t) = 2[1 - e^{-0.52t}]/0.52$.

Let $y(t)$ be the amount of pollutant (measured in km^3) in Lake Sad Times. The rate into lake Sad Times is the same as the rate out of Lake Happy Times, or $0.52x \text{ km}^3/\text{yr}$. The rate out is $52 \text{ km}^3/\text{yr} \times y/100 = 0.52y \text{ km}^3/\text{yr}$. Hence the model equation is

$$y' = 0.52(x - y) = 2[1 - e^{-0.52t}] - 0.52y.$$

This linear equation can also be solved using the integrating factor $u(t) = e^{0.52t}$. With the initial condition $y(0) = 0$ we find the solution

$$\begin{aligned} y(t) &= 2[1 - e^{-0.52t}]/0.52 - 2te^{-0.52t} \\ &= x(t) - 2te^{-0.52t}. \end{aligned}$$

After 3 months, when $t = 1/4$, we have $x(1/4) = 0.4689 \text{ km}^3$ and $y(1/4) = 0.0298 \text{ km}^3$.

- (b) If the factory is shut down, then the flow of pollutant at the rate of $2 \text{ km}^3/\text{yr}$ is stopped. This means that the flow between the lakes and that out of Lake Sad Times will be reduced to $50 \text{ km}^3/\text{yr}$ in order to maintain the volumes. We will start time over at this point and we have the initial conditions $x(0) = x_1 = 0.4689 \text{ km}^3$, and $y(0) = y_1 = 0.0298 \text{ km}^3$. Now there is no flow of pollutant into Lake Happy Times, and the rate out is $x/2 \text{ km}^3/\text{yr}$. Hence the model equation is $x' = -x/2$. The solution is $x(t) = x_1 e^{-t/2}$.

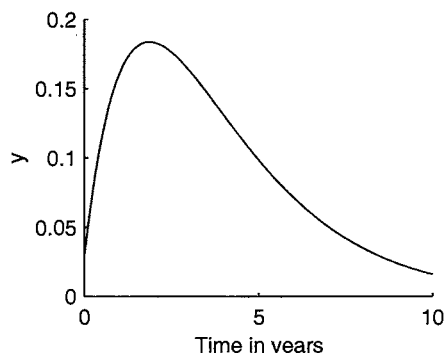
The rate into Lake Sad times is $x/2 \text{ km}^3/\text{yr}$, and the rate out is

$y/2 \text{ km}^3/\text{yr}$. The model equation is $y' = (x - y)/2 = x_1 e^{-t/2}/2 - y/2$. this time we

use the integrating factor $u(t) = e^{t/2}$ and find the solution

$$y(t) = [x_1 t/2 + y_1]e^{-t/2}.$$

The plot of the solution over 10 years is shown in the following figure. It is perhaps a little surprising to see that the level of pollution in Lake Sad Times continues to rise for some time after the factory is closed.



Using a computer or a calculator, we find that $y(t) = y_1/2$ when $t = 10.18$ yrs.

14. Let $x(t)$ represent the amount of salt in Tank I at time t . The rate at which salt enters Tank I is

$$\text{rate in I} = a \text{ lb/gal} \times b \text{ gal/min} = ab \text{ lb/min.}$$

Salt leaves Tank II at

$$\begin{aligned} \text{rate out I} &= x(t)/V \text{ lb/gal} \times b \text{ gal/min} \\ &= (b/V)x(t) \text{ lb/min.} \end{aligned}$$

Consequently,

$$\frac{dx}{dt} = ab - \frac{b}{V}x.$$

This equation is linear with general solution

$$x = aV + Ke^{-(b/V)t}.$$

Initially, there is no salt in Tank I, so $x(0) = 0$ produces $K = -aV$ and

$$x = aV - aVe^{-(b/V)t}.$$

Let $y(t)$ represent the amount of salt in Tank II at time t . Salt enters Tank II at the same rate as it leaves Tank I. Consequently,

$$\text{rate in II} = (b/V)x(t) \text{ lb/min.}$$

Salt leaves Tank II at

$$\begin{aligned} \text{rate out II} &= y(t)/V \text{ lb/gal} \times b \text{ gal/min} \\ &= (b/V)y(t) \text{ lb/min.} \end{aligned}$$

Consequently,

$$\frac{dy}{dt} = \frac{b}{V}x - \frac{b}{V}y.$$

Substitute the solution found for x .

$$\begin{aligned} \frac{dy}{dt} &= \frac{b}{V}(aV - aVe^{-(b/V)t}) - \frac{b}{V}y, \\ \frac{dy}{dt} &= -\frac{b}{V}y + (ab - abe^{-(b/V)t}). \end{aligned}$$

This equation is also linear, with integrating factor $e^{(b/V)t}$, so

$$\begin{aligned} (e^{(b/V)t}y)' &= ab(e^{(b/V)t} - 1), \\ e^{(b/V)t}y &= aVe^{(b/V)t} - abt + L, \\ y &= aV - abte^{-(b/V)t} + Le^{-(b/V)t}. \end{aligned}$$

Initially, there is no salt in Tank II, so $y(0) = 0$ produces $L = -aV$ and

$$y = aV - abte^{-(b/V)t} - aVe^{-(b/V)t}.$$

Section 2.6. Exact Differential Equations

1. $dF = 2ydx + (2x + 2y)dy$

2. $dF = (2x - 2y)dx + (-x + 2y)dy$

3. $dF = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}$

4. $dF = \frac{-xdx - ydy}{(x^2 + y^2)^{3/2}}$

5.

$$dF = \frac{1}{x^2 + y^2}(x^2 y dx + y^3 dx - y dx + x^3 dy + x y^2 dy + x dy)$$

6. $dF = (1/x + 2xy^3)dx + (1/y + 3x^2y^2)dy$

7.

$$dF = \left(\frac{2x}{x^2 + y^2} + 1/y \right) dx + \left(\frac{2y}{x^2 + y^2} - \frac{x}{y^2} \right) dy$$

8. $dF = \frac{ydx - xdy + 4x^2y^3dy + 4y^5dy}{x^2 + y^2}$

9. With $P = 2x + y$ and $Q = x - 6y$, we see that

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$$

so the equation is exact. We solve by setting

$$F(x, y) = \int P(x, y) dx = \int (2x + y) dx = x^2 + xy + \phi(y).$$

To find ϕ , we differentiate

$$Q(x, y) = \frac{\partial F}{\partial y} = x + \phi'(y).$$

Hence $\phi' = -6y$, and we can take $\phi(y) = -3y^2$.
Hence the solution is $F(x, y) = x^2 + xy - 3y^2 = C$.

10. With $P = 1 - y \sin x$ and $Q = \cos x$, we see that

$$\frac{\partial P}{\partial y} = -\sin x = \frac{\partial Q}{\partial x},$$

so the equation is exact. We solve by setting

$$F(x, y) = \int P(x, y) dx = \int (1 - y \sin x) dx = x + y \cos x + \phi(y).$$

To find ϕ , we differentiate

$$Q(x, y) = \frac{\partial F}{\partial y} = \cos x + \phi'(y).$$

Thus $\phi' = 0$, so we can take $\phi = 0$. Hence the solution is $F(x, y) = x + y \cos x = C$.

11. With $P = 1 + \frac{y}{x}$ and $Q = -\frac{1}{x}$, we compute

$$\frac{\partial P}{\partial y} = \frac{1}{x} \neq \frac{\partial Q}{\partial x} = \frac{1}{x^2}.$$

Hence the equation is not exact.

12. With $P = \frac{x}{\sqrt{x^2 + y^2}}$ and $Q = \frac{y}{\sqrt{x^2 + y^2}}$, we compute

$$\frac{\partial P}{\partial y} = \frac{-2xy}{(x^2 + y^2)^{3/2}} = \frac{\partial Q}{\partial x},$$

so the equation is exact. To find the solution we integrate

$$\begin{aligned} F(x, y) &= \int P(x, y) dx \\ &= \int \frac{x}{\sqrt{x^2 + y^2}} dx \\ &= \sqrt{x^2 + y^2} + \phi(y). \end{aligned}$$

To find ϕ , we differentiate

$$Q(x, y) = \frac{\partial F}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \phi'(y).$$

Thus $\phi' = 0$, so we can take $\phi = 0$. Hence the solution is $F(x, y) = \sqrt{x^2 + y^2} = C$.

13. Exact $x^3 + xy - y^3 = C$

14. Not exact

15. Exact $u^2/2 + vu - v^2/2 = C$

16. Exact $\ln(u^2 + v^2) = C$

17. Not exact

18. Exact. $F(u, y) = y \ln u - 2u = C$

19. Exact $x \sin 2t - t^2 = C$

20. Exact $x^2 y^2 + x^4 = C$

21. Not exact.

22. $-x/y + \ln x = C$

23. $x^2 y^2/2 - \ln x + \ln y = C$

24. $\frac{-(y+1)^2}{x^3} = C$

25. $x - (1/2) \ln(x^2 + y^2) = C$

26. $\mu(x) = 1/x^2$. $F(x, y) = \frac{xy^2/2 - y}{x} = C$

27. $\mu(x) = \frac{1}{x}$. $F(x, y) = xy - \ln x - \frac{y^2}{2} = C$.

28. $\mu(y) = 1/\sqrt{y}$. $F(x, y) = 2x\sqrt{y} + (2/3)y^{3/2} = C$

29. $\mu(y) = 1/y^2$. $F(x, y) = \frac{yx + x^2}{y} = C$

30. $F(x, y) = x^2 y^3 = C$

31. $x + y$ and $x - y$ are homogeneous of degree one.

32. $x^2 - xy - y^2$ and $4xy$ are homogeneous of degree two.

33. $x - \sqrt{x^2 + y^2}$ and $-y$ are homogeneous of degree one.

34. $\ln x - \ln y$ and 1 are homogeneous of degree zero.

35. $x^2 - Cx = y^2$

36.

$$F(x, y) = -(1/2) \ln \left(\frac{x^2 + y^2}{x^2} \right) + \arctan(y/x) - \ln x = C$$

37. $F(x, y) = xy + (3/2)x^2 = C$.

38. $y^2 x^2 - 4 \ln y - 2 \ln x = C$

39. $y(x) = \frac{x + Cx^4}{1 - 2Cx^3}$

40. $y(x) = x \ln(C + 2 \ln x)$

41. (a) First,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_0 \sin \theta - \omega}{v_0 \cos \theta}.$$

However, $\cos \theta = x/\sqrt{x^2 + y^2}$ and $\sin \theta = y/\sqrt{x^2 + y^2}$, so

$$\frac{dy}{dx} = \frac{\frac{v_0 y}{\sqrt{x^2 + y^2}} - \omega}{\frac{v_0 x}{\sqrt{x^2 + y^2}}} = \frac{v_0 y - \omega \sqrt{x^2 + y^2}}{v_0 x}.$$

Divide top and bottom by v_0 and replace ω/v_0 with k .

$$\frac{dy}{dx} = \frac{y - \frac{\omega}{v_0} \sqrt{x^2 + y^2}}{x} = \frac{y - k \sqrt{x^2 + y^2}}{x}.$$

(b) Write the equation

$$\frac{dy}{dx} = \frac{y - k \sqrt{x^2 + y^2}}{x}$$

in the form

$$(y - k \sqrt{x^2 + y^2}) dx - x dy = 0.$$

Both terms are homogeneous (degree 1), so make substitutions $y = xv$ and $dy = x dv + v dx$.

$$(xv - k\sqrt{x^2 + x^2v^2}) dx - x(x dv + v dx) = 0$$

After cancelling the common factor x and combining terms,

$$k\sqrt{1+v^2} dx - x dv = 0.$$

Separate variables and integrate.

$$\begin{aligned}\frac{k dx}{x} - \frac{dv}{\sqrt{1+v^2}} &= 0 \\ k \ln x - \ln(\sqrt{1+v^2} + v) &= C\end{aligned}$$

Note the initial condition $(x, y) = (a, 0)$. Because $y = xv$, v must also equal zero at this point. Thus, $(x, v) = (a, 0)$ and

$$\begin{aligned}k \ln a - \ln(\sqrt{1+0^2} + 0) &= C \\ C &= k \ln a.\end{aligned}$$

Therefore,

$$k \ln x - \ln(\sqrt{1+v^2} + v) = k \ln a.$$

Taking the exponential of both sides,

$$\begin{aligned}e^{\ln x^k - \ln(\sqrt{1+v^2} + v)} &= e^{\ln a^k} \\ \frac{x^k}{v + \sqrt{1+v^2}} &= a^k \\ \left(\frac{x}{a}\right)^k &= v + \sqrt{1+v^2}.\end{aligned}$$

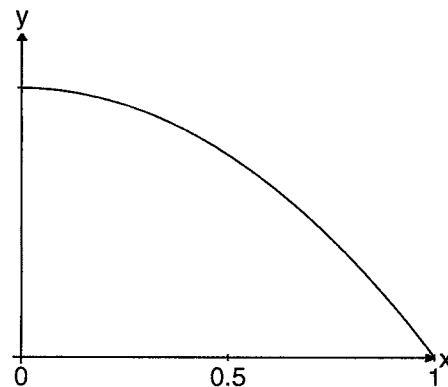
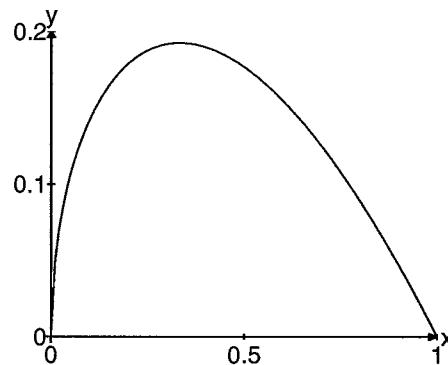
Solve for v .

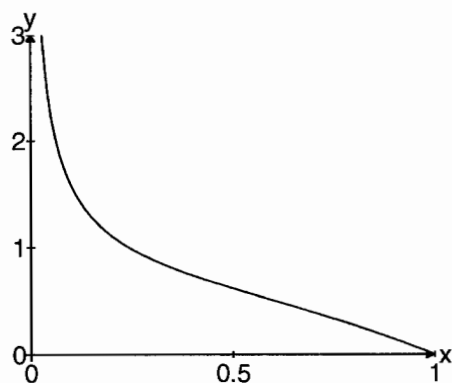
$$\begin{aligned}\left(\frac{x}{a}\right)^k - v &= \sqrt{1+v^2} \\ \left(\frac{x}{a}\right)^{2k} - 2v\left(\frac{x}{a}\right)^k + v^2 &= 1 + v^2 \\ \left(\frac{x}{a}\right)^{2k} - 1 &= 2v\left(\frac{x}{a}\right)^k \\ \frac{1}{2} \left[\left(\frac{x}{a}\right)^k - \left(\frac{x}{a}\right)^{-k} \right] &= v\end{aligned}$$

Finally, recall that $y = xv$, so

$$\begin{aligned}\frac{y}{x} &= \frac{1}{2} \left[\left(\frac{x}{a}\right)^k - \left(\frac{x}{a}\right)^{-k} \right] \\ y &= \frac{a}{2} \left[\left(\frac{x}{a}\right)^{1+k} - \left(\frac{x}{a}\right)^{1-k} \right].\end{aligned}$$

- (c) The following three graphs show the cases where $a = 1$, and $k = 1/2, 1, 3/2$. When $0 < k < 1$, the wind speed is less than that of the goose and the goose flies home. When $k = 1$ the two speeds are equal, and try as he might, the goose can't get home. Instead he approaches a point due north of the nest. When $k > 1$ the wind speed is greater, so the goose loses ground and keeps getting further from the nest.

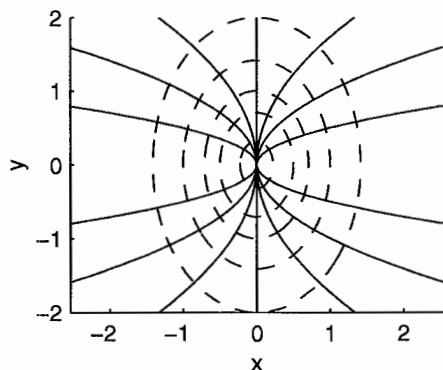




42. The hyperbolas with $F(x, y) = y^2/x = C$ are the solid curves in the following figure. The orthogonal family must satisfy

$$\frac{dy}{dx} = \frac{\partial F}{\partial y} / \frac{\partial F}{\partial x} = -\frac{2x}{y}.$$

The solution to this separable equation is found to be given implicitly by $G(x, y) = 2x^2 + y^2 = C$. These curves are the dashed ellipses in the accompanying figure. They do appear to be orthogonal.



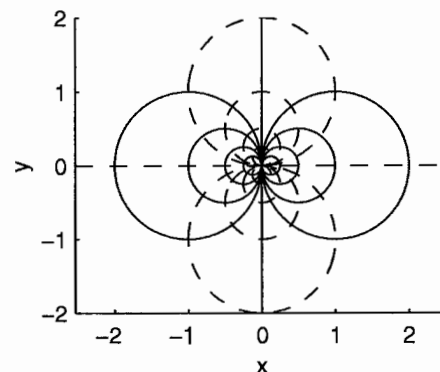
43. (a) The curves are defined by the equation $F(x, y) = x/(x^2 + y^2) = c$. Hence the orthogonal family must satisfy

$$\frac{dy}{dx} = \frac{\partial F}{\partial y} / \frac{\partial F}{\partial x} = \frac{2xy}{x^2 - y^2}.$$

- (b) The differential equation is homogeneous. Solving in the usual way we find that the orthogonal family is defined implicitly by

$$G(x, y) = \frac{y}{x^2 + y^2} = C.$$

The original curves are the solid curves in the following figure, and the orthogonal family is dashed.



44. $\ln(y^2 + x^2) - (2/3)y^3 = C$
 45. $\arctan(y/x) - y^4/4 = C$
 46. Assuming that $m \neq n - 1$, divide both sides of

$$x dy + y dx = x^m y^n dx$$

by $x^n y^n$ to obtain

$$\begin{aligned} \frac{x dy + y dx}{(xy)^n} &= x^{m-n} dx \\ \frac{d((xy)^{1-n})}{1-n} &= x^{m-n} dx. \end{aligned}$$

Thus, because $m - n + 1 \neq 0$,

$$\begin{aligned} \frac{(xy)^{1-n}}{1-n} &= \frac{x^{m-n+1}}{m-n+1} + C \\ (m-n+1)(xy)^{1-n} - (1-n)x^{m-n+1} &= C. \end{aligned}$$

47. $\arctan(y/x) - (1/4)(y^2 + x^2)^2 = C$
 48. $x/y - \ln(xy + 1) = C$

49. $Cxy = \frac{x+y}{x-y}$

50. (a) An exterior angle of a triangle equals the sum of its two remote interior angles, so $\theta = \phi + \alpha$. We're given that $\alpha = \beta$ and ϕ and β are corresponding angles on the same side of a transversal cutting parallel lines, so $\phi = \beta$. Thus, $\theta = \phi + \alpha = \beta + \beta = 2\beta$ and

$$\tan \theta = \tan 2\beta = \frac{2 \tan \beta}{1 - \tan^2 \beta}.$$

However, $\tan \theta = y/x$ and $\tan \beta$ equals the slope of the tangent line to $y = y(x)$ at the point (x, y) ; i.e., $\tan \beta = y'$. Thus,

$$\frac{y}{x} = \frac{2y'}{1 - (y')^2}.$$

- (b) Use the result from part (a) and cross multiply.

$$\begin{aligned} y - y(y')^2 &= 2xy' \\ 0 &= y(y')^2 + 2xy' - y \end{aligned}$$

Use the quadratic formula to solve for y' .

$$y' = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$$

Rearranging,

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$$

becomes

$$\pm \frac{x dx + y dy}{\sqrt{x^2 + y^2}} = dx.$$

The trick now is to recognize that the left-hand side equals $\pm d(\sqrt{x^2 + y^2})$. Thus, when we integrate,

$$\begin{aligned} \pm d(\sqrt{x^2 + y^2}) &= dx \\ \pm \sqrt{x^2 + y^2} &= x + C. \end{aligned}$$

Square, then solve for y^2 .

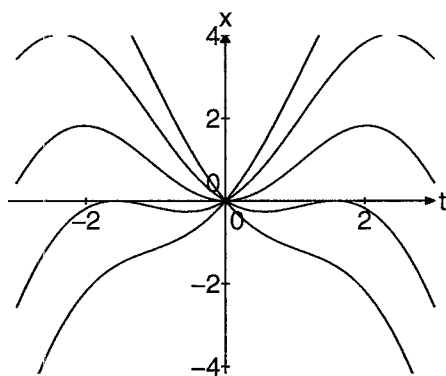
$$\begin{aligned} x^2 + y^2 &= x^2 + 2Cx + C^2 \\ y^2 &= 2Cx + C^2 \end{aligned}$$

This, as was somewhat expected, is the equation of a parabola.

Section 2.7. Existence and Uniqueness of Solutions

1. The right hand side of the equation is $f(t, y) = 4 + y^2$. f is continuous in the whole plane. Its partial derivative $\partial f / \partial y = 2y$ is also continuous on the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.
2. The right hand side of the equation is $f(t, y) = \sqrt{y}$. f is defined only where $y \geq 0$, and it is continuous there. However, $\partial f / \partial y = 1/(2\sqrt{y})$, which is only continuous for $y > 0$. Our initial condition is at $y_0 = 0$, and $t_0 = 4$. There is no rectangle containing (t_0, y_0) where both f and $\partial f / \partial y$ are defined and continuous. Consequently the hypotheses of the theorem are not satisfied.
3. The right hand side of the equation is $f(t, y) = t \tan^{-1} y$, which is continuous in the whole plane. $\partial f / \partial y = t/(1 + y^2)$ is also continuous in the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.
4. The right hand side of the equation is $f(s, \omega) = \omega \sin \omega + s$, which is continuous in the whole plane. $\partial f / \partial \omega = \sin \omega + \omega \cos \omega$ is also continuous in the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

5. The right hand side of the equation is $f(t, x) = t/(x + 1)$, which is continuous in the whole plane, except where $x = -1$. $\partial f/\partial x = -t/(x + 1)^2$ is also continuous in the whole plane, except where $x = -1$. Hence the hypotheses are satisfied in a rectangle containing the initial point $(0, 0)$, so the theorem guarantees a unique solution.
6. The right hand side of the equation is $f(x, y) = y/x + 2$, which is continuous in the whole plane, except where $x = 0$. Since the initial point is $(0, 1)$, f is discontinuous there. Consequently there is no rectangle containing this point in which f is continuous. The hypotheses are not satisfied, so the theorem does not guarantee a unique solution.
7. The equation is linear. The general solution is $y(t) = t \sin t + Ct$. Several solutions are plotted in the following figure.



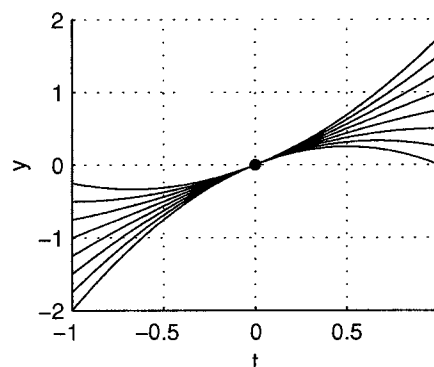
Since every solution satisfies $y(0) = 0$, there is no solution with $y(0) = -3$. If we put the equation into normal form

$$\frac{dy}{dt} = \frac{1}{t}y + t \cos t,$$

we see that the right hand side $f(t, y)$ fails to be continuous at $t = 0$. Consequently the hypotheses of the existence theorem are not satisfied.

8. The equation is linear. The general solution is $y(t) = t + 2Ct^2$. Several solutions are plotted in

the following figure.



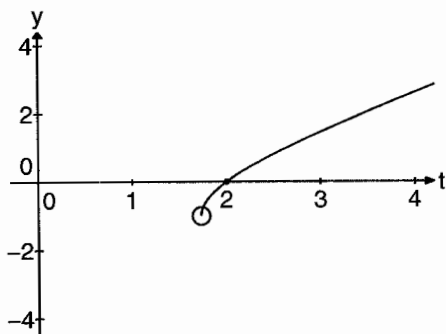
Since the general solution is $y(t) = t + 2Ct^2$, every solution satisfies $y(0) = 0$. There is no solution with $y(0) = 2$. If we put the equation into normal form

$$\frac{dy}{dt} = \frac{2y - t}{t},$$

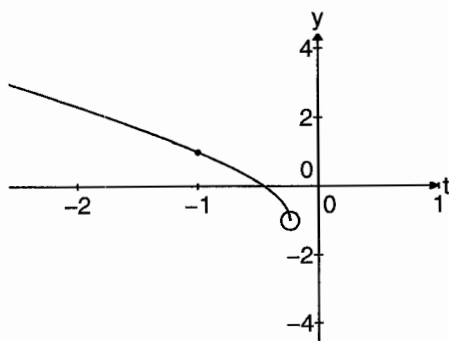
we see that the right hand side $f(t, y) = (2y - t)/t$ fails to be continuous at $t = 0$. Consequently the hypotheses of the existence theorem are not satisfied.

9. The y -derivative of the right hand side $f(t, y) = 3y^{2/3}$ is $2y^{-1/3}$, which is not continuous at $y = 0$. Hence the hypotheses of Theorem 7.16 are not satisfied.
10. The y -derivative of the right hand side $f(t, y) = ty^{1/2}$ is $ty^{-1/2}/2$ which is not continuous at $y = 0$. Hence the hypotheses of Theorem 7.16 are not satisfied.
11. The exact solution is $y(t) = -1 + \sqrt{t^2 - 3}$. The interval of solution is $(\sqrt{3}, \infty)$. The solver has trouble near $\sqrt{3}$. The point where the difficulty arises is

circled in the following figure.

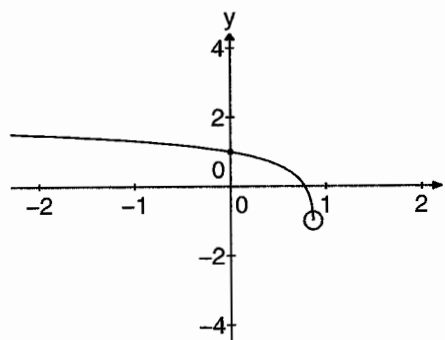


12. The exact solution is $y(t) = -1 + \sqrt{t^2 - 4t - 1}$. The interval of existence is $(-\infty, 2 - \sqrt{5})$. The solver has trouble near $2 - \sqrt{5} \approx -0.2361$. The point where the difficulty arises is circled in the following figure.

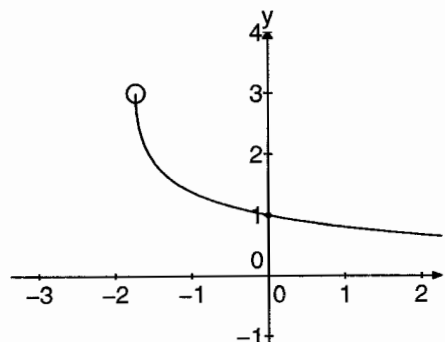


13. The exact solution is $y(t) = -1 + \sqrt{4 + 2 \ln(1 - t)}$. The interval of existence is $(-\infty, 1 - e^{-2})$. The solver has trouble near $1 - e^{-2} \approx 0.8647$. The point where the difficulty arises is circled in the following

figure.

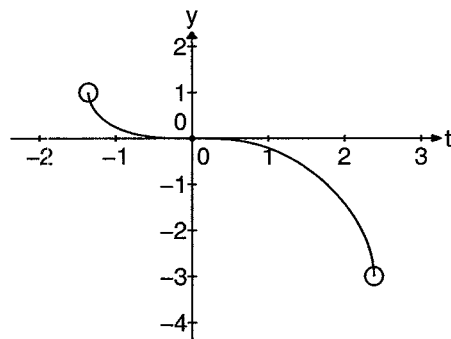


14. The exact solution is $y(t) = 3 - \sqrt{4 + 2 \ln(t + 2) - 2 \ln 2}$. The interval of existence is $(-2 + 2e^{-2}, \infty)$. The solver has trouble near $-2 + 2e^{-2} \approx -1.7293$. The point where the difficulty arises is circled in the following figure.

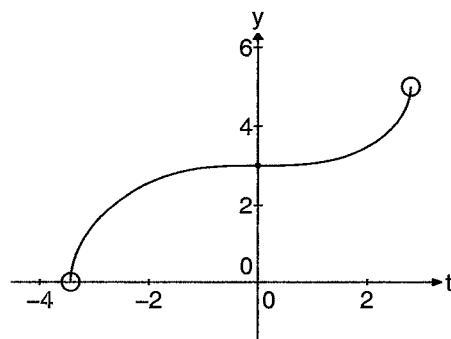


15. The solution is defined implicitly by the equation $y^3/3 + y^2 - 3y = 2t^3/3$. The solver has trouble near $(t_1, 1)$, where $t_1 = -(5/2)^{1/3} \approx -1.3572$, and also near $(t_2, -3)$, where $t_2 = (27/2)^{1/3} \approx 2.3811$. The points where the difficulty arises are circled in the

following figure.

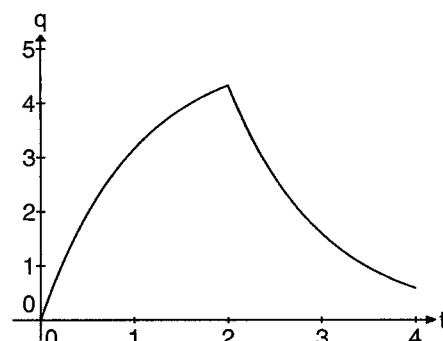


16. The solution is defined implicitly by the equation $2y^3 - 15y^2 + 2t^3 = -81$. The solver that trouble near $(t_1, 0)$, where $t_1 = -(81/2)^{1/3} \approx -3.4341$, and also near $(t_2, 5)$, where $t_2 = 22^{1/3} \approx 2.8020$. The points where the difficulty arises are circled in the following figure.



17. The computed solution is shown in the following figure.

ure.

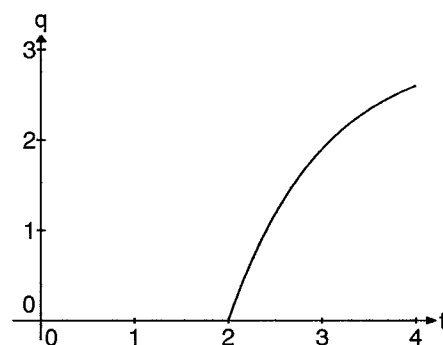


The exact solution is

$$q(t) = \begin{cases} 5 - 5e^{-t}, & \text{if } 0 < t < 2, \\ 5(1 - e^{-2})e^{2-t}, & \text{if } t \geq 2 \end{cases}$$

Hence $q(4) = 5(1 - e^{-2})e^{-2} \approx 0.5851$.

18. The computed solution is shown in the following figure.



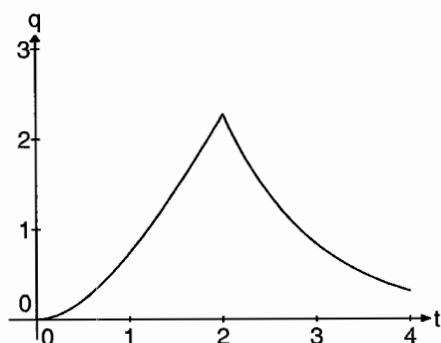
The exact solution is

$$q(t) = \begin{cases} 0, & \text{if } 0 < t < 2, \\ 3(1 - e^{2-t}), & \text{if } t \geq 2 \end{cases}$$

Hence $q(4) = 3(1 - e^{-2}) \approx 2.5940$.

19. The computed solution is shown in the following figure.

ure.

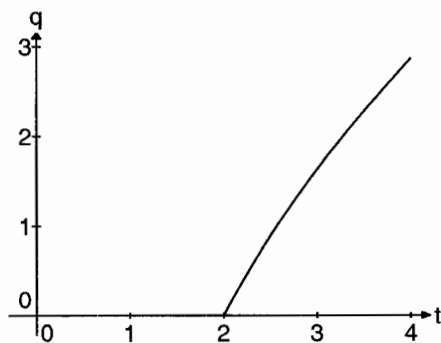


The exact solution is

$$q(t) = \begin{cases} 2(t - 1 + e^{-t}), & \text{if } 0 < t < 2, \\ 2(1 + e^{-2})e^{2-t}, & \text{if } t \geq 2 \end{cases}$$

Hence $q(4) = 2(1 + e^{-2})e^{-2} \approx 0.3073$.

20. The computed solution is shown in the following figure.



The exact solution is

$$q(t) = \begin{cases} 0, & \text{if } 0 < t < 2, \\ t - 1 - e^{2-t}, & \text{if } t \geq 2 \end{cases}$$

Hence $q(4) = 3 - e^{-2} \approx 2.8647$.

21. (a) If

$$y(t) = \begin{cases} 0, & \text{if } t \leq t_0 \\ (t - t_0)^3, & \text{if } t > t_0, \end{cases}$$

then

$$\begin{aligned} y'(t_0^+) &= \lim_{t \rightarrow t_0^+} \frac{y(t) - y(t_0)}{t - t_0} \\ &= \lim_{t \rightarrow t_0^+} \frac{(t - t_0)^3 - 0}{t - t_0} \\ &= \lim_{t \rightarrow t_0^+} (t - t_0)^2 \\ &= 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} y'(t_0^-) &= \lim_{t \rightarrow t_0^-} \frac{y(t) - y(t_0)}{t - t_0} \\ &= \lim_{t \rightarrow t_0^-} \frac{0 - 0}{t - t_0} \\ &= 0. \end{aligned}$$

Therefore, $y'(t_0) = 0$, since both the left and right-hand derivatives equal zero.

- (b) The right hand side of the equation, $f(t, y) = 3y^{2/3}$, is continuous, but $\partial f / \partial y = 2y^{-1/3}$ is not continuous where $y = 0$. Hence the hypotheses of Theorem 7.16 are not satisfied.

22. (a) If

$$y(t) = \begin{cases} 3e^{-2t}, & \text{if } t < 1 \\ 5/2 + (3 - 5e^2/2)e^{-2t}, & \text{if } t \geq 1 \end{cases}$$

then it is easily seen that

$$y'(t) = \begin{cases} -6e^{-2t}, & t < 1 \\ (-6 + 5e^2)e^{-2t}, & t > 1. \end{cases}$$

It remains to find the derivative at $t = 1$. Remember, because of the “cusp” at $t = 1$, we suspect that this derivative will not exist. First,

the derivative from the right,

$$\begin{aligned} y'_+(1) &= \lim_{t \rightarrow 1^+} \frac{y(t) - y(1)}{t - 1} \\ &= \lim_{t \rightarrow 1^+} \frac{1}{t - 1} \left[\left(\frac{5}{2} + \left(3 - \frac{5e^2}{2} \right) e^{-2t} \right) - \left(\frac{5}{2} + \left(3 - \frac{5e^2}{2} \right) e^{-2} \right) \right] \\ &= \lim_{t \rightarrow 1^+} \frac{\left(3 - \frac{5e^2}{2} \right) e^{-2t} - \left(3 - \frac{5e^2}{2} \right) e^{-2}}{t - 1} \end{aligned}$$

Note the indeterminate form $0/0$, so l'Hôpital's rule applies.

$$\begin{aligned} y'_+ &= \lim_{t \rightarrow 1^+} \left[-2 \left(3 - \frac{5e^2}{2} \right) e^{-2t} \right] \\ &= 5 - 6e^{-2} \end{aligned}$$

However, the derivative from the left,

$$\begin{aligned} y'_-(1) &= \lim_{t \rightarrow 1^-} \frac{y(t) - y(1)}{t - 1} \\ &= \lim_{t \rightarrow 1^-} \frac{3e^{-2t} - \left(\frac{5}{2} + \left(3 - \frac{5e^2}{2} \right) e^{-2} \right)}{t - 1} \\ &= \lim_{t \rightarrow 1^-} \frac{3e^{-2t} - 3e^{-2}}{t - 1}. \end{aligned}$$

Again, an indeterminate form $0/0$, so we apply l'Hôpital's rule.

$$\begin{aligned} y'_-(1) &= \lim_{t \rightarrow 1^-} (-6e^{-2t}) \\ &= -6e^{-2}. \end{aligned}$$

- (b) The derivative from the left doesn't equal the derivative from the right. The function $y = y(t)$ is not differentiable at $t = 1$ and cannot be a solution of the differential equation on any interval containing $t = 1$.

- (c) We have that

$$y'(t) = \begin{cases} -6e^{-2t}, & t < 1 \\ (-6 + 5e^2)e^{-2t} & t > 1. \end{cases}$$

If

$$f(t) = \begin{cases} 0, & \text{if } t < 1 \\ 5, & \text{if } t > 1, \end{cases}$$

then

$$-2y + f(t) = \begin{cases} -6e^{-2t}, & t < 1 \\ -5 + (-6 + 5e^2)e^{-2t}, & t > 1. \end{cases}$$

Thus, $y = y(t)$ is a solution of $y' = -2y + f(t)$ on any interval not containing $t = 1$.

23. If

$$y(t) = \begin{cases} 0, & t < 0 \\ t^4, & t \geq 0, \end{cases}$$

then it is easily seen that

$$y'(t) = \begin{cases} 0, & t < 0 \\ 4t^3, & t > 0. \end{cases}$$

It remains to check the existence of $y'(0)$. First, the left-derivative.

$$y'_-(0) = \lim_{t \rightarrow 0^-} \frac{y(t) - y(0)}{t - 0} = \lim_{t \rightarrow 0^-} \frac{0 - 0}{t} = 0.$$

Secondly,

$$y'_+(0) = \lim_{t \rightarrow 0^+} \frac{y(t) - y(0)}{t - 0} = \lim_{t \rightarrow 0^+} \frac{t^4}{t} = \lim_{t \rightarrow 0^+} t^3 = 0.$$

Thus, $y'(0) = 0$ and we can write

$$y'(t) = \begin{cases} 0, & t < 0 \\ 4t^3, & t \geq 0. \end{cases}$$

Now,

$$ty'(t) = \begin{cases} 0, & t < 0 \\ 4t^4, & t \geq 0, \end{cases}$$

and

$$4y(t) = \begin{cases} 0, & t < 0 \\ 4t^4, & t \geq 0, \end{cases}$$

so $y = y(t)$ is a solution of $ty' = 4y$. Finally, $y(0) = 0$. In a similar manner, it is not difficult to show that

$$\omega(t) = \begin{cases} 0, & t < 0 \\ 5t^4, & t \geq 0 \end{cases}$$

is also a solution of the initial value problem $ty' = 4y$, $y(0) = 0$. At first glance, it would appear that we have contradicted uniqueness. However, if $ty' = 4y$ is written in normal form,

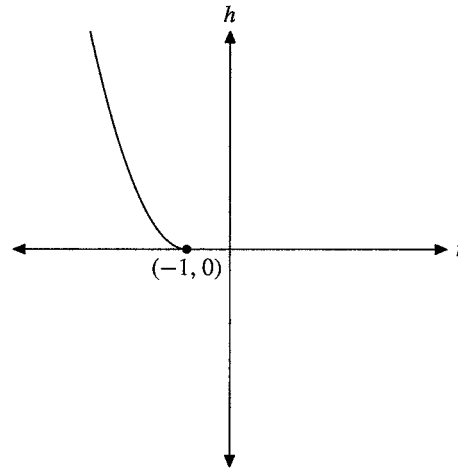
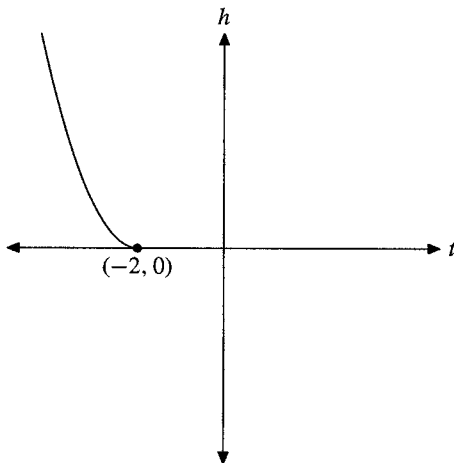
$$y' = \frac{4y}{t},$$

then

$$\frac{\partial}{\partial y} \left(\frac{4y}{t} \right) = \frac{4}{t}$$

is *not* continuous on any rectangular region containing the vertical axis (where $t = 0$), so the hypotheses of the Uniqueness Theorem are not satisfied. There is no contradiction of uniqueness.

24. (a) The point here is the fact that you don't know the moment the water completely drained. Here are two possibilities.



- (b) Let A represent the cross area of the drum and h the height of the water in the drum. Then Δh represents the change in height and $A\Delta h$ the volume of water that has left the drum. A particle of water leaving the drain at speed v travels a distance $v\Delta t$ in a time Δt . Because a is the cross section of the drain, the volume of water leaving the drain in time Δt is $av\Delta t$. Because the water leaving the drum in time Δt must exit the drain,

$$A\Delta h = av\Delta t$$

$$A \frac{\Delta h}{\Delta t} = av.$$

Taking the limit as $\Delta t \rightarrow 0$,

$$A \frac{dh}{dt} = av.$$

Using $v^2 = 2gh$, $v = \sqrt{2gh}$ and

$$\frac{dh}{dt} = -\frac{a}{A} \sqrt{2gh}.$$

The minus sign is present because the drum is *draining*.

- (c) If we let $\omega = \alpha h$ and $s = \beta t$, then by the chain rule

$$\frac{d\omega}{ds} = \frac{d\omega}{dh} \cdot \frac{dh}{dt} \cdot \frac{dt}{ds} = \frac{\alpha}{\beta} \frac{dh}{dt}.$$

Multiply both sides of our equation by α/β ,

$$\frac{\alpha}{\beta} \frac{dh}{dt} = - \left(\frac{a}{A} \right) \left(\frac{\alpha}{\beta} \right) \sqrt{2gh}.$$

Replace $(\alpha/\beta)(dh/dt)$ with $d\omega/ds$ and h with ω/α .

$$\begin{aligned} \frac{d\omega}{ds} &= - \left(\frac{a}{A} \right) \left(\frac{\alpha}{\beta} \right) \sqrt{\frac{2g\omega}{\alpha}} \\ \frac{d\omega}{ds} &= - \frac{1}{\beta} \left(\frac{a}{A} \right) \sqrt{2g\alpha} \sqrt{\omega}. \end{aligned}$$

Let h_0 represent the height of a full tank. This motivates the selection of $\alpha = 1/h_0$ and $\omega = h/h_0$, as $\omega = 0$ when the tank is empty and $\omega = 1$ when the tank is full. Thus,

$$\frac{d\omega}{ds} = - \frac{1}{\beta} \left(\frac{a}{A} \right) \sqrt{\frac{2g}{h_0}} \sqrt{\omega},$$

which motivates the selection of

$$\beta = \left(\frac{a}{A} \right) \sqrt{\frac{2g}{h_0}},$$

which upon substitution, gives us

$$\frac{d\omega}{ds} = -\sqrt{\omega}.$$

(d) Separate the variables and integrate.

$$\begin{aligned} \omega^{-1/2} d\omega &= -ds \\ 2\omega^{1/2} &= -s + C \\ \omega^{1/2} &= \frac{1}{2}(C - s) \\ \omega &= \frac{1}{4}(C - s)^2 \end{aligned}$$

However, as evidenced in part (a), we only want the left half of this parabola. After the drum empties, it remains empty for all time. Thus, for any $C < s_0$,

$$\omega(s) = \begin{cases} \frac{1}{4}(C - s)^2, & s < C \\ 0, & s \geq C, \end{cases}$$

is a solution of $\omega' = -\sqrt{\omega}$, $\omega(s_0) = 0$. Finally, this emergence of multiple solutions does not contradict uniqueness, because in

$$\frac{\partial}{\partial \omega}(-\sqrt{\omega}) = -\frac{1}{2\sqrt{\omega}}$$

is not continuous on any rectangle containing the horizontal axis (defined by $\omega = 0$).

25. The equation $x' = f(t, x)$ satisfies the hypotheses of the uniqueness theorem. Notice that $x_1(0) = x_2(0) = 0$. If they were both solutions $x' = f(t, x)$ near $t = 0$, then by the uniqueness theorem they would have to be equal everywhere. Since they are not, they cannot both be solutions of the differential equation.
26. The equation $x' = f(t, x)$ satisfies the hypotheses of the uniqueness theorem. Notice that $x_1(\pi/2) = x_2(\pi/2) = 0$. If they were both solutions $x' = f(t, x)$ near $t = \pi/2$, then by the uniqueness theorem they would have to be equal everywhere. Since they are not, they cannot both be solutions of the differential equation.
27. Notice that $x_1(t) = 0$ is a solution to the same differential equation with initial value $x_1(0) = 0 < 1 = x(0)$. The right hand side of the differential equation, $f(t, x) = x \cos^2 t$ and $\partial f / \partial x = \cos^2 t$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for x and x_1 cannot cross. Hence we must have $x(t) > x_1(t) = 0$ for all t .
28. Notice that $y_1(t) = 3$ is a solution to the same differential equation with initial value $y_1(1) = 3 > 1 = y(1)$. The right hand side of the differential equation, $f(t, y) = (y - 3)e^{\cos(ty)}$ and $\partial f / \partial y = e^{\cos(ty)}[1 - t(y - 3)\sin(ty)]$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for y and y_1 cannot cross. Hence we must have $y(t) < y_1(t) = 3$ for all t .
29. Notice that the right hand side of the equation is $f(t, y) = (y^2 - 1)e^{ty}$ and f is continuous on the whole plane. Its partial derivative $\partial f / \partial y = 2ye^{ty} + t(y^2 - 1)e^{ty}$ is also continuous on the whole plane. Thus the hypotheses of the uniqueness theorem are satisfied. By direct substitution we discover

that $y_1(t) = -1$ and $y_2(t) = 1$ are both solutions to the differential equation. If y is a solution and satisfies $y(1) = 0$, then $y_1(1) < y(1) < y_2(1)$. By the uniqueness theorem we must have $y_1(t) < y(t) < y_2(t)$ for all t for which y is defined. Hence $-1 < y(t) < 1$ for all t for which y is defined.

30. Notice that $x_1(t) = 0$ and $x_2(t) = 1$ are solutions to the same differential equation with initial values $x_1(0) = 0 < 1/2 = x(0) < 1 = x_2(0)$. The right hand side of the differential equation, $f(t, x) = (x^3 - x)/(1 + t^2x^2)$, and

$$\frac{\partial f}{\partial x} = \frac{(3x^2 - 1)(1 + t^2x^2) - 2t^2x(x^3 - x)}{(1 + t^2x^2)^2},$$

are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for x , x_1 , and x_2 cannot cross. Hence we must have $0 = x_1(t) < x(t) < x_2(t) = 1$ for all t .

31. Notice that $x_1(t) = t^2$ is a solution to the same differential equation with initial value $x_1(0) = 0 < 1 = x(0)$. The right hand side of the differential equation, $f(t, x) = x - t^2 + 2t$ and $\partial f/\partial x = 1$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for x and x_1 cannot cross. Hence we must have $t^2 = x_1(t) < x(t)$ for all t .

32. Notice that $y_1(t) = \cos t$ is a solution to the same differential equation with initial value $y_1(0) = 1 < 2 = y(0)$. The right hand side of the differential equation, $f(t, y) = y^2 - \cos^2 t - \sin t$ and $\partial f/\partial y = 2y$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for y and y_1 cannot cross. Hence we must have $y(t) > y_1(t) = \cos t$ for all t .

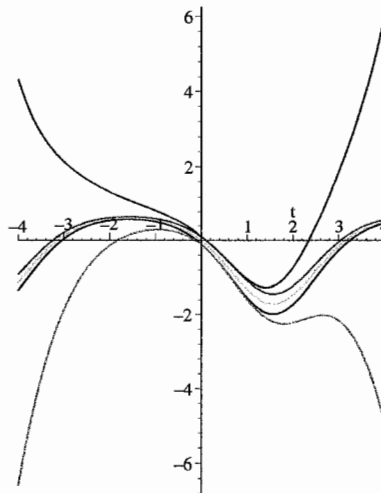
—————x—————

Section 2.8. Dependence of Solutions on Initial Conditions

1. $x(0) = 0.8009$
2. $x(0) = .9084$
3. $x(0) = 0.9596$
4. $x(0) = 0.9826$
5. $x(0) = 0.7275$
6. $x(0) = 0.72897$
7. $x(0) = 0.7290106$
8. $x(0) = 0.729011125$
9. $x(0) = -3.2314$
10. $x(0) = -3.23208$
11. $x(0) = -3.2320923$
12. $x(0) = -3.23092999999$

13. Ten! :-)

14. $1 - e^{\sin t} - (1/10)e^{|t|} \leq y(t) \leq 1 - e^{\sin t} + (1/10)e^{|t|}$

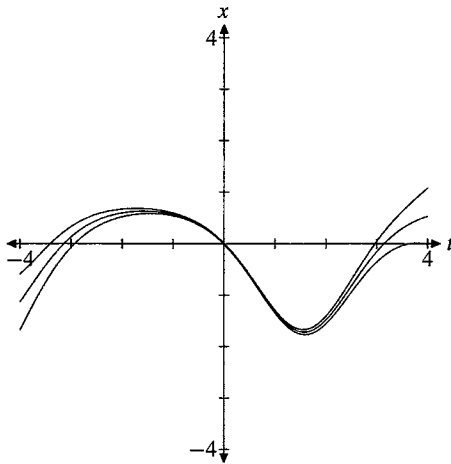


The three middle curves are the solutions to the differential equation corresponding to the initial conditions $x(0) = -.1, 0, .1$, and the outside curves are the graphs of e_L and e_H . Note how the solutions of the differential equation remain inside the graphs of e_L and e_H .

15. The only adjustment from the previous exercise is that we now want $|x_0 - y_0| < 0.01$. This leads to

$$1 - e^{\sin t} - 0.01e^{|t|} \leq y(t) \leq 1 - e^{\sin t} + 0.01e^{|t|}$$

and this image.



16. (a) The right hand side of the equation is $f(t, x) = (x - 1) \cos t$. Thus $\partial f / \partial x = \cos t$, and $\max |\partial f / \partial x| = \max |\cos t| = 1$. Hence Theorem 7.15 predicts that $|x(t) - y(t)| \leq |x(0) - y(0)|e^{|t|}$.
- (b) The equation is separable and linear, and the solutions are $x(t) = 1 - e^{\sin t}$ and $y(t) = 1 - 9e^{\sin t}/10$. Hence the separation is $x(t) - y(t) = e^{\sin t}/10$. Since $\sin t \leq |t|$, we see that
- $$|x(t) - y(t)| = e^{\sin t}/10 \leq e^{|t|}/10 = |x(0) - y(0)|e^{|t|}.$$
- (c) Since $\sin t < |t|$ except at $t = 0$, we have $|x(t) - y(t)| < e^{|t|}/10$, except at $t = 0$.

17. (a) The right hand side of the equation is $f(t, x) = -2x + \sin t$, and $\partial f / \partial x = -2$. Hence $M = \max(|\partial f / \partial x|) = 2$, and Theorem 7.15 predicts that $|y(t) - x(t)| \leq |y(0) - x(0)|e^{M|t|} = |y(0) - x(0)|e^{2|t|}$.

- (b) The equation is linear, and we find that $x(t) = [2 \sin t - \cos t]/5$, and $y(t) = [2 \sin t - \cos t]/5 - e^{-2t}/10$. Hence

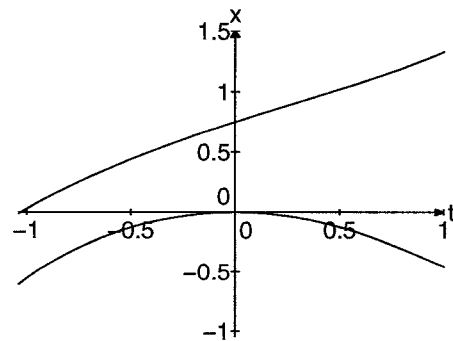
$$x(t) - y(t) = e^{-2t}/10 = |x(0) - y(0)|e^{-2t} \leq |y(0) - x(0)|e^{2|t|}.$$

- (c) Since $e^{-2t} = e^{2|t|}$ for $t < 0$, we see that the maximum predicted error is achieved for all $t < 0$.

18. The right hand side of the equation is $f(t, x) = x^2 - t$, and $\partial f / \partial x = 2x$. On the rectangle R we have $|x| \leq 2$, so $M = \max |\partial f / \partial x| = \max |2x| = 4$. Thus the bound predicted by Theorem 7.15 is

$$|x_1(t) - x_2(t)| \leq |x_1(0) - x_2(0)|e^{4|t|} = 3e^{4|t|}/4.$$

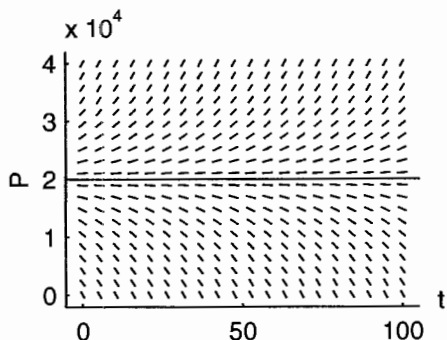
The maximum predicted error is where $|t| = 1$, and it is 40.9486. The two solutions are plotted in the following figure.



The actual bound is about 2, which is much less than 41, the theoretical bound.

Section 2.9. Autonomous Equations and Stability

1. Note that $P' = 0.05P - 1000$ is autonomous, having form $P' = f(P)$. Solving the equation $0 = f(P) = 0.05P - 1000$, we find the equilibrium point $P = 20000$. Thus, $P(t) = 20000$ is an unstable equilibrium solution, as shown in the following figure.

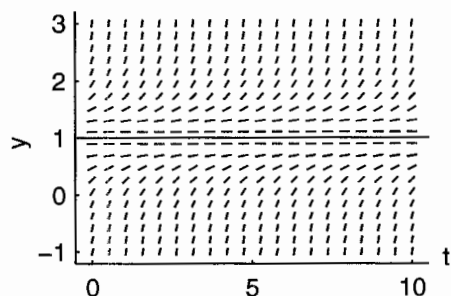


2. Note that $y' = 1 - 2y + y^2$ is autonomous, having form $y' = f(y)$. Solve the equation $f(y) = 0$ to find the equilibrium points.

$$1 - 2y + y^2 = 0$$

$$y = 1.$$

Thus, $y(t) = 1$ is an unstable equilibrium solution, as shown in the following figure.



3. Note that $x' = t^2 - x^2$ is not autonomous, having form $x' = f(t, x)$, where $f(t, x) = t^2 - x^2$. The

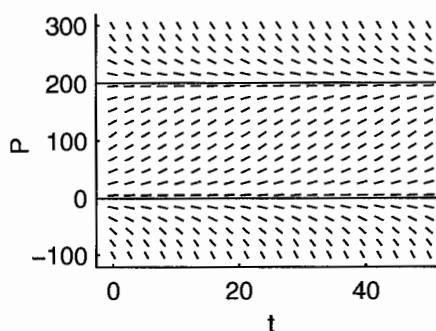
explicit dependence of the right-hand side of this differential equation on the independent variable t causes the equation to be non-autonomous.

4. Note that $P' = 0.13P(1 - P/200)$ is autonomous, having form $P' = f(P)$. Solve the equation $f(P) = 0$ to find the equilibrium points.

$$0.13P \left(1 - \frac{P}{200} \right) = 0$$

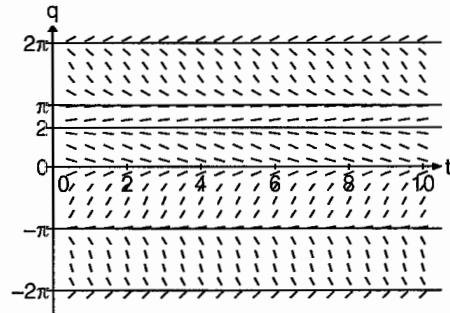
$$P = 0 \quad \text{or} \quad P = 200$$

Thus, $P(t) = 0$ and $P(t) = 200$ are equilibrium solutions, as shown in the following figure. $P = 0$ is unstable and $P = 200$ is asymptotically stable.

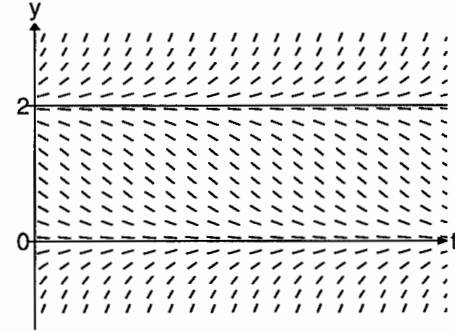


5. The equation is autonomous. The point $q = 2$ is an unstable equilibrium point, as the following figure shows. In addition every solution of $\sin q = 0$ is an equilibrium point. These are the points $k\pi$, where k is any integer, positive or negative. The stability of the equilibrium points alternates between asymptotic

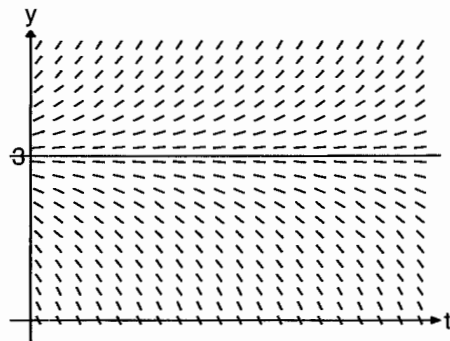
stable and unstable, as is seen in the figure.



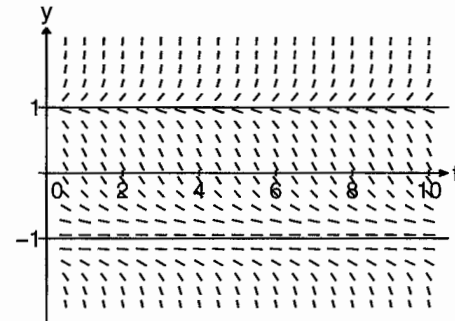
is asymptotically stable and $y = 2$ is unstable.



6. The equation is not autonomous because of the $\cos t$ term.
7. Note that the graph of $f(y)$ intercepts the y -axis at $y = 3$. Consequently, $y = 3$ is an equilibrium point ($f(3) = 0$) and $y(t) = 3$ is an equilibrium solution, shown in the following figure. The solution $y = 3$ is unstable.

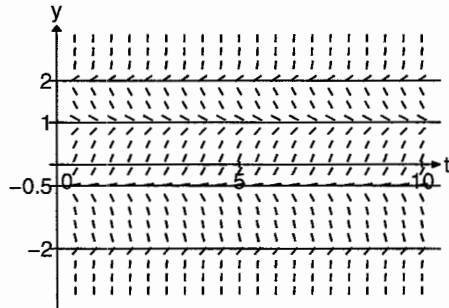


9. Since $f(y)$ has zeros at $y = -1$ and $y = 1$, these are equilibrium points. Correspondingly, $y(t) = -1$ and $y(t) = 1$ are equilibrium solutions, and are plotted in the following figure. Both are unstable.

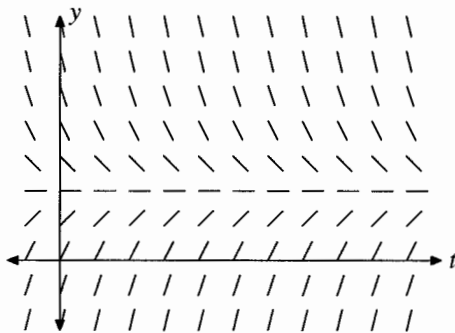


8. Note that the graph of $f(y)$ intercepts the y -axis at $y = 0$ and $y = 2$. Consequently, $y = 0$ and $y = 2$ are equilibrium points ($f(0) = 0$ and $f(2) = 0$) and $y(t) = 0$ and $y(t) = 2$ are equilibrium solutions, shown in the following figure. The solution $y = 0$
10. Since $f(y)$ has zeros at $y = -2$, $y = -1/2$, $y = 1$, and at $y = 2$, all four are equilibrium points. Correspondingly, $y(t) = -2$, $y(t) = -1/2$, $y(t) = 1$, and $y(t) = 2$ are equilibrium solutions, and are plotted in the following figure. $y = -2$ and $y = 1$ are

asymptotically stable and the other two are unstable.



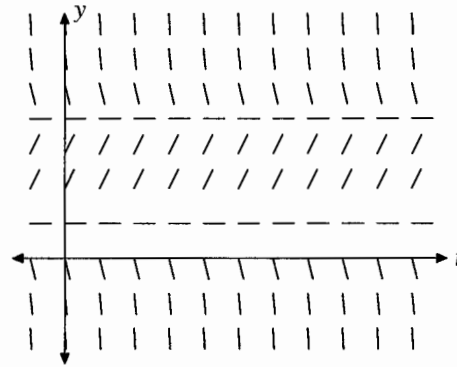
11. Because the differential equation $y' = f(y)$ is autonomous, the slope at any point (t, x) in the direction field does not depend on t , only on y , as shown in the following figure.



The equilibrium point is asymptotically stable.

12. Because the differential equation $y' = f(y)$ is autonomous, the slope at any point (t, x) in the direction field does not depend on t , only on y , as shown

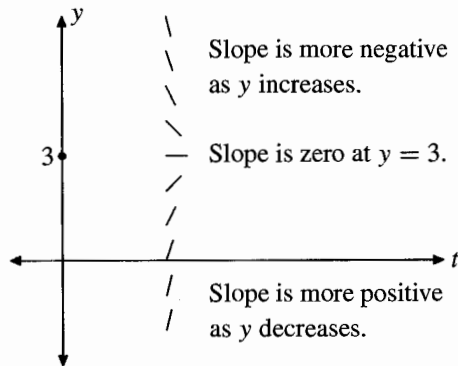
in the following figure.



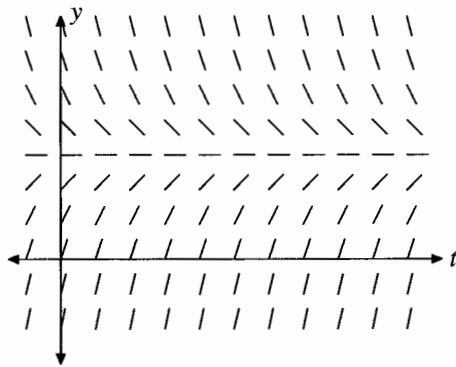
There are two equilibrium points. The smaller of them is unstable and the other is asymptotically stable.

13. The key thing to note is the fact that y' and $f(y)$ are equal. Consequently, the value of $f(y)$ is the slope of the direction line positioned at (t, y) .

- At $y = 3$, $f(y) = 0$ and the slope is zero. Thus $y = 3$ is an equilibrium point. This is shown in the following figure.
- To the right of $y = 3$, note that the graph of f dips below the y -axis. Therefore, as y increases beyond 3, the slope becomes increasingly negative. This is also shown in the following figure.
- To the left of $y = 3$, note that the graph of f rises above the y -axis. Therefore, as y decreases below 3, the slope becomes increasingly positive. This is also shown in the following figure. In particular, this means that the equilibrium point $y = 3$ is asymptotically stable.



Finally, because the equation $y' = f(y)$ is autonomous, the slope of a direction line positioned at (t, y) depends only on y and not on t . Consequently, the rest of the direction field is easily completed, as shown in the next figure.



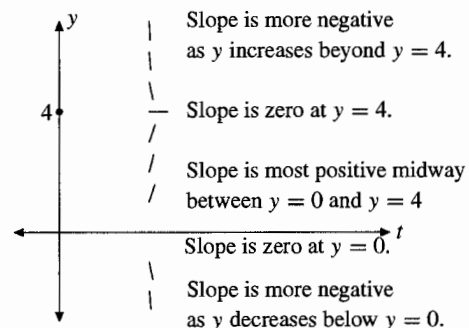
14. The key thing to note is the fact that y' and $f(y)$ are equal. Consequently, the value of $f(y)$ is the slope of the direction line positioned at (t, y) .

- At $y = 0$ and $y = 4$, $f(y) = 0$, so the slope of the direction lines at these y -values is zero. These points are the equilibrium points. This is shown in the following figure.
- To the right of $y = 4$, note that the graph of f dips below the y -axis. Furthermore, as y increases beyond 4, $f(y)$ (the slope of the direction line at (t, y)) becomes increasingly negative. This is also shown in the following figure.

- Between $y = 0$ and $y = 4$, the graph of f lies above the y -axis. Consequently, $f(y)$ is positive for $0 < y < 4$. Moreover, the graph of f has a maximum about halfway between $y = 0$ and $y = 4$. Consequently, the slope of the direction field lines will be positive between $y = 0$ and $y = 4$, with a maximum positive slope occurring about halfway between $y = 0$ and $y = 4$. This is shown in the following figure.

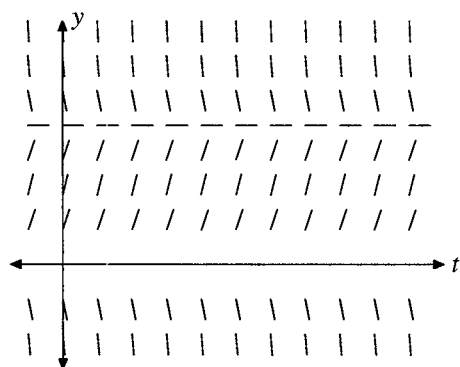
- To the left of $y = 0$, note that the graph of f falls below the y -axis. Furthermore, as y decreases below 0, $f(y)$ (the slope of the direction line at (t, y)) becomes increasingly negative. This is also shown in the next figure.

From these considerations we see that the equilibrium point $y = 0$ is unstable, and $y = 4$ is asymptotically stable.

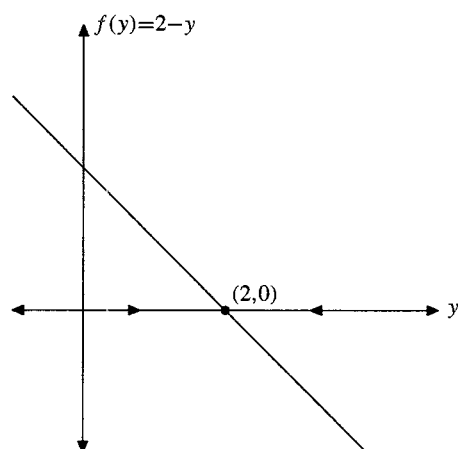


Finally, because the equation $y' = f(y)$ is autonomous, the slope of a direction line positioned at (t, y) depends only on y and not on t . Consequently, the rest of the direction field is easily completed, as

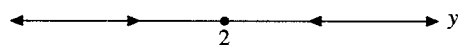
shown in the next figure.



15. (i) In this case, $f(y) = 2 - y$, whose graph is shown in the following figure.

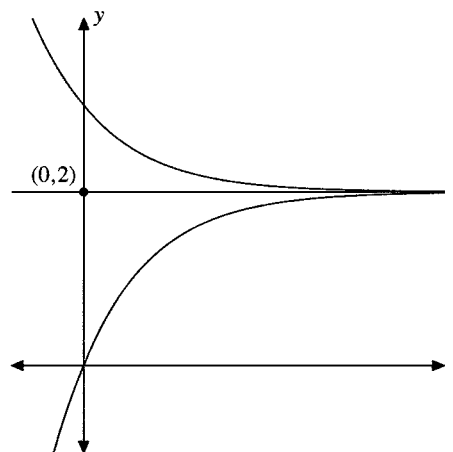


(ii) The phase line is easily captured from the previous figure, and is shown in the following figure.

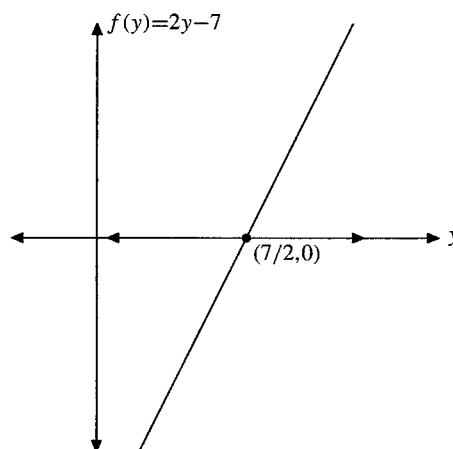


(iii) The phase line in the second figure indicates that solutions increase if $y < 2$ and decrease if $y > 2$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the

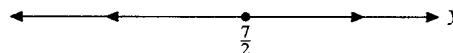
stable equilibrium solution, $y(t) = 2$.



16. (i) In this case, $f(y) = 2y - 7$, whose graph is shown in the next figure.

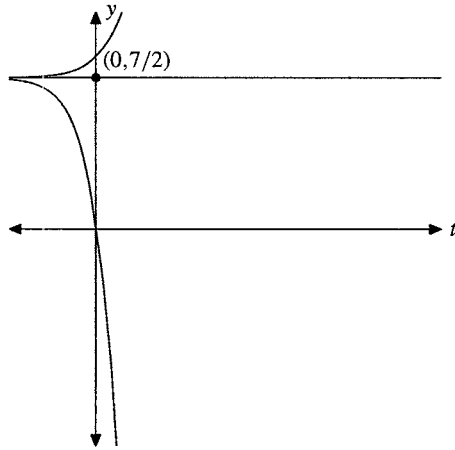


(ii) The phase line is easily captured from this figure, and is shown in next figure.

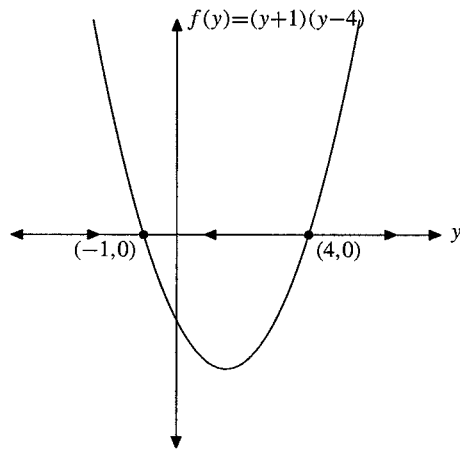


(iii) The phase line in the second figure indicates that solutions decrease if $y < 7/2$ and increase if

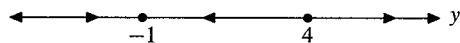
$y > 7/2$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = 7/2$.



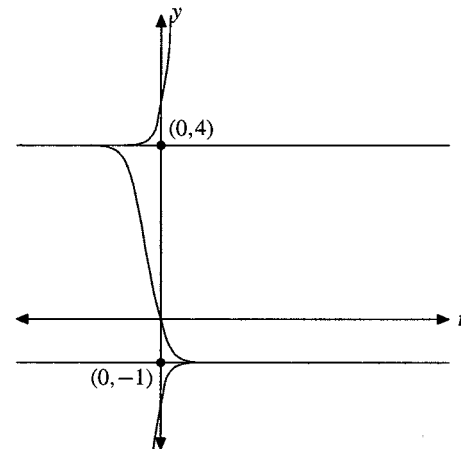
17. (i) In this case, $f(y) = (y+1)(y-4)$, whose graph is shown in the next figure.



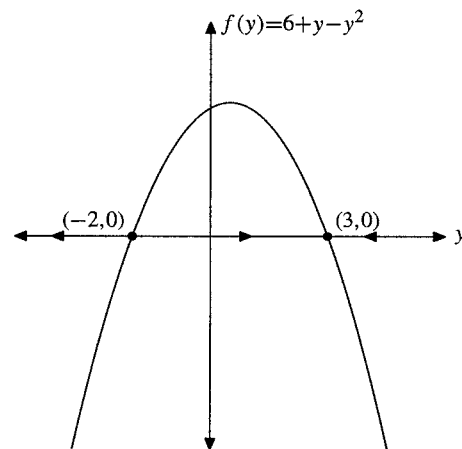
- (ii) The phase line is easily captured from the previous figure, and is shown in the next figure.



(iii) The phase line in the second figure indicates that solutions increase if $y < -1$, decrease for $-1 < y < 4$, and increase if $y > 4$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = 4$, and the stable equilibrium solution, $y(t) = -1$.

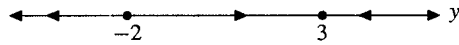


18. (i) In this case, $f(y) = 6 + y - y^2$ factors as $f(y) = (2+y)(3-y)$, whose graph is shown in the next figure.

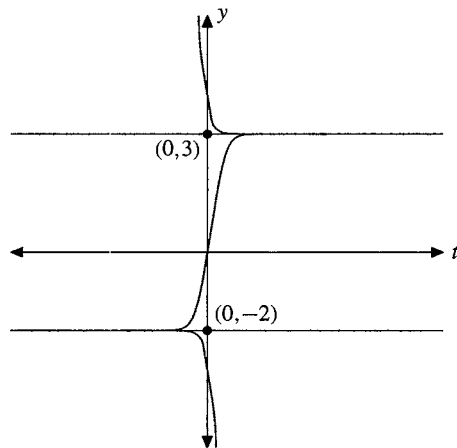


- (ii) The phase line is easily captured from the previ-

ous figure, and is shown in the next figure.

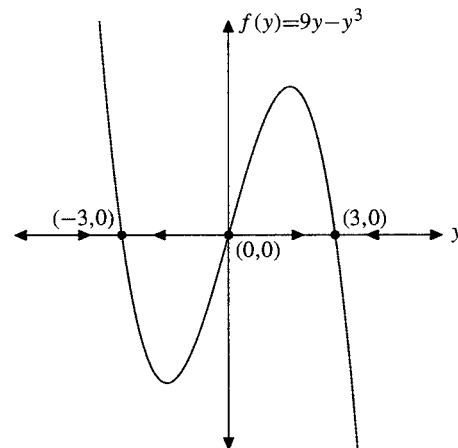


(iii) The phase line in the second figure indicates that solutions decrease if $y < -2$, increase for $-2 < y < 3$, and decrease if $y > 3$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = -2$, and the stable equilibrium solution, $y(t) = 3$.

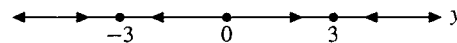


19. (i) In this case, $f(y) = 9y - y^3$ factors as $f(y) = y(y + 3)(y - 3)$, whose graph is shown in the next

figure.

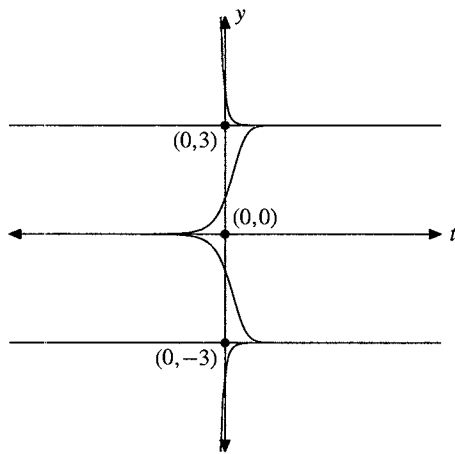


(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

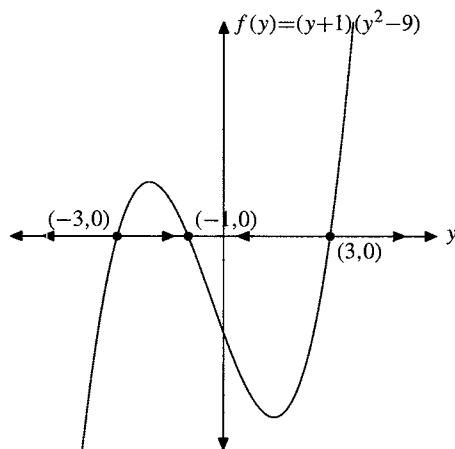


(iii) The phase line in the second figure indicates that solutions increase if $y < -3$, decrease for $-3 < y < 0$, increase if $0 < y < 3$, and decrease for $y > 3$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the stable equilibrium solution, $y(t) = -3$, the unstable equilibrium solution, $y(t) = 0$, and the stable

equilibrium solution, $y(t) = 3$.



20. (i) In this case, $f(y) = (y+1)(y^2-9)$ factors as $f(y) = (y+1)(y-3)(y+3)$, whose graph is shown in the next figure.

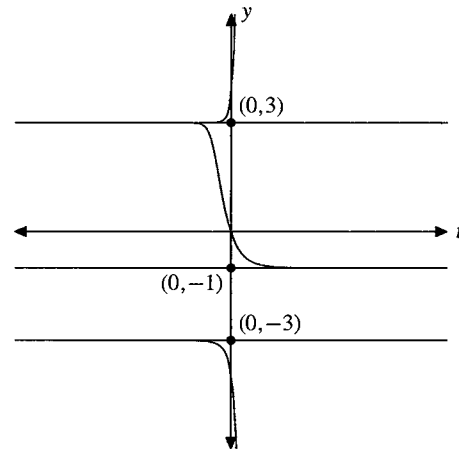


- (ii) The phase line is easily captured from the previous figure, and is shown in the next figure.



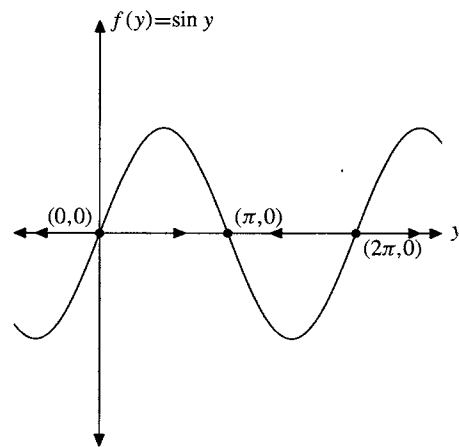
- (iii) The phase line in the second figure indicates that solutions decrease if $y < -3$, increase for

$-3 < y < -1$, decrease if $-1 < y < 3$, and increase for $y > 3$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = -3$, the stable equilibrium solution, $y(t) = -1$, and the unstable equilibrium solution, $y(t) = 3$.

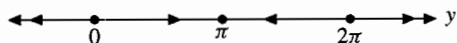


21. Due to the periodic nature of this equation, we sketch only a few regions. You can easily use the periodicity to produce more regions.

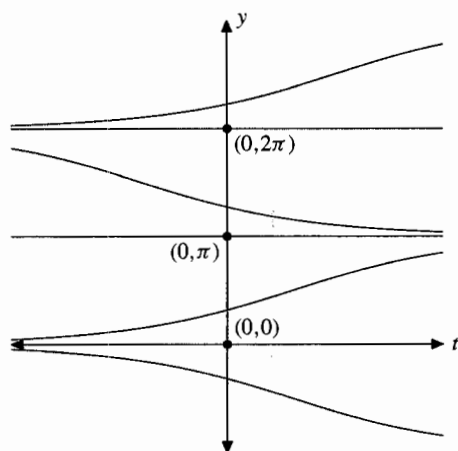
- (i) In this case, $f(y) = \sin y$, whose graph is shown in the next figure.



(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.



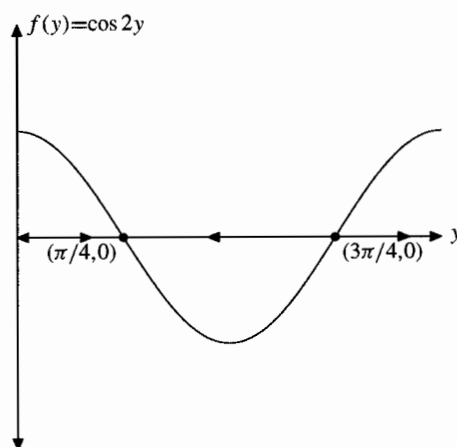
(iii) The phase line in the second figure indicates that solutions decrease if $-\pi < y < 0$, increase for $0 < y < \pi$, decrease if $\pi < y < 2\pi$, and increase for $2\pi < y < 3\pi$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = 0$, the stable equilibrium solution, $y(t) = \pi$, and the unstable equilibrium solution, $y(t) = 2\pi$.



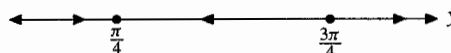
22. Due to the periodic nature of this equation, we sketch only a few regions. You can easily use the periodicity to produce more regions.

(i) In this case, $f(y) = \cos 2y$, whose graph is shown

in the next figure.

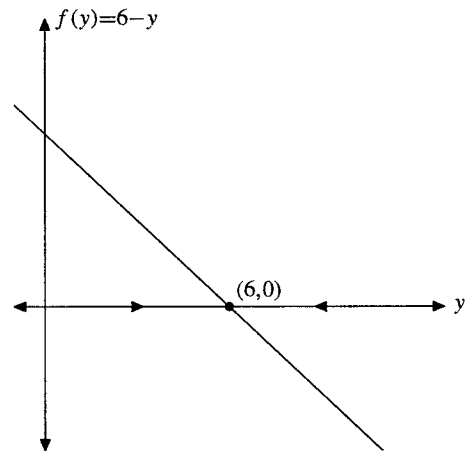
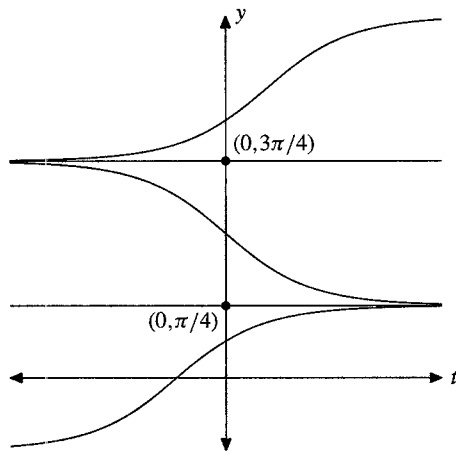


(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.



(iii) The phase line in the second figure indicates that solutions increase if $-\pi/4 < y < \pi/4$, decrease for $\pi/4 < y < 3\pi/4$, and increase if $3\pi/4 < y < 5\pi/4$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the stable equilibrium solution, $y(t) = \pi/4$, and the unstable equilibrium solution,

$$y(t) = 3\pi/4.$$

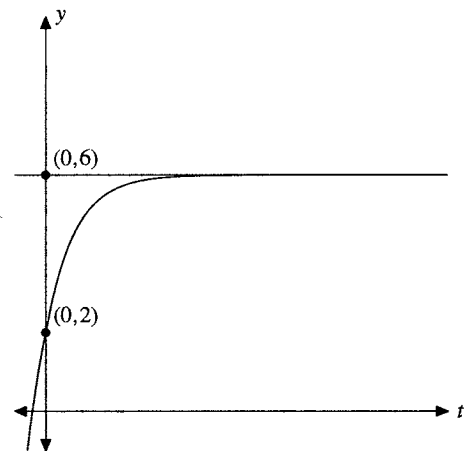


23. The equation is linear, so multiply by the integrating factor and integrate.

$$\begin{aligned}(e^t y)' &= 6e^t \\ e^t y &= 6e^t + C \\ y(t) &= 6 + Ce^{-t}\end{aligned}$$

The initial condition $y(0) = 2$ produces $C = -4$ and $y(t) = 6 - 4e^{-t}$. Now, e^{-t} approaches zero as $t \rightarrow +\infty$, so

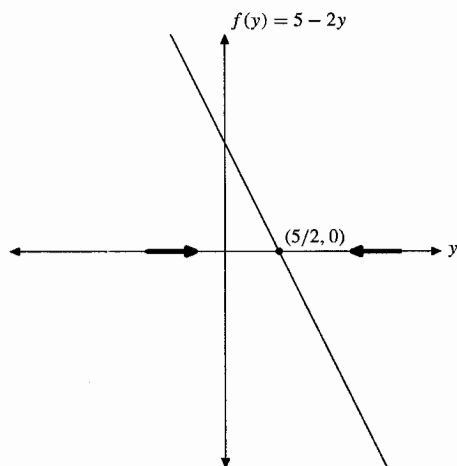
$$\lim_{t \rightarrow +\infty} y(t) = \lim_{t \rightarrow +\infty} (6 - 4e^{-t}) = 6.$$



Compare $y' = f(y)$ with $y' = 6 - y$. Then $f(y) = 6 - y$, whose graph is shown in the first figure below. The phase line on the y -axis in this figure shows that $y = 6$ is a stable equilibrium point, so a trajectory with initial condition $y(0) = 2$ should approach the stable equilibrium solution $y(t) = 6$, as shown in the second figure. This agrees nicely with the analytical solution.

24. Writing the equation as $y' = 5 - 2y$, we see that the right hand side is $f(y) = 5 - 2y$. The graph of f is in the next figure. We have also indicated the direction of the solutions on the y -axis, which shows that $y = 5/2$ is an asymptotically stable equilibrium point. Thus any solution curve will approach

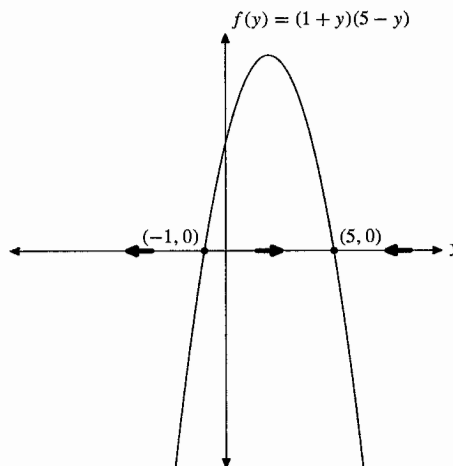
$y = 5/2$ as t increases.



The exact solution can be found since the equations is separable (and linear). With some work we find that it is $y(t) = 5[1 - e^{-2t}]/2$. Clearly the solution has the indicated limiting behavior.

25. The equation has the form $y' = f(y)$, where $f(y) = (1 + y)(5 - y)$. The graph of f is in the next figure. We have also indicated the direction of the solutions on the y -axis. This shows that $y = -1$ is an unstable equilibrium point, and $y = 5$ is an asymptotically stable equilibrium point. Therefore, a solution starting with $y(0) = 2$ will increase and approach $y = 5$

as t increases.



To find the exact solution, we separate variables and use partial fractions to get

$$\frac{1}{6} \left[\frac{1}{1 + y} + \frac{1}{5 - y} \right] dy = dt$$

Integrate,

$$\frac{1}{6} \ln |1 + y| - \frac{1}{6} \ln |5 - y| = t + C,$$

$$\ln |1 + y| - \ln |5 - y| = 6t + 6C,$$

$$\ln \left| \frac{y + 1}{y - 5} \right| = 6t + 6C$$

$$\frac{y + 1}{y - 5} = Ae^{6t},$$

where $A = \pm e^{6C}$. Using the initial condition $y(0) = 2$ we see that $A = -1$, so

$$\frac{y + 1}{y - 5} = -e^{6t}.$$

Solving for y , we find that

$$y(t) = \frac{5e^{6t} - 1}{1 + e^{6t}} = \frac{5 - e^{-6t}}{1 + e^{-6t}}.$$

From this we see that $y(t) \rightarrow 5$ as $t \rightarrow \infty$, agreeing with what we discovered earlier.

26. Separating variables,

$$\frac{dy}{dt} = (3+y)(1-y)$$

$$\frac{dy}{(3+y)(1-y)} = dt.$$

A partial fraction decomposition allows us to continue.

$$\frac{1}{4} \left[\frac{1}{3+y} + \frac{1}{1-y} \right] dy = dt$$

$$\ln |3+y| - \ln |1-y| = 4t + C$$

$$\ln \left| \frac{3+y}{1-y} \right| = 4t + C$$

$$\left| \frac{3+y}{1-y} \right| = e^C e^{4t}$$

$$\frac{3+y}{1-y} = A e^{4t}$$

with $y(0) = 2$,

$$\frac{3+2}{1-2} = A e^{4(0)} \Rightarrow A = -5$$

and

$$\frac{3+y}{1-y} = -5e^{4t}$$

$$3+y = -5e^{4t} + 5ye^{4t}$$

$$3 + 5e^{4t} = y(5e^{4t} - 1)$$

$$y = \frac{3 + 5e^{4t}}{5e^{4t} - 1}.$$

Multiply top and bottom by e^{-4t} .

$$y = \frac{3e^{-4t} + 5}{5 - e^{-4t}}$$

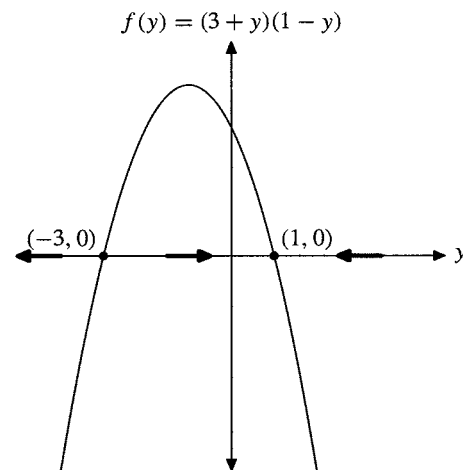
Thus,

$$\lim_{t \rightarrow \infty} y = \frac{0+5}{5-0} = 1.$$

Using qualitative analysis, plot the graph of the right-hand side of

$$\frac{dy}{dt} = (3+y)(1-y)$$

versus y .



Note the equilibrium points at $y = -3$ and $y = 1$. Moreover, note that between -3 and 1 , solutions increase to the stable point at $y = 1$. Thus,

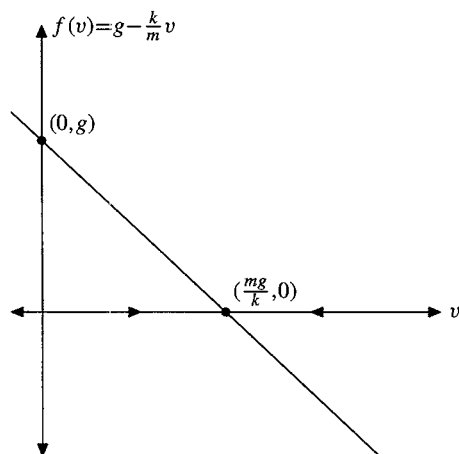
$$\lim_{t \rightarrow \infty} y(t) = 1.$$

27. We have the equation $x' = f(x) = 4 - x^2$. The equilibrium points are at $x = \pm 2$, where $f(x) = 0$. We have $f'(x) = -2x$. Since $f'(-2) = 4 > 0$, $x = -2$ is unstable. Since $f'(2) = -4 < 0$, $x = 2$ is asymptotically stable.
28. We have the equation $x' = f(x) = x(x-1)(x+2)$. The equilibrium points are at $x = 0, 1$, and -2 , where $f(x) = 0$. We have $f'(x) = 3x^2 + 2x - 2$. Since $f'(0) = -2 < 0$, $x = 0$ is asymptotically stable. Because $f'(1) = 3 > 0$, $x = 1$ is unstable. Finally, because $f'(-2) = 2 > 0$, $x = -2$ is also unstable.
29. (a) $f(x) = x^2$, $f(x) = x^3$, or $f(x) = x^4$.
(b) $f(x) = -x^3$, $f(x) = -x^5$, or $f(x) = -x^7$.
30. Notice that we are measuring the displacement as positive below the plane. First divide through by m to get

$$\frac{dv}{dt} = g - \frac{k}{m}v.$$

Note that this equation is autonomous, having form $v' = f(v)$. The graph of f is a line, with slope

$-k/m$ and intercept g , as shown in the following figure.



The phase line on the v -axis in this figure shows that $v = (mg)/k$ is a stable equilibrium point. Our skydiver starts from rest, so the solution trajectory with $v(0) = 0$ should approach the stable equilibrium solution, $v(t) = (mg)/k$. Consequently, the terminal velocity is $(mg)/k$.

31. Let $x(t)$ represent the amount of salt in the tank at time t . The rate at which solution enters the tank is given by

$$\text{Rate In} = 2 \text{ gal/min} \times 3 \text{ lb/gal} = 6 \text{ lb/min.}$$

The rate at which solution leaves the tank is

$$\text{Rate Out} = 2 \text{ gal/min} \times \frac{x}{100} \text{ lb/gal} = \frac{1}{50}x \text{ lb/min.}$$

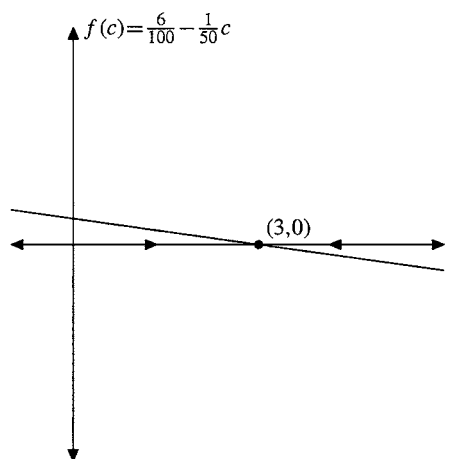
Consequently,

$$\frac{dx}{dt} = 6 - \frac{1}{50}x.$$

Let $c(t)$ represent the concentration of salt in the solution at time t . Thus, $c(t) = x(t)/100$ and $100c' = x'$.

$$\begin{aligned} 100c' &= 6 - \frac{1}{50}(100c) \\ c' &= \frac{6}{100} - \frac{1}{50}c \end{aligned}$$

Let $f(c) = 6/100 - (1/50)c$. Setting $f(c) = 0$ produces the equilibrium point $c = 3$, as shown in the following figure.



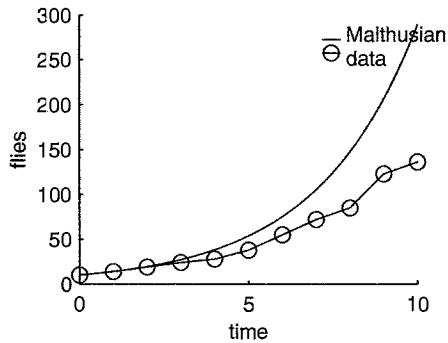
The phase line on the c -axis in this figure shows that $c = 3$ is a stable equilibrium point so a trajectory with initial condition $c(0) = 0$ (the initial concentration of salt is zero) should approach the stable equilibrium solution $c(t) = 3$.

Chapter 3. Modeling and Applications

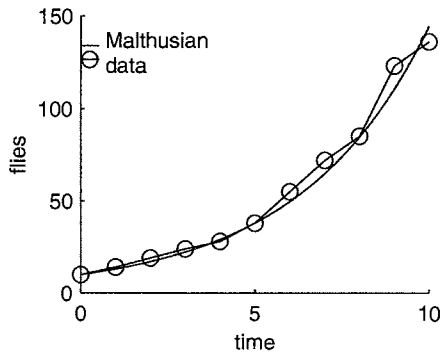
Section 3.1. Modeling Population Growth

1. The equation of the Malthusian model is $P(t) = Ce^{rt}$. Apply the initial condition $P(0) = 100$. Then $100 = Ce^0$, or $C = 100$. Next apply the condition $P(1) = 300$. Then $300 = 100e^r$. Solving gives the reproductive rate $r = \ln 3 \approx 1.0986$. The population after five days is $P(5) = 100e^{5\ln 3} = 24300$.
2. The equation of the Malthusian model is $P(t) = Ce^{rt}$. Apply the cell counts to solve for r and C . $P(1) = 1000$, so $Ce^r = 1000$, i.e. $r = \ln(1000/C)$. Also, $P(2) = 3000$, so $Ce^{2r} = 3000$, i.e. $r = \frac{1}{2} \ln(3000/C)$. Setting these equal and solving, one obtains $C = 1000/3$. Substituting C into $r = \ln(1000/C)$ gives that $r = \ln 3 \approx 1.0986$. In addition $P(0) = C = 1000/3$.
3. Recall the equation of the Malthusian model $P(t) = Ce^{rt}$. The constant C is the initial population $P(0)$, so the tripling condition is stated as $P(10) = P(0)e^{10r} = 3P(0)$. Thus, $r = (1/10) \ln 3 \approx 0.1099$. The population doubles precisely when $P(t) = 2P(0) = P(0)e^{(t \ln 3)/10}$, that is, when $(t \ln 3)/10 = \ln 2$. Solving, one obtains $t = (10 \ln 2)/\ln 3 \approx 6.3093$.
4. The solution of the Malthusian model is $P(t) = Ce^{rt}$. The constant C is the initial population $P(0)$, so the doubling condition is stated as $P(10) = P(0)e^{10r} = 2P(0)$. Thus, $r = (1/10) \ln 2$. To find the value of t so that $P(t) = 10000$, when $P(0) = 1000$, solve $10000 = 1000e^{(r \ln 2)t/10}$ for t . This gives $t = (10 \ln 10)/\ln 2 \approx 33.2193$.
5. The modified Malthusian model will take the form $P' = rP - h$. Multiplying by the integrating factor e^{-rt} and putting the terms involving P on the left gives the equation $P'e^{-rt} - re^{-rt}P = -he^{-rt}$. Solve to obtain $Pe^{-rt} = (h/r)e^{-rt} + C$, where C is a constant. This constant has the value $P(0) - (h/r)$, obtained by substituting $t = 0$. Multiply by e^{rt} and get the solution $P(t) = (h/r) + (P(0) - (h/r))e^{rt}$. The long-time activity of the population depends entirely on the sign of $P(0) - (h/r)$. If it is negative, then the population will die out. If it is positive, the population will grow exponentially. If it is zero, then the population will remain at a constant h/r .
6. First, calculate the reproduction rate r . Under normal Malthusian conditions, the population doubles every eight hours, so $P(8) = 2P(0) = P(0)e^{8r}$. Thus, $r = (1/8) \ln 2 \approx 0.0866$. Now, using the solution to the modified Malthusian model $P' = rP - h$, obtain $P(t) = (h/r) + (P(0) - (h/r))e^{rt}$. Here, $P(0) = 20000$ and $h = 2000$, so $P(t) = (16000/\ln 2) + (20000 - (16000/\ln 2))e^{(t \ln 2)/8} \approx 23083 - 3083e^{0.0866t}$. The culture will be depleted when $(16000/\ln 2) + (20000 - (16000/\ln 2))e^{(t \ln 2)/8} = 0$. Solving for t gives $t = (8/\ln 2) \ln(-16000/(20000 \ln 2 - 16000)) \approx 23.2350$. Now, if $P(0) = 25000$, then the solution becomes $P(t) = (16000/\ln 2) + (25000 - (16000/\ln 2))e^{(t \ln 2)/8} \approx 23083 + 1917e^{0.0866t}$. When t is as above, $P(t) \approx 37421$. Recalling the solution to problem 3.1.5, one will note that this indeed should be the case, as the initial population $P(0)$ is now greater than the critical value h/r .
7. The reproduction rate r is calculated as before by solving $P(4) = 2P(0) = P(0)e^{4r}$. Thus, $r = (1/4) \ln 2 \approx 0.1733$. Using the result from problem 3.1.5, the population is given by $P(t) = (h/r) + (P(0) - (h/r))e^{rt}$. In order for the population to not explode, the coefficient $P(0) - (h/r)$ must not be positive. Hence the harvesting rate must be at least $rP(0)$, which here is $(10000)(1/4) \ln 2 = 2500 \ln 2 \approx 1732.9$.
8. (a) The first data point is $P(0) = 10$, and the second is $P(1) = 14$. The equation of the Malthusian model is $P(t) = Ce^{rt}$. Thus, $C = P(0) = 10$.

10, and $r = \ln(P(1)/10) = \ln(7/5)$. See the following figure for plots.



- (b) Now use the data points $P(0) = 10$ and $P(5) = 38$. Using the equation of the Malthusian model, we get $C = P(0) = 10$ and $r = (1/5) \ln(P(5)/10)$ or $r = (1/5) \ln(3.8)$. See the following figure for plots.

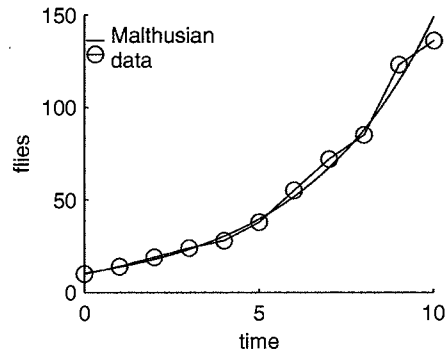


- (c) Using linear regression on the function $\ln P = \ln C + rt$ here gives a system

$$\begin{pmatrix} 11 & 55 \\ 55 & 385 \end{pmatrix} \begin{pmatrix} \ln C \\ r \end{pmatrix} = \begin{pmatrix} 40.4854 \\ 231.5369 \end{pmatrix}$$

The solution to this is $C \approx 10.5626$ and $r \approx$

0.2646. See the following figure for plots.



- (d) The reproductive rate found using only the first two data points (part a) was the least accurate. The rate found using an early data point and a later data point (part b) improved the accuracy. The best estimate of r came from the linear regression in part c, because it used all of the data.
- (e) As long as an accurate value of r is used, the Malthusian model is a good approximation of the data in the table.

9. (a) For simplicity of notation, we will let $\sum$ represent $\sum_{i=1}^n$. Start with

$$S = \sum [y_i - (mx_i + b)]^2 \quad (1.1)$$

and differentiate with respect to m to obtain

$$\frac{\partial S}{\partial m} = 2 \sum [y_i - (mx_i + b)](-x_i).$$

Set this result equal to zero and divide both sides of the resulting equation by -2 to obtain the result

$$0 = \sum [x_i y_i - mx_i^2 - bx_i].$$

Familiar properties of summation allow us to write this in the form

$$m \sum x_i^2 + b \sum x_i = \sum x_i y_i. \quad (1.2)$$

Next, differentiate S , as defined in equation (1.1), with respect to b to obtain

$$\frac{\partial S}{\partial b} = 2 \sum [y_i - (mx_i + b)](-1).$$

Set this result equal to zero then divide both sides of the resulting equation to obtain

$$0 = \sum [y_i - mx_i - b].$$

Familiar properties of summation (recall $\sum_{i=1}^n b = bn$) allows to write this in the form

$$m \sum x_i + bn = \sum y_i. \quad (1.3)$$

Let's eliminate b from equations (1.2) and (1.3). Multiply equation (1.2) by n , then multiply equation (1.3) by $-\sum x_i$ to obtain

$$\begin{aligned} mn \sum x_i^2 + bn \sum x_i &= n \sum x_i y_i \\ -m \left(\sum x_i \right)^2 - bn \sum x_i &= -\sum x_i \sum y_i. \end{aligned}$$

Adding,

$$mn \sum x_i^2 - m \left(\sum x_i \right)^2 = n \sum x_i y_i - \sum x_i \sum y_i.$$

Solving this for m ,

$$m = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i \right)^2}. \quad (1.4)$$

Finally, solve equation (1.3) for b to obtain

$$b = \frac{\sum y_i - m \sum x_i}{n}. \quad (1.5)$$

(b) Now, compare

$$y = mx + b$$

with equation

$$\ln P = \ln C + rt$$

and you will note that we need to associate t_i with x_i , $\ln P_i$ with y_i , r with m , and $\ln C$ with b . With these changes, equation (1.4) becomes

$$r = \frac{n \sum t_i \ln P_i - \sum t_i \sum \ln P_i}{n \sum t_i^2 - \left(\sum t_i \right)^2}, \quad (1.6)$$

and equation (1.5) becomes

$$\ln C = \frac{\sum \ln P_i - r \sum t_i}{n}. \quad (1.7)$$

(c) It's helpful to arrange the data in tabular form.

t_i	P_i	$\ln P_i$	t_i^2	$t_i \ln P_i$
0	10	2.3026	0	0.0000
1	25	3.2189	1	3.2189
2	61	4.1109	4	8.2217
3	144	4.9698	9	14.9094
4	360	5.8861	16	23.5444
10	600	20.4883	30	49.8944

Substitute the appropriate totals from the table into equation (1.6).

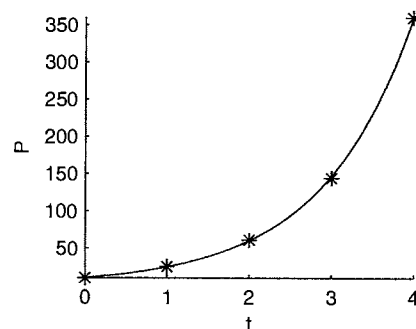
$$r = \frac{5(49.8944) - 10(20.4883)}{5(30) - 10^2} \approx 0.8918$$

Substitute this value of r and the appropriate totals from the table into equation (1.7).

$$\ln C = \frac{20.4883 - 0.8918(10)}{5} \approx 2.3141$$

Thus, $C \approx 10.1154$.

(d) Substituting the values of C and r in $P = Ce^{rt}$, $P = 10.1154e^{0.8918t}$. The plot of the data and the exponential fit follows.



10. To find the minimum and maximum growth rate, use the first derivative test on the growth rate dP/dt . That is, set $d^2P/dt^2 = 0$. Obtain

$$\frac{d^2P}{dt^2} = rP' \left(1 - \frac{2P}{k} \right).$$

Thus, dP/dt reaches an extremum when $P' = 0$ and when $P = k/2$. One wants to show that dP/dt is a maximum when $P = k/2$. So, one checks that d^2P/dt^2 changes sign from positive to

negative at $P = k/2$. When $P = k/2$, $P' = r(k/2)(1 - (k/2)/k) = rk/4$, which is positive. So any possible change in sign of d^2P/dt^2 is in the third factor $(1 - 2P/k)$. This factor does change sign from positive to negative at $P = k/2$, so d^2P/dt^2 changes sign from positive to negative at $P = k/2$. Therefore dP/dt reaches a maximum at $P = k/2$.

11. (a) Let $\omega = \alpha P$ and $s = \beta t$. Then solving for P and t and substituting into the logistic equation, one obtains

$$\frac{\beta d\omega}{\alpha ds} = r \frac{\omega}{\alpha} \left(1 - \frac{\omega}{\alpha K}\right).$$

Isolating $d\omega/ds$ and simplifying, one obtains

$$\frac{d\omega}{ds} = \frac{r}{\beta} \omega - \frac{r}{\alpha\beta K} \omega^2.$$

This is as desired.

- (b) If $\beta = r$ and $\alpha = 1/K$, then we have $d\omega/ds = \omega - \omega^2$.
 (c) Separating variables and using partial fractions, one obtains

$$\left(\frac{1}{\omega} - \frac{1}{1-\omega}\right) d\omega = ds.$$

Thus, $\ln|\omega| - \ln|1-\omega| = s + c$. Exponentiating and combining all constants into a constant a gives

$$\frac{\omega}{1-\omega} = ae^s.$$

Solving for ω gives the desired form

$$\omega = \frac{1}{1 - (-a)e^{-s}}.$$

- (d) Note that $P = \omega K$ and $s = rt$. Thus,

$$P = \frac{K}{1 + ae^{-rt}}.$$

The initial condition $P(t_0) = P_0$ gives $P_0 + P_0 a e^{-rt_0} = K$, so

$$a = \frac{K - P_0}{P_0} e^{rt_0}.$$

Using this in the equation for P , one obtains

$$P = \frac{K P_0}{P_0 + (K - P_0)e^{-r(t-t_0)}}.$$

This is the solution (1.13) given in the text.

12. The carrying capacity $K = 20,000$, and the initial condition $P_0 = 1000$, and it is given that $P(10) = 2000$. Using equation (1.13), one obtains

$$2000 = \frac{(20000)(1000)}{1000 + (19000)e^{-10r}}.$$

Solving gives $r = -(\ln(9/19))/10 \approx 0.0747$. After 25 hours, the population is

$$P(25) = \frac{(20000)(1000)}{1000 + (19000)e^{2.5 \ln(9/19)}} \approx 5084.$$

Now, find t so that $P(t) = K/2 = 10000$. This gives

$$e^{\frac{1}{10} \ln(9/19)t} = \frac{2000 - 1000}{19000}.$$

That is, $t = 10(-\ln(19))/(\ln(9/19)) \approx 39.4055$.

13. The carrying capacity $K = 20,000$, and the initial condition $P_0 = 1000$, and it is given that $P(8) = 1200$. Using equation (1.13), one obtains

$$1200 = \frac{(20000)(1000)}{1000 + (19000)e^{-8r}}.$$

Solving we get

$$r = (-1/8) \ln(188000/((12)(19000))) \approx 0.0241.$$

Now, find t so that $P(t) = 3K/4 = 15000$. This gives $e^{-rt} = 1/57$. That is, $t = (\ln 57)/r \approx 167.6713$.

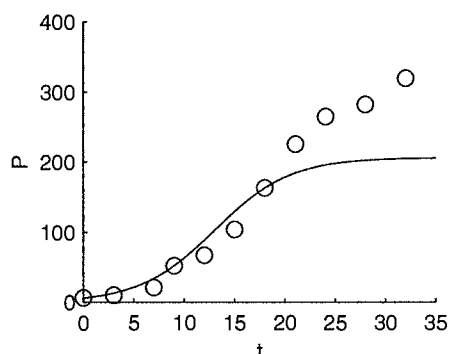
14. (a) Use the following three data points $P_0 = P(0) = 6$, $P_1 = P(9) = 52$, and $P_3 = P(18) = 163$ to estimate r and k : Set $h = 9$. Equation (1.17) gives

$$r = \frac{1}{9} \ln \left(\frac{163(52 - 6)}{6(163 - 52)} \right) = 0.269.$$

Then using this in equation (1.16), one obtains

$$K = \frac{(6)(52)(1 - e^{-0.269(9)})}{6 - 52e^{-0.269(9)}} = 205.90.$$

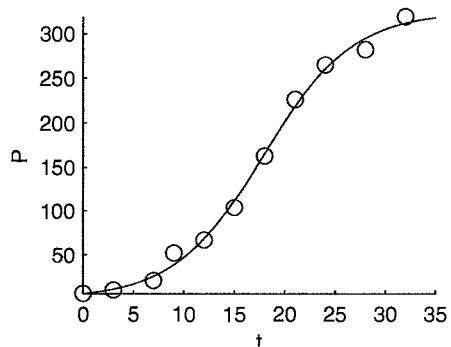
The plot of the logistic solution with these parameters is shown in the following figure.



- (b) To compute the parameters we want to find values of the parameters P_0 , r , and K which minimize

$$\sum_{j=1}^{11} \left[\log \left(\frac{P(t_j)}{P_j} \right) \right]^2,$$

where t_j and P_j are the data from Table 1, and $P(t)$ is the logistic function in equation 3.1.13 with parameters P_0 , r , and K . Using a non-linear least squares minimization program we find that $K = 324.1241$, $P_0 = 5.6047$, and $r = 0.2268$. The plot of the logistic solution with these parameters is shown in the following figure.



- (c) If we use only three points to compute the parameters, the fit of the logistic model is very poor. However, if we use all of the data the fit is quite good.

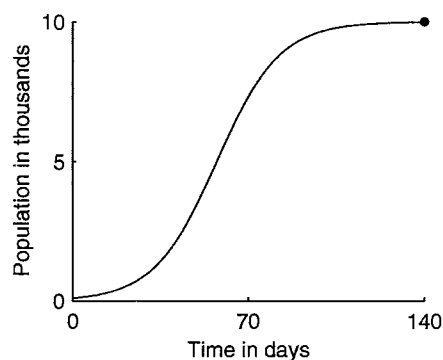
15. The data is shown in the following figure plotted as points. Using this we estimate $K = 9990$. To find r we use the solution to the logistic model

$$P(t) = \frac{K P_0}{P_0 + (K - P_0)e^{-rt}},$$

and solve for

$$r = \frac{1}{t} \ln \left(\frac{P(t)(K - P_0)}{P_0(K - P(t))} \right).$$

If we use $t = 60$, with $P(60) = 5510$, we get $r = 0.08$. The logistic solution with these parameters is the solid curve in the figure.

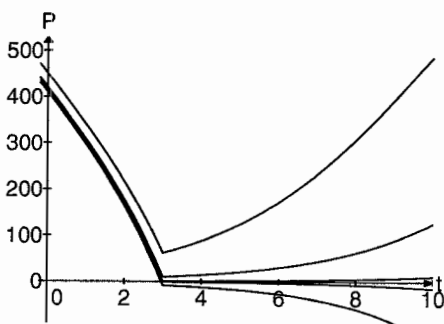


The fit seems to be quite good. However, this method depends on the choice of middle point. If that is an outlier the fit might not be so good as it was in this case.

16. (a) Since the population is measured in thousands, the fishing rate is $h = 0.1$ thousand fish per day. The modified model is $P' = 0.1P(1 - P/10) - 0.1$.
- (b) We look for roots of $0.1P(1 - P/10) - 0.1 = 0$. This leads to the quadratic equation $P^2 - 10P + 10 = 0$, which has roots $P = 5 \pm \sqrt{15}$. Using the graph of $0.1P(1 - P/10) - 0.1$ we see that $P_1 = 5 - \sqrt{15} \approx 1.127$ is an unstable equilibrium point, and $P_2 = 5 + \sqrt{15} \approx 8.873$ is an asymptotically stable equilibrium point.
- (c) We have $P' = 0.1P(1 - P/10) - 0.1 > 0$ for $P_1 < P < P_2$ and negative elsewhere. It

follows using qualitative analysis that for any starting population greater than P_1 , the population tends to the stable equilibrium at P_2 . For any starting population smaller than P_1 , the population decreases until it dies out. In particular a population of 1000 is doomed while a population of 2000 tends to P_2 .

17. (a) In the following figure, the five curves are given by initial conditions $P(0) = 410$, $P(0) = 414$, $P(0) = 415$, $P(0) = 420$, and $P(0) = 450$. Notice that the solution curve with initial condition $P(0) = 415$ never reaches zero, although the curve with initial condition $P(0) = 414$ does. Thus, the critical population is between 414 and 415.



- (b) The critical initial population will be the initial condition for a solution curve which has $P = 0$ when $t = 3$. Writing the differential equation for $t < 3$ gives

$$\frac{dP}{dt} = 0.38P - 0.00038P^2 - 200.$$

Separate variables.

$$\frac{dP}{-0.00038P^2 + 0.38P - 200} = dt.$$

Integrating this, one obtains

$$t + C = \frac{2}{\sqrt{.1596}} \tan^{-1} \left(\frac{-0.00076P + 0.38}{\sqrt{.1596}} \right).$$

Using conditions $P(3) = 0$, one obtains

$$3 + C = \frac{2}{\sqrt{.1596}} \tan^{-1} \left(\frac{0.38}{\sqrt{.1596}} \right)$$

or $C = 0.8067$. Now setting $t = 0$ and solving for $P = P_0$, one obtains $P_0 = 414.5557$.

18. We will use days and 1000 individuals as our units. We are given that the carrying capacity is $K = 10$, $P(0) = 1$, and with $t_1 = 2.3/24$,

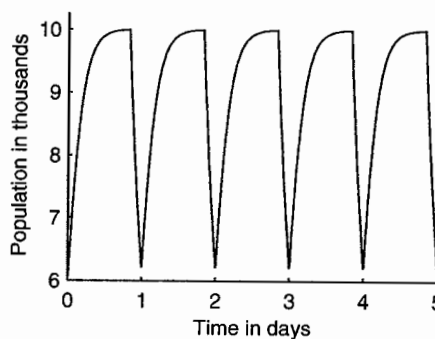
$$2 = P(t_1) = \frac{K P_0}{P_0 + (K - P_0)e^{-rt_1}}.$$

Solving for r , we get $r = 8.4619$.

The harvesting rate during the last four hours of each day is $1500/\text{hr} = 36,000/\text{day}$. Hence the harvesting rate is

$$h(t) = \begin{cases} 0, & \text{in the first 20 hours} \\ 36, & \text{in the last 4 hours} \end{cases}$$

of each day. Using a square wave function to model this in our solver we get the following graph of the solution.



The population at the end of each day is approximately 10,000.

19. (a) The harvesting rate is $0.01P$, so the modified model is

$$P' = 0.1P(1 - P/10) - 0.01P.$$

- (b) The model equation simplifies to $P' = 0.01P(9 - P)$. This is another logistic equation. Thus $P = 0$ is an unstable equilibrium point, and $P = 9$ is an asymptotically stable equilibrium point, representing the carrying capacity.

- (c) The population tends to the carrying capacity $P = 9$.
20. The harvesting rate is γP , so the modified model is $P' = rP(1 - P/K) - \gamma P$. This can be written as
- $$P' = (r - \gamma)P(1 - P/K_\gamma),$$
- where the new carrying capacity is $K_\gamma = (r - \gamma)K/r$. If $\gamma < r$, this is again a logistic equation with carrying capacity K_γ , so all populations tend to this value. If $\gamma > r$, then $K_\gamma < 0$, and we see that $P' < 0$ whenever $P > 0$. Thus any real population tends to 0. The population dies out.
21. Refer to the answer to Exercise 19. The units of γP are individuals/ unit time. It represents the number of individuals harvested in a unit of time. For this reason it is called the yield. Notice that for values of $\gamma < r$ the model is another logistic equation with carrying capacity $K_\gamma = (r - \gamma)K/r$. For populations at the carrying capacity the yield is $\gamma K_\gamma = \gamma(r - \gamma)K/r$. This function of γ is maximized when $\gamma = r/2$. The carrying capacity for this value of γ is $K/2$ and the yield is $rK/4$.

Section 3.2. Models and the Real World

- We have $P_0 = P(1790) = 3929827$, and $P(1800) = 5305925 = P_0 e^{10r}$. Solving the last equation for r we get $r = \ln(5305925/3929827)/10 \approx 0.03$.
- We have $h = 60$, and the three data points $P_0 = P(1790) = 3929$, $P_1 = P(1850) = 23192$, and $P_2 = P(1910) = 91972$. Thus using (1.19) we get

$$r = \frac{1}{h} \ln \left(\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)} \right) = 0.0313.$$

With this value we use the first equation in (1.18) to find the carrying capacity

$$K = \frac{P_0 P_1 (1 - e^{-rh})}{P_0 - P_1 e^{-rh}} = 197274.$$

- Answers will vary depending on the data chosen.

Section 3.3. Personal Finance

- (a) Let $P(t)$ represent the balance in the account after t years. Because the initial deposit at time $t = 0$ is $P = \$1200$,

$$P' = 0.05P, \quad P(0) = 1200.$$

Separate the variables and solve for P , then use the initial condition to produce

$$P(t) = 1200e^{0.05t}.$$

Thus, the amount in the account after 10 years is

$$P(10) = 1200e^{0.05(10)} \approx \$1978.50.$$

(b) Solve $P(t) = 5000$, or

$$\begin{aligned} 1200e^{0.05t} &= 5000 \\ e^{(1/20)t} &= \frac{25}{6} \\ t &= 20 \ln \frac{25}{6} \\ t &\approx 28.5 \text{ yr} \end{aligned}$$

2. Let $P(t)$ represent the balance in the account t years after the initial investment. Let P_0 represent the initial investment, i.e., $P(0) = P_0$. Consequently,

$$P' = rP, \quad P(0) = P_0.$$

Separate variables and use the initial condition to produce

$$P = P_0 e^{rt}.$$

Because the initial investment doubles in five years,

$$\begin{aligned} 2P_0 &= P_0 e^{r(5)}, \\ 2 &= e^{5r}, \\ r &= \frac{\ln 2}{5}, \\ r &\approx 0.1386, \end{aligned}$$

or, 13.86%.

3. Let $P(t)$ represent the balance in the account t years after the initial investment. Let r represent the annual rate, d the yearly deposit, and P_0 the initial investment. Then,

$$P' = rP + d, \quad P(0) = P_0.$$

This equation is linear, with integrating factor e^{-rt} . Consequently,

$$\begin{aligned} (e^{-rt}P)' &= de^{-rt}, \\ e^{-rt}P &= -\frac{d}{r}e^{-rt} + C, \\ P &= -\frac{d}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 + d/r$ and

$$P(t) = -\frac{d}{r} + \left(P_0 + \frac{d}{r}\right)e^{rt}.$$

Thus, the amount in the account after 10 years is

$$\begin{aligned} P(10) &= -\frac{1200}{0.06} + \left(5000 + \frac{1200}{0.06}\right)e^{0.06(10)} \\ &\approx \$25,553. \end{aligned}$$

4. Let $P(t)$ represent the balance in the account t years after Jason's day of birth. Let r represent the annual rate, d the annual deposit. Since no initial investment is required,

$$P' = rP + d, \quad P(0) = 0.$$

This equation is linear, with integrating factor e^{-rt} . Consequently,

$$\begin{aligned} (e^{-rt}P)' &= de^{-rt}, \\ e^{-rt}P &= -\frac{d}{r}e^{-rt} + C, \\ P &= -\frac{d}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = 0$ to produce $C = d/r$ and

$$P(t) = \frac{d}{r}(e^{rt} - 1).$$

Because $P(18) = 50000$,

$$\begin{aligned} 50000 &= \frac{d}{r}(e^{r(18)} - 1), \\ d &= \frac{50000r}{e^{18r} - 1}, \\ d &= \frac{50000(0.0625)}{e^{18(0.0625)} - 1}, \\ d &\approx \$1,502.25 \end{aligned}$$

5. Let $P(t)$ represent the balance in the account after t years. Let r represent the annual rate, w the yearly withdrawal, and P_0 the amount of the inheritance. Then

$$P' = rP - w, \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt} . Consequently,

$$\begin{aligned} (e^{-rt}P)' &= -we^{-rt}, \\ e^{-rt}P &= \frac{w}{r}e^{-rt} + C, \\ P &= \frac{w}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, to find when the funds are depleted, set $P(t) = 0$.

$$0 = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt},$$

$$e^{rt} = \frac{w/r}{w/r - P_0},$$

$$t = \frac{1}{r} \ln \frac{w/r}{w/r - P_0}.$$

Thus, the account will be depleted in

$$t = \frac{1}{0.05} \ln \frac{8000/0.05}{8000/0.05 - 50000} \approx 7.5 \text{ years.}$$

6. Let $P(t)$ represent the loan balance after t years. Let r represent the annual rate, w the annual payment, and P_0 the amount of the loan. Then

$$P' = rP - w \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt} . Consequently,

$$(e^{-rt}P)' = -we^{-rt},$$

$$e^{-rt}P = \frac{w}{r}e^{-rt} + C,$$

$$P = \frac{w}{r} + Ce^{rt}.$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, the loan is exhausted at the end of four years. Consequently, $P(4) = 0$, so

$$0 = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(4)},$$

$$\frac{w}{r}(e^{4r} - 1) = P_0e^{4r},$$

$$P_0 = \frac{w}{r}(1 - e^{-4r})$$

$$P_0 = \frac{(225)(12)}{0.08}(1 - e^{-4(0.08)}),$$

$$P_0 \approx \$9,242.47$$

7. Let $P(t)$ represent the loan balance after t years. Let r represent the annual rate, w the annual payment, and P_0 the amount of the loan. Then

$$P' = rP - w \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt} . Consequently,

$$(e^{-rt}P)' = -we^{-rt},$$

$$e^{-rt}P = \frac{w}{r}e^{-rt} + C,$$

$$P = \frac{w}{r} + Ce^{rt}.$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, the loan is exhausted at the end of 30 years. Consequently, $P(30) = 0$, so

$$0 = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(30)},$$

$$\frac{w}{r}(e^{30r} - 1) = P_0e^{30r},$$

$$w = \frac{rP_0}{1 - e^{-30r}},$$

$$w = \frac{0.08(100000)}{1 - e^{-30(0.08)}}$$

$$w \approx \$8,798.15$$

8. Let $P(t)$ represent the loan balance after t years. Let r represent the annual interest rate, w the annual payment, and P_0 the amount of the loan. Then

$$P' = rP - w, \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt} . Consequently,

$$(e^{-rt}P)' = -we^{-rt},$$

$$e^{-rt}P = \frac{w}{r}e^{-rt} + C,$$

$$P = \frac{w}{r} + Ce^{rt}.$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, \$1,000 per month makes \$12,000 per year, so $w = 12000$. Furthermore, the term of the loan is 30 years, so $P(30) = 0$ and

$$\begin{aligned} 0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(30)}, \\ we^{30r} - w &= P_0 re^{30r}, \\ P_0 &= \frac{w(e^{30r} - 1)}{re^{30r}}, \\ P_0 &= \frac{12000(e^{30(0.0725)} - 1)}{0.0725e^{30(0.0725)}}, \\ P_0 &\approx \$146,713. \end{aligned}$$

9. Let $S(t)$ represent José's salary after t years. Consequently,

$$S' = 0.01S, \quad S(0) = 28,$$

where S is measured in *thousands* of dollars. Therefore, José's salary is given by

$$S(t) = 28e^{0.01t}.$$

Let $P(t)$ represent the balance of the account after t years. Let ρ represent the fixed percentage of José's salary that is deposited in the account on an annual basis. Thus,

$$P' = 0.06P + 28\rho e^{0.01t}, \quad P(0) = 2.5,$$

where $P(t)$ is also measured in *thousands* of dollars. Multiply by the integrating factor, $e^{-0.06t}$, and integrate.

$$\begin{aligned} (e^{-0.06t}P)' &= 28\rho e^{-0.05t} \\ e^{-0.06t}P &= \frac{28}{-0.05}\rho e^{-0.05t} + C \\ P &= -560\rho e^{0.01t} + Ce^{0.06t} \end{aligned}$$

The initial condition $P(0) = 2.5$ produces $C = 2.5 + 560\rho$ and

$$P = -560\rho e^{0.01t} + (2.5 + 560\rho)e^{0.06t}.$$

Now, José wants \$50,000 in the account at the end of 20 years, so $P(20) = 50$ and

$$\begin{aligned} 50 &= -560\rho e^{0.01(20)} + (2.5 + 560\rho)e^{0.06(20)}, \\ \rho &= \frac{50 - 2.5e^{1.2}}{560(e^{1.2} - e^{0.2})}, \\ \rho &\approx 0.035, \end{aligned}$$

or 3.5%.

10. Let $P(t)$ represent the balance after t years, r the annual rate, and d the annual deposit. Thus,

$$P' = rP + d, \quad P(0) = 1000.$$

However, in this case the annual deposit is a function of time. The initial deposit is \$1,000 and this is increased by \$500 each year. Consequently, $d(t) = 1000 + 500t$ and

$$P' = rP + (1000 + 500t), \quad P(0) = 1000.$$

Multiply by the integrating factor, e^{-rt} , and integrate.

$$\begin{aligned} (e^{-rt}P)' &= (1000 + 500t)e^{-rt} \\ e^{-rt}P &= \left(-\frac{1000 + 500t}{r} - \frac{500}{r^2}\right)e^{-rt} + C \\ P &= -\frac{1000 + 500t}{r} - \frac{500}{r^2} + Ce^{rt} \end{aligned}$$

The initial condition $P(0) = 1000$ gives $C = 1000 + 1000/r + 500/r^2$ and

$$\begin{aligned} P &= -\frac{1000 + 500t}{r} - \frac{500}{r^2} \\ &\quad + \left(1000 + \frac{1000}{r} + \frac{500}{r^2}\right)e^{rt}. \end{aligned}$$

At ten years,

$$\begin{aligned} P(10) &= -\frac{1000 + 500(10)}{0.06} - \frac{500}{(0.06)^2} \\ &\quad + \left(1000 + \frac{1000}{0.06} + \frac{500}{(0.06)^2}\right)e^{0.06(10)}, \\ P(10) &\approx \$46,373.93. \end{aligned}$$

11. (a) Let $P(n)$ represent the balance at the end of n compounding periods, I the annual interest rate, m the number of compounding periods per year, and P_0 the initial investment. Thus,

$$P(n+1) = \left(1 + \frac{I}{m}\right)P(n), \quad P(0) = P_0.$$

(b) Compare

$$P(n+1) = \left(1 + \frac{I}{m}\right) P(n), \quad P(0) = P_0.$$

with

$$a(n+1) = ra(n), \quad a(0) = a_0,$$

and note that $r = 1 + I/m$ and $a_0 = P_0$. Consequently,

$$a(n) = a_0 r^n$$

becomes

$$P(n) = P_0 \left(1 + \frac{I}{m}\right)^n.$$

12. (a) Let $P(t)$ represent the balance at the end of t years. Let r represent the annual rate and P_0 the initial investment. Thus,

$$P' = rP, \quad P(0) = P_0.$$

Consequently,

$$\begin{aligned} P(t) &= P_0 e^{rt}, \\ P(10) &= 2000e^{0.06(10)}, \\ P(10) &\approx \$3,644.24 \end{aligned}$$

- (b) In semiannual case, $m = 2$. Furthermore, there are 20 compounding periods in 10 years, so

$$P(20) = 2000 \left(1 + \frac{0.06}{2}\right)^{20} \approx \$3,612.22.$$

In the monthly case, $m = 12$. There are 120 compounding periods in 10 years, so

$$P(120) = 2000 \left(1 + \frac{0.06}{12}\right)^{120} \approx \$3,638.79.$$

In the daily case, $m = 365$. There are 3650 compounding periods in 10 years, so

$$P(3650) = 2000 \left(1 + \frac{0.06}{365}\right)^{3650} \approx \$3,644.06.$$

13. (a) The series $1 + r + r^2 + \cdots + r^{n-1}$ is geometric, finite, with sum $(1-r^n)/(1-r)$. Consequently,

$$a(n) = a_0 r^n + b(1 + r + r^2 + \cdots + r^{n-1}),$$

$$a(n) = a_0 r^n + b \left(\frac{1-r^n}{1-r}\right),$$

$$a(n) = a_0 r^n + \frac{b}{1-r} - \frac{b}{1-r} r^n,$$

$$a(n) = \left(a_0 - \frac{b}{1-r}\right) r^n + \frac{b}{1-r}.$$

(b) Comparing

$$P(n+1) = \left(1 + \frac{I}{m}\right) P(n) + d, \quad P(0) = P_0.$$

with

$$a(n+1) = ra(n) + b, \quad a(0) = a_0$$

shows us that $r = 1 + I/m$, $b = d$, and $a_0 = P_0$. Consequently, the solution

$$a(n) = \left(a_0 - \frac{b}{1-r}\right) r^n + \frac{b}{1-r}$$

becomes

$$\begin{aligned} P(n) &= \left(P_0 - \frac{d}{1 - (1 + I/m)}\right) \left(1 + \frac{I}{m}\right)^n \\ &\quad + \frac{d}{1 - (1 + I/m)}, \\ P(n) &= \left(P_0 + \frac{md}{I}\right) \left(1 + \frac{I}{m}\right)^n - \frac{md}{I}. \end{aligned}$$

14. (a) Let $P(t)$ represent the balance after t years, r the annual interest rate, d the yearly deposit, and P_0 the initial investment. Thus,

$$P' = rP + d, \quad P(0) = P_0.$$

Multiply by the integrating factor, e^{-rt} , and integrate.

$$(e^{-rt} P)' = de^{-rt}$$

$$e^{-rt} P = -\frac{d}{r} e^{-rt} + C$$

$$P(t) = -\frac{d}{r} + C e^{rt}$$

The initial condition $P(0) = P_0$ gives $C = P_0 + d/r$ and

$$P(t) = -\frac{d}{r} + \left(P_0 + \frac{d}{r}\right)e^{rt}.$$

After 10 years,

$$\begin{aligned} P(10) &= -\frac{1000}{0.05} + \left(2000 + \frac{1000}{0.05}\right)e^{0.05(10)} \\ &\approx \$16,271.87. \end{aligned}$$

- (b) Let $P(n)$ represent the balance after n compounding periods, I the annual interest rate, m the number of compounding periods per year, and P_0 the initial investment. Then,

$$P(n+1) = \left(1 + \frac{I}{m}\right)P(n) + d, \quad P(0) = P_0,$$

with solution (from exercise 13)

$$P(n) = \left(P_0 + \frac{md}{I}\right)\left(1 + \frac{I}{m}\right)^n - \frac{md}{I}.$$

In ten years, there are 40 compounding periods. Thus,

$$\begin{aligned} P(40) &= \left(2000 + \frac{(4)(250)}{0.05}\right)\left(1 + \frac{0.05}{4}\right)^{40} \\ &\quad - \frac{(4)(250)}{0.05} \\ &\approx \$16,159.63. \end{aligned}$$

- (c) One significant change is the deposit at the end of each compounding period (at the end of each day). Because Demetrios is depositing \$1,000 per year, his daily deposit is $d = \$1000/365$. Next, there are 3,650 compounding periods in the ten-year period. Thus,

$$\begin{aligned} P(3650) &= \left(2000 + \frac{(365)(1000/365)}{0.05}\right) \\ &\quad \times \left(1 + \frac{0.05}{365}\right)^{3650} \\ &\quad - \frac{(365)(1000/365)}{0.05}, \\ &\approx \$16,270.63. \end{aligned}$$

15. (a) Let $P(t)$ represent the balance of the loan after t years, I the annual interest rate, w the yearly payment, and P_0 the amount of the loan. Then,

$$P' = rP - w, \quad P(0) = P_0.$$

Multiply by the integrating factor, e^{-rt} , and integrate.

$$\begin{aligned} (e^{-rt}P)' &= -we^{-rt} \\ e^{-rt}P &= \frac{w}{r}e^{-rt} + C \\ P &= \frac{w}{r} + Ce^{-rt} \end{aligned}$$

The initial condition $P(0) = P_0$ gives $C = P_0 - w/r$ and

$$P = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{-rt}.$$

Because the loan expires in 5 years, $P(5) = 0$ and

$$\begin{aligned} 0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(5)}, \\ w &= \frac{P_0 r}{1 - e^{-5r}}, \\ w &= \frac{(12000)(0.08)}{1 - e^{-5(0.08)}}, \\ w &\approx \$2,911.91, \end{aligned}$$

or \$242.66 per month.

- (b) Let $P(n)$ represent the balance on the loan after n compounding periods, I the annual rate, m the number of compounding periods per year, w the monthly payment, and P_0 the initial amount of the loan. Then, in a manner similar to that in Exercise 13, the progress of the loan is modeled by the first order difference equation

$$P(n+1) = \left(1 + \frac{I}{m}\right)P(n) - w, \quad P(0) = P_0,$$

with solution

$$P(n) = \left(P_0 - \frac{mw}{I}\right)\left(1 + \frac{I}{m}\right)^n + \frac{mw}{I}.$$

There are 60 compounding periods in 5 years,

so $P(60) = 0$ and

$$0 = \left(P_0 - \frac{mw}{I}\right) \left(1 + \frac{I}{m}\right)^{60} + \frac{mw}{I},$$

$$w = \frac{P_0 I (1 + I/m)^{60}}{m((1 + I/m)^{60} - 1)},$$

$$w = \frac{P_0 I}{m(1 - (1 + I/m)^{-60})},$$

$$w = \frac{(12000)(0.08)}{12(1 - (1 + 0.08/12)^{-60})},$$

$$w \approx \$243.32.$$

————— × —————

Section 3.4. Electrical Circuits

1. The model equation is $Q' + Q/2 = 5$. The solution is $Q(t) = 10 - 10e^{-t/2}$.
2. The model equation is $Q' + Q/2 = 5e^{-t/10}$. The solution is

$$Q(t) = 25(e^{-t/10} - e^{-t/2})/2.$$
3. The model equation is $Q' + Q/2 = 5 \sin 2t$. The solution is

$$Q(t) = (-40 \cos 2t + 10 \sin 2t + 40e^{-t/2})/17.$$
4. The model equation is $Q' + Q/2 = 5 \cos 3t$. The solution is

$$Q(t) = (10 \cos 3t + 60 \sin 3t - 10e^{-t/2})/37.$$
5. The model equation is $Q' + Q/2 = 5 - t/20$. The solution is

$$Q(t) = 51(1 - e^{-t/2})/5 - t/10.$$
6. The model equation is $Q' + Q/2 = 5(1 - e^{-t/10})$. The solution is

$$Q(t) = 10 - (25e^{-t/10} - 5e^{-t/2})/2.$$
7. The model equation is $I' + I/10 = 1$. The solution is $I(t) = 10(1 - e^{-t/10})$.
8. The model equation is $I' = e^{-t/10} - I/10$. The solution is $I(t) = te^{-t/10}$.
9. The model equation is $I' + I/10 = 5 \sin 2\pi t$. The solution is

$$I(t) = \frac{50(-20\pi \cos 2\pi t + 20\pi e^{-t/10} + \sin 2\pi t)}{1 + 400\pi^2}.$$
10. The model equation is $I' + I/10 = 4 \cos 3t$. The solution is

$$I(t) = (40 \cos 3t + 1200 \sin 3t - 40e^{-t/10})/901.$$
11. The model equation is $I' + I/10 = 10 - 2t$. The solution is

$$I(t) = 300 - 20t - 300e^{-t/10}.$$
12. The model equation is $I' + I/10 = 10(1 - e^{-t/20})$. The solution is

$$I(t) = 100(1 - 2e^{-t/20} + e^{-t/10}).$$
13. $Q(t) = EC(1 - e^{-t/RC})$
14. $I(t) = (E + (RI_0 - E)e^{-Rt/L})/R$

15. The model equation, $Q' + Q/20 = (1/2)e^{-t/100}$ with $Q(0) = 0$, has solution

$$Q(t) = 25(e^{-t/100} - e^{-t/20})/2.$$

We find the maximum by setting the derivative

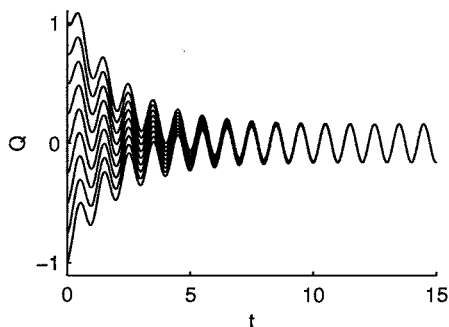
$$Q'(t) = 25 \left[\frac{1}{20}e^{-t/20} - \frac{1}{100}e^{-t/100} \right] / 2 = 0,$$

and solving for t . It has a maximum at $t = 25 \ln 5 \approx 40.24$. The maximum value is $Q(25 \ln 5) = 6.69$.

16. The model equation, $I' + I/10 = (1/2)e^{-t/20}$ with $I(0) = 0$, has solution $I = 10(e^{-t/20} - e^{-t/10})$. The solution is maximized at $t = 20 \ln 2 \approx 13.86$ with maximum value $I(20 \ln 2) = 2.5$

17. (a) 23.87
 (b) The model equation, $Q' + Q/10 = 10$ with $Q(0) = 0$, has solution $Q = 100(1 - e^{-t/10})$ for $0 \leq t \leq 5$. Note that $Q(5) = Q_5 = 100(1 - e^{-1/2})$. Then for the next 5 seconds, the model equation is $Q' = -Q/10$ with $Q(5) = Q_5$. The solution to this equation is $Q = Q_5 e^{-(t-5)/10}$. At $t = 10$ seconds, the amount of charge is $Q = Q_5 e^{-1/2} \approx 23.87$.

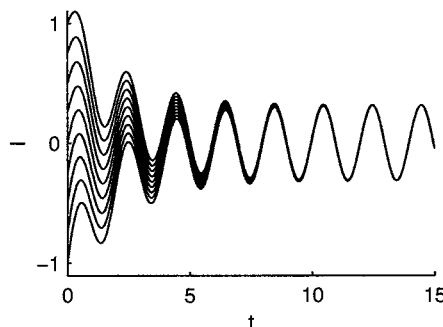
18. (a) The nine solution are plotted below.



The steady state solution is a particular solution to the inhomogeneous equation, which all solutions converge to regardless of their initial condition. Note that this is the case for all nine solutions corresponding to the nine different initial conditions.

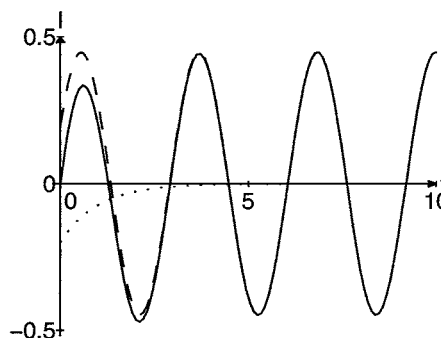
- (b) The frequency of the force is $2\pi/2\pi = 1$ Hz. This also appears to be the period of the steady-state response.

19. The nine solutions are plotted below.



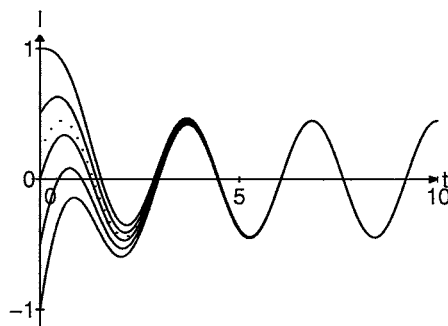
The frequency is of the force is $2\pi/2 = \pi$ Hz. This also appears to be the frequency of the steady-state response.

20. (a) In the following plot, the solution is the solid curve, the transient response is the dotted curve, and the steady-state solution is dashed.



- (b) In the following figure, the steady-state solu-

tion is the dotted curve.



All solutions converge to the steady state solution as time gets large.

21. If we use $I = Q'$, we get the differential equation

$$Q' + \frac{1}{RC}Q = \frac{E}{C} \sin \omega t.$$

This linear equation can be solved using the integrating factor $e^{t/RC}$. We get the solution

$$Q(t) = \frac{EC}{1 + R^2C^2\omega^2} (\sin \omega t - RC\omega \cos \omega t) + Ae^{-t/RC},$$

where A is an arbitrary constant. The term $Ae^{-t/RC}$ dies out as t increases, so it is the transient term. The first term does not die out and is therefore the steady-state solution.

22. By the capacitance law, the voltage drop across the capacitor is $V = Q/C$. Thus $Q = CV$. If we substitute this into the given equation we get the result. The equation is linear, and it can be solved using the integrating factor $e^{t/RC}$. The general solution is

$$V(t) = \frac{E}{1 + R^2C^2\omega^2} (RC\omega \sin \omega t + \cos \omega t) + Ae^{-t/RC},$$

where A is an arbitrary constant. The term $Ae^{-t/RC}$ dies out as t increases, so it is the transient term. The first term does not die out and is therefore the steady-state solution.

23. By the chain rule

$$\frac{dI}{dt} = \frac{dI}{dQ} \frac{dQ}{dt} = I \frac{dI}{dt}.$$

Hence the equation becomes

$$LI \frac{dI}{dQ} + Q/C = 0.$$

This separable equation has the solution $I = \sqrt{k - Q^2/LC}$ where k is a constant. Since $I = dQ/dt$, the equation now becomes

$$\frac{dQ}{dt} = \sqrt{k - Q^2/LC}.$$

This is another separable equation, and

$$\frac{dQ}{\sqrt{k - Q^2/LC}} = dt$$

The integral on the left involves an arcsine and the solution is

$$Q(t) = k_1 \sqrt{LC} \sin \left(\frac{t}{\sqrt{LC}} + k_2 \right).$$

Since $I = dQ/dt$, we also obtain

$$I(t) = k_1 \cos \left(\frac{t}{\sqrt{LC}} + k_2 \right),$$

where k_1 and k_2 are constants.

Chapter 4. Second-Order Equations

Section 4.1. Definitions and Examples

1. Compare

$$y'' + 3y' + 5y = 3 \cos 2t$$

with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = 3$, $q(t) = 5$, and $g(t) = 3 \cos 2t$. Hence, the equation is linear and inhomogeneous.

2. Divide both sides of $t^2 y'' = 4y' - \sin t$ by t^2 , then rearrange to obtain

$$y'' - \frac{4}{t^2}y' = -\frac{\sin t}{t^2}.$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = -4/t^2$, $q(t) = 0$, and $g(t) = -(\sin t)/t^2$. Hence, the equation is linear and inhomogeneous.

3. Expand $t^2 y'' + (1 - y)y' = \cos 2t$ to obtain

$$t^2 y'' + y' - yy' = \cos 2t.$$

Note that the term yy' is nonlinear. Hence, this equation is nonlinear.

4. Divide both sides of $ty'' + (\sin t)y' = 4y - \cos 5t$ by t , then rearrange to obtain

$$y'' + \frac{\sin t}{t}y' - \frac{4}{t}y = -\frac{\cos 5t}{t}$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = (\sin t)/t$, $q(t) = -4/t$, and $g(t) = -(\cos 5t)/t$. Hence, the equation is linear and inhomogeneous.

5. In

$$t^2 y'' + 4yy' = 0,$$

note that the term $4yy'$ is nonlinear. Hence, this equation is nonlinear.

6. Compare

$$y'' + 4y' + 7y = 3e^{-t} \sin t$$

with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = 4$, $q(t) = 7$, and $g(t) = 3e^{-t} \sin t$. Hence, the equation is linear and inhomogeneous.

7. In

$$y'' + 3y' + 4 \sin y = 0$$

note that the term $4 \sin y$ is nonlinear. Hence, this equation is nonlinear.

8. Divide both sides of $(1 - t^2)y'' = 3y$ by $1 - t^2$, then rearrange the terms to obtain

$$y'' - \frac{3}{1 - t^2}y = 0.$$

Compare with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = 0$, $q(t) = -3/(1 - t^2)$, and $g(t) = 0$. Hence, the equation is linear and homogeneous.

9. Use $k = (mg)/x$ to determine the spring constant.

$$k = \frac{2 \text{ kg} \times 9.8 \text{ m/s}^2}{0.5 \text{ m}} = 39.2 \text{ N/m}.$$

Using the model

$$my'' + \mu y' + ky = F(t),$$

we note that there is no damping ($\mu = 0$) and there is no driving force ($F(t) = 0$), so the equation becomes

$$my'' + ky = 0.$$

With $m = 2 \text{ kg}$ and $k = 39.2 \text{ N/m}$, this becomes

$$2y'' + 39.2y = 0.$$

Because the initial displacement was 12 cm, $y(0) = 0.12$ (assuming an orientation where y is positive in the downward direction). Because the mass is released from rest, the initial velocity is zero and $y'(0) = 0$.

10. Use $k = (mg)/x$ to determine the spring constant.

$$k = \frac{5 \text{ kg} \times 9.8 \text{ m/s}^2}{0.75 \text{ m}} = 65.3 \text{ N/m}.$$

Using the model

$$my'' + \mu y' + ky = F(t),$$

we note that there is no damping ($\mu = 0$) and there is no driving force ($F(t) = 0$), so the equation becomes

$$my'' + ky = 0.$$

With $m = 5 \text{ kg}$ and $k = 65.3 \text{ N/m}$, this becomes

$$5y'' + 65.3y = 0.$$

Because the initial displacement was 36 cm upward, $y(0) = -0.36$ (assuming an orientation where y is positive in the downward direction). Because the mass is given an initial downward velocity of 0.45 m/s, $y'(0) = 0.45$. Because we've chosen the downward direction to represent positive y -values, downward velocities are positive.

11. The period of the driving force is 2 seconds. Thus, the circular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ rad/s}.$$

Because the amplitude is $A = 0.5 \text{ m}$, and the spring is initially displaced 0.5 m upward (remember, upward is negative) by the driving force, the driving force can be described with

$$F(t) = -0.5 \cos \pi t$$

Now, $m = 2 \text{ kg}$, $k = 39.2 \text{ N/m}$, and the damping force is given by $R(v) = -0.05v$. This makes the damping constant $\mu = 0.05$. Thus, the equation

$$my'' + \mu y' + ky = F(t)$$

becomes

$$2y'' + 0.05y' + 39.2y = -0.5 \cos \pi t$$

From Exercise 9, the initial conditions are $y(0) = 0.12$ and $y'(0) = 0$.

12. The period of the driving force is 4 seconds. Thus, the circular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ rad/s}.$$

Because the amplitude is $A = 0.25 \text{ m}$, and the spring is initially displaced 0.25 m downward (remember, upward is negative) by the driving force, the driving force can be described with

$$F(t) = 0.25 \cos \frac{\pi t}{2}$$

Now, $m = 5 \text{ kg}$, $k = 65.3 \text{ N/m}$, and the damping force is given by $R(v) = -0.125v$. This makes the damping constant $\mu = 0.125$. Thus, the equation

$$my'' + \mu y' + ky = F(t)$$

becomes

$$5y'' + 0.125y' + 65.3y = 0.25 \cos \frac{\pi t}{2}$$

From Exercise 10, the initial conditions are $y(0) = -0.36$ and $y'(0) = 0.45$.

13. If
- $y_1(t) = e^{3t}$
- , then

$$\begin{aligned} y_1'' - y_1' - 6y_1 &= (e^{3t})'' - (e^{3t})' - 6(e^{3t}) \\ &= 9e^{3t} - 3e^{3t} - 6e^{3t} \\ &= 0. \end{aligned}$$

If $y_2(t) = e^{-2t}$, then

$$\begin{aligned} y_2'' - y_2' - 6y_2 &= (e^{-2t})'' - (e^{-2t})' - 6(e^{-2t}) \\ &= 4e^{-2t} + 2e^{-2t} - 6e^{-2t} \\ &= 0. \end{aligned}$$

Finally, if $y(t) = C_1e^{3t} + C_2e^{-2t}$, then

$$\begin{aligned} y'' - y' - 6y &= (C_1e^{3t} + C_2e^{-2t})'' - (C_1e^{3t} + C_2e^{-2t})' \\ &\quad - 6(C_1e^{3t} + C_2e^{-2t}) \\ &= 9C_1e^{3t} + 4C_2e^{-2t} - 3C_1e^{3t} + 2C_2e^{-2t} \\ &\quad - 6C_1e^{3t} - 6C_2e^{-2t} \\ &= 0. \end{aligned}$$

14. If
- $y_1(t) = \cos 2t$
- , then

$$\begin{aligned} y_1'' + 4y_1 &= (\cos 2t)'' + 4(\cos 2t) \\ &= -4\cos 2t + 4\cos 2t \\ &= 0. \end{aligned}$$

If $y_2(t) = \sin 2t$, then

$$\begin{aligned} y_2'' + 4y_2 &= (\sin 2t)'' + 4(\sin 2t) \\ &= -4\sin 2t + 4\sin 2t \\ &= 0. \end{aligned}$$

Finally, if $y(t) = C_1 \cos 2t + C_2 \sin 2t$, then

$$\begin{aligned} y'' + 4y &= (C_1 \cos 2t + C_2 \sin 2t)'' \\ &\quad + 4(C_1 \cos 2t + C_2 \sin 2t) \\ &= -4C_1 \cos 2t - 4C_2 \sin 2t \\ &\quad + 4C_1 \cos 2t + 4C_2 \sin 2t \\ &= 0. \end{aligned}$$

15. If
- $y_1(t) = e^t \cos t$
- , then

$$\begin{aligned} y_1'(t) &= e^t(-\sin t) + e^t \cos t \\ &= e^t(\cos t - \sin t), \\ y_1''(t) &= e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) \\ &= -2e^t \sin t, \end{aligned}$$

and

$$\begin{aligned} y_1'' - 2y_1' + 2y_1 &= -2e^t \sin t - 2e^t(\cos t - \sin t) + 2e^t \cos t \\ &= 0. \end{aligned}$$

If $y_2(t) = e^t \sin t$, then

$$\begin{aligned} y_2'(t) &= e^t \cos t + e^t \sin t \\ &= e^t(\cos t + \sin t), \\ y_2''(t) &= e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) \\ &= 2e^t \cos t, \end{aligned}$$

and

$$\begin{aligned} y_2'' - 2y_2' + 2y_2 &= 2e^t \cos t - 2e^t(\cos t + \sin t) + 2e^t \sin t \\ &= 0. \end{aligned}$$

Finally, if $y(t) = C_1e^t \cos t + C_2e^t \sin t$, or if $y(t) = e^t(C_1 \cos t + C_2 \sin t)$, then

$$\begin{aligned} y'(t) &= e^t(-C_1 \sin t + C_2 \cos t) \\ &\quad + e^t(C_1 \cos t + C_2 \sin t) \\ &= e^t((C_1 + C_2) \cos t + (-C_1 + C_2) \sin t), \\ y''(t) &= e^t((C_1 + C_2)(-\sin t) + (-C_1 + C_2) \cos t) \\ &\quad + e^t((C_1 + C_2) \cos t + (-C_1 + C_2) \sin t) \\ &= 2e^t(C_2 \cos t - C_1 \sin t), \end{aligned}$$

and

$$\begin{aligned} y'' - 2y' + 2y &= 2e^t(C_2 \cos t - C_1 \sin t) \\ &\quad - 2e^t((C_1 + C_2) \cos t + (-C_1 + C_2) \sin t) \\ &\quad + 2e^t(C_1 \cos t + C_2 \sin t) \\ &= 0. \end{aligned}$$

16. If
- $y_1(t) = e^{-2t}$
- , then

$$\begin{aligned} y_1'' + 4y_1' + 4y_1 &= (e^{-2t})'' + 4(e^{-2t})' + 4(e^{-2t}) \\ &= 4e^{-2t} - 8e^{-2t} + 4e^{-2t} \\ &= 0. \end{aligned}$$

If $y_2(t) = te^{-2t}$, then

$$y_2'(t) = e^{-2t} - 2te^{-2t}, \quad \text{and} \\ y_2''(t) = -4e^{-2t} + 4te^{-2t}.$$

Thus,

$$y_2'' + 4y_2' + 4y_2 = -4e^{-2t} + 4te^{-2t} \\ + 4e^{-2t} - 8te^{-2t} + 4te^{-2t} \\ = 0.$$

Finally, if $y(t) = C_1e^{-2t} + C_2te^{-2t}$, then

$$y'(t) = -2C_1e^{-2t} + C_2e^{-2t} - 2C_2te^{-2t}, \quad \text{and} \\ y''(t) = 4C_1e^{-2t} - 4C_2e^{-2t} + 4C_2te^{-2t},$$

then

$$y'' + 4y' + 4y = 4C_1e^{-2t} - 4C_2e^{-2t} + 4C_2te^{-2t} \\ - 8C_1e^{-2t} + 4C_2e^{-2t} - 8C_2te^{-2t} \\ + 4C_1e^{-2t} + 4C_2te^{-2t} \\ = 0.$$

17. We leave it to our readers to first check that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ are solutions of $y'' - y' - 2y = 0$. Next note that

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-t}}{e^{2t}} = e^{-3t} \neq c,$$

where c is some constant. Therefore $y_1(t)$ is not a constant multiple of $y_2(t)$ and the solutions are linearly independent. Further,

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^t,$$

which is never equal to zero. Therefore, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent and form a fundamental set of solutions.

18. We leave it to our readers to first check that $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ are solutions of $y'' + 9y = 0$. Next note that

$$\frac{y_1(t)}{y_2(t)} = \frac{\cos 3t}{\sin 3t} = \cot 3t \neq c,$$

where c is some constant. Therefore $y_1(t)$ is not a constant multiple of $y_2(t)$ and the solutions are linearly independent. Further,

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \cos 3t & \sin 3t \\ -3 \sin 3t & 3 \cos 3t \end{vmatrix} \\ = 3 \cos^2 3t + 3 \sin^2 3t = 3,$$

which is never equal to zero. Therefore, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent and form a fundamental set of solutions.

19. We leave it to our readers to first check that $y_1(t) = e^{-2t} \cos 3t$ and $y_2(t) = e^{-2t} \sin 3t$ are solutions of $y'' + 4y' + 13y = 0$. Next note that

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-2t} \cos 3t}{e^{-2t} \sin 3t} = \cot 3t \neq c,$$

where c is some constant. Therefore $y_1(t)$ is not a constant multiple of $y_2(t)$ and the solutions are linearly independent. Further,

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) \\ = e^{-2t} \cos 3t \times e^{-2t} (3 \cos 3t - 2 \sin 3t) \\ - e^{-2t} \sin 3t \times e^{-2t} (-3 \sin 3t - 2 \cos 3t) \\ = e^{-4t} (3 \cos^2 3t - 2 \cos 3t \sin 3t) \\ = 3e^{-4t},$$

which is never equal to zero. Therefore, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent and form a fundamental set of solutions.

20. We leave it to our readers to first check that $y_1(t) = e^{-3t}$ and $y_2(t) = te^{-3t}$ are solutions of $y'' + 6y' + 9y = 0$. Next note that

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-3t}}{te^{-3t}} = \frac{1}{t} \neq c,$$

where c is some constant. Therefore $y_1(t)$ is not a constant multiple of $y_2(t)$ and the solutions are linearly independent. Further,

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t} - 3te^{-3t} \end{vmatrix} = e^{-6t},$$

which is never equal to zero. Therefore, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent and form a fundamental set of solutions.

21. If $y_1(t) = t^2$ and $y_2(t) = t|t|$, then

$$\frac{y_1(t)}{y_2(t)} = \frac{t^2}{t|t|} = \begin{cases} -1, & t < 0, \\ 1, & t > 0. \end{cases}$$

Thus, $y_1(t)$ is not a constant multiple of $y_2(t)$ on $(-\infty, +\infty)$. However, the Wronskian

$$\begin{aligned} W(t) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} t^2 & t|t| \\ 2t & 2|t| \end{vmatrix} \\ &= 2t^2|t| - 2t^2|t| \\ &= 0. \end{aligned}$$

Thus, the Wronskian is identically zero on $(-\infty, +\infty)$, seemingly contradicting Proposition 1.27, that is, until one realizes that the hypothesis that requires that y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ is unsatisfied.

22. If $y_1(t) = e^t$, then

$$y'' + 2y' - 3y = e^t + 2e^t - 3e^t = 0,$$

and if $y_2(t) = e^{-3t}$, then

$$y'' + 2y' - 3y = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0,$$

Furthermore,

$$\frac{y_1(t)}{y_2(t)} = \frac{e^t}{e^{-3t}} = e^{4t},$$

which is nonconstant. Thus, y_1 is not a constant multiple of y_2 and the solutions $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions. Thus, the general solution of $y'' + 2y' - 3y = 0$ is

$$y(t) = C_1 e^t + C_2 e^{-3t},$$

and its derivative is

$$y'(t) = C_1 e^t - 3C_2 e^{-3t}.$$

The initial conditions, $y(0) = 1$ and $y'(0) = -2$ lead to the equations

$$\begin{aligned} 1 &= C_1 + C_2 \\ -2 &= C_1 - 3C_2 \end{aligned}$$

and the constants $C_1 = 1/4$ and $C_2 = 3/4$. Thus, the solution of the initial value problem is

$$y(t) = \frac{1}{4}e^t + \frac{3}{4}e^{-3t}.$$

23. If $y_1(t) = \cos 4t$, then

$$y_1'' + 16y_1 = -16 \cos 4t + 16 \cos 4t = 0,$$

and if $y_2(t) = \sin 4t$, then

$$y_2'' + 16y_2 = -16 \sin 4t + 16 \sin 4t = 0.$$

Furthermore,

$$\frac{y_1(t)}{y_2(t)} = \frac{\cos 4t}{\sin 4t} = \cot 4t,$$

which is nonconstant. Thus, y_1 is not a constant multiple of y_2 and the solutions $y_1(t) = \cos 4t$ and $y_2(t) = \sin 4t$ form a fundamental set of solutions. Thus, the general solution of $y'' + 16y = 0$ is

$$y(t) = C_1 \cos 4t + C_2 \sin 4t,$$

and its derivative is

$$y'(t) = -4C_1 \sin 4t + 4C_2 \cos 4t.$$

The initial conditions, $y(0) = 2$ and $y'(0) = -1$ lead to the equations

$$\begin{aligned} 2 &= C_1 \\ -1 &= 4C_2 \end{aligned}$$

and the constants $C_1 = 2$ and $C_2 = -1/4$. Thus, the solution of the initial value problem is

$$y(t) = 2 \cos 4t - \frac{1}{4} \sin 4t.$$

24. If
- $y_1(t) = e^{-t} \cos 2t$
- , then

$$\begin{aligned} y_1'(t) &= -e^{-t} \cos 2t - 2e^{-t} \sin 2t, \quad \text{and} \\ y_1''(t) &= -3e^{-t} \cos 2t + 4e^{-t} \sin 2t. \end{aligned}$$

Thus,

$$\begin{aligned} y_1'' + 2y_1' + 5y_1 &= -3e^{-t} \cos 2t + 4e^{-t} \sin 2t \\ &\quad - 2e^{-t} \cos 2t - 4e^{-t} \sin 2t + 5e^{-t} \cos 2t \\ &= 0. \end{aligned}$$

If $y_2(t) = e^{-t} \sin 2t$, then

$$\begin{aligned} y_2'(t) &= -e^{-t} \sin 2t + 2e^{-t} \cos 2t, \quad \text{and} \\ y_2''(t) &= -3e^{-t} \sin 2t - 4e^{-t} \cos 2t. \end{aligned}$$

Thus,

$$\begin{aligned} y_2'' + 2y_2' + 5y_2 &= -3e^{-t} \sin 2t - 4e^{-t} \cos 2t \\ &\quad - 2e^{-t} \sin 2t + 4e^{-t} \cos 2t + 5e^{-t} \sin 2t \\ &= 0. \end{aligned}$$

Furthermore,

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-t} \cos 2t}{e^{-t} \sin 2t} = \cot 2t,$$

which is nonconstant. Thus, y_1 is not a constant multiple of y_2 and the solutions $y_1(t) = e^{-t} \cos 2t$ and $y_2(t) = e^{-t} \sin 2t$ form a fundamental set of solutions. Thus, the general solution of $y'' + 2y' + 5y = 0$ is

$$y(t) = C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t,$$

and its derivative is

$$\begin{aligned} y'(t) &= -C_1 e^{-t} \cos 2t - 2C_1 e^{-t} \sin 2t \\ &\quad - C_2 e^{-t} \sin 2t + 2C_2 e^{-t} \cos 2t. \end{aligned}$$

The initial conditions, $y(0) = -1$ and $y'(0) = 0$ lead to the equations

$$\begin{aligned} -1 &= C_1 \\ 0 &= -C_1 + 2C_2 \end{aligned}$$

and the constants $C_1 = -1$ and $C_2 = -1/2$. Thus, the solution of the initial value problem is

$$y(t) = -e^{-t} \cos 2t - \frac{1}{2} e^{-t} \sin 2t.$$

25. If
- $y_1(t) = e^{-4t}$
- , then

$$y_1'' + 8y_1' + 16y_1 = 16e^{-4t} + 8(-4e^{-4t}) + 16e^{-4t} = 0.$$

Thus, y_1 is a solution. If $y_2(t) = te^{-4t}$, then

$$\begin{aligned} y_2'(t) &= e^{-4t}(1 - 4t), \quad \text{and} \\ y_2''(t) &= e^{-4t}(-8 + 16t). \end{aligned}$$

Thus,

$$\begin{aligned} y_2'' + 8y_2' + 16y_2 &= e^{-4t}(-8 + 16t) + 8e^{-4t}(1 - 4t) + 16te^{-4t} \\ &= 0. \end{aligned}$$

Thus, y_2 is a solution. Because

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-4t}}{te^{-4t}} = \frac{1}{t}$$

is nonconstant, y_1 and y_2 are independent and form a fundamental set of solutions. Thus, the general solution is

$$y(t) = C_1 e^{-4t} + C_2 t e^{-4t}.$$

Substituting the initial condition $y(0) = 2$ provides $C_1 = 2$. After a bit of work, the derivative of the general solution is

$$y'(t) = e^{-4t}((C_2 - 4C_1) - 4C_2 t).$$

Substituting the initial condition $y'(0) = -1$ gives

$$-1 = C_2 - 4C_1,$$

and the fact that $C_1 = 2$ provides us with

$$\begin{aligned} -1 &= C_2 - 8 \\ C_2 &= 7. \end{aligned}$$

Thus, the solution is

$$y(t) = 2e^{-4t} + 7te^{-4t}.$$

26. (a) If
- $y_1(t) = t^2$
- , then

$$t^2 y_1'' + t y_1' - 4y_1 = 2t^2 + 2t^2 - 4t^2 = 0.$$

(b) Let $y_2(t) = v(t)y_1(t) = v(t)t^2$. Then,

$$y_2' = v't^2 + 2vt, \quad \text{and} \\ y_2'' = v''t^2 + 4v't + 2v.$$

Thus,

$$\begin{aligned} 0 &= t^2 y_2'' + t y_2' - 4y_2 \\ &= t^2(v''t^2 + 4v't + 2v) \\ &\quad + t(v't^2 + 2vt) - 4vt^2 \\ &= t^4 v'' + 5t^3 v'. \end{aligned}$$

Because t is not identically zero,

$$\begin{aligned} t v'' + 5v' &= 0 \\ \frac{v''}{v'} &= -\frac{5}{t} \\ \ln v' &= -5 \ln t = \ln t^{-5} \\ v' &= t^{-5} \\ v &= -\frac{1}{4}t^{-4}. \end{aligned}$$

Thus,

$$y_2(t) = v(t)y_1(t) = \left(-\frac{1}{4}t^{-4}\right)t^2 = -\frac{1}{4}t^{-2}.$$

Note that any constant multiple of this solution is also a solution, so choose $y_2(t) = t^{-2}$ and check this in the differential equation.

$$\begin{aligned} t^2 y_2'' + t y_2' - 4y_2 &= t^2(6t^{-4}) + t(-2t^{-3}) \\ &\quad - 4(t^{-2}) \\ &= 6t^{-2} - 2t^{-2} - 4t^{-2} \\ &= 0. \end{aligned}$$

Note further that

$$\frac{y_1(t)}{y_2(t)} = \frac{t^2}{t^{-2}} = t^4,$$

which is nonconstant. Therefore, y_1 and y_2 are independent and form a fundamental set of solutions. The general solution is

$$y(t) = C_1 t^2 + C_2 t^{-2}.$$

27. If $y_1(t) = t$, then

$$t^2 y_1'' - 2t y_1' + 2y_1 = t^2(0) - 2t(1) + 2t = 0.$$

Thus, $y_1(t) = t$ is a solution. Let $y_2(t) = v(t)y_1(t) = v(t)t$, where v is to be determined. Then,

$$\begin{aligned} y_2 &= tv \\ y_2' &= tv' + v \\ y_2'' &= tv'' + 2v'. \end{aligned}$$

Substituting these into the differential equation,

$$\begin{aligned} 0 &= t^2 y_2'' - 2t y_2' + 2y_2 \\ &= t^2(tv'' + 2v') - 2t(tv' + v) + 2tv \\ &= t^3 v''. \end{aligned}$$

Because t^3 is not identically equal to zero, $v'' = 0$ and

$$\begin{aligned} v' &= k_1 \\ v &= k_1 t + k_2. \end{aligned}$$

Choose $k_1 = 1$ and $k_2 = 0$, which gives $v(t) = t$ and $y_2(t) = v(t)y_1(t) = tt = t^2$. Checking, we see that

$$t^2 y_2'' - 2t y_2' + 2y_2 = t^2(2) - 2t(2t) + 2(t^2) = 0,$$

making $y_2(t) = t^2$ a solution. Note that

$$\frac{y_1(t)}{y_2(t)} = \frac{t}{t^2} = \frac{1}{t}$$

is nonconstant, so y_1 and y_2 are independent and form a fundamental set of solutions. Thus, the general solution is

$$y(t) = C_1 t + C_2 t^2.$$

28. If $y_1(t) = t$, then

$$t^2 y_1'' + t y_1' - y_1 = t^2(0) + t(1) - t = 0.$$

Thus, $y_1(t) = t$ is a solution. Let $y_2(t) = v(t)y_1(t) = v(t)t$, where v is to be determined. Then,

$$\begin{aligned} y_2 &= tv \\ y_2' &= tv' + v \\ y_2'' &= tv'' + 2v'. \end{aligned}$$

Substituting these into the differential equation,

$$\begin{aligned} 0 &= t^2 y_2'' + t y_2' - y_2 \\ &= t^2 (t v'' + 2v') + t(t v' + v) - t v \\ &= t^3 v'' + 3t^2 v' \\ &= t^2 (t v'' + 3v'). \end{aligned}$$

Because t^2 is not identically equal to zero, $t v'' + 3v' = 0$ and

$$\begin{aligned} \frac{v''}{v'} &= -\frac{3}{t} \\ \ln v' &= -3 \ln t = \ln t^{-3} \\ v' &= t^{-3} \\ v &= -\frac{1}{2} t^{-2}. \end{aligned}$$

Thus,

$$y_2(t) = v(t) y_1(t) = \left(-\frac{1}{2} t^{-2}\right) t = -\frac{1}{2} t^{-1}.$$

Note that any constant multiple of this solution is also a solution, so choose $y_2(t) = t^{-1}$ and check this in the differential equation.

$$\begin{aligned} t^2 y_2'' + t y_2' - y_2 &= t^2 (2t^{-3}) + t(-t^{-2}) - t^{-1} \\ &= 2t^{-1} - t^{-1} - t^{-1} \\ &= 0. \end{aligned}$$

Note that

$$\frac{y_1(t)}{y_2(t)} = \frac{t}{t^{-1}} = t^2$$

is nonconstant, so y_1 and y_2 are independent and form a fundamental set of solutions. Thus, the general solution is

$$y(t) = C_1 t + C_2 t^{-1}.$$

29. If $y_1(t) = t$, then

$$t^2 y_1'' - 3t y_1' + 3y_1 = t^2(0) - 3t(1) + 3t = 0,$$

making $y_1(t) = t$ a solution. Let $y_2(t) = v(t) y_1(t) = t v(t)$, with v to be determined. Thus,

$$\begin{aligned} y_2 &= t v \\ y_2' &= t v' + v \\ y_2'' &= 2v' + t v''. \end{aligned}$$

Substituting these into the differential equation,

$$\begin{aligned} 0 &= t^2 y_2'' - 3t y_2' + 3y_2 \\ &= t^2 (2v' + t v'') - 3t (t v' + v) + 3t v \\ &= t^2 (t v'' - v'). \end{aligned}$$

Because t^2 is not identically zero,

$$\begin{aligned} t v'' - v' &= 0 \\ \frac{v''}{v'} &= \frac{1}{t} \\ \ln v' &= \ln t \\ v' &= t \\ v &= \frac{1}{2} t^2. \end{aligned}$$

Thus,

$$y_2(t) = v(t) y_1(t) = \frac{1}{2} t^2(t) = \frac{1}{2} t^3.$$

Readers should check that this is a solution of the differential equation. Further,

$$\frac{y_1(t)}{y_2(t)} = \frac{t}{(1/2)t^3} = \frac{2}{t^2}$$

is nonconstant, so y_1 and y_2 are independent and form a fundamental set of solutions. Thus, the general solution is

$$y(t) = C_1 t + C_2 t^3,$$

where we have absorbed the $1/2$ into the arbitrary constant C_2 .

30. If $y_1(t) = 1/t$, then

$$t^2 y_1'' + 4t y_1' + 2y_1 = t^2 (2t^{-3}) + 4t(-t^{-2}) + 2t^{-1} = 0,$$

making $y_1(t) = 1/t$ a solution. Let $y_2(t) = v(t) y_1(t) = v(t) t^{-1}$, with v to be determined. Thus,

$$\begin{aligned} y_2 &= t^{-1} v \\ y_2' &= v' t^{-1} - v t^{-2} \\ y_2'' &= v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}. \end{aligned}$$

Substituting these into the differential equation,

$$\begin{aligned} 0 &= t^2 y_2'' + 4t y_2' + 2y_2 \\ &= t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) \\ &\quad + 4t(v't^{-1} - vt^{-2}) + 2vt^{-1} \\ &= tv'' + 2v'. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{v''}{v'} &= -\frac{2}{t} \\ \ln v' &= -2 \ln t = \ln t^{-2} \\ v' &= t^{-2} \\ v &= -t^{-1}. \end{aligned}$$

Thus,

$$y_2(t) = v(t)y_1(t) = (-t^{-1})(t^{-1}) = -t^{-2}.$$

Readers should check that this is a solution of the differential equation. Further,

$$\frac{y_1(t)}{y_2(t)} = \frac{t^{-1}}{-t^{-2}} = -t$$

is nonconstant, so y_1 and y_2 are independent and form a fundamental set of solutions. Thus, the general solution is

$$y(t) = C_1 t^{-1} + C_2 t^{-2},$$

where we have absorbed the minus sign into the arbitrary constant C_2 .

Section 4.2. Second Order Equations and Systems

1. First, solve the differential equation $y'' + 2y' - 3y = 0$ for the highest derivative of y present in the equation.

$$y'' = -2y' + 3y$$

Next, set $v = y'$. Then

$$v' = y'' = -2y' + 3y = -2v + 3y.$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -2v + 3y \end{aligned}$$

2. First, solve the differential equation $4y'' + 4y' + y = 0$ for the highest derivative of y present in the equation.

$$y'' = -y' - \frac{1}{4}y$$

Next, set $v = y'$. Then

$$v' = y'' = -y' - \frac{1}{4}y = -v - \frac{1}{4}y.$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -v - \frac{1}{4}y \end{aligned}$$

3. First, solve the differential equation $y'' + 3y' + 4y = 2 \cos 2t$ for the highest derivative of y present in the equation.

$$y'' = -3y' - 4y + 2 \cos 2t$$

Next, set $v = y'$. Then

$$\begin{aligned} v' = y'' &= -3y' - 4y + 2 \cos 2t \\ &= -3v - 4y + 2 \cos 2t. \end{aligned}$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -3v - 4y + 2 \cos 2t \end{aligned}$$

4. First, solve the differential equation $y'' + 2y' + 2y = \sin 2\pi t$ for the highest derivative of y present in the equation.

$$y'' = -2y' - 2y + \sin 2\pi t$$

Next, set $v = y'$. Then

$$\begin{aligned} v' = y'' &= -2y' - 2y + \sin 2\pi t \\ &= -2v - 2y + \sin 2\pi t. \end{aligned}$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -2v - 2y + \sin 2\pi t \end{aligned}$$

5. First, solve the differential equation $y'' + \mu(t^2 - 1)y' + y = 0$ for the highest derivative of y present in the equation.

$$y'' = -\mu(t^2 - 1)y' - y$$

Next, set $v = y'$. Then

$$v' = y'' = -\mu(t^2 - 1)y' - y = -\mu(t^2 - 1)v - y.$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -\mu(t^2 - 1)v - y \end{aligned}$$

6. First, solve the differential equation $y'' + cy' - ay + by^3 = A \cos \omega t$ for the highest derivative of y present in the equation.

$$y'' = -cy' + ay - by^3 + A \cos \omega t$$

Next, set $v = y'$. Then

$$\begin{aligned} v' = y'' &= -cy' + ay - by^3 + A \cos \omega t \\ &= -cv + ay - by^3 + A \cos \omega t. \end{aligned}$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -cv + ay - by^3 + A \cos \omega t \end{aligned}$$

7. First, solve the differential equation $LQ'' + RQ' + (1/C)Q = E(t)$ for the highest derivative of Q present in the equation.

$$Q'' = -\frac{R}{L}Q' - \frac{1}{LC}Q + \frac{1}{L}E(t)$$

Next, set $I = Q'$. Then

$$\begin{aligned} I' = Q'' &= -\frac{R}{L}Q' - \frac{1}{LC}Q + \frac{1}{L}E(t) \\ &= -\frac{R}{L}I - \frac{1}{LC}Q + \frac{1}{L}E(t). \end{aligned}$$

Thus, we now have the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + \frac{1}{L}E(t) \end{aligned}$$

8. Given $y'' = 2y' - 3y + 2 \cos 3t$, set $x_1 = y$ and $x_2 = y'$ and differentiate.

$$\begin{aligned} x_1' &= y' \\ x_2' = y'' &= 2y' - 3y + 2 \cos 3t. \end{aligned}$$

Replace all occurrence of y with x_1 and y' with x_2 .

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= 2x_2 - 3x_1 + 2 \cos 3t. \end{aligned}$$

9. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -\frac{\mu}{m}v - \frac{k}{m}y \end{aligned}$$

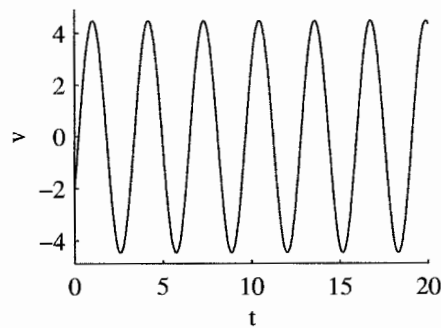
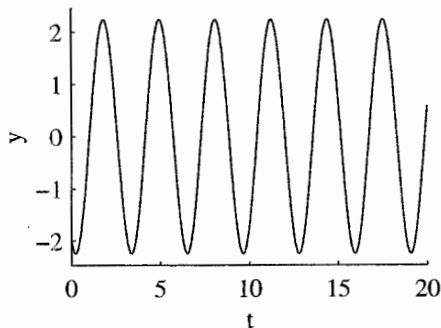
Now, if $m = 1$ kg, $\mu = 0$ kg/s, $k = 4$ kg/s², $y(0) = -2$ m, and $y'(0) = -2$ m/s, then

$$\begin{aligned} y' &= v \\ v' &= -4y, \end{aligned}$$

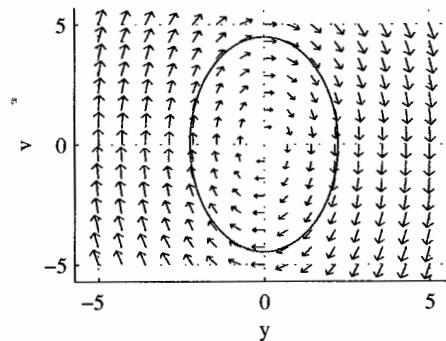
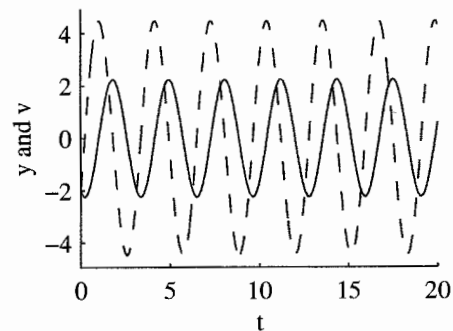
and

$$\begin{aligned}y(0) &= -2 \\v(0) = y'(0) &= -2.\end{aligned}$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.



10. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned}y' &= v \\v' &= -\frac{\mu}{m}v - \frac{k}{m}y\end{aligned}$$

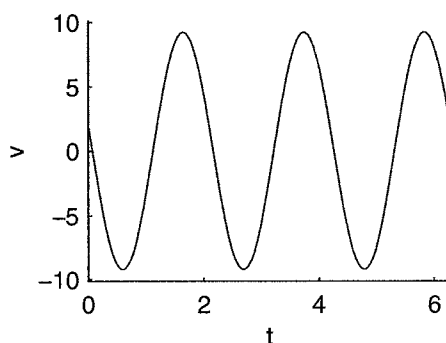
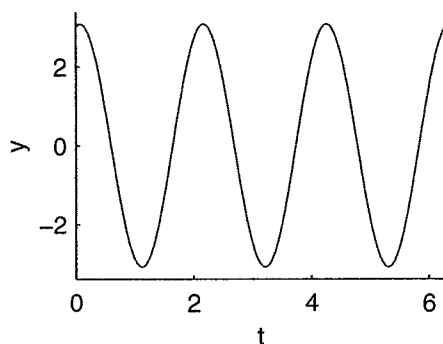
Now, if $m = 1$ kg, $\mu = 0$ kg/s, $k = 9$ kg/s², $y(0) = 3$ m, and $y'(0) = 2$ m/s, then

$$\begin{aligned}y' &= v \\v' &= -9y,\end{aligned}$$

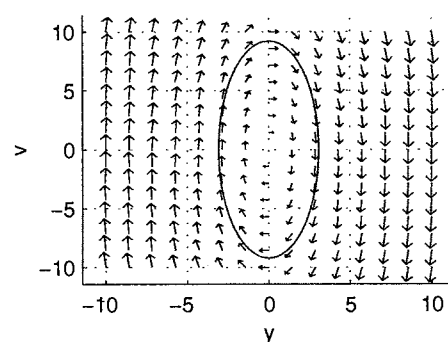
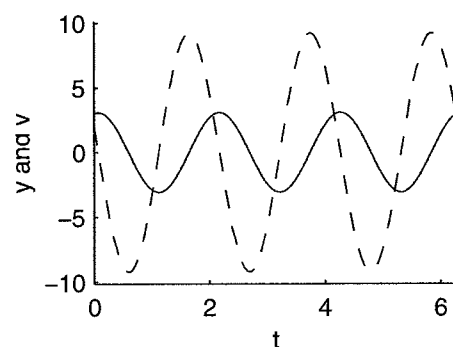
and

$$\begin{aligned}y(0) &= 3 \\v(0) = y'(0) &= 2.\end{aligned}$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.



11. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned}y' &= v \\v' &= -\frac{\mu}{m}v - \frac{k}{m}y\end{aligned}$$

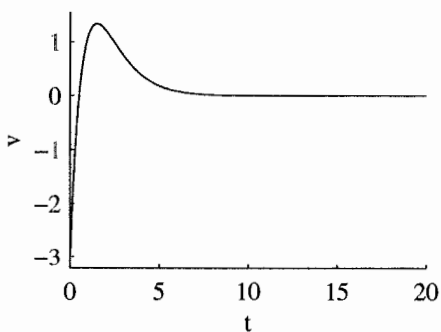
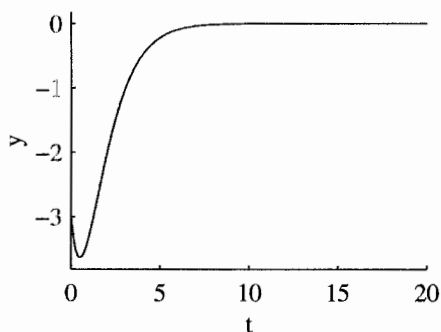
Now, if $m = 1$ kg, $\mu = 2$ kg/s, $k = 1$ kg/s², $y(0) = -3$ m, and $y'(0) = -2$ m/s, then

$$\begin{aligned}y' &= v \\v' &= -2v - y,\end{aligned}$$

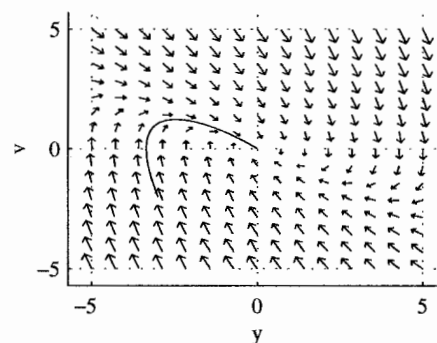
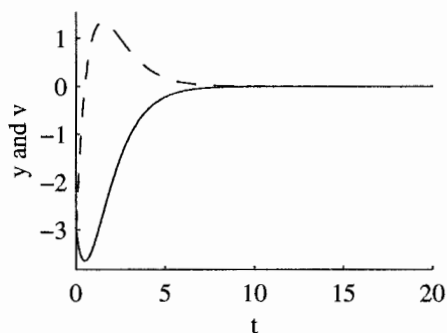
and

$$\begin{aligned}y(0) &= -3 \\v(0) = y'(0) &= -2.\end{aligned}$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.



12. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$y' = v$$

$$v' = -\frac{\mu}{m}v - \frac{k}{m}y$$

Now, if $m = 4$ kg, $\mu = 4$ kg/s, $k = 1$ kg/s², $y(0) = 3$ m, and $y'(0) = 1$ m/s, then

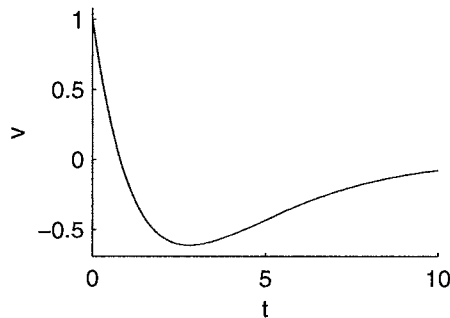
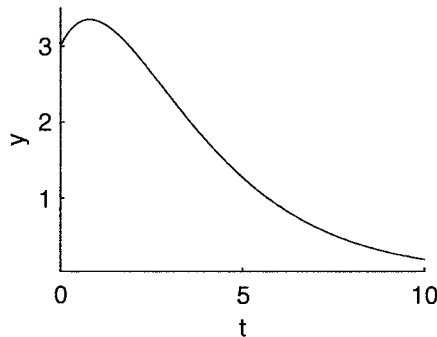
$$y' = v$$

$$v' = -v - \frac{1}{4}y,$$

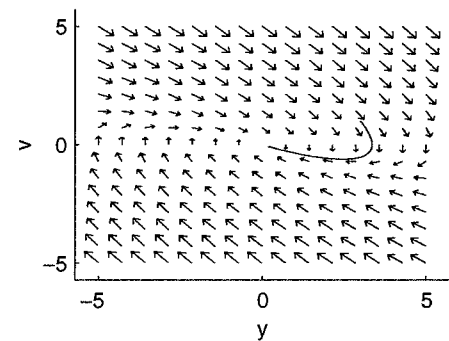
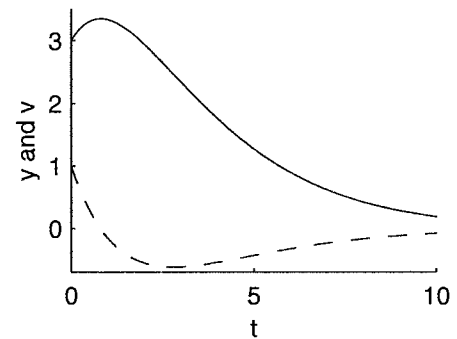
and

$$\begin{aligned}y(0) &= 3 \\v(0) = y'(0) &= 1.\end{aligned}$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.



13. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned}y' &= v \\v' &= -\frac{\mu}{m}v - \frac{k}{m}y\end{aligned}$$

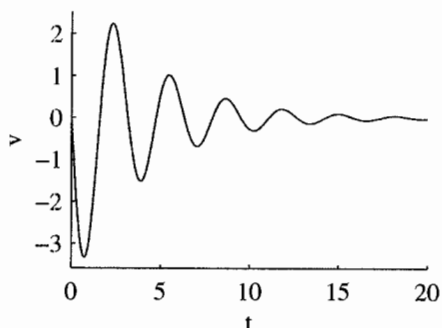
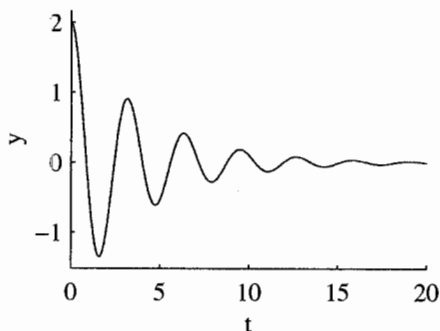
Now, if $m = 1$ kg, $\mu = 0.5$ kg/s, $k = 4$ kg/s², $y(0) = 2$ m, and $y'(0) = 0$ m/s, then

$$\begin{aligned}y' &= v \\v' &= -0.5v - 4y,\end{aligned}$$

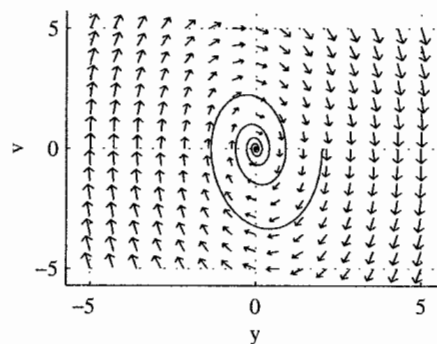
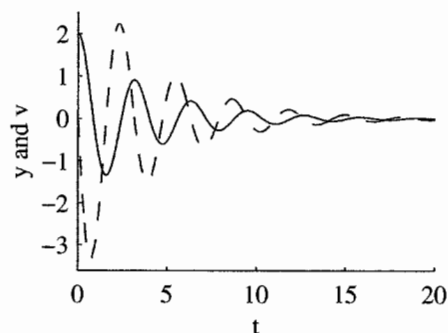
and

$$\begin{aligned} y(0) &= 2 \\ v(0) &= y'(0) = 0. \end{aligned}$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.



14. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -\frac{\mu}{m}v - \frac{k}{m}y \end{aligned}$$

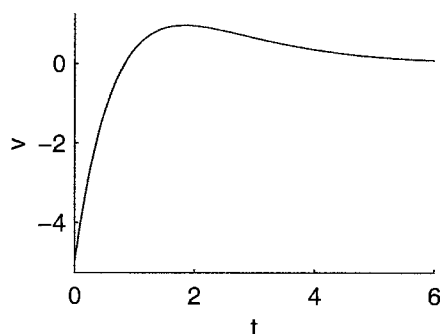
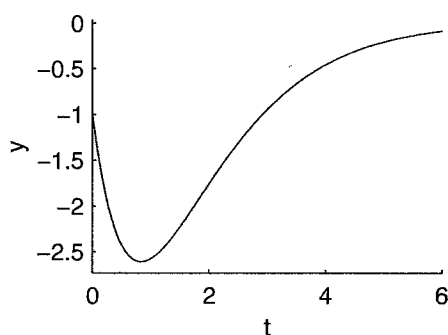
Now, if $m = 1$ kg, $\mu = 2$ kg/s, $k = 1$ kg/s², $y(0) = -1$ m, and $y'(0) = -5$ m/s, then

$$\begin{aligned} y' &= v \\ v' &= -2v - y, \end{aligned}$$

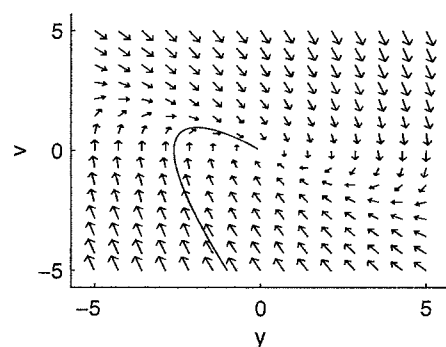
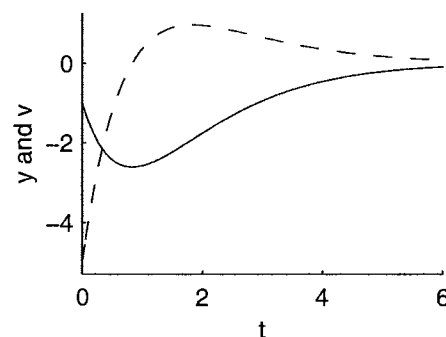
and

$$\begin{aligned}y(0) &= -1 \\v(0) = y'(0) &= -5.\end{aligned}$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.



15. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$\begin{aligned}y' &= v \\v' &= -\frac{\mu}{m}v - \frac{k}{m}y\end{aligned}$$

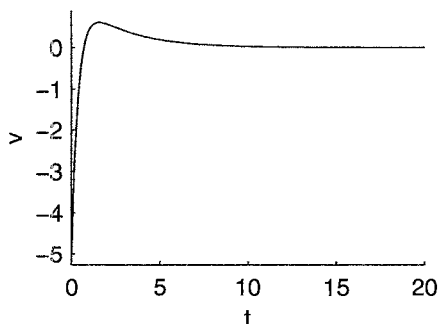
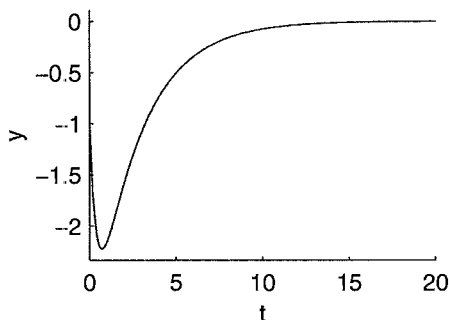
Now, if $m = 1$ kg, $\mu = 3$ kg/s, $k = 1$ kg/s², $y(0) = -1$ m, and $y'(0) = -5$ m/s, then

$$\begin{aligned}y' &= v \\v' &= -3v - y,\end{aligned}$$

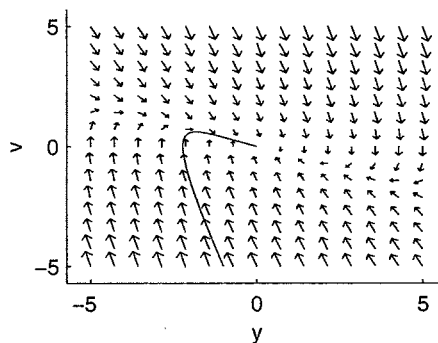
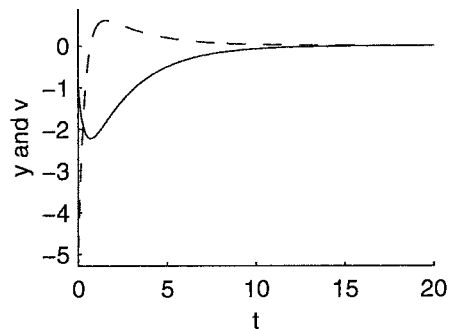
and

$$\begin{aligned}y(0) &= -1 \\v(0) = y'(0) &= -5.\end{aligned}$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.



16. Our solver requires that we first change the second order equation $my'' + \mu y' + ky = 0$ into a system of two first order equations. If we let $y' = v$, then

$$v' = y'' = -\frac{\mu}{m}y' - \frac{k}{m}y = -\frac{\mu}{m}v - \frac{k}{m}y,$$

which leads to the following system of first order equations.

$$y' = v$$

$$v' = -\frac{\mu}{m}v - \frac{k}{m}y$$

Now, if $m = 1$ kg, $\mu = 0.2$ kg/s, $k = 1$ kg/s², $y(0) = -3$ m, and $y'(0) = -2$ m/s, then

$$y' = v$$

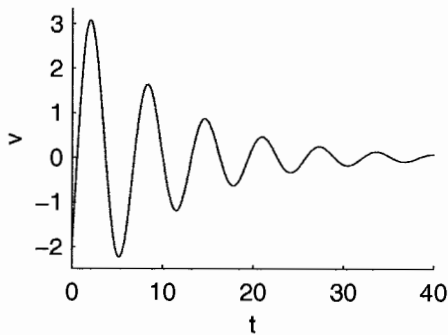
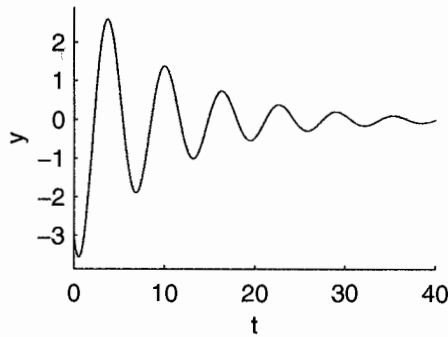
$$v' = -0.2v - y,$$

and

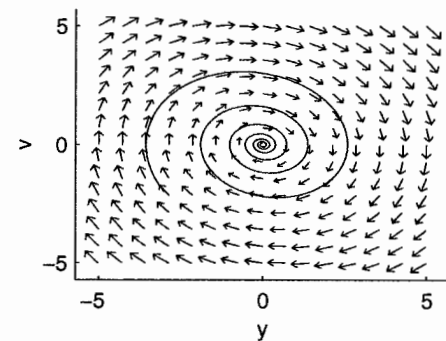
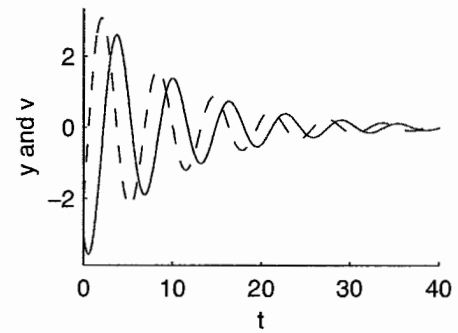
$$y(0) = -3$$

$$v(0) = y'(0) = -2.$$

Entering this system in our solver, we generate plots of position versus time and velocity versus time.



Our solver also generates a combined plot of both position and velocity versus time, and a plot of velocity versus position in the phase plane.

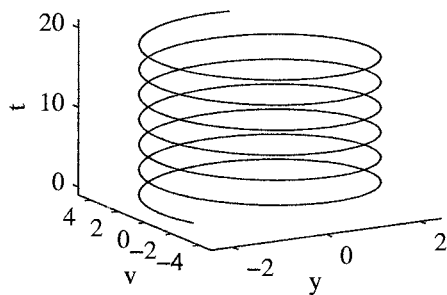


17. (a) In Figure 3, the peaks of the curve $t \rightarrow y(t)$ occur where $y' = v = 0$, and these are the points where the curve $t \rightarrow v(t)$ crosses the t -axis.
- (b) In Figure 3, the peaks of the curve $t \rightarrow v(t)$ occur slightly before the curve $t \rightarrow y(t)$ crosses the t -axis. This is because the velocity has maximum magnitude where $v' = a = y'' = 0$. Since the damping constant $\mu > 0$, at these points the differential equation $my'' + \mu y' + ky = 0$ gives $y = -\mu v/k$. Thus $y \neq 0$ and has a sign opposite to that of v .
18. (a) These represent the maximum and minimum displacements of the mass.
- (b) When the trajectory crosses the v -axis, the displacement y is zero. These crossings represent

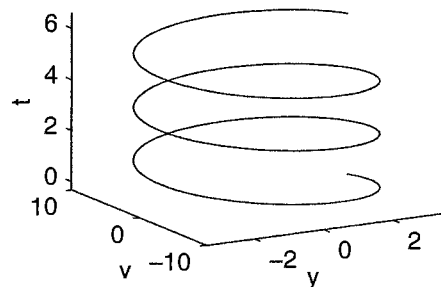
the moments when the mass passes through its equilibrium position.

- (c) This is the natural decay of the spring mass system over time. The oscillations become smaller and smaller, until the mass is finally comes to rest at it equilibrium point.

19. A plot of $t \rightarrow (y(t), v(t), t)$ for the system in Exercise 9.

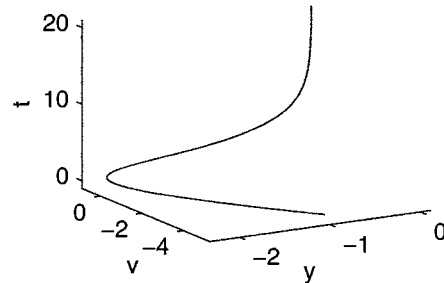


20. A plot of $t \rightarrow (y(t), v(t), t)$ for the system in Exercise 10.

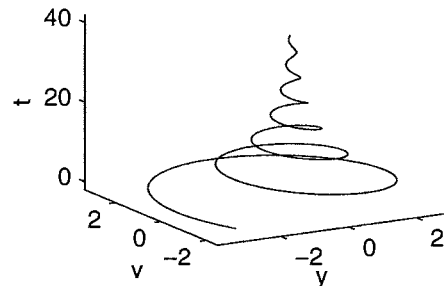


21. A plot of $t \rightarrow (y(t), v(t), t)$ for the system in Exercise

cise 15.



22. A plot of $t \rightarrow (y(t), v(t), t)$ for the system in Exercise 16.



23. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2 \cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$\begin{aligned} I' = Q'' &= -\frac{R}{L}Q' - \frac{1}{LC}Q + 2 \cos 2t \\ &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t, \end{aligned}$$

which leads to the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t \end{aligned}$$

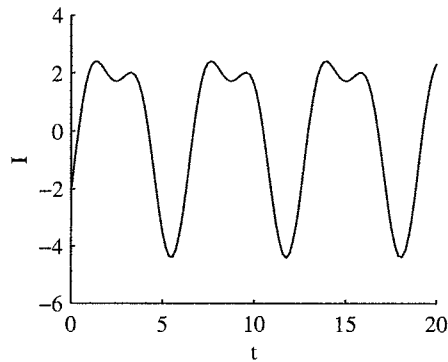
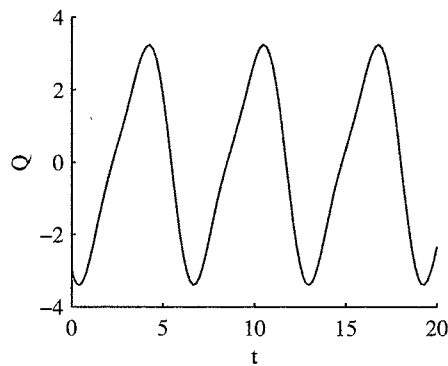
Now, if $L = 1$ H, $R = 0$ Ω , $C = 1$ F, $Q(0) = -3$ C, and $I(0) = -2$ A, then

$$\begin{aligned} Q' &= I \\ I' &= -Q + 2 \cos 2t, \end{aligned}$$

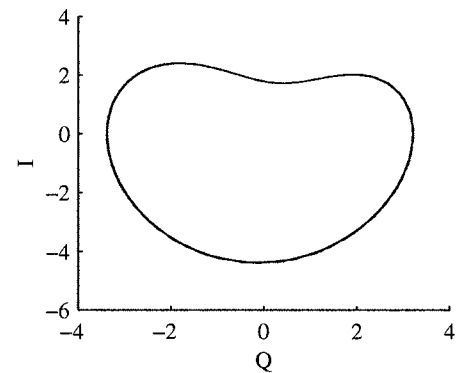
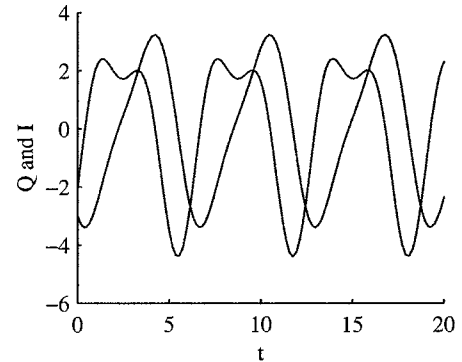
and

$$\begin{aligned} Q(0) &= -3 \\ I(0) &= -2. \end{aligned}$$

Entering this system in our solver, we generate plots of charge versus time and current versus time.



Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.



24. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2 \cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$\begin{aligned} I' = Q'' &= -\frac{R}{L}Q' - \frac{1}{LC}Q + 2 \cos 2t \\ &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t, \end{aligned}$$

which leads to the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t \end{aligned}$$

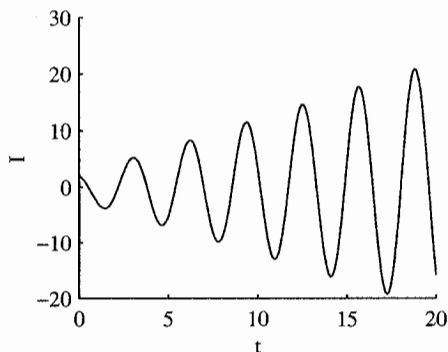
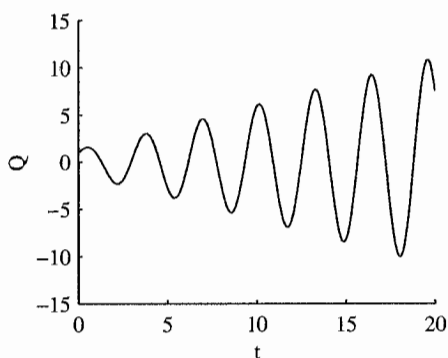
Now, if $L = 1 \text{ H}$, $R = 0 \Omega$, $C = 1/4 \text{ F}$, $Q(0) = 1 \text{ C}$, and $I(0) = 2 \text{ A}$, then

$$\begin{aligned} Q' &= I \\ I' &= -4Q + 2 \cos 2t, \end{aligned}$$

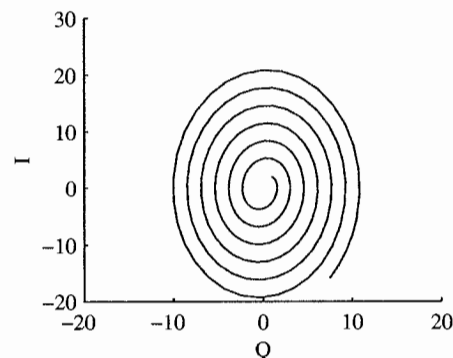
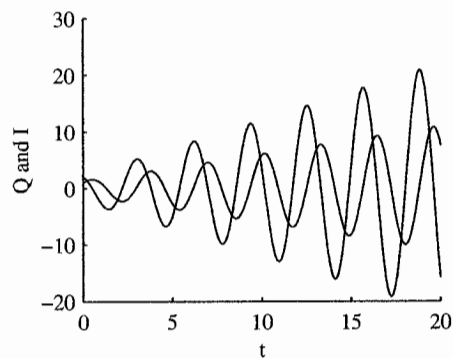
and

$$\begin{aligned} Q(0) &= 1 \\ I(0) &= 2. \end{aligned}$$

Entering this system in our solver, we generate plots of charge versus time and current versus time.



Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.



25. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2 \cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$\begin{aligned} I' = Q'' &= -\frac{R}{L}Q' - \frac{1}{LC}Q + 2 \cos 2t \\ &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t, \end{aligned}$$

which leads to the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t \end{aligned}$$

Now, if $L = 1 \text{ H}$, $R = 5 \Omega$, $C = 1 \text{ F}$, $Q(0) = 1 \text{ C}$,

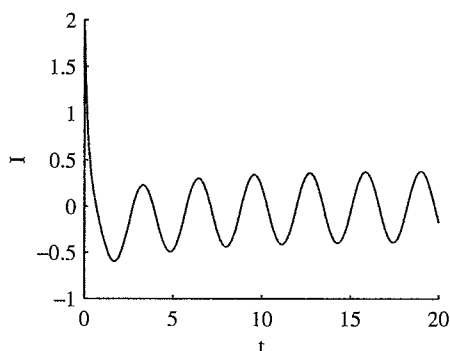
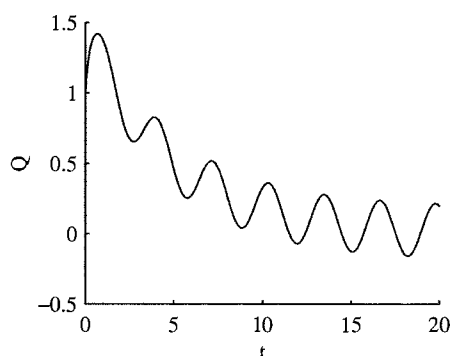
and $I(0) = 2$ A, then

$$\begin{aligned} Q' &= I \\ I' &= -5I - Q + 2 \cos 2t, \end{aligned}$$

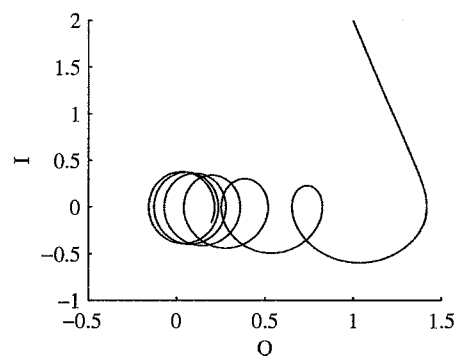
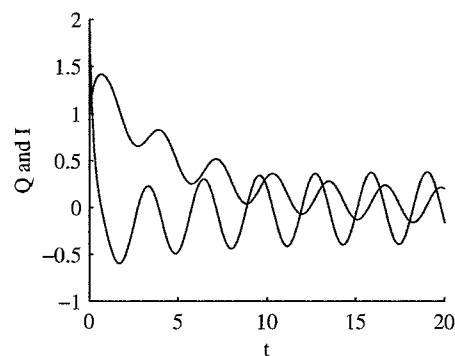
and

$$\begin{aligned} Q(0) &= 1 \\ I(0) &= 2. \end{aligned}$$

Entering this system in our solver, we generate plots of charge versus time and current versus time.



Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.



26. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2 \cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$\begin{aligned} I' = Q'' &= -\frac{R}{L}Q' - \frac{1}{LC}Q + 2 \cos 2t \\ &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t, \end{aligned}$$

which leads to the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t \end{aligned}$$

Now, if $L = 2$ H, $R = 4 \Omega$, $C = 1$ F, $Q(0) = -1$ C,

and $I(0) = -2$ A, then

$$Q' = I$$

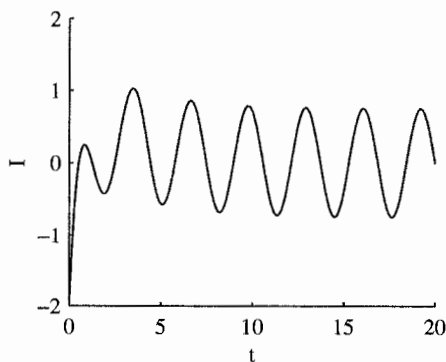
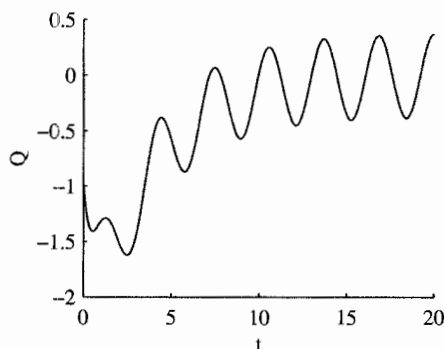
$$I' = -2I - \frac{1}{2}Q + 2 \cos 2t,$$

and

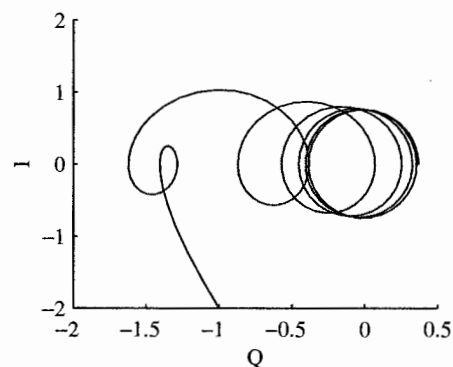
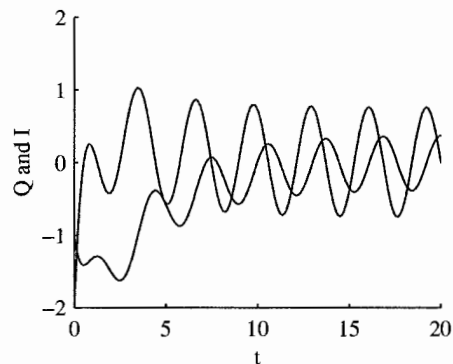
$$Q(0) = -1$$

$$I(0) = -2.$$

Entering this system in our solver, we generate plots of charge versus time and current versus time.



Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.



27. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2 \cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$\begin{aligned} I' = Q'' &= -\frac{R}{L}Q' - \frac{1}{LC}Q + 2 \cos 2t \\ &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t, \end{aligned}$$

which leads to the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t \end{aligned}$$

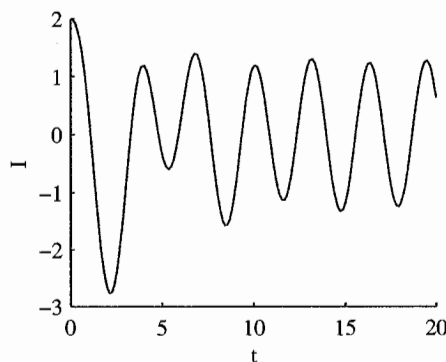
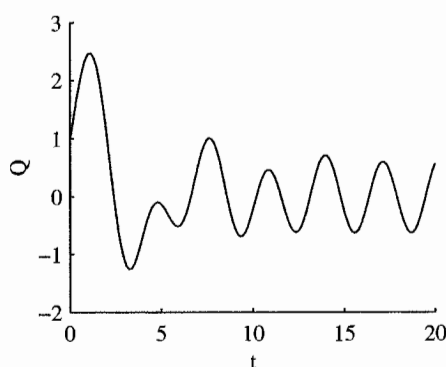
Now, if $L = 1$ H, $R = 0.5 \Omega$, $C = 1$ F, $Q(0) = 1$ C, and $I(0) = 2$ A, then

$$\begin{aligned} Q' &= I \\ I' &= -0.5I - Q + 2 \cos 2t, \end{aligned}$$

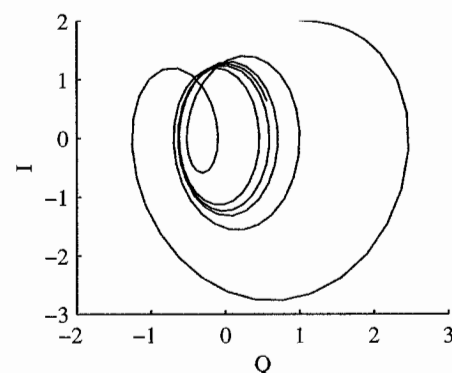
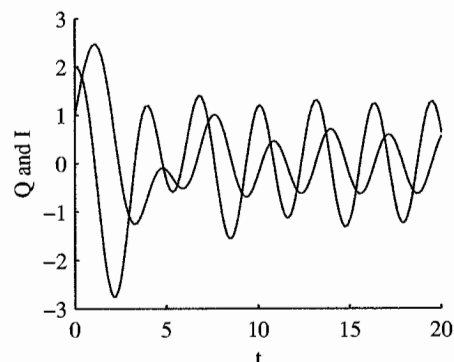
and

$$\begin{aligned} Q(0) &= 1 \\ I(0) &= 2. \end{aligned}$$

Entering this system in our solver, we generate plots of charge versus time and current versus time.



Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.



28. Our solver requires that we first change the second order equation $LQ'' + RQ' + (1/C)Q = 2 \cos 2t$ into a system of two first order equations. If we let $Q' = I$, then

$$\begin{aligned} I' = Q'' &= -\frac{R}{L}Q' - \frac{1}{LC}Q + 2 \cos 2t \\ &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t, \end{aligned}$$

which leads to the following system of first order equations.

$$\begin{aligned} Q' &= I \\ I' &= -\frac{R}{L}I - \frac{1}{LC}Q + 2 \cos 2t \end{aligned}$$

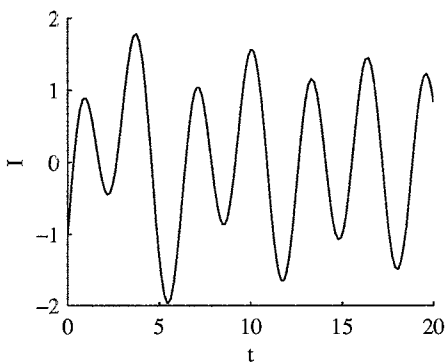
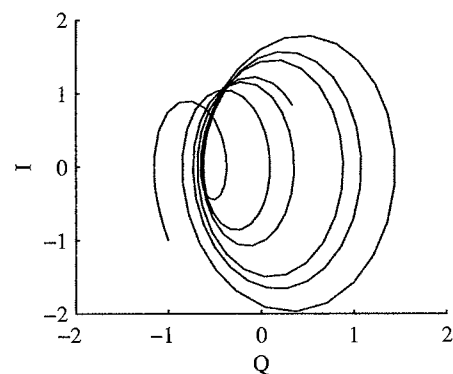
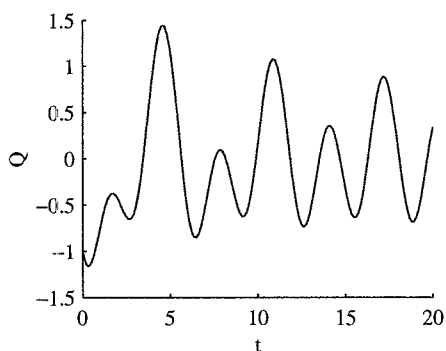
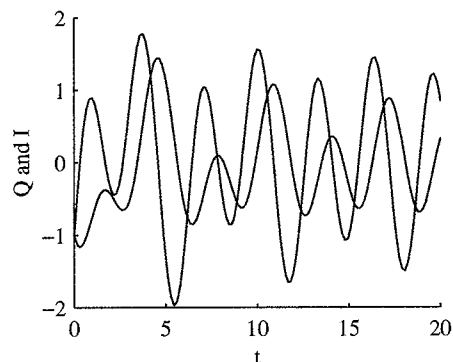
Now, if $L = 1 \text{ H}$, $R = 0.2 \Omega$, $C = 1 \text{ F}$, $Q(0) = -1 \text{ C}$, and $I(0) = -1 \text{ A}$, then

$$\begin{aligned} Q' &= I \\ I' &= -0.2I - Q + 2 \cos 2t, \end{aligned}$$

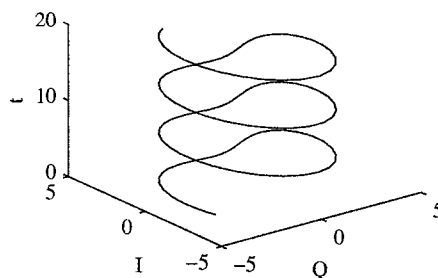
and

$$\begin{aligned} Q(0) &= -1 \\ I(0) &= -1. \end{aligned}$$

Entering this system in our solver, we generate plots of charge versus time and current versus time.

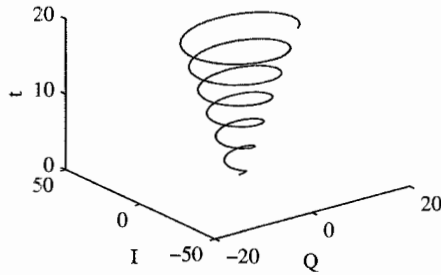


29. A plot of $t \rightarrow (Q(t), I(t), t)$ for the circuit of Exercise 23.

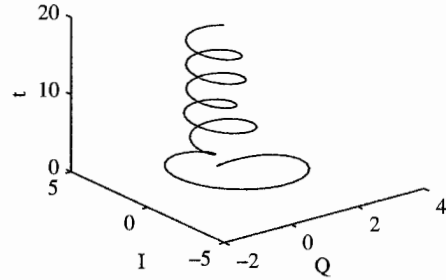


Our solver also generates a combined plot of both charge and current versus time, and a plot of current versus charge in the phase plane.

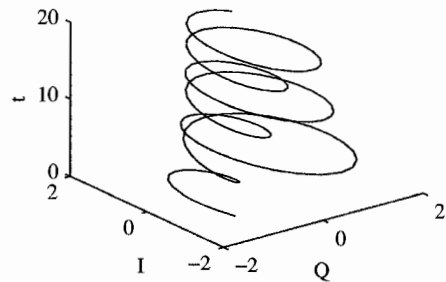
30. A plot of $t \rightarrow (Q(t), I(t), t)$ for the circuit of Exercise 24.



cise 27.



32. A plot of $t \rightarrow (Q(t), I(t), t)$ for the circuit of Exercise 28.



31. A plot of $t \rightarrow (Q(t), I(t), t)$ for the circuit of Exer-

_____x_____

Section 4.3. Linear Homogeneous Equations with Constant Coefficients

1. Let $y = e^{\lambda t}$ in $y'' - y' - 2y = 0$ to obtain

$$\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - 2e^{\lambda t} = 0$$

$$e^{\lambda t} (\lambda^2 - \lambda - 2) = 0.$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0,$$

and roots $\lambda = 2$ and $\lambda = -1$. Because the roots are distinct, the solutions $y_1(t) = e^{2t}$ and $y_2(t) = e^{-t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{2t} + C_2 e^{-t}.$$

2. Let
- $y = e^{\lambda t}$
- in
- $2y'' - 3y' - 2y = 0$
- to obtain

$$\begin{aligned} 2\lambda^2 e^{\lambda t} - 3\lambda e^{\lambda t} - 2e^{\lambda t} &= 0 \\ e^{\lambda t}(2\lambda^2 - 3\lambda - 2) &= 0. \end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\begin{aligned} 2\lambda^2 - 3\lambda - 2 &= 0 \\ (2\lambda + 1)(\lambda - 2) &= 0, \end{aligned}$$

and roots $\lambda = -1/2$ and $\lambda = 2$. Because the roots are distinct, the solutions $y_1(t) = e^{-t/2}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-t/2} + C_2 e^{2t}.$$

3. Let
- $y = e^{\lambda t}$
- in
- $y'' + 5y' + 6y = 0$
- to obtain

$$\begin{aligned} \lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6e^{\lambda t} &= 0, \\ e^{\lambda t}(\lambda^2 + 5\lambda + 6) &= 0. \end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\begin{aligned} \lambda^2 + 5\lambda + 6 &= 0, \\ (\lambda + 3)(\lambda + 2) &= 0, \end{aligned}$$

and roots $\lambda = -3$ and $\lambda = -2$. Because the roots are distinct, the solution $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-2t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-3t} + C_2 e^{-2t}.$$

4. Let
- $y = e^{\lambda t}$
- in
- $y'' + y' - 12y = 0$
- to obtain

$$\begin{aligned} \lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 12e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 + \lambda - 12) &= 0. \end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\begin{aligned} \lambda^2 + \lambda - 12 &= 0 \\ (\lambda + 4)(\lambda - 3) &= 0, \end{aligned}$$

and roots $\lambda = -4$ and $\lambda = 3$. Because the roots are distinct, the solutions $y_1(t) = e^{-4t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-4t} + C_2 e^{3t}.$$

5. Let
- $y = e^{\lambda t}$
- in
- $2y'' - y' - y = 0$
- to obtain

$$\begin{aligned} 2\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - e^{\lambda t} &= 0, \\ e^{\lambda t}(2\lambda^2 - \lambda - 1) &= 0. \end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\begin{aligned} 2\lambda^2 - \lambda - 1 &= 0, \\ (2\lambda + 1)(\lambda - 1) &= 0, \end{aligned}$$

and roots $\lambda = -1/2$ and $\lambda = 1$. Because the roots are distinct, the solutions $y_1(t) = e^{-(1/2)t}$ and $y_2(t) = e^t$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-(1/2)t} + C_2 e^t.$$

6. Let
- $y = e^{\lambda t}$
- in
- $6y'' + y' - y = 0$
- to obtain

$$\begin{aligned} 6\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - e^{\lambda t} &= 0 \\ e^{\lambda t}(6\lambda^2 + \lambda - 1) &= 0. \end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\begin{aligned} 6\lambda^2 + \lambda - 1 &= 0 \\ (3\lambda - 1)(2\lambda + 1) &= 0, \end{aligned}$$

and roots $\lambda = 1/3$ and $\lambda = -1/2$. Because the roots are distinct, the solutions $y_1(t) = e^{(1/3)t}$ and $y_2(t) = e^{-(1/2)t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{t/3} + C_2 e^{-t/2}.$$

7. Let
- $y = e^{\lambda t}$
- in
- $3y'' - 2y' - y = 0$
- to obtain

$$\begin{aligned} 3\lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} - e^{\lambda t} &= 0 \\ e^{\lambda t}(3\lambda^2 - 2\lambda - 1) &= 0. \end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\begin{aligned} 3\lambda^2 - 2\lambda - 1 &= 0 \\ (3\lambda + 1)(\lambda - 1) &= 0, \end{aligned}$$

and roots $\lambda = -1/3$ and $\lambda = 1$. Because the roots are distinct, the solutions $y_1(t) = e^{-t/3}$ and $y_2(t) = e^t$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-t/3} + C_2 e^t.$$

8. Let $y = e^{\lambda t}$ in $6y'' + 5y' - 6y = 0$ to obtain

$$\begin{aligned} 6\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} - 6e^{\lambda t} &= 0, \\ e^{\lambda t}(6\lambda^2 + 5\lambda - 6) &= 0. \end{aligned}$$

Because $e^{\lambda t} \neq 0$, we arrive at the logistic equation

$$\begin{aligned} 6\lambda^2 + 5\lambda - 6 &= 0 \\ (3\lambda - 2)(2\lambda + 3) &= 0, \end{aligned}$$

and roots $\lambda = 2/3$ and $\lambda = -3/2$. Because the roots are distinct, the solutions $y_1(t) = e^{(2/3)t}$ and $y_2(t) = e^{-(3/2)t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-(3/2)t} + C_2 e^{(2/3)t}.$$

9. If $y'' + y = 0$, then the characteristic equation is

$$\lambda^2 + 1 = 0.$$

The roots of the characteristic equation are $\pm i$, leading to the complex solutions

$$z(t) = e^{it} \quad \text{and} \quad \bar{z}(t) = e^{-it}.$$

However, by Euler's identity,

$$z(t) = \cos t + i \sin t,$$

and the real and imaginary parts of z lead to a fundamental set of real solutions, $y_1(t) = \cos t$ and $y_2(t) = \sin t$. Hence, the general solution is

$$y(t) = C_1 \cos t + C_2 \sin t.$$

10. If $y'' + 4y = 0$, then the characteristic equation is

$$\lambda^2 + 4 = 0.$$

The roots of the characteristic equation are $\pm 2i$, leading to the complex solutions

$$z(t) = e^{2it} \quad \text{and} \quad \bar{z}(t) = e^{-2it}.$$

However, by Euler's identity,

$$z(t) = \cos 2t + i \sin 2t,$$

and the real and imaginary parts of z lead to a fundamental set of real solutions $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$. Hence, the general solution is

$$y(t) = C_1 \cos 2t + C_2 \sin 2t.$$

11. If $y'' + 4y' + 5y = 0$, then the characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0.$$

The roots of the characteristic equation are

$$z = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i,$$

leading to the complex solutions

$$z(t) = e^{(-2+i)t} \quad \text{and} \quad \bar{z}(t) = e^{(-2-i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-2t} e^{it} = e^{-2t} (\cos t + i \sin t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions, $y_1(t) = e^{-2t} \cos t$ and $y_2(t) = e^{-2t} \sin t$. Hence, the general solution is

$$\begin{aligned} y(t) &= C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t \\ &= e^{-2t} (C_1 \cos t + C_2 \sin t). \end{aligned}$$

12. If $y'' + 2y' + 17y = 0$, then the characteristic equation is

$$\lambda^2 + 2\lambda + 17 = 0.$$

The roots of the characteristic equation are $-1 \pm 4i$, leading to the complex solutions

$$z(t) = e^{(-1+4i)t} \quad \text{and} \quad \bar{z}(t) = e^{(-1-4i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-t} e^{4it} = e^{-t} (\cos 4t + i \sin 4t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions $y_1(t) = e^{-t} \cos 4t$ and $y_2(t) = e^{-t} \sin 4t$. Hence the general solution is

$$y(t) = C_1 e^{-t} \cos 4t + C_2 e^{-t} \sin 4t.$$

13. If
- $y'' + 2y = 0$
- , then the characteristic equation is

$$\lambda^2 + 2 = 0.$$

The roots of the characteristic equation are $\pm\sqrt{2}i$, leading to the complex solutions

$$z(t) = e^{\sqrt{2}it} \quad \text{and} \quad \bar{z}(t) = e^{-\sqrt{2}it}.$$

However, by Euler's identity,

$$z(t) = \cos \sqrt{2}t + i \sin \sqrt{2}t,$$

and the real and imaginary parts of z lead to a fundamental set of real solutions, $y_1(t) = \cos \sqrt{2}t$ and $y_2(t) = \sin \sqrt{2}t$. Hence, the general solution is

$$y(t) = C_1 \cos \sqrt{2}t + C_2 \sin \sqrt{2}t.$$

14. If
- $y'' + 2y' + 3y = 0$
- , then the characteristic equation is

$$\lambda^2 + 2\lambda + 3 = 0.$$

The roots of the characteristic equation are $-1 \pm \sqrt{2}i$, leading to the complex solutions

$$z(t) = e^{(-1+\sqrt{2}i)t} \quad \text{and} \quad \bar{z}(t) = e^{(-1-\sqrt{2}i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-t} e^{\sqrt{2}it} = e^{-t} (\cos \sqrt{2}t + i \sin \sqrt{2}t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions $y_1(t) = e^{-t} \cos \sqrt{2}t$ and $y_2(t) = e^{-t} \sin \sqrt{2}t$. Hence the general solution is

$$y(t) = C_1 e^{-t} \cos \sqrt{2}t + C_2 e^{-t} \sin \sqrt{2}t.$$

15. If
- $y'' - 2y' + 4y = 0$
- , then the characteristic equation is

$$\lambda^2 - 2\lambda + 4 = 0.$$

The roots of the characteristic equation are $1 \pm \sqrt{3}i$, leading to the complex solutions

$$z(t) = e^{(1+\sqrt{3}i)t} \quad \text{and} \quad \bar{z}(t) = e^{(1-\sqrt{3}i)t}.$$

However, by Euler's identity,

$$z(t) = e^t e^{\sqrt{3}it} = e^t (\cos \sqrt{3}t + i \sin \sqrt{3}t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions $y_1(t) = e^t \cos \sqrt{3}t$ and $y_2(t) = e^t \sin \sqrt{3}t$. Hence the general solution is

$$y(t) = C_1 e^t \cos \sqrt{3}t + C_2 e^t \sin \sqrt{3}t.$$

16. If
- $y'' + 2y' + 2y = 0$
- , then the characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0.$$

The roots of the characteristic equation are $-1 \pm i$, leading to the complex solutions

$$z(t) = e^{(-1+i)t} \quad \text{and} \quad \bar{z}(t) = e^{(-1-i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-t} e^{it} = e^{-t} (\cos t + i \sin t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions, $y_1(t) = e^{-t} \cos t$ and $y_2(t) = e^{-t} \sin t$. Hence, the general solution is

$$y(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t.$$

17. If
- $y'' - 4y' + 4y = 0$
- , then the characteristic equation is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = 2$. Therefore, $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{2t} + C_2 t e^{2t} = (C_1 + C_2 t) e^{2t}.$$

18. If
- $y'' - 6y' + 9y = 0$
- , then the characteristic equation is

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0.$$

Hence the characteristic equation has a repeated root, $\lambda = 3$. Therefore, $y_1(t) = e^{3t}$ and $y_2(t) = te^{3t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{3t} + C_2 t e^{3t} = (C_1 + C_2 t) e^{3t}.$$

19. If
- $4y'' + 4y' + y = 0$
- , then the characteristic equation is

$$4\lambda^2 + 4\lambda + 1 = (2\lambda + 1)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = -1/2$. Therefore, $y_1(t) = e^{-(1/2)t}$ and $y_2(t) = te^{-(1/2)t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-t/2} + C_2 t e^{-t/2} = (C_1 + C_2 t) e^{-t/2}.$$

20. If $4y'' + 12y' + 9y = 0$, then the characteristic equation is

$$4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2 = 0.$$

Hence, the characteristic equation has a repeated root, $\lambda = -3/2$. Therefore, $y_1(t) = e^{-(3/2)t}$ and $y_2(t) = te^{-(3/2)t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-(3/2)t} + C_2 t e^{-(3/2)t} = (C_1 + C_2 t) e^{-(3/2)t}.$$

21. If $16y'' + 8y' + y = 0$, then the characteristic equation is

$$16\lambda^2 + 8\lambda + 1 = (4\lambda + 1)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = -1/4$. Therefore, $y_1(t) = e^{-(1/4)t}$ and $y_2(t) = te^{-(1/4)t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-t/4} + C_2 t e^{-t/4} = (C_1 + C_2 t) e^{-t/4}.$$

22. If $y'' + 4y' + 4y = 0$, then the characteristic equation is

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = -2$. Therefore, $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t} = (C_1 + C_2 t) e^{-2t}.$$

23. If $16y'' + 24y' + 9y = 0$, then the characteristic equation is

$$16\lambda^2 + 24\lambda + 9 = (4\lambda + 3)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = -3/4$. Therefore, $y_1(t) = e^{-(3/4)t}$ and $y_2(t) = te^{-(3/4)t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-3t/4} + C_2 t e^{-3t/4} = (C_1 + C_2 t) e^{-3t/4}.$$

24. If $y'' + 8y' + 16y = 0$, then the characteristic equation is

$$\lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0.$$

Hence, the characteristic equation has a repeated root, $\lambda = -4$. Therefore, $y_1(t) = e^{-4t}$ and $y_2(t) = te^{-4t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-4t} + C_2 t e^{-4t} = (C_1 + C_2 t) e^{-4t}.$$

25. If $y'' - y' - 2y = 0$, then the characteristic equation is

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0,$$

with roots $\lambda = 2$ and $\lambda = -1$. This leads to the general solution

$$y(t) = C_1 e^{2t} + C_2 e^{-t}.$$

Using the initial condition $y(0) = -1$ provides

$$-1 = C_1 + C_2.$$

Differentiating the general solution,

$$y'(t) = 2C_1 e^{2t} - C_2 e^{-t},$$

then using the initial condition $y'(0) = 2$ leads to

$$2 = 2C_1 - C_2.$$

These equations yield $C_1 = 1/3$ and $C_2 = -4/3$, giving the particular solution

$$y(t) = \frac{1}{3} e^{2t} - \frac{4}{3} e^{-t}.$$

26. If $10y'' - y' - 3y = 0$, then the characteristic equation is

$$10\lambda^2 - \lambda - 3 = (2\lambda + 1)(5\lambda - 3) = 0,$$

with roots $\lambda = -1/2$ and $\lambda = 3/5$. This leads to the general solution

$$y(t) = C_1 e^{-t/2} + C_2 e^{3t/5}.$$

Using the initial condition $y(0) = 1$ provides

$$1 = C_1 + C_2.$$

Differentiating the general solution,

$$y'(t) = -\frac{1}{2} C_1 e^{-t/2} + \frac{3}{5} C_2 e^{3t/5},$$

then using the initial condition $y'(0) = 0$ leads to

$$0 = -\frac{1}{2}C_1 + \frac{3}{5}C_2.$$

These equations yield $C_1 = 6/11$ and $C_2 = 5/11$, giving the particular solution

$$y(t) = \frac{6}{11}e^{-t/2} + \frac{5}{11}e^{3t/5}.$$

27. If $y'' - 2y' + 17y = 0$, then the characteristic equation is

$$\lambda^2 - 2\lambda + 17 = 0,$$

with roots $1 \pm 4i$. The complex solution,

$$z(t) = e^{(1+4i)t} = e^t(\cos 4t + i \sin 4t)$$

leads to a fundamental set of real solutions and the general solution

$$y(t) = e^t(C_1 \cos 4t + C_2 \sin 4t).$$

The initial condition $y(0) = -2$ provides

$$-2 = C_1.$$

Differentiating the general solution,

$$y'(t) = e^t(C_1 \cos 4t + C_2 \sin 4t) + e^t(-4C_1 \sin 4t + 4C_2 \cos 4t),$$

then using the initial condition $y'(0) = 3$ leads to

$$3 = C_1 + 4C_2.$$

These equations yield $C_1 = -2$ and $C_2 = 5/4$, giving the solution

$$y(t) = e^t(-2 \cos 4t + (5/4) \sin 4t).$$

28. If $y'' + 25y = 0$, then the characteristic equation is $\lambda^2 + 25 = 0$, which has roots $\lambda = \pm 5i$. This leads to a pair of complex solutions

$$z(t) = e^{5it} \quad \text{and} \quad \bar{z}(t) = e^{-5it}.$$

By Euler's identity,

$$z(t) = \cos 5t + i \sin 5t,$$

leading to a fundamental set of real solutions $y_1(t) = \cos 5t$ and $y_2(t) = \sin 5t$ and the general solution

$$y(t) = C_1 \cos 5t + C_2 \sin 5t.$$

Next, $y(0) = 1$ provides $C_1 = 1$. Differentiating the general solution,

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t,$$

and using the initial condition $y'(0) = -1$ provides $-1 = 5C_2$, or $C_2 = -1/5$. Thus, the particular solution is

$$y(t) = \cos 5t - \frac{1}{5} \sin 5t.$$

29. If $y'' + 10y' + 25y = 0$, then the characteristic equation is

$$\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0,$$

with repeated root $\lambda = -5$. This leads to the fundamental set of solutions $y_1(t) = e^{-5t}$ and $y_2(t) = te^{-5t}$ and the general solution is

$$y(t) = C_1 e^{-5t} + C_2 t e^{-5t} = (C_1 + C_2 t) e^{-5t}.$$

Using the initial condition $y(0) = 2$ leads to

$$2 = C_1.$$

Differentiating the general solution,

$$y'(t) = C_2 e^{-5t} - 5(C_1 + C_2 t) e^{-5t},$$

then the initial condition $y'(0) = -1$ leads to

$$-1 = C_2 - 5C_1.$$

Thus, $C_1 = 2$ and $C_2 = 9$ and the final solution is

$$y(t) = (2 + 9t) e^{-5t}.$$

30. If $y'' - 2y' - 3y = 0$, then the characteristic equation is

$$\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0,$$

with roots $\lambda = 3$ and $\lambda = -1$. This leads to the general solution

$$y(t) = C_1 e^{3t} + C_2 e^{-t}.$$

Using the initial condition $y(0) = 2$ provides

$$2 = C_1 + C_2.$$

Differentiating the general solution,

$$y'(t) = 3C_1e^{3t} - C_2e^{-t},$$

then using the initial condition $y'(0) = -3$ leads to

$$-3 = 3C_1 - C_2.$$

These equations yield $C_1 = -1/4$ and $C_2 = 9/4$, giving the particular solution

$$y(t) = -\frac{1}{4}e^{3t} + \frac{9}{4}e^{-t}.$$

31. If $y'' + 2y' + 3y = 0$, then the characteristic equation is $\lambda^2 + 2\lambda + 3 = 0$ with roots $z = -1 \pm \sqrt{2}i$. The complex solution

$$z(t) = e^{(-1+\sqrt{2}i)t} = e^{-t}(\cos \sqrt{2}t + i \sin \sqrt{2}t)$$

provides a fundamental set of real solutions, $y_1(t) = e^{-t} \cos \sqrt{2}t$ and $y_2(t) = e^{-t} \sin \sqrt{2}t$, and the general solution

$$y(t) = e^{-t}(C_1 \cos \sqrt{2}t + C_2 \sin \sqrt{2}t).$$

The initial condition $y(0) = 1$ provides

$$1 = C_1.$$

Differentiating the general solution,

$$y'(t) = e^{-t}(-C_1\sqrt{2} \cos \sqrt{2}t + C_2\sqrt{2} \sin \sqrt{2}t) - e^{-t}(C_1 \sin \sqrt{2}t + C_2 \cos \sqrt{2}t),$$

and the initial condition $y'(0) = 0$ provides

$$0 = -C_1 + \sqrt{2}C_2.$$

Thus, $C_1 = 1$ and $C_2 = \sqrt{2}/2$, giving the solution

$$y(t) = e^{-t} \left(\cos \sqrt{2}t + \frac{\sqrt{2}}{2} \sin \sqrt{2}t \right).$$

32. If $y'' - 4y' - 5y = 0$, then the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0,$$

with roots $\lambda = 5$ and $\lambda = -1$. This leads to the general solution

$$y(t) = C_1e^{5t} + C_2e^{-t}.$$

Differentiating the general solution,

$$y'(t) = 5C_1e^{5t} - C_2e^{-t},$$

then using $y'(1) = -1$,

$$-1 = 5C_1e^5 - C_2e^{-1}.$$

Thus, $C_1 = -(1/3)e^{-5}$ and $C_2 = -(2/3)e$. Thus, the final solution is

$$y(t) = -\frac{1}{3}e^{-5}e^{5t} - \frac{2}{3}ee^{-t} = -\frac{1}{3}e^{5(t-1)} - \frac{2}{3}e^{-(t-1)}.$$

33. If $8y'' + 2y' - y = 0$, then the characteristic equation is

$$8\lambda^2 + 2\lambda - 1 = (4\lambda - 1)(2\lambda + 1) = 0,$$

with roots $\lambda = 1/4$ and $\lambda = -1/2$. This leads to the general solution

$$y(t) = C_1e^{t/4} + C_2e^{-t/2}.$$

Using the initial condition $y(-1) = 1$ provides

$$1 = C_1e^{-1/4} + C_2e^{1/2}. \quad (*)$$

Differentiating the general solution,

$$y'(t) = \frac{1}{4}C_1e^{t/4} - \frac{1}{2}C_2e^{-t/2},$$

then using the initial condition $y'(-1) = 0$ leads to

$$0 = \frac{1}{4}C_1e^{-1/4} - \frac{1}{2}C_2e^{1/2}. \quad (**)$$

Subtracting 4 times equation (**) from equation (*), provides $1 = 3C_2e^{1/2}$, so $C_2 = (1/3)e^{-1/2}$. Substituting this result in equation (**) yields $C_1 = (2/3)e^{1/4}$. Thus, the particular solution is

$$y(t) = \frac{2}{3}e^{1/4}e^{t/4} + \frac{1}{3}e^{-1/2}e^{-t/2}$$

$$y(t) = \frac{2}{3}e^{(t+1)/4} + \frac{1}{3}e^{-(t+1)/2}.$$

34. If $4y'' + y = 0$, then the characteristic equation is $4\lambda^2 + 1 = 0$ with roots $\pm i/2$. The complex solution

$$z(t) = e^{(i/2)t} = \cos \frac{t}{2} + i \sin \frac{t}{2}$$

provides a fundamental set of real solutions, $y_1(t) = \cos(t/2)$ and $y_2(t) = \sin(t/2)$, leading to the general solution

$$y(t) = C_1 \cos \frac{t}{2} + C_2 \sin \frac{t}{2}.$$

The initial condition $y(1) = 0$ provides

$$0 = C_1 \cos \frac{1}{2} + C_2 \sin \frac{1}{2}.$$

Differentiating the general solution,

$$y'(t) = -\frac{1}{2}C_1 \sin \frac{t}{2} + \frac{1}{2}C_2 \cos \frac{t}{2}.$$

The initial condition $y'(1) = -2$ provides

$$-2 = -\frac{1}{2}C_1 \sin \frac{1}{2} + \frac{1}{2}C_2 \cos \frac{1}{2}.$$

Thus, $C_1 = 4 \sin(1/2)$ and $C_2 = -4 \cos(1/2)$, and the final solution is

$$\begin{aligned} y(t) &= 4 \sin \frac{1}{2} \cos \frac{t}{2} - 4 \cos \frac{1}{2} \sin \frac{t}{2}, \\ &= 4 \sin \left(\frac{1}{2} - \frac{t}{2} \right), \\ &= -4 \sin \frac{1}{2} (t - 1). \end{aligned}$$

35. If $y'' + 12y' + 36y = 0$, then the characteristic equation is

$$\lambda^2 + 12\lambda + 36 = (\lambda + 6)^2 = 0,$$

with double root $\lambda = -6$. This leads to a fundamental set of solutions, $y_1(t) = e^{-6t}$ and $y_2(t) = te^{-6t}$, and the general solution

$$y(t) = (C_1 + C_2t)e^{-6t}.$$

Using the initial condition $y(1) = 0$ provides

$$0 = (C_1 + C_2)e^{-6}.$$

Thus, $C_1 = -C_2$. Differentiating the general solution,

$$\begin{aligned} y'(t) &= (C_1 + C_2t)(-6e^{-6t}) + C_2e^{-6t} \\ &= e^{-6t}((C_2 - 6C_1) - 6C_2t), \end{aligned}$$

then using the initial condition $y'(1) = -1$ leads to

$$\begin{aligned} -1 &= e^{-6}((C_2 - 6C_1) - 6C_2) \\ -1 &= e^{-6}(-6C_1 - 5C_2). \end{aligned}$$

Substituting $C_1 = -C_2$ in this last equation,

$$\begin{aligned} -1 &= e^{-6}(-6(-C_2) - 5C_2) \\ C_2 &= -e^6. \end{aligned}$$

Then, $C_1 = -C_2 = e^6$. Thus, the particular solution is

$$\begin{aligned} y(t) &= (e^6 - e^6t)e^{-6t} \\ &= (1 - t)e^{-6(t-1)}. \end{aligned}$$

36. If $y'' - 4y' + 13y = 0$, then the characteristic equation is $\lambda^2 - 4\lambda + 13 = 0$ with roots $z = 2 \pm 3i$. The complex solution

$$z(t) = e^{(2+3i)t} = e^{2t}(\cos 3t + i \sin 3t)$$

provides a fundamental set of real solutions, $y_1(t) = e^{2t} \cos 3t$ and $y_2(t) = e^{2t} \sin 3t$, and the general solution

$$y(t) = e^{2t}(C_1 \cos 3t + C_2 \sin 3t).$$

The initial condition $y(0) = 4$ provides

$$4 = C_1.$$

Differentiating the general solution,

$$\begin{aligned} y'(t) &= 2e^{2t}(C_1 \cos 3t + C_2 \sin 3t) \\ &\quad + e^{2t}(-3C_1 \sin 3t + 3C_2 \cos 3t), \end{aligned}$$

and the initial condition $y'(0) = 0$ provides

$$0 = 2C_1 + 3C_2.$$

Thus, $C_1 = 4$ and $C_2 = -8/3$, giving the solution

$$y(t) = e^{2t} \left(4 \cos 3t - \frac{8}{3} \sin 3t \right).$$

37. (a) If λ_1 and λ_2 are characteristic roots of the characteristic equation

$$\lambda^2 + p\lambda + q = 0,$$

then the characteristic equation factors as follows.

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

Expanding this result, we obtain

$$\lambda_2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0.$$

When this is compared with the characteristic equation,

$$\lambda_1\lambda_2 = q.$$

- (b) Similarly, when coefficients are compared,

$$\lambda_1 + \lambda_2 = -p.$$

38. If the equation $y'' + py' + qy = 0$ has characteristic equation $\lambda^2 + p\lambda + q = 0$, and the characteristic equation has a double root, $\lambda = \lambda_1$, then we know

that $\lambda_1^2 + p\lambda_1 + q = 0$. Secondly, because

$$\begin{aligned} 0 &= \lambda^2 + p\lambda + q \\ &= (\lambda - \lambda_1)(\lambda - \lambda_1) \\ &= \lambda^2 - 2\lambda_1\lambda + \lambda_1^2, \end{aligned}$$

we can compare coefficients and note that $-2\lambda_1 = p$, or $2\lambda_1 + p = 0$. If $y(t) = te^{\lambda_1 t}$, then

$$\begin{aligned} y'(t) &= (\lambda_1 t + 1)e^{\lambda_1 t} \\ y''(t) &= (2\lambda_1 + \lambda_1^2 t)e^{\lambda_1 t}. \end{aligned}$$

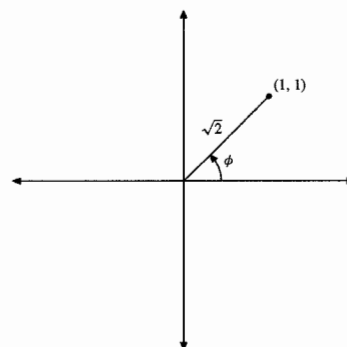
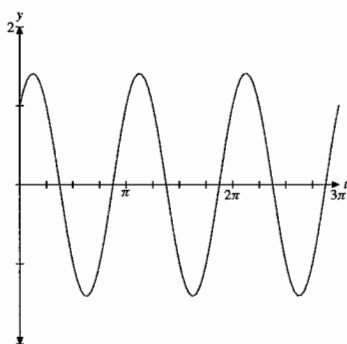
Now,

$$\begin{aligned} y'' + py' + qy &= (2\lambda_1 + \lambda_1^2 t)e^{\lambda_1 t} + p(\lambda_1 t + 1)e^{\lambda_1 t} + qte^{\lambda_1 t} \\ &= e^{\lambda_1 t}(2\lambda_1 + p) + t(\lambda_1^2 + p\lambda_1 + q)e^{\lambda_1 t} \\ &= 0, \end{aligned}$$

where we've used the facts that $2\lambda_1 + p = 0$ and $\lambda_1^2 + p\lambda_1 + q = 0$. Thus, $y(t) = te^{\lambda_1 t}$ is a solution of $y'' + py' + qy = 0$.

Section 4.4. Harmonic Motion

1. The first of two images contains the plot of $y = \cos 2t + \sin 2t$ on the interval $[0, 3\pi]$.



Plot the coefficients of

$$y = 1 \cdot \cos 2t + 1 \cdot \sin 2t$$

in the first quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\begin{aligned}\tan \phi &= \frac{1}{1} = 1 \\ \phi &= \frac{\pi}{4}.\end{aligned}$$

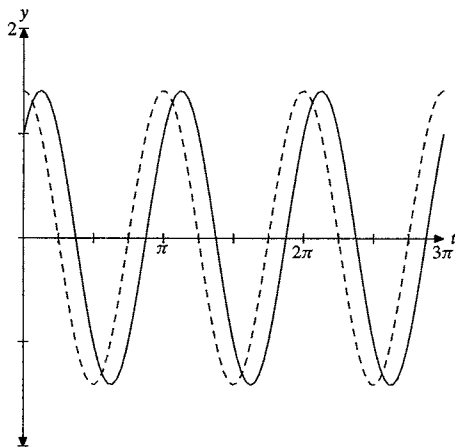
Factor out the magnitude $\sqrt{2}$ as follows.

$$y = \sqrt{2} \left(\frac{1}{\sqrt{2}} \cos 2t + \frac{1}{\sqrt{2}} \sin 2t \right)$$

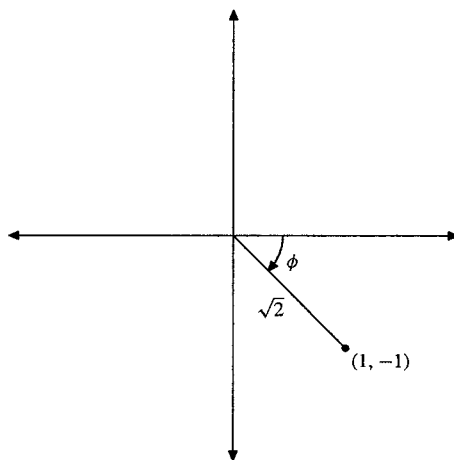
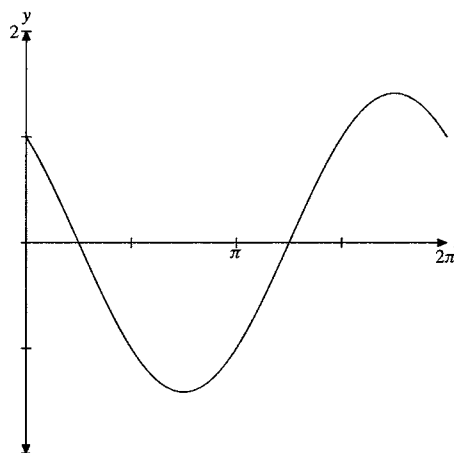
But $\cos \phi = 1/\sqrt{2}$ and $\sin \phi = 1/\sqrt{2}$, so we can write

$$\begin{aligned}y &= \sqrt{2} (\cos \phi \cos 2t + \sin \phi \sin 2t) \\ y &= \sqrt{2} \cos(2t - \phi) \\ y &= \sqrt{2} \cos \left(2t - \frac{\pi}{4} \right) \\ y &= \sqrt{2} \cos 2 \left(t - \frac{\pi}{8} \right).\end{aligned}$$

Hence, the curve has amplitude $\sqrt{2}$, period $T = \pi$, and is shifted to the right $\pi/8$ units. This is clearly shown in the following image where the dashed curve is the unshifted $y = \sqrt{2} \cos 2t$.



2. The first of two images contains the plot of $y = \cos t - \sin t$ on the interval $[0, 2\pi]$.



Plot the coefficients of

$$y = 1 \cos t - 1 \sin t$$

in the fourth quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\begin{aligned}\tan \phi &= \frac{-1}{1} = -1 \\ \phi &= -\frac{\pi}{4}\end{aligned}$$

Factor out the magnitude $\sqrt{2}$ as follows.

$$y = \sqrt{2} \left(\frac{1}{\sqrt{2}} \cos t - \frac{1}{\sqrt{2}} \sin t \right)$$

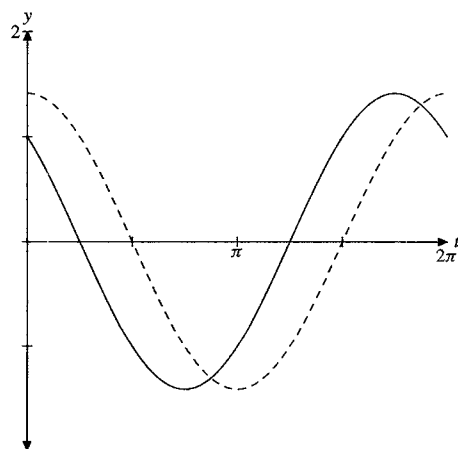
Because $\cos \phi = 1/\sqrt{2}$ and $\sin \phi = -1/\sqrt{2}$, we can write

$$y = \sqrt{2}(\cos \phi \cos t + \sin \phi \sin t)$$

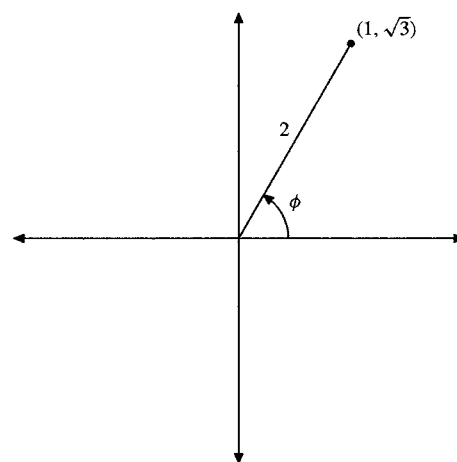
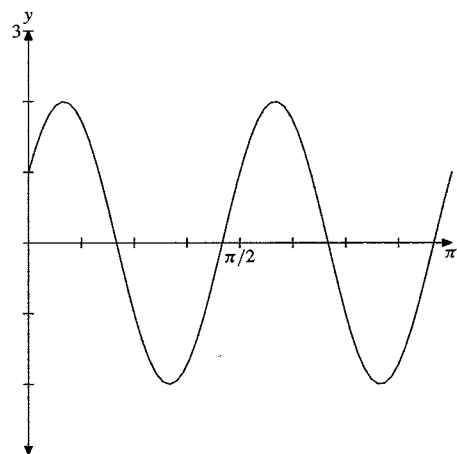
$$y = \sqrt{2} \cos(t - \phi)$$

$$y = \sqrt{2} \cos \left(t + \frac{\pi}{4} \right).$$

Hence, the curve has amplitude $\sqrt{2}$, period 2π , and it is shifted to the left $\pi/4$ units. This is clearly shown in the following image where the dashed curves is the unshifted $y = \sqrt{2} \cos t$.



3. The first of two images contains the plot of $y = \cos 4t + \sqrt{3} \sin 4t$ on the interval $[0, \pi]$.



Plot the coefficients of

$$y = 1 \cdot \cos 4t + \sqrt{3} \cdot \sin 4t$$

in the first quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\tan \phi = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\phi = \frac{\pi}{3}.$$

Factor out the magnitude 2 as follows.

$$y = 2 \left(\frac{1}{2} \cos 4t + \frac{\sqrt{3}}{2} \sin 4t \right)$$

But $\cos \phi = 1/2$ and $\sin \phi = \sqrt{3}/2$, so we can write

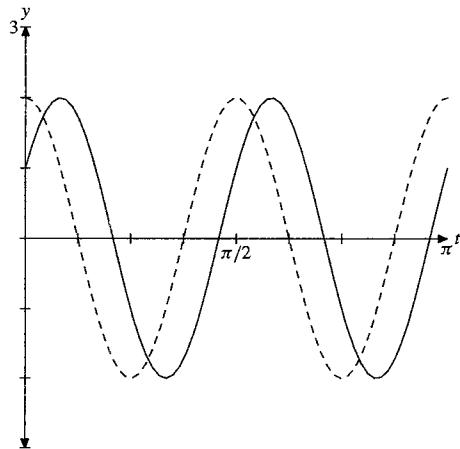
$$y = 2 (\cos \phi \cos 4t + \sin \phi \sin 4t)$$

$$y = 2 \cos(4t - \phi)$$

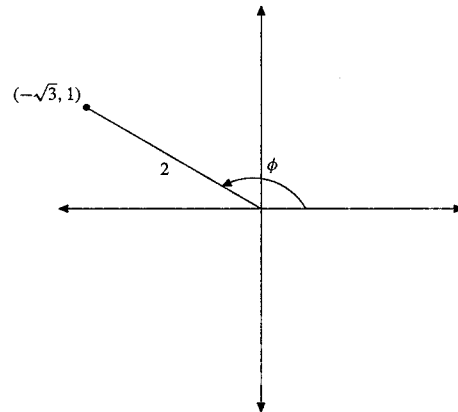
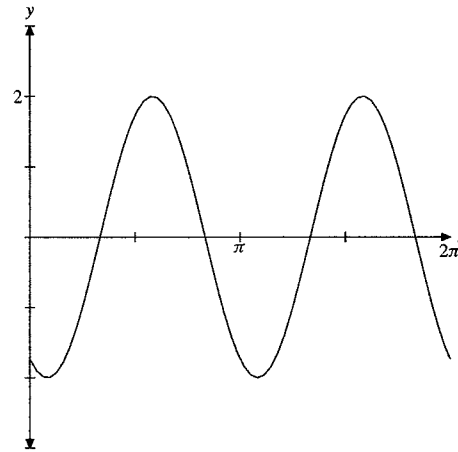
$$y = 2 \cos \left(4t - \frac{\pi}{3} \right)$$

$$y = 2 \cos 4 \left(t - \frac{\pi}{12} \right).$$

Hence, the curve has amplitude 2, period $T = \pi/2$, and is shifted to the right $\pi/12$ units. This is clearly shown in the following image where the dashed curve is the unshifted $y = 2 \cos 4t$.



4. The first two images contains the plot of $y = -\sqrt{3} \cos 2t + \sin 2t$ on the interval $[0, 2\pi]$.



Plot the coefficients of

$$y = -\sqrt{3} \cdot \cos 2t + 1 \cdot \sin 2t$$

in the second quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\tan \phi = -\frac{1}{\sqrt{3}}$$

$$\phi = \frac{5\pi}{6}$$

Factor out the magnitude 2 as follows.

$$y = 2 \left(-\frac{\sqrt{3}}{2} \cos 2t + \frac{1}{2} \sin 2t \right)$$

But $\cos \phi = -\sqrt{3}/2$ and $\sin \phi = 1/2$, so we can write

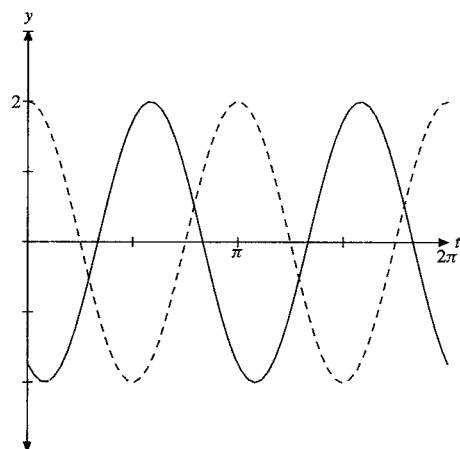
$$y = 2(\cos \phi \cos 2t + \sin \phi \sin 2t)$$

$$y = 2 \cos(2t - \phi)$$

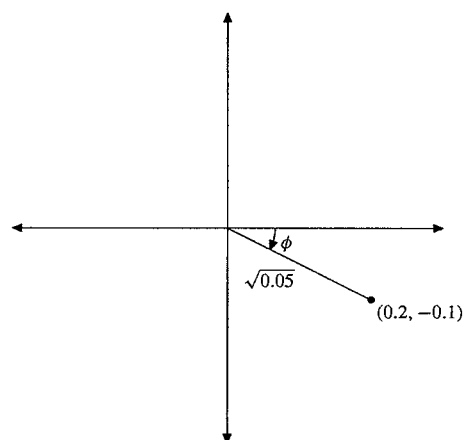
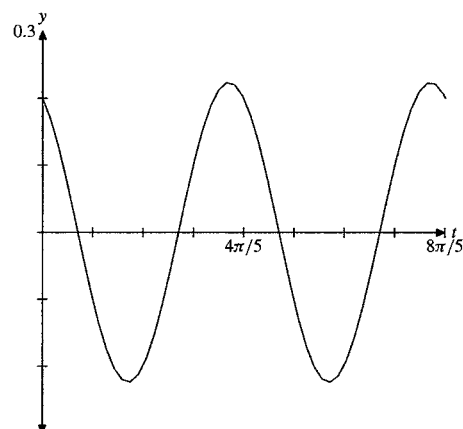
$$y = 2 \cos\left(2t - \frac{5\pi}{6}\right)$$

$$y = 2 \cos 2\left(t - \frac{5\pi}{12}\right).$$

Hence, the curve has amplitude 2, period $T = 2\pi/2 = \pi$, and is shifted to the right $5\pi/12$ units. This is clearly seen in the following image where the dashed curve is the unshifted $y = 2 \cos 2t$.



5. The first of the two images contains the plot of $y = 0.2 \cos 2.5t - 0.1 \sin 2.5t$ on the interval $[0, 8\pi/5]$.



Plot the coefficients of

$$y = 0.2 \cos 2.5t - 0.1 \sin 2.5t$$

in the fourth quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\tan \phi = \frac{-1}{2} = -\frac{1}{2}$$

$$\phi \approx -0.4636.$$

Factor out the magnitude 2 as follows.

$$y = \sqrt{0.05} \left(\frac{0.2}{\sqrt{0.05}} \cos 2.5t - \frac{0.1}{\sqrt{0.05}} \sin 2.5t \right)$$

But $\cos \phi = 0.2/\sqrt{0.05}$ and $\sin \phi = -0.1/\sqrt{0.05}$, so we can write

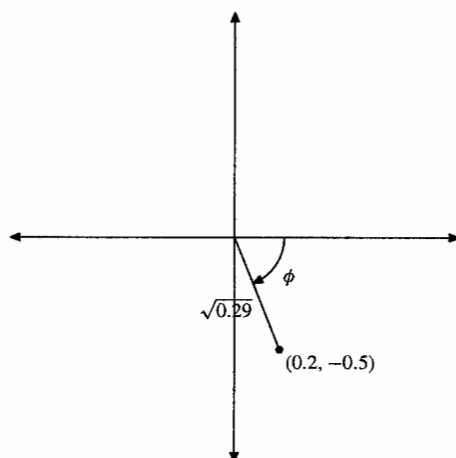
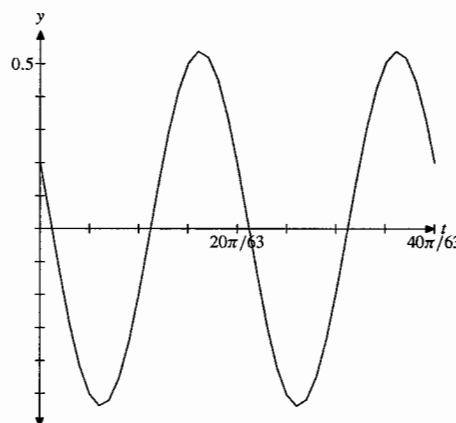
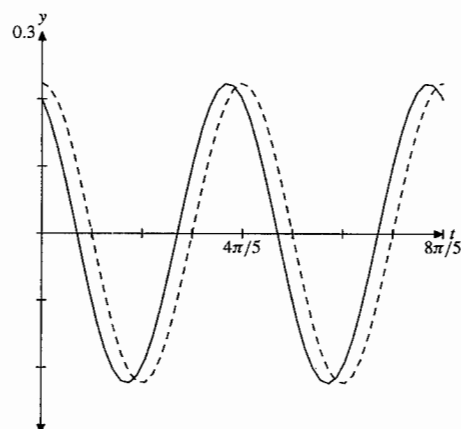
$$y = \sqrt{0.05} (\cos \phi \cos 2.5t + \sin \phi \sin 2.5t)$$

$$y = 0.2236 \cos(2.5t - \phi)$$

$$y = 0.2236 \cos(2.5t + 0.4636)$$

$$y = 0.2236 \cos 2.5(t + 0.1855).$$

Hence, the curve has amplitude 0.2236, period $T = 2\pi/2.5 = 4\pi/5$, and is shifted to the left 0.1855 units. This is clearly shown in the image below, where the dashed curve is the unshifted $y = 0.2236 \cos 2.5t$.



Plot the coefficients of

$$y = 0.2 \cos 6.3t - 0.5 \sin 6.3t$$

in the fourth quadrant, calculate the magnitude of the vector, and mark the angle. The angle ϕ is easily calculated.

$$\tan \phi = \frac{-0.5}{0.2} = -\frac{5}{2}$$

$$\phi \approx -1.1071$$

Factor out the magnitude $\sqrt{0.29}$ as follows.

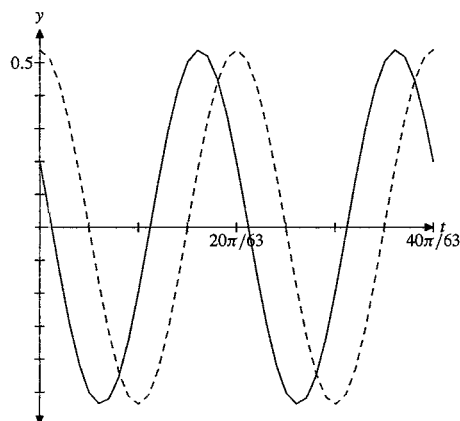
$$y = \sqrt{0.29} \left(\frac{0.2}{\sqrt{0.29}} \cos 6.3t - \frac{0.5}{\sqrt{0.29}} \sin 6.3t \right)$$

6. The first of two images contains the plot of $y = 0.2 \cos 6.3t - 0.5 \sin 6.3t$ on $[0, 2]$.

But $\cos \phi = 0.2/\sqrt{0.29}$ and $\sin \phi = -0.5/\sqrt{0.29}$, so we can write

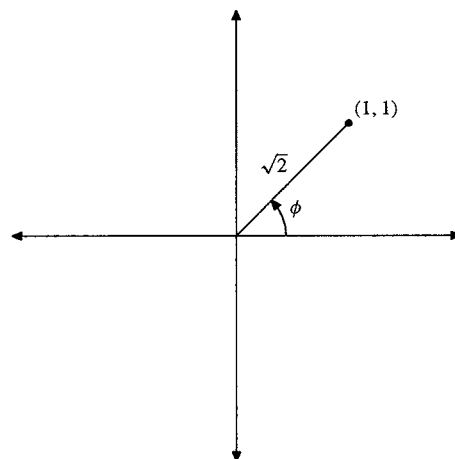
$$\begin{aligned} y &= \sqrt{0.29}(\cos \phi \cos 6.3t + \sin \phi \sin 6.3t) \\ y &= 0.5385 \cos(6.3t - \phi) \\ y &= 0.5385 \cos(6.3t + 1.1903) \\ y &= 0.5385 \cos 6.3(t + 0.1889) \end{aligned}$$

Hence, the curve has amplitude 0.5385, period $2\pi/6.3 = 20\pi/63$, and is shifted to the left 0.1889 units. This is clearly shown in the image below where the dashed curve is unshifted $y = 0.5385 \cos 6.3t$.



7. Plot the coefficients of $1 \cdot \cos 5t + 1 \cdot \sin 5t$ in the first quadrant, calculate the magnitude of the vector,

and mark the angle.



The angle ϕ is easily calculated.

$$\begin{aligned} \tan \phi &= 1 \\ \phi &= \frac{\pi}{4} \end{aligned}$$

Factor out the magnitude as follows:

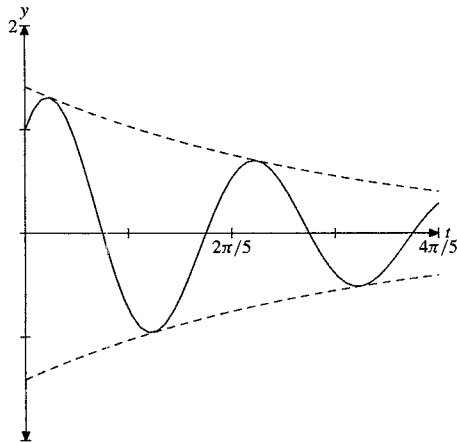
$$y = \sqrt{2}e^{-t/2} \left(\frac{1}{\sqrt{2}} \cos 5t + \frac{1}{\sqrt{2}} \sin 5t \right)$$

But $\cos \phi = 1/\sqrt{2}$ and $\sin \phi = 1/\sqrt{2}$, so we can write

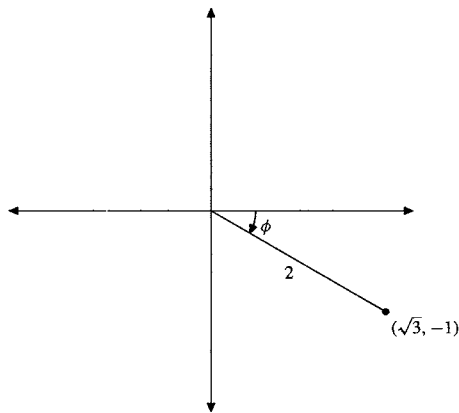
$$\begin{aligned} y &= \sqrt{2}e^{-t/2} (\cos \phi \cos 5t + \sin \phi \sin 5t) \\ y &= \sqrt{2}e^{-t/2} \cos(5t - \phi) \\ y &= \sqrt{2}e^{-t/2} \cos(5t - \pi/4). \end{aligned}$$

Thus, the amplitude is $\sqrt{2}e^{-t/2}$ which, along with $-\sqrt{2}e^{-t/2}$, clearly bound the graph of $y =$

$$\sqrt{2}e^{-t/2} \cos(5t - \pi/4).$$



8. Plot the coefficients of $\sqrt{3} \cos 4t - 1 \sin 4t$ in the fourth quadrant, calculate the magnitude of the vector, and mark the angle.



The angle ϕ is easily calculated.

$$\tan \phi = -\frac{1}{\sqrt{3}}$$

$$\phi = -\frac{\pi}{6}$$

Factor out the magnitude as follows.

$$y = 2e^{-t/4} \left(\frac{\sqrt{3}}{2} \cos 4t - \frac{1}{2} \sin 4t \right)$$

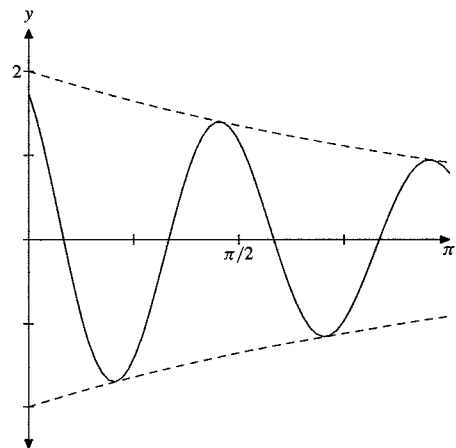
But $\cos \phi = \sqrt{3}/2$ and $\sin \phi = -1/2$, so we can write

$$y = 2e^{-t/4} (\cos \phi \cos 4t + \sin \phi \sin 4t)$$

$$y = 2e^{-t/4} \cos(4t - \phi)$$

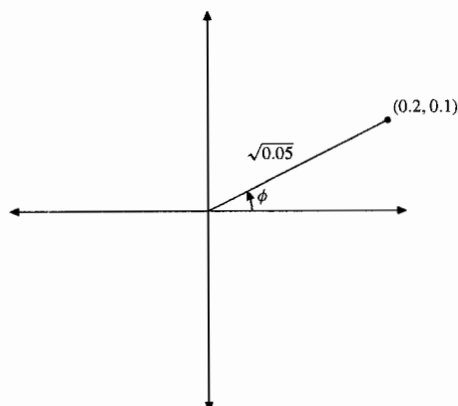
$$y = 2e^{-t/4} \cos\left(4t + \frac{\pi}{6}\right).$$

Thus, the amplitude is $2e^{-t/4}$ which, along with $-2e^{-t/4}$, clearly bound the graph of $y = 2e^{-t/4} \cos(4t + \pi/6)$.



9. Plot the coefficients of $0.2 \cdot \cos 2t + 0.1 \cdot \sin 2t$ in the first quadrant, calculate the magnitude of the vector,

and mark the angle.



The angle ϕ is easily calculated.

$$\tan \phi = \frac{0.1}{0.2} = \frac{1}{2}$$

$$\phi = 0.4636$$

Factor out the magnitude as follows:

$$y = \sqrt{0.05}e^{-0.1t} \left(\frac{0.2}{\sqrt{0.05}} \cos 2t + \frac{0.1}{\sqrt{0.05}} \sin 2t \right)$$

But $\cos \phi = 0.2/\sqrt{0.05}$ and $\sin \phi = 0.1/\sqrt{0.05}$, so we can write

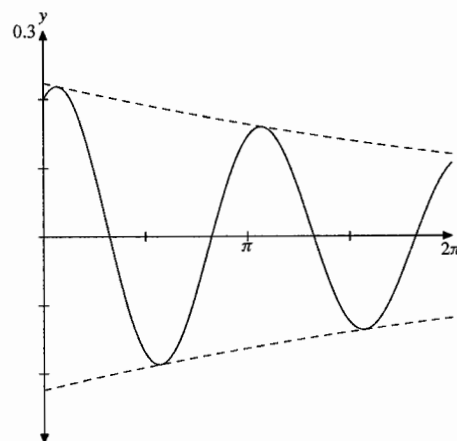
$$y = \sqrt{0.05}e^{-0.1t} (\cos \phi \cos 2t + \sin \phi \sin 2t)$$

$$y = 0.2236e^{-0.1t} \cos(2t - \phi)$$

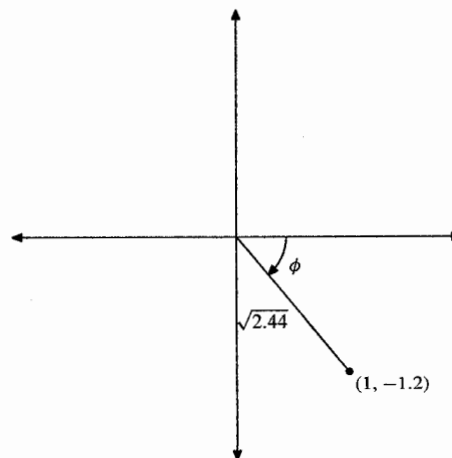
$$y = 0.2236e^{-0.1t} \cos(2t - 0.4636).$$

Thus, the amplitude is $0.2236e^{-0.1t}$ which, along with $-0.2236e^{-0.1t}$, clearly bound the graph of $y =$

$$0.2236e^{-0.1t} \cos(2t - 0.4636).$$



10. Plot the coefficients of $1 \cos 4.2t - 1.2 \sin 4.2t$ in the fourth quadrant, calculate the magnitude of the vector, and mark the angles.



The angle ϕ is easily calculated.

$$\tan \phi = \frac{-1.2}{1} = -\frac{12}{10} = -\frac{6}{5}$$

$$\phi = -0.8761$$

Factor out the magnitude as follows.

$$y = \sqrt{2.44}e^{-0.2t} \left(\frac{1}{\sqrt{2.44}} \cos 4.2t - \frac{1.2}{\sqrt{2.44}} \sin 4.2t \right)$$

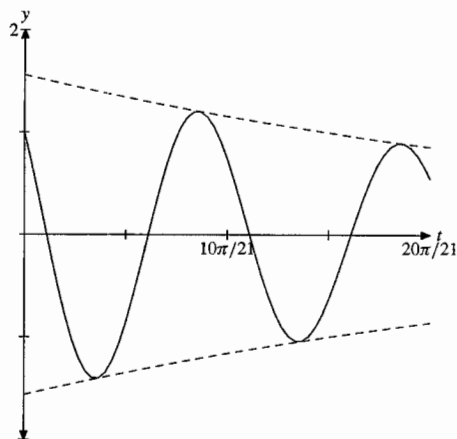
But $\cos \phi = 1/\sqrt{2.44}$ and $\sin \phi = -1.2/\sqrt{2.44}$, so we can write

$$y = \sqrt{2.44}e^{-0.2t} (\cos \phi \cos 4.2t + \sin \phi \sin 4.2t)$$

$$y = \sqrt{2.44}e^{-0.2t} \cos(4.2t - \phi)$$

$$y = 1.5620e^{-0.2t} \cos(4.2t + 0.8761)$$

Thus, the amplitude is $1.5620e^{-0.2t}$ which, along with $-1.5620e^{-0.2t}$, clearly bound the graph of $y = 1.5620e^{-0.2t} \cos(4.2t + 0.8761)$.



11. Substitute $m = 0.2$ kg and $k = 5$ kg/s² in $my'' + ky = 0$ to obtain $0.2y'' + 5y = 0$ or

$$y'' + 25y = 0.$$

The characteristic equation is $\lambda^2 + 25 = 0$, with zeros $\lambda = \pm 5i$, so

$$z(t) = e^{5it} = \cos 5t + i \sin 5t$$

is a complex solution. The real and imaginary parts of this solution form a fundamental set of real solutions, giving the general solution

$$y(t) = C_1 \cos 5t + C_2 \sin 5t.$$

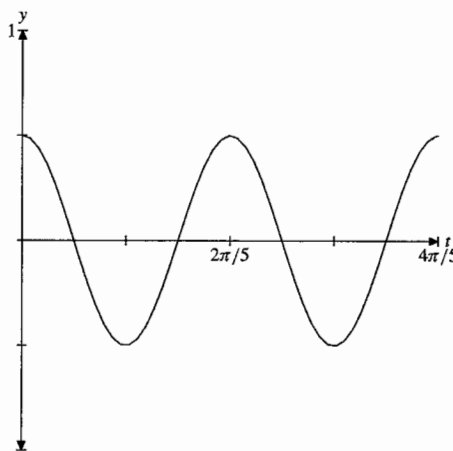
The initial displacement is 0.5 m, so $y(0) = 0.5$ and $C_1 = 0.5$. Differentiating,

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t.$$

The system is released from rest, so $y'(0) = 0$ and $C_2 = 0$. Thus, the solution is

$$y(t) = 0.5 \cos 5t,$$

which has amplitude 0.5, frequency 5 rad/s, and zero phase.



12. Substitute $m = 0.1$ kg and $k = 3.6$ kg/s² in $my'' + ky = 0$ to obtain $0.1y'' + 3.6y = 0$, or

$$y'' + 36y = 0.$$

The characteristic equation is $\lambda^2 + 36 = 0$, with zeros $\lambda = \pm 6i$, so

$$z(t) = e^{6it} = \cos 6t + i \sin 6t$$

is a complex solution. The real and imaginary parts of this solution form a fundamental set of real solutions, giving the general solution

$$y(t) = C_1 \cos 6t + C_2 \sin 6t.$$

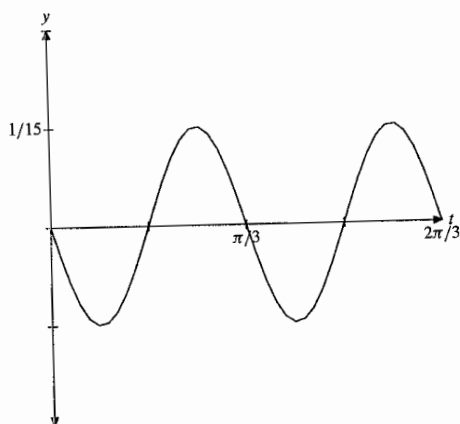
The initial displacement is zero, so $y(0) = 0$ and $C_1 = 0$. Differentiating,

$$y'(t) = -6C_1 \sin 6t + 6C_2 \cos 6t.$$

The initial velocity is -0.4 m/s, so $y'(0) = -0.4$, leading to $C_2 = -1/15$. Thus, the solution is

$$y(t) = -\frac{1}{15} \sin 6t,$$

which has amplitude $1/15$, frequency 6 rad/s, and zero phase.



13. The system $(2/5)x'' + kx = 0$, or $x'' + (5k/2)x = 0$, is equivalent to

$$x'' + \omega_0^2 x = 0,$$

with $\omega_0^2 = 5k/2$. The characteristic equation is $\lambda^2 + \omega_0^2 = 0$ with roots $\lambda = \pm \omega_0 i$. Thus, we have complex solution

$$z(t) = e^{i\omega_0 t} = \cos \omega_0 t + i \sin \omega_0 t.$$

The real and imaginary parts of this solution form a fundamental set of solutions and provide the general solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

This solution is periodic with period $T = 2\pi/\omega_0$. Thus, if it is known that the period is $\pi/2$, then

$$\begin{aligned} T &= \frac{2\pi}{\omega_0} \\ \frac{\pi}{2} &= \frac{2\pi}{\omega_0} \\ \omega_0 &= 4. \end{aligned}$$

But $\omega_0^2 = 5k/2$,

$$\begin{aligned} 4^2 &= \frac{5k}{2} \\ k &= \frac{32}{5}. \end{aligned}$$

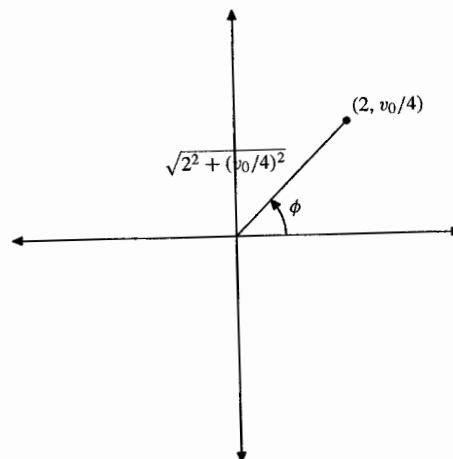
The initial condition $x(0) = 2$ gives $C_1 = 2$. Differentiating $x(t)$,

$$x'(t) = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t.$$

The initial condition $x'(0) = v_0$ leads to $C_2 = v_0/\omega_0$ and the solution

$$x(t) = 2 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t = 2 \cos 4t + \frac{v_0}{4} \sin 4t.$$

Plot the coefficients, calculate the magnitude of the vector, and mark the angle.



As in Exercises 1–6, factoring out the magnitude will place the solution in the form

$$x(t) = \sqrt{4 + v_0^2/16} \cos(4t - \phi),$$

where $\tan \phi = v_0/8$. If it is known that the amplitude is 2 , then

$$\begin{aligned} \sqrt{4 + v_0^2/16} &= 2 \\ 4 + v_0^2/16 &= 4 \\ v_0 &= 0. \end{aligned}$$

14. The system $mx'' + kx = 0$, or $x'' + (k/m)x = 0$, is equivalent to

$$x'' + \omega_0^2 x = 0,$$

with $\omega_0^2 = k/m$. The characteristic equation is $\lambda^2 + \omega_0^2 = 0$, with roots $\lambda = \pm \omega_0 i$. Thus, we have complex solution

$$z(t) = e^{i\omega_0 t} = \cos \omega_0 t + i \sin \omega_0 t.$$

The real and imaginary parts of this solution form a fundamental set of solutions and provide the general solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

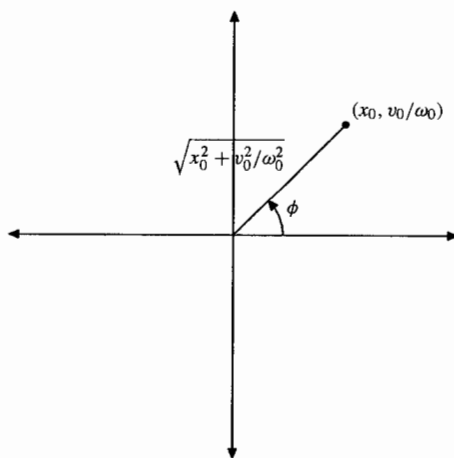
The initial condition $x(0) = x_0$ gives $C_1 = x_0$. Differentiate.

$$x'(t) = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t.$$

The initial condition $x'(0) = v_0$ gives $C_2 = v_0/\omega_0$. Thus, the solution is

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t.$$

Plot the coefficients, calculate the magnitude of the vector, and mark the angle.



The amplitude of the motion is

$$\begin{aligned} A &= \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}} = \sqrt{x_0^2 + \frac{v_0^2}{k/m}} \\ &= \sqrt{x_0^2 + \frac{mv_0^2}{k}}. \end{aligned}$$

The tangent of the angle ϕ is easily calculated.

$$\tan \phi = \frac{v_0}{x_0 \omega_0}$$

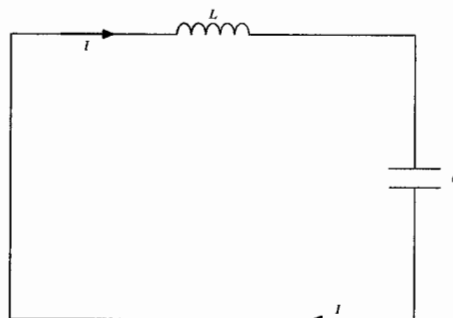
Factor out the magnitude as follows.

$$x(t) = A \left(\frac{x_0}{A} \cos \omega_0 t + \frac{v_0/\omega_0}{A} \sin \omega_0 t \right)$$

But $\cos \phi = x_0/A$ and $\sin \phi = (v_0/\omega_0)/A$, so we can write

$$\begin{aligned} x(t) &= A(\cos \phi \sin \omega_0 t + \sin \phi \cos \omega_0 t) \\ &= A \cos(\omega_0 t - \phi). \end{aligned}$$

15. (a) In an LC circuit,



the sum of the voltage drops around the closed loop is zero. The voltage drop across the inductor is LI' , where $L = 6 \mu\text{H}$ and $I(t)$ is the current, and the voltage drop across the capacitor is $(1/C)Q$, where $C = 2 \mu\text{F}$ and $Q(t)$ is the charge on the capacitor. Thus, we can write

$$LI' + \frac{1}{C}Q = 0,$$

or, since $I = dQ/dt$,

$$LQ'' + \frac{1}{C}Q = 0.$$

To find the initial charge on the capacitor, use

$$\begin{aligned}V_C &= \frac{1}{C}Q \\Q &= CV_C \\Q &= (2 \times 10^{-6} \text{ F})(20 \text{ V}) \\Q &= 4.0 \times 10^{-5} \text{ coulombs.}\end{aligned}$$

- (b) If the initial current is zero, then $Q'(0) = I(0) = 0$. Thus, $LQ'' + (1/C)Q = 0$, or

$$Q'' + \frac{1}{LC}Q = 0,$$

with the initial conditions $Q(0) = 4.0 \times 10^{-5}$, and $Q'(0) = 0$. This last equation is equivalent to

$$Q'' + \omega_0^2 Q = 0,$$

where $\omega_0^2 = 1/(LC)$. This has characteristic equation $\lambda^2 + \omega_0^2 = 0$ and zeros $\lambda = \pm i\omega_0$, leading to the complex solution

$$z(t) = e^{i\omega_0 t} = \cos \omega_0 t + i \sin \omega_0 t.$$

The real and imaginary parts of this solution form a fundamental set of solutions, leading to the general solution

$$Q(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

The initial condition $Q(0) = 4.0 \times 10^{-5}$ gives $C_1 = 4.0 \times 10^{-5}$. Differentiating $Q(t)$,

$$Q'(t) = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t.$$

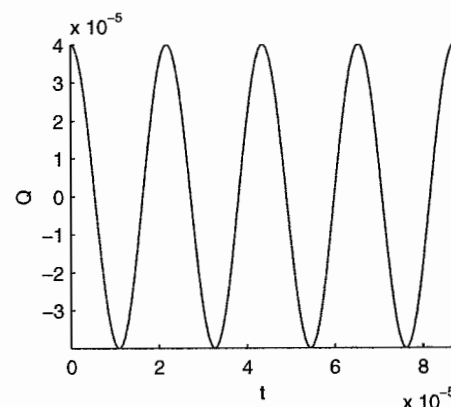
The initial condition $Q'(0) = 0$ gives $C_2 = 0$ and the solution

$$Q(t) = (4.0 \times 10^{-5}) \cos \omega_0 t.$$

Thus, the amplitude is 4.0×10^{-5} . Because $\omega_0^2 = 1/(LC)$,

$$\begin{aligned}\omega_0 &= \sqrt{\frac{1}{LC}} \\&= \sqrt{\frac{1}{(6 \times 10^{-6})(2 \times 10^{-6})}} \\&\approx 2.887 \times 10^5 \text{ rad/s.}\end{aligned}$$

The phase is zero. A plot of charge versus time follows.



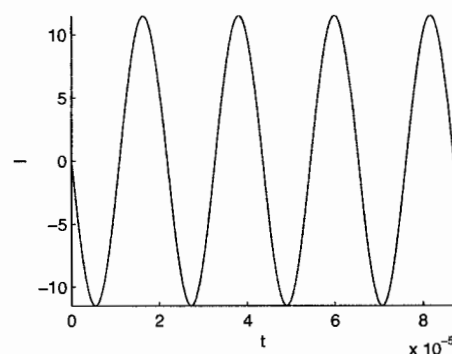
To find the current, differentiate the charge formula with respect to time.

$$I(t) = Q'(t) = -(4.0 \times 10^{-5})\omega_0 \sin \omega_0 t$$

Thus,

$$I(t) = -11.548 \sin \omega_0 t,$$

making the amplitude 11.548 and the frequency $\omega_0 \approx 2.887 \times 10^5$ rad/s. The phase is zero. A plot of current versus time follows.



16. (a) By Hooke's Law,

$$k = \frac{F}{y}$$

$$k = \frac{mg}{y}$$

$$k = \frac{(1 \text{ kg})(9.8 \text{ m/s}^2)}{4.9 \text{ m}}$$

$$k = 2 \text{ N/m}$$

- (b) With
- $m = 1 \text{ kg}$
- ,
- $\mu = 3 \text{ kg/s}$
- , and
- $k = 2 \text{ N/m}$
- ,
- $my'' + \mu y' + ky = 0$
- becomes

$$y'' + 3y' + 2y = 0,$$

with characteristic equation $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ and zeros $\lambda_1 = -1$ and $\lambda_2 = -2$. Thus, the solution is

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

The initial displacement is $y(0) = -1 \text{ m}$, so

$$-1 = C_1 + C_2.$$

Differentiate.

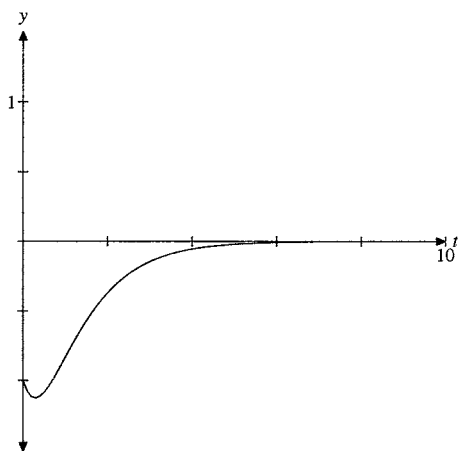
$$y'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$$

The initial velocity is $y'(0) = -1 \text{ m/s}$, so

$$-1 = -C_1 - 2C_2.$$

Thus, $C_1 = -3$ and $C_2 = 2$, and the solution is

$$y(t) = -3e^{-t} + 2e^{-2t}.$$



17. Suppose that the spring-mass system
- $my'' + \mu y' + ky = 0$
- is overdamped. Then,

$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0$$

$$y'' + 2cy' + \omega_0^2 y = 0,$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The characteristic equation is $\lambda^2 + 2c\lambda + \omega_0^2 = 0$ and the zeros are

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \quad \text{and}$$

$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

In order that the system be overdamped, we need $c^2 - \omega_0^2 > 0$. Of course, this condition results in $\lambda_1 < \lambda_2 < 0$ and the general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

To determine how many times this solution can cross the t -axis, set $y(t) = 0$ and solve for t .

$$0 = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$0 = e^{\lambda_1 t} (C_1 + C_2 e^{(\lambda_2 - \lambda_1)t})$$

Of course, $e^{\lambda_1 t}$ is never zero, so this leads to

$$C_1 + C_2 e^{(\lambda_2 - \lambda_1)t} = 0$$

$$e^{(\lambda_2 - \lambda_1)t} = -\frac{C_1}{C_2}.$$

If $C_1/C_2 > 0$, then there are no crossings, but if $C_1/C_2 < 0$, there will be exactly one crossing of the t -axis. The challenge is to find initial conditions y_0 and v_0 that lead to zero and one crossings. But $y(0) = y_0$ gives

$$y_0 = C_1 + C_2.$$

Differentiating $y(t)$,

$$y'(t) = C_1 \lambda_1 e^{\lambda_1 t} + C_2 \lambda_2 e^{\lambda_2 t},$$

the initial condition $y'(0) = v_0$ provides

$$v_0 = C_1 \lambda_1 + C_2 \lambda_2,$$

and this system is easily solved for C_1 and C_2 ,

$$C_1 = \frac{\lambda_2 y_0 - v_0}{\lambda_2 - \lambda_1} \quad \text{and} \quad C_2 = \frac{v_0 - \lambda_1 y_0}{\lambda_2 - \lambda_1},$$

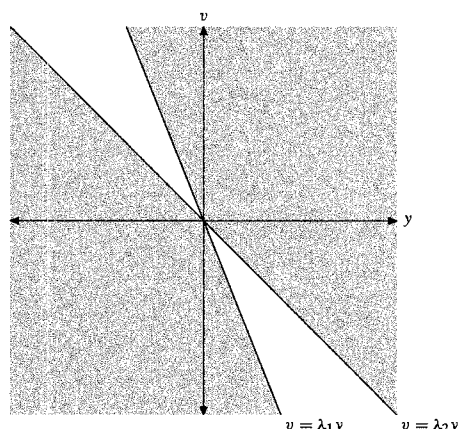
which gives us

$$\frac{C_1}{C_2} = \frac{\lambda_2 y_0 - v_0}{v_0 - \lambda_1 y_0}.$$

Because there is exactly one zero crossing if $C_1/C_2 < 0$, we see two possible ways that this can happen:

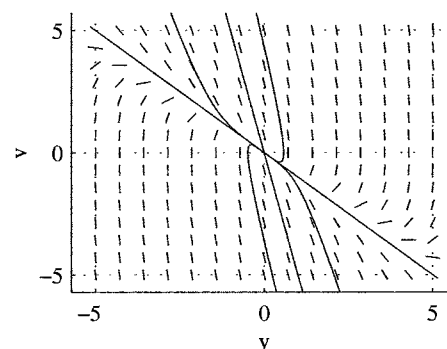
- "Case 1:" We must have $\lambda_2 y_0 - v_0 < 0$ and $v_0 - \lambda_1 y_0 > 0$, which leads to the condition that $v_0 > \lambda_2 y_0$ and $v_0 > \lambda_1 y_0$, or
- "Case 2:" We must have $\lambda_2 y_0 - v_0 > 0$ and $v_0 - \lambda_1 y_0 < 0$, which leads to the condition that $v_0 < \lambda_2 y_0$ and $v_0 < \lambda_1 y_0$.

These conditions are easy to analyze if you think geometrically.

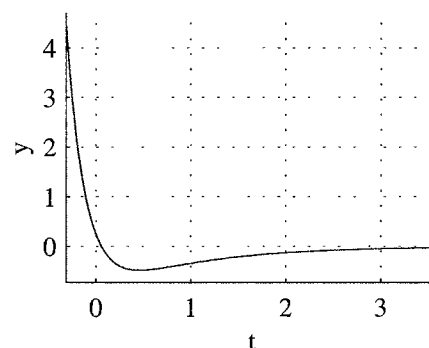


Thus, there will be one zero crossing if you select an initial condition from the shaded region. This is not especially intuitive until you give it a try in your solver. For example, the spring mass system $y'' + 6y' + 5y = 0$, with characteristic equation $\lambda^2 + 6\lambda + 5 = 0$ and zeros $\lambda_1 = -5$ and $\lambda_2 = -1$ is overdamped. If we sketch the lines $v = -5y$ and $v = -y$ on our phase portrait, note, that as predicted,

those initial conditions starting in the shaded region cross $y = 0$ exactly once.



Of course, what we really want is a plot of y versus t .



Note that the graph crosses the t -axis exactly once. Finally, by picking initial conditions from the unshaded region, you can find a solution that does not cross the time axis.

Further refinement of the above arguments is possible. For example, how would you limit the initial conditions so that the solution crosses at some time $t > 0$?

18. By Hooke's Law,

$$\begin{aligned}k &= \frac{F}{y} = \frac{mg}{y} \\k &= \frac{(0.05 \text{ kg})(9.8 \text{ m/s}^2)}{(0.2 \text{ m})} \\k &= 2.45 \text{ N/m}.\end{aligned}$$

Hence, $my'' + \mu y' + ky = 0$ becomes $0.05y'' + \mu y' + 2.45y = 0$, or

$$y'' + 20\mu y' + 49y = 0.$$

This system has characteristic equation $\lambda^2 + 20\mu\lambda + 49 = 0$, with zeros given by

$$\lambda = \frac{-20\mu \pm \sqrt{400\mu^2 - 196}}{2}.$$

The system is critically damped if it has one single, repeated root. This happens only if

$$\begin{aligned}400\mu^2 - 196 &= 0 \\ \mu^2 &= \frac{196}{400} \\ \mu &= \frac{14}{20}.\end{aligned}$$

Thus, $\mu = 7/10$, and our equation becomes

$$y'' + 14y' + 49y = 0,$$

whose characteristic equation $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2 = 0$ has a repeated root $\lambda = -7$. Thus, the general solution is

$$y(t) = (C_1 + C_2t)e^{-7t}.$$

If we assume that the mass is displaced in a downward direction, $y(0) = -0.15 \text{ m}$ and $C_1 = -0.15$. Differentiate.

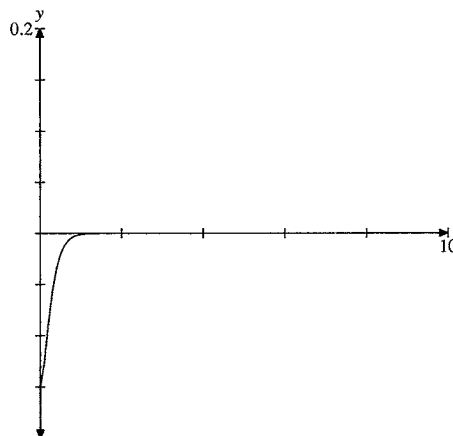
$$y'(t) = C_2e^{-7t} - 7(C_1 + C_2t)e^{-7t}$$

If the mass is released from rest, then $y'(0) = 0$ and

$$0 = C_2 - 7C_1.$$

Thus, $C_2 = -1.05$ and the solution is

$$y(t) = -(0.15 + 1.05t)e^{-7t}.$$



19. Suppose that the spring-mass system $my'' + \mu y' + ky = 0$ is critically damped. Then,

$$\begin{aligned}y'' + \frac{\mu}{m}y' + \frac{k}{m}y &= 0 \\ y'' + 2cy' + \omega_0^2y &= 0,\end{aligned}$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The characteristic equation is $\lambda^2 + 2c\lambda + \omega_0^2 = 0$ and the zeros are

$$\begin{aligned}\lambda_1 &= -c - \sqrt{c^2 - \omega_0^2} \quad \text{and} \\ \lambda_2 &= -c + \sqrt{c^2 - \omega_0^2}.\end{aligned}$$

In order that the system be critically damped, we need $c^2 - \omega_0^2 = 0$. Of course, this condition results in $\lambda_1 = \lambda_2 = -c$ and the general solution is

$$y(t) = (C_1 + C_2t)e^{-ct}.$$

To determine how many times this solution can cross the t -axis, set $y(t) = 0$ and solve for t .

$$0 = (C_1 + C_2t)e^{-ct}$$

Of course, e^{-ct} is never zero, so this leads to

$$\begin{aligned}C_1 + C_2t &= 0 \\ t &= -\frac{C_1}{C_2}.\end{aligned}$$

Thus, the solution will always cross the t -axis (except when $C_1 \neq 0$ and $C_2 = 0$), but let's find only those crossings when $t > 0$, which necessitates that $C_1/C_2 < 0$. The challenge is to find initial conditions y_0 and v_0 that lead to a crossing when $t > 0$. But $y(0) = y_0$ gives

$$y_0 = C_1.$$

Differentiating $y(t)$,

$$y'(t) = ((C_2 - cC_1) - cC_2t)e^{-ct},$$

the initial condition $y'(0) = v_0$ provides

$$v_0 = C_2 - cC_1,$$

and this system is easily solved for C_1 and C_2 ,

$$C_1 = y_0 \quad \text{and} \quad C_2 = v_0 + cy_0,$$

which gives us

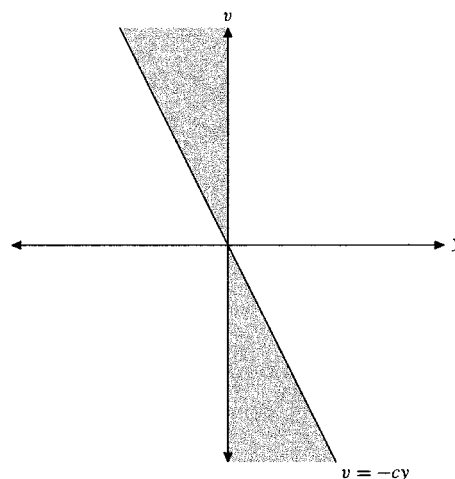
$$\frac{C_1}{C_2} = \frac{y_0}{v_0 + cy_0}.$$

Because there is a zero crossing at $t > 0$ if $C_1/C_2 < 0$, we see two possible ways that this can happen:

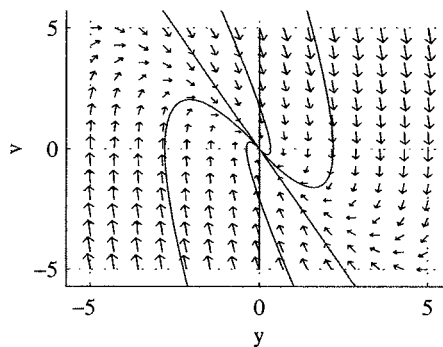
- "Case 1:" We must have $y_0 > 0$ and $v_0 + cy_0 < 0$, which leads to the condition that $y_0 > 0$ and $v_0 < -cy_0$, or
- "Case 2:" We must have $y_0 < 0$ and $v_0 + cy_0 > 0$, which leads to the condition that $y_0 < 0$ and $v_0 > -cy_0$.

These conditions are easy to analyze if you think

geometrically.

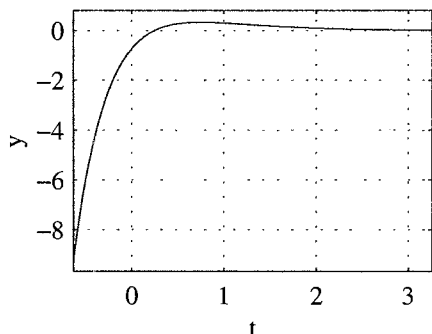


Thus, there will be one zero crossing if you select an initial condition from the shaded region. This is not especially intuitive until you give it a try in your solver. For example, the spring mass system $y'' + 4y' + 4y = 0$, with characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ and repeated zero $\lambda = -2$ is critically damped. If we sketch the lines $v = -2y$ on our phase portrait, note, that as predicted, those initial conditions starting in the shaded region cross $y = 0$ exactly once at $t > 0$.



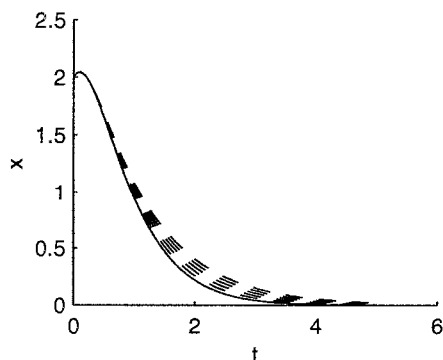
Of course, what we really want is a plot of y versus

t.



Note that the graph crosses the t -axis exactly once. Finally, by picking initial conditions from the unshaded region, you will note that this solution also crosses the y -axis exactly once, but at $t < 0$.

20. (a) The system $x'' + \mu x' + 4x = 0$ has characteristic equation $\lambda^2 + \mu\lambda + 4 = 0$. If $\mu = 4$, this becomes $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$, and there is one repeated root, hence the critically damped case.
- (b) The critically damped solution (solid line in figure) approaches the t -axis faster than any of the other overdamped solutions.



An overdamped screen door will shut on its own without slamming. A critically damped door will shut as fast as possible without slamming.

21. (a) Suppose that $mx'' + \mu x' + kx = 0$ is overdamped. We can write

$$x'' + \frac{\mu}{m}x' + \frac{k}{m}x = 0$$

$$x'' + 2cx' + \omega_0^2 x = 0,$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The system has characteristic equation $\lambda^2 + 2c\lambda + \omega_0^2 = 0$ and zeros

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \quad \text{and}$$

$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

If we set $\gamma = \sqrt{c^2 - \omega_0^2}$, then

$$\lambda_1 = -c - \gamma \quad \text{and} \quad \lambda_2 = -c + \gamma,$$

and $\lambda_2 - \lambda_1 = 2\gamma$. If the system is overdamped, note that

$$c^2 - \omega_0^2 > 0$$

$$\left(\frac{\mu}{2m}\right)^2 > \frac{k}{m}$$

$$\mu^2 > 4mk$$

$$\mu > 2\sqrt{mk}.$$

The general solution is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

The initial condition $x(0) = 0$ gives $0 = C_1 + C_2$ and $C_1 = -C_2$. Differentiating $x(t)$,

$$x'(t) = C_1 \lambda_1 e^{\lambda_1 t} + C_2 \lambda_2 e^{\lambda_2 t},$$

and the initial condition $x'(0) = v_0$ provides $v_0 = C_1 \lambda_1 + C_2 \lambda_2$. This system is easily solved for

$$C_1 = \frac{v_0}{\lambda_1 - \lambda_2} = -\frac{v_0}{2\gamma} \quad \text{and}$$

$$C_2 = \frac{-v_0}{\lambda_1 - \lambda_2} = \frac{v_0}{2\gamma},$$

so

$$\begin{aligned} x(t) &= -\frac{v_0}{2\gamma}(e^{\lambda_1 t} - e^{\lambda_2 t}) \\ &= -\frac{v_0}{2\gamma}(e^{(-c-\gamma)t} - e^{(-c+\gamma)t}) \\ &= \frac{v_0}{\gamma} \cdot e^{-ct} \cdot \frac{e^{\gamma t} - e^{-\gamma t}}{2} \\ &= \frac{v_0}{\gamma} e^{-ct} \sinh \gamma t. \end{aligned}$$

But, $c = \mu/(2m)$, so

$$x(t) = \frac{v_0}{\gamma} e^{-\mu t/(2m)} \sinh \gamma t,$$

where $\gamma = \sqrt{c^2 - \omega_0^2} = \sqrt{\mu^2 - 4km}/(2m)$.

- (b) The initial position is zero. The initial velocity is in the downward direction. Hence, due to the fact that the system is overdamped, we know that the solution has already crossed the t -axis once, and the motion is downward, after which the solution will approach the t -axis asymptotically. Therefore, the mass will reach its lowest point the *first* time the velocity is zero. Differentiating $x(t)$,

$$\begin{aligned} v(t) &= x'(t) \\ &= \frac{v_0}{\gamma} e^{-\mu t/(2m)} (\cosh \gamma t)(\gamma) \\ &\quad + \frac{v_0}{\gamma} \left(-\frac{\mu}{2m}\right) e^{-\mu t/(2m)} \sinh \gamma t \\ &= v_0 e^{\mu t/(2m)} \left(\cosh \gamma t - \frac{\mu}{2m\gamma} \sinh \gamma t \right). \end{aligned}$$

Setting the velocity equal to zero, it must be the case that

$$\begin{aligned} \cosh \gamma t - \frac{\mu}{2m\gamma} \sinh \gamma t &= 0 \\ \frac{\mu}{2m\gamma} \sinh \gamma t &= \cosh \gamma t \\ \tanh \gamma t &= \frac{2m\gamma}{\mu}. \end{aligned}$$

Taking the inverse hyperbolic tangent,

$$\begin{aligned} \gamma t &= \tanh^{-1} \left(\frac{2m\gamma}{\mu} \right) \\ t &= \frac{1}{\gamma} \tanh^{-1} \left(\frac{2m\gamma}{\mu} \right). \end{aligned}$$

- (c) If the system is critically damped, then $c^2 - \omega_0^2 = 0$ and the characteristic equation produces a repeated zero, $\lambda = -c$. Thus, the general solution is

$$x(t) = (C_1 + C_2 t) e^{-ct}.$$

The initial condition $x(0) = 0$ produces $C_1 = 0$. The derivative of $x(t)$ is

$$\begin{aligned} v(t) &= x'(t) \\ &= (C_1 + C_2 t)(-c e^{-ct}) + C_2 e^{-ct} \\ &= ((C_2 - cC_1) - cC_2 t) e^{-ct}. \end{aligned}$$

The initial condition $x'(0) = v_0$ gives $v_0 = C_2 - cC_1$, so the fact that $C_1 = 0$ produces $C_2 = v_0$ and the solution

$$x(t) = v_0 t e^{-ct}$$

But $c = \mu/(2m)$, so

$$x(t) = v_0 t e^{-\mu t/(2m)}$$

and

$$\begin{aligned} v(t) &= x'(t) \\ &= v_0 t \left(-\frac{\mu}{2m}\right) e^{-\mu t/(2m)} + v_0 e^{-\mu t/(2m)} \\ &= v_0 e^{-\mu t/(2m)} \left(-\frac{\mu t}{2m} + 1\right). \end{aligned}$$

Because the system is critically damped, and the solution has already crossed the t -axis, the solution will travel downward, then approach the t -axis asymptotically. Hence, the solution reaches its lowest point when the velocity *first* equals zero. This occurs when

$$\begin{aligned} \frac{\mu t}{2m} &= 1 \\ t &= \frac{2m}{\mu}. \end{aligned}$$

22. The force exerted by the viscous solution is proportional to the velocity of the mass, but opposite the motion. Thus

$$R(v) = -\mu v.$$

If the force is 0.3 N when the velocity is 0.2 m/s,

$$\begin{aligned}\mu &= -\frac{R(v)}{v} \\ \mu &= -\frac{-0.3 \text{ N}}{0.2 \text{ m/s}} \\ \mu &= 1.5 \text{ kg/s}.\end{aligned}$$

If the mass is 0.1 kg (100g) and the spring constant is 9.8 kg/s², then $my'' + \mu y' + ky = 0$ becomes

$$\begin{aligned}0.1y'' + 1.5y' + 9.8y &= 0 \\ y'' + 15y' + 98y &= 0\end{aligned}$$

The characteristic equation is $\lambda^2 + 15\lambda + 98 = 0$ with zeros $\lambda = (-15 \pm \sqrt{167}i)/2 = -(15/2) \pm i\omega$, where $\omega = \sqrt{167}/2$. The general solution to the homogeneous equation is

$$y(t) = e^{-15t/2} (C_1 \cos \omega t + C_2 \sin \omega t).$$

Let's assume that the mass is displaced upward 0.1 m (10 cm). Thus, $C_1 = 1/10$. Differentiate.

$$\begin{aligned}y'(t) &= e^{-15t/2} (-\omega C_1 \sin \omega t + \omega C_2 \cos \omega t) \\ &\quad - \frac{15}{2} e^{-15t/2} (C_1 \cos \omega t + C_2 \sin \omega t).\end{aligned}$$

If the mass is released from rest, then $y'(0) = 0$ and

$$0 = \omega C_2 - \frac{15}{2} C_1 = \omega C_2 - \frac{3}{4}$$

so $C_2 = 3/4\omega$. Thus, the solution is

$$y(t) = e^{-15t/2} \left(\frac{1}{10} \cos \omega t + \frac{3}{4\omega} \sin \omega t \right).$$

Therefore, the amplitude is

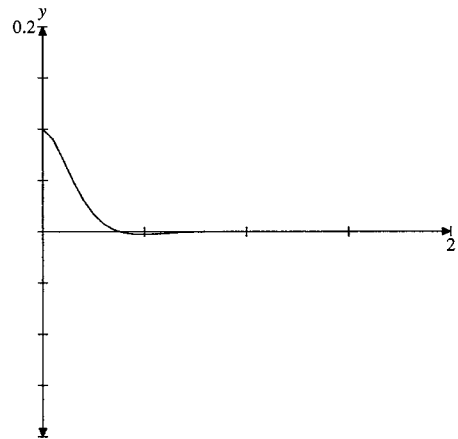
$$A = \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{3}{4\omega}\right)^2} \approx 0.1532$$

and

$$\begin{aligned}\tan \phi &= \frac{3\sqrt{167}/334}{1/10} \\ \phi &\approx 0.8597.\end{aligned}$$

Thus, the solution can be written

$$y(t) \approx 0.1532 e^{-7.5t} \cos \left(\frac{\sqrt{167}t}{2} - 0.8597 \right).$$



23. The voltage across the capacitor is given by $V_C = (1/C)Q$. Because the initial voltage across the capacitor is $V_C(0) = 50$ V,

$$Q(0) = C V_C(0) = (0.008 \text{ F})(50 \text{ V}) = 0.4 \text{ C},$$

where C is the abbreviation for coulombs. The sum of the voltage drops around the LRC circuit equals zero, so $LI' + RI + (1/C)Q = 0$. Because $I = Q'$, this equation becomes $LQ'' + RQ' + (1/C)Q = 0$. Substituting $L = 4$ H, $R = 20$ Ω , and $C = 0.008$ F, we get $4Q'' + 20Q' + 125Q = 0$. With no initial current this equation becomes

$$Q'' + 5Q' + \frac{125}{4}Q = 0,$$

with initial conditions $Q(0) = 0.4$ and $Q'(0) = I(0) = 0$. The characteristic equation is $\lambda^2 + 5\lambda + (125/4) = 0$ with zeros $\lambda = -5/2 \pm 5i$. This leads to the complex solution

$$z(t) = e^{(-5/2+5i)t} = e^{-5t/2} (\cos 5t + i \sin 5t).$$

The real and imaginary parts of this solution provide a fundamental set of solutions and the general solution

$$Q(t) = e^{-5t/2} (C_1 \cos 5t + C_2 \sin 5t).$$

The initial condition $Q(0) = 0.4$ gives $C_1 = 2/5$. Differentiating $Q(t)$,

$$Q'(t) = e^{-5t/2}(-5C_1 \sin 5t + 5C_2 \cos 5t) - \frac{5}{2}e^{-5t/2}(C_1 \cos 5t + C_2 \sin 5t).$$

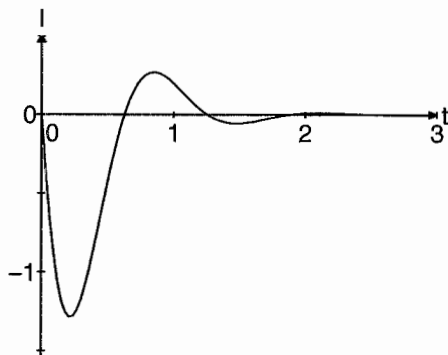
The initial condition $Q'(0) = 0$ leads to $0 = 5C_2 - (5/2)C_1$, and because $C_1 = 2/5$, we have $C_2 = 1/5$ and the solution

$$Q(t) = \frac{1}{5}e^{-5t/2}(2 \cos 5t + \sin 5t).$$

We find the current by differentiating Q to get

$$I(t) = -\frac{5}{2}e^{-5t/2} \sin 5t.$$

Since $-\sin 5t = \cos(5t + \pi/2)$, we see that the phase is $\phi = -\pi/2$. Clearly the amplitude is $5/2$, and the frequency is 5 rad/s .



24. If a 10 kg mass displaces 1 m, then

$$\begin{aligned} k &= \frac{F}{y} = \frac{mg}{y} \\ k &= \frac{(10 \text{ kg})(9.8 \text{ m/s}^2)}{1 \text{ m}} \\ k &= 98 \text{ N/m.} \end{aligned}$$

If the damping constant is $\mu = 20 \text{ kg/s}$, then $my'' + \mu y' + ky = 0$ becomes

$$\begin{aligned} 10y'' + 20y' + 98y &= 0 \\ y'' + 2y' + 9.8y &= 0 \end{aligned}$$

This has characteristic equation $\lambda^2 + 2\lambda + 9.8 = 0$ and roots that are approximately equal to $\lambda = -1 \pm 2.9665i$. Thus the general solution is given approximately by

$$y(t) = e^{-t}(C_1 \cos 2.9665t + C_2 \sin 2.9665t).$$

The initial displacement is zero, so $y(0) = 0$ and $C_1 = 0$. Differentiate.

$$\begin{aligned} y'(t) &= e^{-t}(-2.9665C_1 \sin 2.9665t \\ &\quad + 2.9665C_2 \cos 2.9665t) \\ &\quad - e^{-t}(C_1 \cos 2.9665t + C_2 \sin 2.9665t). \end{aligned}$$

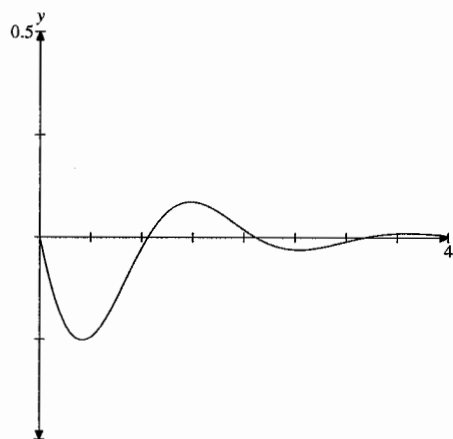
If the initial velocity is $y'(0) = -1.2 \text{ m/s}$, then

$$-1.2 = 2.9665C_2 - C_1.$$

Thus, $C_2 \approx 0.4045$ and

$$\begin{aligned} y(t) &= -0.4045e^{-t} \sin 2.9665t \\ &= -0.4045e^{-t} \cos\left(2.9665t + \frac{\pi}{2}\right). \end{aligned}$$

Thus, the amplitude is 0.4045 , the frequency is 2.9665 rad/s , and the phase is $\phi = -\pi/2$.



25. The voltage across the capacitor is given by $V_C = (1/C)Q$. Because the initial voltage across the capacitor is $V_C(0) = 1 \text{ V}$,

$$Q(0) = CV_C(0) = (0.02 \text{ F})(1 \text{ V}) = 0.02 \text{ C},$$

where C is the abbreviation for coulombs. The sum of the voltage drops around the LRC circuit equals zero, so $LI' + RI + (1/C)Q = 0$. Because $I = Q'$, this equation becomes $LQ'' + RQ' + (1/C)Q = 0$. Substituting $L = 10$ H, $R = 40$ Ω , and $C = 0.02$ F, we get $10Q'' + 40Q' + 50Q = 0$. With no initial current this equation becomes

$$Q'' + 4Q' + 5Q = 0,$$

with initial conditions $Q(0) = 0.02$ and $Q'(0) = I(0) = 0$. The characteristic equation is $\lambda^2 + 4\lambda + 5 = 0$ with zeros $\lambda = -2 \pm i$. This leads to the complex solution

$$z(t) = e^{(-2+i)t} = e^{-2t}(\cos t + i \sin t).$$

The real and imaginary parts of this solution provide a fundamental set of solutions and the general solution

$$Q(t) = e^{-2t}(C_1 \cos t + C_2 \sin t).$$

The initial condition $Q(0) = 0.02$ gives $C_1 = 1/50$. Differentiating $Q(t)$,

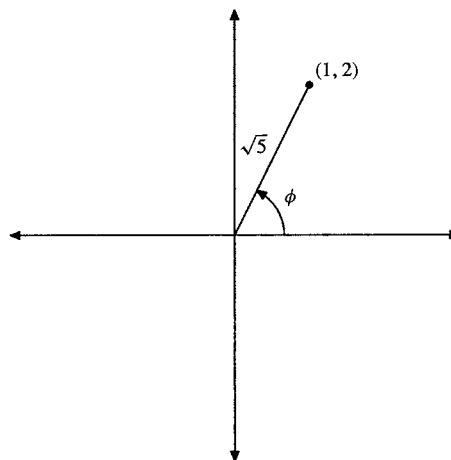
$$Q'(t) = e^{-2t}(-C_1 \sin t + C_2 \cos t) - 2e^{-2t}(C_1 \cos t + C_2 \sin t).$$

The initial condition $Q'(0) = 0$ leads to $0 = C_2 - 2C_1$, and because $C_1 = 1/50$, we have $C_2 = 1/25$ and the solution

$$Q(t) = \frac{1}{50}e^{-2t}(\cos t + 2 \sin t).$$

Plotting the coefficients of $\cos t$ and $\sin t$ and mark-

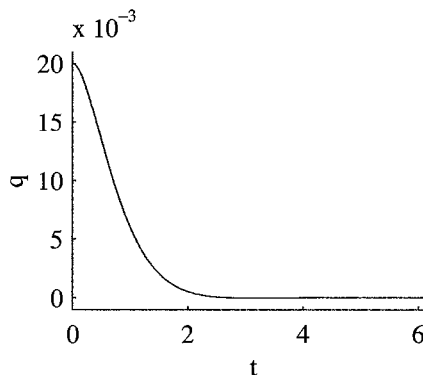
ing the angle,



we see that $\tan \phi = 2$, so $\phi = \arctan 2 \approx 1.1071$. Factoring out the magnitude of the radial vector,

$$\begin{aligned} Q(t) &= \frac{\sqrt{5}}{50}e^{-2t}\left(\frac{1}{\sqrt{5}}\cos t + \frac{2}{\sqrt{5}}\sin t\right) \\ &= \frac{\sqrt{5}}{50}e^{-2t}(\cos \phi \cos t + \sin \phi \sin t) \\ &= \frac{\sqrt{5}}{50}e^{-2t}\cos(t - 1.1071). \end{aligned}$$

Thus, the amplitude is $\sqrt{5}/50$, the frequency is 1 rad/s, and the phase is 1.1071.



Section 4.5. Inhomogeneous Equations; the Method of Undetermined Coefficients

1. Let $y(t) = Ae^{-3t}$. Then

$$y'(t) = -3Ae^{-3t}$$

$$y''(t) = 9Ae^{-3t},$$

and $y'' + 3y' + 2y = 4e^{-3t}$ becomes

$$9Ae^{-3t} + 3(-3Ae^{-3t}) + 2(Ae^{-3t}) = 4e^{-3t}$$

$$2A = 4$$

$$A = 2.$$

Thus, $y = 2e^{-3t}$ is a particular solution.

2. Let $y(t) = Ae^{-t}$. Then

$$y'(t) = -Ae^{-t}$$

$$y''(t) = Ae^{-t},$$

and $y'' + 6y' + 8y = -3e^{-t}$ becomes

$$Ae^{-t} + 6(-Ae^{-t}) + 8(Ae^{-t}) = -3e^{-t}$$

$$3A = -3$$

$$A = -1.$$

Thus, $y = -e^{-t}$ is a particular solution.

3. Let $y(t) = Ae^{-t}$. Then

$$y'(t) = -Ae^{-t}$$

$$y''(t) = Ae^{-t},$$

and $y'' + 2y' + 5y = 12e^{-t}$ becomes

$$Ae^{-t} + 2(-Ae^{-t}) + 5(Ae^{-t}) = 12e^{-t}$$

$$4A = 12$$

$$A = 3.$$

Thus, $y = 3e^{-t}$ is a particular solution.

4. Let $y(t) = Ae^{2t}$. Then,

$$y'(t) = 2Ae^{2t}$$

$$y''(t) = 4Ae^{2t},$$

and $y'' + 3y' - 18y = 18e^{2t}$ becomes

$$4Ae^{2t} + 3(2Ae^{2t}) - 18(Ae^{2t}) = 18e^{2t}$$

$$-8A = 18$$

$$A = -\frac{9}{4}.$$

Thus, $y = -(9/4)e^{2t}$ is a particular solution.

5. Let $y_p = a \cos 3t + b \sin 3t$. Then

$$y'_p = -3a \sin 3t + 3b \cos 3t$$

$$y''_p = -9a \cos 3t - 9b \sin 3t,$$

and the equation $y'' + 4y = \cos 3t$ becomes

$$-9a \cos 3t - 9b \sin 3t + 4(a \cos 3t + b \sin 3t)$$

$$= \cos 3t,$$

or, equivalently,

$$-5a \cos 3t - 5b \sin 3t = \cos 3t.$$

Thus, $-5a = 1$ and $-5b = 0$ lead to $a = -1/5$ and $b = 0$ and the particular solution $y_p = -1/5 \cos 3t$.

6. Let $y_p = a \cos 2t + b \sin 2t$. Then

$$y'_p = -2a \sin 2t + 2b \cos 2t$$

$$y''_p = -4a \cos 2t - 4b \sin 2t,$$

and the equation $y'' + 9y = \sin 2t$ becomes

$$-4a \cos 2t - 4b \sin 2t + 9(a \cos 2t + b \sin 2t)$$

$$= \sin 2t,$$

or, equivalently,

$$5a \cos 2t + 5b \sin 2t = \sin 2t.$$

Thus, $5a = 0$ and $5b = 1$ lead to $a = 0$ and $b = 1/5$ and the particular solution $y_p = (1/5) \sin 2t$.

7. Let $y_p = a \cos 2t + b \sin 2t$. Then

$$y'_p = -2a \sin 2t + 2b \cos 2t$$

$$y''_p = -4a \cos 2t - 4b \sin 2t,$$

and substituting these results into $y'' + 7y' + 6y = 3 \sin 2t$ leads to

$$(2a + 14b) \cos 2t + (-14a + 2b) \sin 2t = 3 \sin 2t.$$

Thus,

$$2a + 14b = 0$$

$$-14a + 2b = 3,$$

leading to $a = -21/100$ and $b = 3/100$. Thus, the particular solution is $y_p = -(21/100) \cos 2t + (3/100) \sin 2t$.

8. Let
- $y_p = a \cos 3t + b \sin 3t$
- . Then

$$\begin{aligned}y_p' &= -3a \sin 3t + 3b \cos 3t \\y_p'' &= -9a \cos 3t - 9b \sin 3t,\end{aligned}$$

and the equation $y'' + 7y' + 10y = -4 \sin 3t$ becomes

$$(a + 21b) \cos 3t + (-21a + b) \sin 3t = -4 \sin 3t.$$

Thus,

$$\begin{aligned}a + 21b &= 0 \\-21a + b &= -4,\end{aligned}$$

leading to $a = 42/221$ and $b = -2/221$ and the particular solution $y_p = (42/221) \cos 3t - (2/221) \sin 3t$.

9. If
- $z(t) = x(t) + iy(t)$
- , then
- $z'' + pz' + qz = Ae^{i\omega t}$
- becomes

$$\begin{aligned}(x'' + iy'') + p(x' + iy') + q(x + iy) \\&= A(\cos \omega t + i \sin \omega t) \\(x'' + px' + qx) + i(y'' + py' + qy) \\&= A \cos \omega t + iA \sin \omega t.\end{aligned}$$

Comparing real and imaginary parts,

$$\begin{aligned}x'' + px' + qx &= A \cos \omega t \\y'' + py' + qy &= A \sin \omega t.\end{aligned}$$

Therefore, if $z(t) = x(t) + iy(t)$ is a solution of $z'' + pz' + qz = Ae^{i\omega t}$, then the real and imaginary parts of z are solutions of $x'' + px' + qx = A \cos \omega t$ and $y'' + py' + qy = A \sin \omega t$, respectively.

10. Let
- $z = Ae^{i3t}$
- . Then

$$\begin{aligned}z' &= 3iAe^{i3t} \\z'' &= (3i)^2 Ae^{i3t},\end{aligned}$$

and $z'' + 4z = e^{i3t}$ leads to

$$\begin{aligned}(3i)^2 Ae^{i3t} + 4Ae^{i3t} &= e^{i3t} \\((3i)^2 + 4)A &= 1 \\A &= -\frac{1}{5}.\end{aligned}$$

Thus, $z = -(1/5)e^{i3t}$ is a particular solution of $z'' + 4z = e^{i3t}$ and the real part of this solution $y = -(1/5) \cos 3t$, is a solution of $y'' + 4y = \cos 3t$.

11. Let
- $z = Ae^{i2t}$
- . Then

$$\begin{aligned}z' &= 2iAe^{i2t} \\z'' &= (2i)^2 Ae^{i2t},\end{aligned}$$

and $z'' + 9z = e^{i2t}$ leads to

$$\begin{aligned}(2i)^2 Ae^{i2t} + 9Ae^{i2t} &= e^{i2t} \\((2i)^2 + 9)A &= 1 \\A &= \frac{1}{5}.\end{aligned}$$

Thus, $z = (1/5)e^{i2t}$ is a particular solution of $z'' + 9z = e^{i2t}$ and the imaginary part of this solution, $y = (1/5) \sin 2t$, is the solution of $y'' + 9y = \sin 2t$.

12. Let
- $z = Ae^{i2t}$
- . Then

$$\begin{aligned}z' &= 2iAe^{i2t} \\z'' &= (2i)^2 Ae^{i2t},\end{aligned}$$

and $z'' + 7z' + 6z = 3e^{i2t}$ leads to

$$\begin{aligned}(2i)^2 Ae^{i2t} + 7(2i)Ae^{i2t} + 6Ae^{i2t} &= 3e^{i2t} \\((2i)^2 + 7(2i) + 6)A &= 3 \\A &= \frac{3}{2 + 14i} \\A &= \frac{3}{100} - \frac{21}{100}i.\end{aligned}$$

Thus,

$$\begin{aligned}z &= \left(\frac{3}{100} - \frac{21}{100}i \right) e^{i2t} \\&= \left(\frac{3}{100} - \frac{21}{100}i \right) (\cos 2t + i \sin 2t) \\&= \left(\frac{3}{100} \cos 2t + \frac{21}{100} \sin 2t \right) \\&\quad + i \left(-\frac{21}{100} \cos 2t + \frac{3}{100} \sin 2t \right)\end{aligned}$$

is a solution of $z'' + 7z' + 6z = 3e^{i2t}$. The imaginary part,

$$y = -\frac{21}{100} \cos 2t + \frac{3}{100} \sin 2t$$

is a solution of $y'' + 7y' + 6y = 3 \sin 2t$.

13. Let
- $z = Ae^{i3t}$
- . Then

$$\begin{aligned} z' &= (3i)Ae^{i3t} \\ z'' &= (3i)^2 Ae^{i3t}, \end{aligned}$$

and $z'' + 7z' + 10z = -4e^{i3t}$ leads to

$$\begin{aligned} (3i)^2 Ae^{i3t} + 7(3i)Ae^{i3t} + 10Ae^{i3t} &= -4e^{i3t} \\ ((3i)^2 + 7(3i) + 10)A &= -4 \end{aligned}$$

Solving for A , we get

$$\begin{aligned} A &= -\frac{4}{1 + 21i} \\ &= -\frac{2}{221} + \frac{42}{221}i. \end{aligned}$$

Thus,

$$\begin{aligned} z &= \left(-\frac{2}{221} + \frac{42}{221}i\right)e^{i3t} \\ &= \left(-\frac{2}{221} + \frac{42}{221}i\right)(\cos 3t + i \sin 3t) \\ &= \left(-\frac{2}{221} \cos 3t - \frac{42}{221} \sin 3t\right) \\ &\quad + i \left(\frac{42}{221} \cos 3t - \frac{2}{221} \sin 3t\right) \end{aligned}$$

is a solution of $z'' + 7z' + 10z = -4e^{i3t}$. The imaginary part,

$$y = \frac{42}{221} \cos 3t - \frac{2}{221} \sin 3t,$$

is a solution of $y'' + 7y' + 10y = -4 \sin 3t$.

14. Let
- $y = at + b$
- . Then
- $y' = a$
- ,
- $y'' = 0$
- and
- $y'' + 5y' + 4y = 2 + 3t$
- leads to

$$\begin{aligned} 5a + 4(at + b) &= 2 + 3t \\ 4at + (5a + 4b) &= 3t + 2. \end{aligned}$$

Thus, $4a = 3$ and $5a + 4b = 2$, which gives $a = 3/4$ and $b = -7/16$. Therefore, a particular solution is $y = (3/4)t - 7/16$.

15. Let
- $y = at + b$
- . Then,
- $y' = a$
- ,
- $y'' = 0$
- , and
- $y'' + 6y' + 8y = 2t - 3$
- leads to
- $8at + (6a + 8b) = 2t - 3$
- . Thus,
- $8a = 2$
- and
- $6a + 8b = -3$
- , which gives
- $a = 1/4$
- and
- $b = -9/16$
- . Therefore, the particular solution is
- $y = (1/4)t - 9/16$
- .

16. Let
- $y = at^2 + bt + c$
- . Then substituting

$$\begin{aligned} y' &= 2at + b \\ y'' &= 2a \end{aligned}$$

in $y'' + 5y' + 6y = 4 - t^2$, arranging terms,

$$\begin{aligned} 2a + 5(2at + b) + 6(at^2 + bt + c) &= 4 - t^2 \\ 6at^2 + (10a + 6b)t + (2a + 5b + 6c) &= -t^2 + 4, \end{aligned}$$

and comparing coefficients,

$$\begin{aligned} 6a &= -1 \\ 10a + 6b &= 0 \\ 2a + 5b + 6c &= 4, \end{aligned}$$

provides $a = -1/6$, $b = 5/18$, and $c = 53/108$. Thus, a particular solution is

$$y = -\frac{1}{6}t^2 + \frac{5}{18}t + \frac{53}{108}.$$

17. Let
- $y = at^3 + bt^2 + ct + d$
- . Then, substituting

$$\begin{aligned} y' &= 3at^2 + 2bt + c \\ y'' &= 6at + 2b \end{aligned}$$

into $y'' + 3y' + 4y = t^3$, arranging terms,

$$\begin{aligned} 4at^3 + (9a + 4b)t^2 + (6a + 6b + 4c)t \\ + (2b + 3c + 4d) &= t^3, \end{aligned}$$

and comparing coefficients,

$$\begin{aligned} 4a &= 1 \\ 9a + 4b &= 0 \\ 6a + 6b + 4c &= 0 \\ 2b + 3c + 4d &= 0, \end{aligned}$$

provides $a = 1/4$, $b = -9/16$, $c = 15/32$, and $d = -9/128$. Thus, a particular solution is

$$y = \frac{1}{4}t^3 - \frac{9}{16}t^2 + \frac{15}{32}t - \frac{9}{128}.$$

18. The homogeneous equation $y'' + 3y' + 2y = 0$ has characteristic equation $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ with zeros $\lambda_1 = -1$ and $\lambda_2 = -2$. This leads to the homogeneous solution

$$y_h = C_1 e^{-t} + C_2 e^{-2t}.$$

The particular solution $y_p = Ae^{-4t}$ has derivatives $y'_p = -4Ae^{-4t}$ and $y''_p = 16Ae^{-4t}$, which when substituted into the equation $y'' + 3y' + 2y = 3e^{-4t}$ provides

$$16Ae^{-4t} + 3(-4Ae^{-4t}) + 2Ae^{-4t} = 3e^{-4t}$$

$$6A = 3$$

$$A = \frac{1}{2}.$$

Thus, a particular solution is $y_p = (1/2)e^{-4t}$. This leads to the general solution

$$y = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{2} e^{-4t}.$$

The initial condition $y(0) = 1$ provides

$$1 = C_1 + C_2 + \frac{1}{2}.$$

Differentiating,

$$y' = -C_1 e^{-t} - 2C_2 e^{-2t} - 2e^{-4t}.$$

The initial condition $y'(0) = 0$ provides

$$0 = -C_1 - 2C_2 - 2.$$

This system has solution $C_1 = 3$ and $C_2 = -5/2$, leading to the solution

$$y = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}.$$

19. The homogeneous equation $y'' - 4y' - 5y = 0$ has characteristic equation $\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$ with zeros $\lambda_1 = 5$ and $\lambda_2 = -1$. This leads to the homogeneous solution

$$y_h = C_1 e^{5t} + C_2 e^{-t}.$$

The particular solution $y_p = Ae^{-2t}$ has derivatives $y'_p = -2Ae^{-2t}$ and $y''_p = 4Ae^{-2t}$, which when substituted into the equation $y'' - 4y' - 5y = 4e^{-2t}$ provides

$$4Ae^{-2t} + 8Ae^{-2t} - 5Ae^{-2t} = 4e^{-2t}$$

$$7A = 4$$

$$A = \frac{4}{7}.$$

Thus, a particular solution is $y_p = (4/7)e^{-2t}$. This leads to the general solution

$$y = C_1 e^{5t} + C_2 e^{-t} + \frac{4}{7} e^{-2t}.$$

The initial condition $y(0) = 0$ gives

$$0 = C_1 + C_2 + \frac{4}{7}.$$

Differentiating,

$$y' = 5C_1 e^{5t} - C_2 e^{-t} - \frac{8}{7} e^{-2t}.$$

The initial condition $y'(0) = -1$ provides

$$-1 = 5C_1 - C_2 - \frac{8}{7}.$$

This system has solution $C_1 = -1/14$ and $C_2 = -1/2$, leading to the solution

$$y = -\frac{1}{14} e^{5t} - \frac{1}{2} e^{-t} + \frac{4}{7} e^{-2t}.$$

20. The homogeneous equation $y'' + 2y' + 2y = 0$ has characteristic equation $\lambda^2 + 2\lambda + 2 = 0$ with zeros $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. This leads to the homogeneous solution

$$y_h = e^{-t}(C_1 \cos t + C_2 \sin t).$$

The particular solution $z = Ae^{i2t}$ has derivatives

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

which, when inserted in the complex equation $z'' + 2z' + 2z = 2e^{i2t}$, gives

$$\begin{aligned} ((2i)^2 + 2(2i) + 2) A e^{i2t} &= 2e^{i2t} \\ A &= \frac{2}{-2 + 4i} \\ A &= -\frac{1}{5} - \frac{2}{5}i. \end{aligned}$$

This gives the particular solution

$$\begin{aligned} z &= \left(-\frac{1}{5} - \frac{2}{5}i\right) e^{i2t} \\ z &= \left(-\frac{1}{5} - \frac{2}{5}i\right) (\cos 2t + i \sin 2t). \end{aligned}$$

The real part of this solution,

$$y_p = -\frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t,$$

is a particular solution of $y'' + 2y' + 2y = 2 \cos 2t$. Thus, the general solution is

$$y = e^{-t}(C_1 \cos t + C_2 \sin t) - \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t.$$

The initial condition $y(0) = -2$ gives $-2 = C_1 - 1/5$. Differentiating,

$$\begin{aligned} y' &= e^{-t}(-C_1 \sin t + C_2 \cos t) - \\ &\quad e^{-t}(C_1 \cos t + C_2 \sin t) + \frac{2}{5} \sin 2t + \frac{4}{5} \cos 2t. \end{aligned}$$

The initial condition $y'(0) = 0$ provides

$$0 = C_2 - C_1 + \frac{4}{5}.$$

This system has solution $C_1 = -9/5$ and $C_2 = -13/5$. Therefore, the solution is

$$\begin{aligned} y &= e^{-t} \left(-\frac{9}{5} \cos t - \frac{13}{5} \sin t \right) \\ &\quad - \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t. \end{aligned}$$

21. The homogeneous equation $y'' - 2y' + 5y = 0$ has characteristic equation $\lambda^2 - 2\lambda + 5 = 0$ with zeros

$\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. This leads to the homogeneous solution

$$y_h = e^t(C_1 \cos 2t + C_2 \sin 2t).$$

The particular solution $z = Ae^{it}$ has derivatives

$$\begin{aligned} z' &= iAe^{it} \\ z'' &= i^2 e^{it}, \end{aligned}$$

which, when inserted in the complex equation $z'' - 2z' + 5z = 3e^{it}$, gives

$$\begin{aligned} i^2 Ae^{it} - 2iAe^{it} + 5Ae^{it} &= 3e^{it} \\ (i^2 - 2i + 5)A &= 3 \end{aligned}$$

$$\begin{aligned} A &= \frac{3}{4 - 2i} \\ A &= \frac{3}{5} + \frac{3}{10}i. \end{aligned}$$

This gives the particular solution

$$\begin{aligned} z &= \left(\frac{3}{5} + \frac{3}{10}i\right) e^{it} \\ z &= \left(\frac{3}{5} \cos t + \frac{3}{10}i\right) (\cos t + i \sin t). \end{aligned}$$

The real part of this solution is a particular solution of $y'' - 2y' + 5y = 3 \cos t$.

$$y_p = \frac{3}{5} \cos t - \frac{3}{10} \sin t$$

Thus, the general solution is

$$y = e^t(C_1 \cos 2t + C_2 \sin 2t) + \frac{3}{5} \cos t - \frac{3}{10} \sin t.$$

The initial condition $y(0) = 0$ gives $0 = C_1 + 3/5$. Differentiating,

$$\begin{aligned} y' &= e^t(-2C_1 \sin 2t + 2C_2 \cos 2t) \\ &\quad + e^t(C_1 \cos 2t + C_2 \sin 2t) \\ &\quad - \frac{3}{5} \sin t - \frac{3}{10} \cos t. \end{aligned}$$

The initial condition $y'(0) = -2$ provides $-2 = 2C_2 + C_1 - 3/10$. The solution of this system is

$C_1 = -3/5$ and $C_2 = -11/20$. Therefore, the solution is

$$y = e^t \left(-\frac{3}{5} \cos 2t - \frac{11}{20} \sin 2t \right) + \frac{3}{5} \cos t - \frac{3}{10} \sin t.$$

22. The homogeneous equation $y'' + 4y' + 4y = 0$ has characteristic equation $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$ and repeated root $\lambda = -2$. Thus the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^{-2t}.$$

The particular solution $y_p = at + b$ has derivatives $y'_p = a$ and $y''_p = 0$, which when substituted in $y'' + 4y' + 4y = 4 - t$,

$$\begin{aligned} 4a + 4(at + b) &= 4 - t \\ 4at + (4a + 4b) &= -t + 4. \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} 4a &= -1 \\ 4a + 4b &= 4, \end{aligned}$$

which has solution $a = -1/4$ and $b = 5/4$. Thus, the general solution is

$$y = (C_1 + C_2 t)e^{-2t} - \frac{1}{4}t + \frac{5}{4}.$$

The initial condition $y(0) = -1$ provides

$$-1 = C_1 + \frac{5}{4}.$$

Differentiate.

$$y' = C_2 e^{-2t} - 2e^{-2t}(C_1 + C_2 t) - \frac{1}{4}$$

The initial condition $y'(0) = 0$ provides

$$0 = C_1 - 2C_1 - \frac{1}{4}.$$

Thus, $C_1 = -9/4$ and $C_2 = -17/4$ and

$$y = \left(-\frac{9}{4} - \frac{17}{4}t \right) e^{-2t} - \frac{1}{4}t + \frac{5}{4}.$$

23. The homogeneous equation $y'' - 2y' + y = 0$ has characteristic equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, with repeated zero $\lambda = 1$. Thus, the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^t.$$

The particular solution $y_p = at^3 + bt^2 + ct + d$ has derivatives

$$\begin{aligned} y'_p &= 3at^2 + 2bt + c \\ y''_p &= 6at + 2b, \end{aligned}$$

which when substituted in $y'' - 2y' + y = t^3$, rearranging, yields

$$at^3 + (-6a + b)t^2 + (6a - 4b + c)t + (2b - 2c + d) = t^3.$$

Thus,

$$\begin{aligned} a &= 1 \\ -6a + b &= 0 \\ 6a - 4b + c &= 0 \\ 2b - 2c + d &= 0, \end{aligned}$$

which has solution $a = 1$, $b = 6$, $c = 18$, and $d = 24$. Thus, the general solution is

$$y = (C_1 + C_2 t)e^t + t^3 + 6t^2 + 18t + 24.$$

The initial condition $y(0) = 1$ gives $1 = C_1 + 24$. Differentiating,

$$y' = C_2 e^t + (C_1 + C_2 t)e^t + 3t^2 + 12t + 18.$$

The initial condition $y'(0) = 0$ gives $0 = C_2 + C_1 + 18$. The system has solution $C_1 = -23$ and $C_2 = 5$. Therefore, the solution is

$$y = (-23 + 5t)e^t + t^3 + 6t^2 + 18t + 24.$$

24. The homogeneous equation $y'' - 3y' - 10y = 0$ has characteristic equation $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$ with zeros $\lambda_1 = 5$ and $\lambda_2 = -2$. Thus, the homogeneous solution is

$$y_h = C_1 e^{5t} + C_2 e^{-2t}.$$

Thus, the forcing term of $y'' - 3y' - 10y = 3e^{-2t}$ is a solution of the homogeneous equation. Substitute $y_p = Ate^{-2t}$ and its derivatives

$$\begin{aligned} y'_p &= Ae^{-2t}(1 - 2t) \\ y''_p &= (-4 - 4t)Ae^{-2t} \end{aligned}$$

into the equation.

$$\begin{aligned} & (-4 + 4t)Ae^{-2t} - 3Ae^{-2t}(1 - 2t) - 10Ate^{-2t} \\ & = 3e^{-2t} \\ & [(-4 + 4t) - 3(1 - 2t) - 10t]Ae^{-2t} \\ & = 3e^{-2t} \end{aligned}$$

Solving for A , we get $A = 3/7$. Thus, $y = -(3/7)te^{-2t}$ is a particular solution.

25. The homogeneous equation $y'' - y' - 2y = 0$ has characteristic equation $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$ with zeros $\lambda_1 = 2$ and $\lambda_2 = -1$. Thus, the homogeneous solution is

$$y_h = C_1e^{2t} + C_2e^{-t}.$$

The forcing term of $y'' - y' - 2y = 2e^{-t}$ is also a solution of the homogeneous equation, so we multiply by a factor of t and try $y_p = Ate^{-t}$, which has derivatives

$$\begin{aligned} y' &= Ae^{-t}(1 - t) \\ y'' &= Ae^{-t}(t - 2). \end{aligned}$$

Substituting these in $y'' - y' - 2y = 2e^{-t}$ gives

$$\begin{aligned} Ae^{-t}(t - 2) - Ae^{-t}(1 - t) - 2Ate^{-t} &= 2e^{-t} \\ A(t - 2 - 1 + t - 2t) &= 2 \\ -3A &= 2 \\ A &= -\frac{2}{3}. \end{aligned}$$

Thus, a particular solution is $y_p = -(2/3)te^{-t}$.

26. The homogeneous equation $z'' + 4z = 0$ has characteristic equation $\lambda^2 + 4 = 0$ and zeros $\pm 2i$. Thus, the homogeneous solution is

$$z_h = C_1e^{2it} + C_2e^{-2it}.$$

The forcing term of $z'' + 4z = 4e^{i2t}$ is also a solution of the homogeneous equation, so multiply by a factor of t and try $z_p = Ate^{i2t}$. The particular solution has derivatives

$$\begin{aligned} z'_p &= Ae^{i2t}(1 + 2it) \\ z''_p &= 4Ae^{i2t}(i - t). \end{aligned}$$

If these are substituted in $z'' + 4z = 4e^{i2t}$, then

$$\begin{aligned} 4Ae^{i2t}(i - t) + 4Ate^{i2t} &= 4e^{i2t} \\ (4(i - t) + 4t)A &= 4 \\ A &= \frac{1}{i} = -i. \end{aligned}$$

Hence,

$$\begin{aligned} z_p &= -ite^{i2t} \\ &= it(\cos 2t + i \sin 2t) \\ &= t \sin 2t + it \cos 2t. \end{aligned}$$

The real part of this solution is a particular solution of $y'' + 4y = 4 \cos 2t$.

$$y_p = t \sin 2t$$

27. The homogeneous equation $z'' + 9z = 0$ has characteristic equation $\lambda^2 + 9 = 0$, which has zeros $\lambda_1 = 3i$ and $\lambda_2 = -3i$. Thus, the homogeneous solution is

$$z_h = C_1e^{3it} + C_2e^{-3it}.$$

The forcing term of $z'' + 9z = e^{3it}$ is also a solution of the homogeneous equation, so multiply by a factor of t and try the particular solution $z_p = Ate^{3it}$. The particular solution has derivatives

$$\begin{aligned} z'_p &= Ae^{3it}(1 + 3it) \\ z''_p &= 3iAe^{3it}(2 + 3it). \end{aligned}$$

If these are substituted into $z'' + 9z = e^{3it}$, then

$$\begin{aligned} 3iAe^{3it}(2 + 3it) + 9Ate^{3it} &= e^{3it} \\ 3iA(2 + 3it) + 9At &= 1 \\ 6iA &= 1 \\ A &= -\frac{1}{6}i. \end{aligned}$$

Thus,

$$\begin{aligned} z_p &= -\frac{1}{6}ite^{3it} \\ &= -\frac{1}{6}it(\cos 3t + i \sin 3t) \\ &= \frac{1}{6}t \sin 3t + i \left(-\frac{1}{6}t \cos 3t \right). \end{aligned}$$

The imaginary part of this solution is a particular solution of $y'' + 9y = \sin 3t$.

$$y_p = -\frac{1}{6}t \cos 3t$$

28. The homogeneous equation $y'' + 4y' + 4y = 0$ has characteristic equation $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$ and homogeneous solution

$$y_h = (C_1 + C_2 t)e^{-2t}.$$

The forcing term of $y'' + 4y' + 4y = 2e^{-2t}$ and $At e^{-2t}$ are solutions of the homogeneous equation, so multiply by another factor of t and try $y_p = At^2 e^{-2t}$. The derivatives of y_p are

$$\begin{aligned} y_p' &= 2Ae^{-2t}(t - t^2) \\ y_p'' &= 2Ae^{-2t}(2t^2 - 4t + 1). \end{aligned}$$

Substituting these in $y'' + 4y' + 4y = 2e^{-2t}$,

$$\begin{aligned} &2Ae^{-2t}(2t^2 - 4t + 1) + 8Ae^{-2t} \\ &\quad + (t - t^2) + 4At^2 e^{-2t} \\ &= 2e^{-2t} \\ &[2(2t^2 - 4t + 1) + 8(t - t^2) + 4t^2] A \\ &= 2 \end{aligned}$$

Solving for A shows that $A = 1$. Thus, $y_p = t^2 e^{-2t}$ is a particular solution.

29. The homogeneous equation $y'' + 6y' + 9y = 0$ has characteristic equation $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$ and repeated zero $\lambda = -3$. Thus, the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^{-3t}.$$

The forcing term of $y'' + 6y' + 9y = 5e^{-3t}$ is a solution of the homogeneous equation, as is te^{-3t} . Consequently, try $y_p = At^2 e^{-3t}$, which has derivatives

$$\begin{aligned} y_p' &= -3At^2 e^{-3t} + 2Ate^{-3t} \\ y_p'' &= 9At^2 e^{-3t} - 12Ate^{-3t} + 2Ae^{-3t}. \end{aligned}$$

Substituting these into $y'' + 6y' + 9y = 5e^{-3t}$ yields

$$\begin{aligned} 2Ae^{-3t} &= 5e^{-3t} \\ 2A &= 5 \\ A &= \frac{5}{2}. \end{aligned}$$

Therefore, $y_p = (5/2)t^2 e^{-3t}$ is a solution of $y'' + 6y' + 9y = 5e^{-3t}$.

30. Assume that

$$\begin{aligned} y_f'' + py_f' + qy_f &= f(t), \\ y_g'' + py_g' + qy_g &= g(t). \end{aligned}$$

Multiply the first equation by α , the second by β .

$$\begin{aligned} \alpha y_f'' + \alpha p y_f' + \alpha q y_f &= \alpha f(t), \\ \beta y_g'' + \beta p y_g' + \beta q y_g &= \beta g(t). \end{aligned}$$

Add these equations together.

$$\begin{aligned} (\alpha y_f'' + \beta y_g'') + p(\alpha y_f' + \beta y_g') + q(\alpha y_f + \beta y_g) \\ = \alpha f(t) + \beta g(t) \end{aligned}$$

Using the linearity of differentiation, we can write

$$\begin{aligned} (\alpha y_f + \beta y_g)'' + p(\alpha y_f + \beta y_g)' + q(\alpha y_f + \beta y_g) \\ = \alpha f(t) + \beta g(t) \end{aligned}$$

Thus, $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of $y'' + py' + qy = \alpha f(t) + \beta g(t)$.

31. It is easy to see that $y = 1$ is a solution of $y'' + 2y' + 2y = 2$. Next, $z_p = Ae^{i2t}$ has derivatives

$$\begin{aligned} z_p' &= 2iAe^{i2t} \\ z_p'' &= (2i)^2 Ae^{i2t}. \end{aligned}$$

Substituting these into $z'' + 2z' + 2z = e^{i2t}$,

$$\begin{aligned} (2i)^2 Ae^{i2t} + 2(2i)Ae^{i2t} + 2Ae^{i2t} &= e^{i2t} \\ ((2i)^2 + 2(2i) + 2)A &= 1 \\ A &= \frac{1}{-2 + 4i} \\ A &= -\frac{1}{10} - \frac{1}{5}i. \end{aligned}$$

Thus,

$$\begin{aligned} z_p &= \left(-\frac{1}{10} - \frac{1}{5}i\right)e^{i2t} \\ &= \left(-\frac{1}{10} - \frac{1}{5}i\right)(\cos 2t + i \sin 2t) \\ &= \left(-\frac{1}{10} \cos 2t + \frac{1}{5} \sin 2t\right) \\ &\quad + i\left(-\frac{1}{5} \cos 2t - \frac{1}{10} \sin 2t\right). \end{aligned}$$

The real part of this, $y = -(1/10)\cos 2t + (1/5)\sin 2t$, is a solution of $y'' + 2y' + 2y = \cos 2t$. Thus,

$$y = 1 - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t$$

is a solution of $y'' + 2y' + 2y = 2 + \cos 2t$.

32. First, let $y = at + b$, with derivatives $y' = a$ and $y'' = 0$, be a solution of $y'' - y = t$. Substituting,

$$-at - b = t,$$

so $a = -1$ and $b = 0$ and $y(t) = -t$ a solution of $y'' - y = t$. Next, the homogeneous equation $y'' - y = 0$ has characteristic $\lambda^2 - 1 = 0$ and zeros $\lambda_1 = -1$ and $\lambda_2 = 1$, giving the homogeneous solution

$$y_h = C_1e^{-t} + C_2e^t.$$

Because the inhomogeneous part of $y'' - y = e^{-t}$ is also a solution of the homogeneous, use $y_p = Ate^{-t}$ and its derivatives,

$$\begin{aligned}y'_p &= Ae^{-t}(1 - t), \\y''_p &= Ae^{-t}(t - 2)\end{aligned}$$

in the equation $y'' - y = e^{-t}$. Thus,

$$\begin{aligned}Ae^{-t}(t - 2) - Ate^{-t} &= e^{-t} \\[(t - 2) - t]A &= 1 \\-2A &= 1 \\A &= -\frac{1}{2},\end{aligned}$$

and $y(t) = -(1/2)te^{-t}$ is a solution of $y'' - y = e^{-t}$. Therefore,

$$\begin{aligned}y &= -t - \left(-\frac{1}{2}te^{-t}\right) \\y &= -t + \frac{1}{2}te^{-t}\end{aligned}$$

is a solution of $y'' - y = t - e^{-t}$.

33. First, let $y = at + b$, with derivatives $y' = a$ and $y'' = 0$, be a solution of $y'' + 25y = 2 + 3t$. Substituting,

$$25at + 25b = 2 + 3t,$$

which, upon equating coefficients, leads to $a = 2/25$ and $b = 3/25$ and the particular solution $y = (2/25)t + 3/25$. Next, the homogeneous equation $z'' + 25z = 0$ has characteristic $\lambda^2 + 25 = 0$ and zeros $\lambda = \pm 5i$, giving the complex solution

$$z = C_1e^{5it} + C_2e^{-5it}.$$

Because $z = e^{5it}$ is a solution of the homogeneous equation, we try $z = Ate^{5it}$ as a solution of $z'' + 25z = e^{5it}$. The derivatives

$$\begin{aligned}z' &= (5i)Ate^{5it} + Ae^{5it} \\z'' &= (5i)^2Ate^{5it} + 2(5i)Ae^{5it},\end{aligned}$$

when substituted in $z'' + 25z = e^{5it}$, provide

$$\begin{aligned}(5i)^2Ate^{5it} + 2(5i)Ae^{5it} + 25Ate^{5it} &= e^{5it} \\10iA &= 1 \\A &= -\frac{1}{10}i.\end{aligned}$$

Thus,

$$z = -\frac{1}{10}ite^{5it} = -\frac{1}{10}it(\cos 5t + i\sin 5t).$$

The real part of this solution, $y = (1/10)t \sin 5t$, is a solution of $y'' + 25y = \cos 5t$. Thus,

$$y = (2/25)t + 3/25 + (1/10)t \sin 5t$$

is a solution of $y'' + 25y = 2 + 3t + \cos 5t$.

34. It is easy to see that $y = 3$ is a solution of $y'' + 2y' + y = 3$. Next, the homogeneous equation $y'' + 2y' + y = 0$ has characteristic equation $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$ and homogeneous solution

$$y_h = (C_1 + C_2t)e^{-t}.$$

Note that $y = Ae^{-t}$ and $y = Ate^{-t}$ are solutions of the homogeneous equation, so substitute $y = At^2e^{-t}$ and its derivatives

$$\begin{aligned}y'_p &= Ate^{-t}(2 - t) \\y''_p &= Ae^{-t}(t^2 - 4t + 2)\end{aligned}$$

in the equation $y'' + 2y' + y = e^{-t}$.

$$\begin{aligned}Ae^{-t}(t^2 - 4t + 2) + 2Ate^{-t}(2 - t) + At^2e^{-t} \\= e^{-t},\end{aligned}$$

or $[(t^2 - 4t + 2) + 2t(2 - t) + t^2]A = 1$. Hence $A = 1/2$ and $y = (1/2)t^2e^{-t}$ is a solution of $y'' + 2y' + y = e^{-t}$. Therefore,

$$y = 3 - \frac{1}{2}t^2e^{-t}$$

is a solution of $y'' + 2y' + y = 3 - e^{-t}$.

35. Substitute $z = Ae^{i2t}$ and its derivatives

$$\begin{aligned} z' &= (2i)Ae^{i2t} \\ z'' &= (2i)^2Ae^{i2t} \end{aligned}$$

into $z'' + 4z' + 3z = e^{i2t}$.

$$\begin{aligned} (2i)^2Ae^{i2t} + 4(2i)Ae^{i2t} + 3Ae^{i2t} &= e^{i2t} \\ ((2i)^2 + 4(2i) + 3)A &= 1 \\ A &= -\frac{1}{65} - \frac{8}{65}i. \end{aligned}$$

Thus,

$$\begin{aligned} z &= \left(-\frac{1}{65} - \frac{8}{65}i\right)e^{i2t} \\ &= \left(-\frac{1}{65} - \frac{8}{65}i\right)(\cos 2t + i \sin 2t) \\ &= \left(-\frac{1}{65} \cos 2t + \frac{8}{65} \sin 2t\right) \\ &\quad + i\left(-\frac{8}{65} \cos 2t - \frac{1}{65} \sin 2t\right). \end{aligned}$$

Thus, $y = -(1/65)\cos 2t + (8/65)\sin 2t$ is a solution of $y'' + 4y' + 3y = \cos 2t$ and $y = -(8/65)\cos 2t - (1/65)\sin 2t$ is a solution of $y'' + 4y' + 3y = \sin 2t$. This means that

$$\begin{aligned} y &= \left(-\frac{1}{65} \cos 2t + \frac{8}{65} \sin 2t\right) \\ &\quad + 3\left(-\frac{8}{65} \cos 2t - \frac{1}{65} \sin 2t\right) \\ &= -\frac{5}{13} \cos 2t + \frac{1}{13} \sin 2t \end{aligned}$$

is a solution of $y'' + 4y' + 3y = \cos 2t + 3 \sin 2t$.

36. Substitute $z = Ae^{it}$ and its derivatives,

$$\begin{aligned} z' &= iAe^{it} \\ z'' &= i^2Ae^{it} \end{aligned}$$

in $z'' + 2z' + 2z = e^{it}$.

$$\begin{aligned} i^2Ae^{it} + 2iAe^{it} + 2Ae^{it} &= e^{it} \\ (i^2 + 2i + 2)A &= 1 \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{1 + 2i} \\ A &= \frac{1}{5} - \frac{2}{5}i \end{aligned}$$

Thus,

$$\begin{aligned} z &= \left(\frac{1}{5} - \frac{2}{5}i\right)e^{it} \\ z &= \left(\frac{1}{5} - \frac{2}{5}i\right)(\cos t + i \sin t) \end{aligned}$$

is a solution of $z'' + 2z' + 2z = e^{it}$. The real and imaginary parts,

$$\begin{aligned} y_1(t) &= \frac{1}{5} \cos t + \frac{2}{5} \sin t \quad \text{and} \\ y_2(t) &= -\frac{2}{5} \cos t + \frac{1}{5} \sin t, \end{aligned}$$

are solutions of

$$\begin{aligned} y_1'' + 2y_1' + 2y_1 &= \cos t \quad \text{and} \\ y_2'' + 2y_2' + 2y_2 &= \sin t, \end{aligned}$$

respectively. Therefore,

$$\begin{aligned} y &= 3\left(\frac{1}{5} \cos t + \frac{2}{5} \sin t\right) - \left(-\frac{1}{5} \cos t + \frac{1}{5} \sin t\right) \\ y &= \cos t + \sin t \end{aligned}$$

is a solution of $y'' + 2y' + 2y = 3 \cos t - \sin t$.

37. The homogeneous equation $y'' + 4y' + 4y = 0$ has characteristic equation $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$ with repeated zero $\lambda = -2$ and solution $y = (C_1 + C_2t)e^{-2t}$. Thus, both e^{-2t} and te^{-2t} are solutions of the homogeneous equation. Thus, try $y = At^2e^{-2t}$ as a particular solution of $y'' + 4y' + 4y = e^{-2t}$.

$$\begin{aligned} y' &= -2At^2e^{-2t} + 2Ate^{-2t} \\ y'' &= 4At^2e^{-2t} - 8Ate^{-2t} + 2Ae^{-2t} \end{aligned}$$

Substitute y and its derivatives in $y'' + 4y' + 4y = e^{-2t}$ to get

$$2Ae^{-2t} = e^{-2t}$$

$$2A = 1$$

$$A = \frac{1}{2}.$$

Thus, $y = (1/2)t^2e^{-2t}$ is a solution of $y'' + 4y' + 4y = e^{-2t}$. Next, let $z = Ae^{i2t}$ and

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2Ae^{i2t}.$$

Substituting in $z'' + 4z' + 4z = e^{i2t}$,

$$(2i)^2Ae^{i2t} + 4(2i)Ae^{i2t} + 4Ae^{i2t} = e^{i2t}$$

$$((2i)^2 + 4(2i) + 4)A = 1$$

$$A = -\frac{1}{8}i,$$

so

$$z = -\frac{1}{8}ie^{i2t} = -\frac{1}{8}i(\cos 2t + i \sin 2t).$$

The imaginary part of this solution, $y = -(1/8)\cos 2t$, is a solution of $y'' + 4y' + 4y = \sin 2t$. Thus,

$$y = \frac{1}{2}t^2e^{-2t} - \frac{1}{8}\cos 2t$$

is a solution of $y'' + 4y' + 4y = e^{-2t} + \sin 2t$.

38. Substitute $y = Ae^{-4t}$ and its derivatives

$$y' = -4Ae^{-4t}$$

$$y'' = 16Ae^{-4t}$$

in $y'' + 16y = e^{-4t}$.

$$16Ae^{-4t} + 16Ae^{-4t} = e^{-4t}$$

$$32A = 1$$

$$A = \frac{1}{32}$$

Thus, $y = (1/32)e^{-4t}$ is a solution of $y'' + 16y = e^{-4t}$. The homogeneous equation $y'' + 16y = 0$ has characteristic $\lambda^2 + 16 = 0$ with zeros $\pm 4i$. Hence,

the inhomogeneous part of $z'' + 16z = e^{i4t}$ is also a solution of the homogeneous equation

$$z_h = C_1e^{4it} + C_2e^{-4it}.$$

Substitute $z = At e^{i4t}$ and its derivatives

$$z' = Ae^{i4t}(1 + 4it)$$

$$z'' = 8Ae^{i4t}(1 - 2t)$$

into $z'' + 16z = e^{i4t}$.

$$8e^{i4t}(i - 2t) + 16At e^{i4t} = e^{i4t}$$

$$[8(i - 2t) + 16t]A = 1$$

$$8iA = 1$$

$$A = -\frac{1}{8}i$$

Thus,

$$z = -\frac{1}{8}it e^{i4t}$$

$$= -\frac{1}{8}it(\cos 4t + i \sin 4t).$$

The imaginary part of this solution,

$$y = -\frac{1}{8}t \cos 4t$$

is a solution of $y'' + 16y = \sin 4t$. Thus,

$$y = \frac{1}{32}e^{-4t} + 3\left(-\frac{1}{8}t \cos 4t\right)$$

$$y = \frac{1}{32}e^{-4t} - \frac{3}{8}t \cos 4t$$

is a solution of $y'' + 16y = e^{-4t} = e^{-4t} + 3 \sin 4t$.

39. If $y = (at + b)e^{-4t}$, then

$$y' = e^{-4t}(a - 4at - 4b)$$

$$y'' = e^{-4t}(-8a + 16at + 16b),$$

and substituting in $y'' + 3y' + 2y = te^{-4t}$ gives

$$(-8a + 16at + 16b) + 3(a - 4at - 4b)$$

$$+ 2(at + b)$$

$$= t$$

or $6at + (-5a + 6b) = t$. Comparing coefficients, $6a = 1$ and $-5a + 6b = 0$. This gives $a = 1/6$ and $b = 5/36$ and $y = ((1/6)t + 5/36)e^{-4t}$ as a solution of $y'' + 3y' + 2y = te^{-4t}$.

40. Substitute
- $y = (at + b)e^{-3t}$
- and its derivatives

$$y' = e^{-3t}((a - 3b) - 3at)$$

$$y'' = 3e^{-3t}((-2a + 3b) + 3at)$$

into $y'' - 3y' + 2y = te^{-3t}$.

$$\begin{aligned} & 3e^{-3t}((-2a + 3b) + 3at) \\ & - 3e^{-3t}((a - 3b) - 3at) \\ & + 2e^{-3t}(at + b) \\ & = te^{-3t} \end{aligned}$$

or $20at + (-9a + 20b) = t$ Comparing coefficients,

$$20a = 1$$

$$-9a + 20b = 0,$$

so $a = 1/20$ and $b = 9/400$ and the solution is

$$y = \left(\frac{1}{20}t + \frac{9}{400}\right)e^{-3t}.$$

41. Let
- $y = (at^2 + bt + c)e^{-2t}$
- . Then,

$$y' = e^{-2t}(-2at^2 + (2a - 2b)t + (b - 2c))$$

$$y'' = e^{-2t}(4at^2 + (-8a + 4b)t + (2a - 4b + 4c)).$$

Substituting in $y'' + 2y' + y = t^2e^{-2t}$,

$$\begin{aligned} & (4at^2 + (-8a + 4b)t + (2a - 4b + 4c)) \\ & + 2(-2at^2 + (2a - 2b)t + (b - 2c)) \\ & + (at^2 + bt + c) \\ & = t^2, \end{aligned}$$

or $at^2 + (-4a + b)t + (2a - 2b + c) = t^2$. Comparing coefficients,

$$a = 1$$

$$-4a + b = 0$$

$$2a - 2b + c = 0$$

and $a = 1$, $b = 4$, and $c = 6$. Thus, $y = (t^2 + 4t + 6)e^{-2t}$ is a solution of $y'' + 2y' + y = t^2e^{-2t}$.

42. The homogeneous equation
- $y'' + 5y' + 4y = 0$
- has characteristic equation
- $\lambda^2 + 5\lambda + 4 = (\lambda + 4)(\lambda + 1) = 0$
- and solution

$$y_h = C_1e^{-4t} + C_2e^{-t}.$$

Thus, instead of $y = (at + b)e^{-t}$, we will try $y = t(at + b)e^{-t}$. After some work, the derivatives are found.

$$y' = e^{-t}(-at^2 + (2a - b)t + b)$$

$$y'' = e^{-t}(at^2 + (-4a + b)t + (2a - 2b))$$

Substitute these into the equation $y'' + 5y' + 4y = te^{-t}$. After some computation,

$$6at + (2a + 3b) = t.$$

Comparing coefficients,

$$6a = 1$$

$$2a + 3b = 0,$$

so $a = 1/6$ and $b = -1/9$. Thus,

$$y = t\left(\frac{1}{6}t - \frac{1}{9}\right)e^{-t}$$

is a solution of $y'' + 5y' + 4y = te^{-t}$.

43. The homogeneous equation
- $y'' + 3y' + 2y = 0$
- has characteristic equation
- $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$
- and solution

$$y = C_1e^{-t} + C_2e^{-2t}.$$

Thus, instead of $y = (at^2 + bt + c)e^{-2t}$, we will try $y = t(at^2 + bt + c)e^{-2t}$. After some work, the derivatives are found.

$$y' = e^{-2t}(-2at^3 + (3a - 2b)t^2 + (2b - 2c)t + c)$$

$$y'' = e^{-2t}(4at^3 + (-12a + 4b)t^2$$

$$+ (6a - 8b + 4c)t + (2b - 4c))$$

Substitute these into the differential equation $y'' + 3y' + 2y = t^2e^{-2t}$. After some computation,

$$-3at^2 + (6a - 2b)t + (2b - c) = t^2.$$

Comparing coefficients,

$$-3a = 1$$

$$6a - 2b = 0$$

$$2b - c = 0$$

leading to $a = -1/3$, $b = -1$, and $c = -2$. Thus,

$$y = t\left(-\frac{1}{3}t^2 - t - 2\right)e^{-2t}$$

is a solution of $y'' + 3y' + 2y = t^2e^{-2t}$.

44. Substitute $y_p = e^{-2t}(a \cos t + b \sin t)$ and its derivatives

$$y'_p = e^{-2t}((-2a + b) \cos t + (-a - 2b) \sin t)$$

$$y''_p = e^{-2t}((3a - 4b) \cos t + (4a + 3b) \sin t)$$

into the equation $y'' + 2y' + 2y = e^{-2t} \sin t$.

$$\begin{aligned} & e^{-2t}((3a - 4b) \cos t + (4a + 3b) \sin t) \\ & + 2e^{-2t}((-2a + b) \cos t + (-a - 2b) \sin t) \\ & + 2e^{-2t}(a \cos t + b \sin t) \\ & = e^{-2t} \sin t, \end{aligned}$$

or

$$(a - 2b) \cos t + (2a + b) \sin t = 0 \cos t + 1 \sin t$$

Comparing coefficients,

$$a - 2b = 0$$

$$2a + b = 1,$$

so $a = 2/5$ and $b = 1/5$ and the solution is

$$y_p = e^{-2t} \left(\frac{2}{5} \cos t + \frac{1}{5} \sin t \right).$$

45. If $z(t) = x(t) + iy(t)$ is a solution of $z'' + pz' + qz = Ae^{(a+bi)t}$, then

$$(x'' + iy'') + p(x' + iy') + q(x + iy) = Ae^{at}e^{ibt}$$

or

$$\begin{aligned} & (x'' + px' + qx) + i(y'' + py' + qy) \\ & = Ae^{at} \cos bt + iAe^{at} \sin bt. \end{aligned}$$

Equating real and imaginary parts,

$$x'' + px' + qx = Ae^{at} \cos bt$$

$$y'' + py' + qy = Ae^{at} \sin bt.$$

46. Substitute $z_p = Ae^{(-2+i)t}$ and its derivatives

$$z'_p = A(-2 + i)e^{(-2+i)t}$$

$$z''_p = A(-2 + i)^2 e^{(-2+i)t}$$

into $z'' + 2z' + 2z = e^{(-2+i)t}$.

$$\begin{aligned} & A(-2 + i)^2 e^{(-2+i)t} + 2A(-2 + i)e^{(-2+i)t} \\ & + 2Ae^{(-2+i)t} \\ & = e^{(-2+i)t} \end{aligned}$$

or $[(-2 + i)^2 + 2(-2 + i) + 2]A = 1$. Solving for A we get $A = (1 + 2i)/5$. Thus,

$$z_p = \left(\frac{1}{5} + \frac{2}{5}i \right) e^{(-2+i)t}$$

$$z_p = e^{-2t} \left(\frac{1}{5} + \frac{2}{5}i \right) (\cos t + i \sin t),$$

and the imaginary parts of z_p ,

$$y = e^{-2t} \left(\frac{2}{5} \cos t + \frac{1}{5} \sin t \right),$$

is a solution of $y'' + 2y' = e^{-2t} \sin t$.

47. If $z(t) = x(t) = iy(t)$ is a solution of $z'' + z' + z = te^{it}$, then

$$\begin{aligned} & (x'' + iy'') + (x' + iy') + (x + iy) \\ & = t(\cos t + i \sin t) \\ & (x'' + x' + x) + i(y'' + y' + y) \\ & = t \cos t + it \sin t. \end{aligned}$$

Comparing imaginary parts,

$$y'' + y' + y = t \sin t.$$

Thus, if $z = (at + b)e^{it}$ and its derivatives,

$$z' = e^{it}(a + i(at + b))$$

$$z'' = e^{it}(-at - b + 2ai)$$

are substituted into $z'' + z' + z = te^{it}$, then

$$\begin{aligned} & (-at - b + 2ai) + (a + i(at + b)) \\ & + (at + b) \\ & = t, \end{aligned}$$

or $iat + ((1 + 2i)a + ib) = t$. Thus,

$$ia = 1$$

$$(1 + 2i)a + ib = 0,$$

and $a = -i$ and $b = 1 + 2i$. Thus, a particular solution of $z'' + z' + z = te^{it}$ is

$$\begin{aligned} z &= (-it + (1 + 2i))e^{it} \\ &= (1 + i(2 - t))(\cos t + i \sin t) \\ &= (\cos t - (2 - t) \sin t) + i((2 - t) \cos t + \sin t). \end{aligned}$$

Therefore, the imaginary part, $y = (2 - t) \cos t + \sin t$, is a solution of $y'' + y' + y = t \sin t$.

48. Suppose that $y_f(t)$ is a solution to the linear equation $y_f'' + py_f' + qy_f = f(t)$ and $y_g(t)$ is a solution to

$y_g'' + py_g' + qy_g = g(t)$. If $y(t) = \alpha y_f(t) + \beta y_g(t)$, then

$$\begin{aligned} y'' + py' + qy &= (\alpha y_f(t) + \beta y_g(t))'' \\ &\quad + p(\alpha y_f(t) + \beta y_g(t))' \\ &\quad + q(\alpha y_f(t) + \beta y_g(t)) \\ &= \alpha[y_f'' + py_f' + qy_f] \\ &\quad + \beta[y_g'' + py_g' + qy_g] \\ &= \alpha f(t) + \beta g(t). \end{aligned}$$

————— × —————

Section 4.6. Variation of Parameters

1. The homogeneous equation $y'' + 9y = 0$ has $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ as a fundamental set of solutions. The Wronskian is

$$\begin{aligned} W(\cos 3t, \sin 3t) &= \begin{vmatrix} \cos 3t & \sin 3t \\ -3 \sin 3t & 3 \cos 3t \end{vmatrix} \\ &= 3 \cos^2 3t + 3 \sin^2 3t \\ &= 3. \end{aligned}$$

Form the solution

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned} v_1' &= \frac{-y_2 g(t)}{W(y_1, y_2)} \\ &= \frac{-\sin 3t \tan 3t}{3} \\ &= -\frac{1}{3} \cdot \frac{\sin^2 3t}{\cos 3t} \\ &= -\frac{1}{3} \cdot \frac{1 - \cos^2 3t}{\cos 3t} \\ &= -\frac{1}{3} (\sec 3t - \cos 3t). \end{aligned}$$

Thus,

$$v_1 = -\frac{1}{9} \ln |\sec 3t + \tan 3t| + \frac{1}{9} \sin 3t.$$

Secondly,

$$\begin{aligned} v_2' &= \frac{y_1 g(t)}{W(y_1, y_2)} \\ &= \frac{\cos 3t \tan 3t}{3} \\ &= \frac{1}{3} \sin 3t. \end{aligned}$$

Thus,

$$v_2 = -\frac{1}{9} \cos 3t.$$

Inserting these results in $y_p = v_1 y_1 + v_2 y_2$,

$$\begin{aligned} y_p &= \left[-\frac{1}{9} \ln |\sec 3t + \tan 3t| + \frac{1}{9} \sin 3t \right] \cos 3t \\ &\quad + \left(-\frac{1}{9} \cos 3t \right) \sin 3t \\ &= -\frac{1}{9} \cos 3t \ln |\sec 3t + \tan 3t| \end{aligned}$$

is the particular solution we seek.

2. A fundamental system of solutions to the homogeneous equation is $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$. We look for a solution of the form

$$y = v_1 y_1 + v_2 y_2 = v_1 \cos 2t + v_2 \sin 2t.$$

Differentiating we get

$$y' = v_1' \cos 2t + v_2' \sin 2t - 2v_1 \sin 2t + 2v_2 \cos 2t.$$

To simplify future calculations we set

$$v_1' \cos 2t + v_2' \sin 2t = 0,$$

so $y' = -2v_1 \cos 2t + 2v_2 \sin 2t$, and

$$y'' = -2v_1' \cos 2t + 2v_2' \sin 2t \\ - 4[v_1 \cos 2t + v_2 \sin 2t].$$

Adding, we get

$$y'' + 4y = -2v_1' \cos 2t + 2v_2' \sin 2t \\ = \sec 2t.$$

We must solve the system

$$v_1' \cos 2t + v_2' \sin 2t = 0, \\ -2v_1' \cos 2t + 2v_2' \sin 2t = \sec 2t.$$

The solutions are

$$v_1' = -\frac{1}{2} \tan 2t \quad \text{and} \quad v_2' = \frac{1}{2}.$$

Integrating we get

$$v_1(t) = \frac{1}{4} \ln(\cos 2t) \quad \text{and} \quad v_2(t) = \frac{t}{2}.$$

Thus the solution is

$$y(t) = v_1 \cos 2t + v_2 \sin 2t \\ = \frac{1}{4} \cos 2t \cdot \ln(\cos 2t) + \frac{t}{2} \sin 2t.$$

3. The homogeneous equation $y'' - y = 0$ has $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ as a fundamental set of solutions. The Wronskian is

$$W(e^t, e^{-t}) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} \\ = -2.$$

Form the solution

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are to be determined. Indeed,

$$v_1' = \frac{-y_2 g(t)}{W(y_1, y_2)} \\ = \frac{-e^{-t}(t+3)}{-2} \\ = \frac{1}{2} e^{-t}(t+3).$$

Integrating by parts,

$$v_1 = -\frac{1}{2} e^{-t}(t+3) - \frac{1}{2} e^{-t}.$$

Secondly,

$$v_2' = \frac{y_1 g(t)}{W(y_1, y_2)} \\ = \frac{e^t(t+3)}{-2} \\ = -\frac{1}{2} e^t(t+3).$$

Integrating by parts,

$$v_2 = -\frac{1}{2} e^t(t+3) + \frac{1}{2} e^t.$$

Inserting these results in $y_p = v_1 y_1 + v_2 y_2$,

$$y_p = \left(-\frac{1}{2} e^{-t}(t+3) - \frac{1}{2} e^{-t} \right) e^t \\ + \left(-\frac{1}{2} e^t(t+3) + \frac{1}{2} e^t \right) e^{-t} \\ = -\frac{1}{2}(t+3) - \frac{1}{2} - \frac{1}{2}(t+3) + \frac{1}{2} \\ = -(t+3)$$

is the particular solution we seek.

4. A fundamental system of solutions to the homogeneous equation is $x_1(t) = e^{-t}$ and $x_2(t) = e^{3t}$. We look for a solution of the form

$$x(t) = v_1(t)x_1(t) + v_2(t)x_2(t) = v_1(t)e^{-t} + v_2(t)e^{3t}.$$

Differentiating we get

$$x' = v_1' e^{-t} + v_2' e^{3t} - v_1 e^{-t} + 3v_2 e^{3t}.$$

To simplify future calculations we set

$$v_1' e^{-t} + v_2' e^{3t} = 0.$$

Then $x' = -v_1 e^{-t} + 3v_2 e^{3t}$, and

$$x'' = -v_1' e^{-t} + 3v_2' e^{3t} + v_1 e^{-t} + 9v_2 e^{3t}.$$

Adding, we get

$$\begin{aligned}x'' - 2x' - 3x &= -v_1'e^{-t} + 3v_2'e^{3t} \\ &= 4e^{3t}.\end{aligned}$$

Thus we must solve the system

$$\begin{aligned}v_1'e^{-t} + v_2'e^{3t} &= 0 \\ -v_1'e^{-t} + 3v_2'e^{3t} &= 4e^{3t}.\end{aligned}$$

The solutions are

$$v_1' = -e^{4t} \quad \text{and} \quad v_2' = 1.$$

Integrating we get

$$v_1(t) = -\frac{1}{4}e^{4t} \quad \text{and} \quad v_2(t) = t.$$

Thus the solution is

$$x(t) = -\frac{1}{4}e^{3t} + te^{3t}.$$

5. The homogeneous equation $y'' - 2y' + y = 0$ has $y_1(t) = e^t$ and $y_2(t) = te^t$ as a fundamental set of solutions. The Wronskian is

$$\begin{aligned}W(e^t, te^t) &= \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} \\ &= e^{2t}.\end{aligned}$$

Form the solution

$$y_p = v_1y_1 + v_2y_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned}v_1' &= \frac{-y_2g(t)}{W(y_1, y_2)} \\ &= \frac{(-te^t)e^t}{e^{2t}} \\ &= -t.\end{aligned}$$

Thus,

$$v_1 = -\frac{1}{2}t^2.$$

Secondly,

$$\begin{aligned}v_2' &= \frac{y_1g(t)}{W(y_1, y_2)} \\ &= \frac{e^t e^t}{e^{2t}} \\ &= 1\end{aligned}$$

Thus,

$$v_2 = t.$$

Inserting these results in $y_p = v_1y_1 + v_2y_2$,

$$\begin{aligned}y_p &= -\frac{1}{2}t^2e^t + t(te^t) \\ &= -\frac{1}{2}t^2e^t + t^2e^t \\ &= \frac{1}{2}t^2e^t.\end{aligned}$$

is the particular solution we seek.

6. A fundamental system of solutions to the homogeneous equation is $x_1(t) = e^{2t}$ and $x_2(t) = te^{2t}$. We look for a solution of the form

$$x(t) = v_1(t)x_1(t) + v_2(t)x_2(t) = v_1(t)e^{2t} + v_2(t)te^{2t}.$$

Differentiating we get

$$x' = v_1'e^{2t} + v_2'te^{2t} + 2v_1e^{2t} + v_2(1+2t)e^{2t}.$$

To simplify future calculations we set

$$v_1'e^{2t} + v_2'te^{2t} = 0.$$

Then $x' = 2v_1e^{2t} + v_2(1+2t)e^{2t}$, and

$$x'' = 2v_1'e^{2t} + v_2'(1+2t)e^{2t} + 4v_1e^{2t} + v_2(4+4t)e^{2t}.$$

Adding, we get

$$\begin{aligned}x'' - 4x' + 4x &= 2v_1'e^{2t} + v_2'(1+2t)e^{2t} \\ &= e^{2t}.\end{aligned}$$

Thus we must solve the system

$$\begin{aligned}v_1'e^{2t} + v_2'te^{2t} &= 0, \\ 2v_1'e^{2t} + v_2'(1+2t)e^{2t} &= e^{2t}.\end{aligned}$$

The solutions are

$$v_1' = -t \quad \text{and} \quad v_2' = 1.$$

Integrating we get

$$v_1(t) = -t^2/2 \quad \text{and} \quad v_2(t) = t.$$

Thus the solution is

$$x(t) = \frac{1}{2}t^2e^{2t}.$$

7. The homogeneous equation $x'' + x = 0$ has $x_1(t) = \cos t$ and $x_2(t) = \sin t$ as a fundamental set of solutions. The Wronskian is

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.$$

Form the solution

$$x_p = v_1 x_1 + v_2 x_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned} v_1' &= \frac{-\sin t \tan^2 t}{1} \\ &= -\sin t (\sec^2 t - 1) \\ &= -\sec t \tan t + \sin t. \end{aligned}$$

Thus,

$$v_1 = -\sec t - \cos t.$$

Secondly,

$$\begin{aligned} v_2' &= \frac{\cos t \tan^2 t}{1} \\ &= \cos t (\sec^2 t - 1) \\ &= \sec t - \cos t. \end{aligned}$$

Thus,

$$v_2 = \ln |\sec t + \tan t| - \sin t.$$

Inserting these results in $x_p = v_1 x_1 + v_2 x_2$,

$$\begin{aligned} x_p &= (-\sec t - \cos t) \cos t \\ &\quad + (\ln |\sec t + \tan t| - \sin t) \sin t \\ &= -1 - \cos^2 t + \sin t \ln |\sec t + \tan t| - \sin^2 t \\ &= -2 + \sin t \ln |\sec t + \tan t| \end{aligned}$$

is the particular solution we seek.

8. A fundamental system of solutions to the homogeneous equation is $x_1(t) = \cos t$ and $x_2(t) = \sin t$. We look for a solution of the form

$$\begin{aligned} x(t) &= v_1(t)x_1(t) + v_2(t)x_2(t) \\ &= v_1(t)\cos t + v_2(t)\sin t. \end{aligned}$$

Differentiating we get

$$x' = v_1' \cos t + v_2' \sin t - v_1 \sin t + v_2 \cos t.$$

To simplify future calculations we set

$$v_1' \cos t + v_2' \sin t = 0.$$

Then $x' = -v_1 \sin t + v_2 \cos t$, and

$$x'' = -v_1' \sin t + v_2' \cos t - v_1 \cos t - v_2 \sin t.$$

Adding, we get

$$\begin{aligned} x'' + x &= -v_1' \sin t + v_2' \cos t \\ &= \sec^2 t. \end{aligned}$$

Thus we must solve the system

$$\begin{aligned} v_1' \cos t + v_2' \sin t &= 0, \\ -v_1' \sin t + v_2' \cos t &= \sec^2 t. \end{aligned}$$

The solutions are

$$v_1' = -\sin t \sec^2 t \quad \text{and} \quad v_2' = \sec t.$$

Integrating we get

$$v_1(t) = -\sec t \quad \text{and} \quad v_2(t) = \ln(\sec t + \tan t).$$

Thus the solution is

$$\begin{aligned} x(t) &= v_1(t)\cos t + v_2(t)\sin t \\ &= -\sec t \cos t + \sin t \cdot \ln(\sec t + \tan t) \\ &= -1 + \sin t \cdot \ln(\sec t + \tan t). \end{aligned}$$

9. A fundamental set of solutions to the homogeneous equation is $x_1(t) = \cos t$ and $x_2(t) = \sin t$. We look for a solution of the form

$$\begin{aligned} x(t) &= v_1(t)x_1(t) + v_2(t)x_2(t) \\ &= v_1(t)\cos t + v_2(t)\sin t. \end{aligned}$$

Differentiating we get

$$x' = v_1' \cos t + v_2' \sin t - v_1 \sin t + v_2 \cos t.$$

To simplify future calculations we set

$$v_1' \cos t + v_2' \sin t = 0.$$

Then $x' = -v_1 \sin t + v_2 \cos t$, and

$$x'' = -v_1' \sin t + v_2' \cos t - v_1 \cos t - v_2 \sin t.$$

Adding, we get

$$\begin{aligned}x'' + x &= -v'_1 \sin t + v'_2 \cos t \\ &= \tan^3 t.\end{aligned}$$

Thus we must solve the system

$$\begin{aligned}v'_1 \cos t + v'_2 \sin t &= 0, \\ -v'_1 \sin t + v'_2 \cos t &= \tan^3 t.\end{aligned}$$

The solutions are

$$v'_1 = -\sin t \tan^3 t \quad \text{and} \quad v'_2 = \cos t \tan^3 t.$$

Integrating we get

$$\begin{aligned}v_1(t) &= -\int \sin t \tan^3 t \, dt \\ &= \int \frac{\sin^4 t}{\cos^3 t} \, dt \\ &= -\int \frac{1 - 2\cos^2 t + \cos^4 t}{\cos^3 t} \, dt \\ &= -\int (\sec^3 t - 2\sec t + \cos t) \, dt.\end{aligned}$$

Using integration by parts,

$$\begin{aligned}\int \sec^3 t \, dt &= \int \sec t \sec^2 t \, dt \\ &= \sec t \tan t - \int \sec t \tan^2 t \, dt.\end{aligned}$$

Using the identity $1 + \tan^2 t = \sec^2 t$ reveals

$$\int \sec^3 t \, dt = \frac{1}{2} \sec t \tan t + \frac{1}{2} \ln |\sec t \tan t|.$$

Thus,

$$\begin{aligned}v_1(t) &= -\frac{1}{2} \sec t \tan t - \frac{1}{2} \ln |\sec t + \tan t| \\ &\quad + 2 \ln |\sec t + \tan t| - \sin t \\ &= -\frac{1}{2} \sec t \tan t + \frac{3}{2} \ln |\sec t + \tan t| - \sin t.\end{aligned}$$

Integrating again,

$$\begin{aligned}v_2(t) &= \int \cos t \tan^3 t \, dt \\ &= \int \frac{\sin^3 t}{\cos^2 t} \, dt \\ &= \int \frac{\sin t (1 - \cos^2 t)}{\cos^2 t} \, dt \\ &= \int \left(\frac{\sin t}{\cos^2 t} - \sin t \right) \, dt \\ &= \int (\sec t \tan t - \sin t) \, dt \\ &= \sec t + \cos t.\end{aligned}$$

Thus, the solution is

$$\begin{aligned}x(t) &= v_1(t) \cos t + v_2(t) \sin t \\ &= \left(-\frac{1}{2} \sec t \tan t + \frac{3}{2} \ln |\sec t + \tan t| \right. \\ &\quad \left. + \tan t - \sin t \right) \cos t \\ &\quad + (\sec t + \cos t) \sin t \\ &= -\frac{1}{2} \tan t + \frac{3}{2} (\cos t) \ln |\sec t + \tan t| + \tan t \\ &= \frac{1}{2} \tan t + \frac{3}{2} (\cos t) (\ln |\sec t + \tan t|).\end{aligned}$$

10. A fundamental set of solutions to the homogeneous equation are $x_1(t) = \cos t$ and $x_2(t) = \sin t$. We will look for a solution of the form

$$\begin{aligned}x(t) &= v_1(t)x_1(t) + v_2(t)x_2(t) \\ &= v_1(t) \cos t + v_2(t) \sin t.\end{aligned}$$

Differentiating, we get

$$x' = v'_1 \cos t + v'_2 \sin t - v_1 \sin t + v_2 \cos t.$$

To simplify future calculations we set

$$v'_1 \cos t + v'_2 \sin t = 0.$$

Then, $x' = -v_1 \sin t + v_2 \cos t$, and

$$x'' = -v'_1 \sin t + v'_2 \cos t - v_1 \cos t - v_2 \sin t.$$

Adding, we get

$$\begin{aligned}x'' + x &= -v'_1 \sin t + v'_2 \cos t \\ &= \sec^3 t.\end{aligned}$$

Thus, we must solve the system.

$$\begin{aligned}v_1' \cos t + v_2' \sin t &= 0 \\ -v_1' \sin t + v_2' \cos t &= \sec^3 t.\end{aligned}$$

The solutions are

$$v_1' = -\sin t \sec^3 t \quad \text{and} \quad v_2' = \sec^2 t.$$

Integrating, we get

$$\begin{aligned}v_1(t) &= -\int \sin t \sec^3 t \, dt \\ &= -\int \frac{\sin t}{\cos^3 t} \, dt \\ &= -\frac{1}{2}(\cos t)^{-2} \\ &= -\frac{1}{2} \sec^2 t.\end{aligned}$$

Integrating again,

$$\begin{aligned}v_2(t) &= \int \sec^2 t \, dt \\ &= \tan t.\end{aligned}$$

Thus, the solution is

$$\begin{aligned}x(t) &= v_1(t) \cos t + v_2(t) \sin t \\ &= \left(-\frac{1}{2} \sec^2 t\right) \cos t + \tan t \sin t \\ &= -\frac{1}{2} \sec t + \tan t \sin t.\end{aligned}$$

11. The homogeneous equation $y'' + y = 0$ has $y_1(t) = \cos t$ and $y_2(t) = \sin t$ as a fundamental set of solutions. The Wronskian is

$$\begin{aligned}W(\cos t, \sin t) &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\ &= 1.\end{aligned}$$

Form the solution

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are to be determined. Indeed,

$$\begin{aligned}v_1' &= \frac{-\sin t(\tan t + \sin t + 1)}{1} \\ &= -\frac{\sin^2 t}{\cos t} - \sin^2 t - \sin t \\ &= \frac{\cos^2 t - 1}{\cos t} - \frac{1 - \cos 2t}{2} - \sin t \\ &= \cos t - \sec t - \frac{1}{2} + \frac{1}{2} \cos 2t - \sin t.\end{aligned}$$

Thus,

$$v_1 = \sin t - \ln |\sec t + \tan t| - \frac{1}{2}t + \frac{1}{4} \sin 2t + \cos t.$$

Secondly,

$$\begin{aligned}v_2' &= \frac{\cos t(\tan t + \sin t + 1)}{1} \\ &= \sin t + \cos t \sin t + \cos t \\ &= \sin t + \frac{1}{2} \sin 2t + \cos t.\end{aligned}$$

Thus,

$$v_2 = -\cos t - \frac{1}{4} \cos 2t + \sin t.$$

Inserting these results in $y_p = v_1 y_1 + v_2 y_2$,

$$\begin{aligned}y_p &= \left(\sin t - \ln |\sec t + \tan t| - \frac{1}{2}t \right. \\ &\quad \left. + \frac{1}{4} \sin 2t + \cos t \right) \cos t \\ &\quad + \left(-\cos t - \frac{1}{4} \cos 2t + \sin t \right) \sin t \\ &= 1 - \frac{1}{2}t \cos t - \cos t \ln |\sec t + \tan t| \\ &\quad + \frac{1}{4}(\sin 2t \cos t - \sin t \cos 2t) \\ &= 1 - \frac{1}{2}t \cos t - \cos t \ln |\sec t + \tan t| \\ &\quad + \frac{1}{4} \sin(2t - t) \\ &= 1 - \frac{1}{2}t \cos t - \cos t \ln |\sec t + \tan t| \\ &\quad + \frac{1}{4} \sin t\end{aligned}$$

is the particular solution we seek.

12. A fundamental system of solutions to the homogeneous equation is $y_1(t) = \cos t$ and $y_2(t) = \sin t$. We look for a solution of the form

$$\begin{aligned} y(t) &= v_1(t)y_1(t) + v_2(t)y_2(t) \\ &= v_1(t)\cos t + v_2(t)\sin t. \end{aligned}$$

Differentiating we get

$$y' = v_1' \cos t + v_2' \sin t - v_1 \sin t + v_2 \cos t.$$

To simplify future calculations we set

$$v_1' \cos t + v_2' \sin t = 0.$$

Then $y' = -v_1 \sin t + v_2 \cos t$, and

$$y'' = -v_1' \sin t + v_2' \cos t - v_1 \cos t - v_2 \sin t.$$

Adding, we get

$$\begin{aligned} y'' + y &= -v_1' \sin t + v_2' \cos t \\ &= \sec t + \cos t - 1. \end{aligned}$$

Thus we must solve the system

$$\begin{aligned} v_1' \cos t + v_2' \sin t &= 0, \\ -v_1' \sin t + v_2' \cos t &= \sec t + \cos t - 1. \end{aligned}$$

The solutions are

$$\begin{aligned} v_1' &= -\tan t - \sin t \cos t + \sin t \quad \text{and} \\ v_2' &= 1 + \cos^2 t - \cos t. \end{aligned}$$

Integrating we get

$$\begin{aligned} v_1(t) &= \ln(\cos t) = \frac{1}{2} \cos^2 t - \cos t \quad \text{and} \\ v_2(t) &= \frac{3}{2}t + \frac{1}{2} \sin t \cos t - \sin t. \end{aligned}$$

Thus the solution is

$$\begin{aligned} y(t) &= v_1(t)\cos t + v_2(t)\sin t \\ &= \cos t \cdot \ln(\cos t) + \frac{1}{2} \cos t + \frac{3}{2} \sin t - 1. \end{aligned}$$

13. The formulae given in the text depend upon the fact that the coefficient of y'' is 1. We start by dividing our equation by t^2 .

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = \frac{1}{t^3}.$$

First, check $y_1(t) = t$ is a solution.

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = (0) + \frac{3}{t}(1) - \frac{3}{t^2}(t) = 0.$$

Check that $y_2(t) = t^{-3}$ is a solution.

$$\begin{aligned} y'' + \frac{3}{t}y' - \frac{3}{t^2}y &= (12t^{-5}) + \frac{3}{t}(-3t^{-4}) - \frac{3}{t^2}(t^{-3}) \\ &= 12t^{-5} - 9t^{-5} - 3t^{-5} \\ &= 0. \end{aligned}$$

Calculate the Wronskian.

$$W(t, t^{-3}) = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}.$$

Next,

$$\begin{aligned} v_1' &= \frac{-y_2 g(t)}{W} \\ &= \frac{-t^{-3}t^{-3}}{-4t^{-3}} \\ &= \frac{1}{4}t^{-3}. \end{aligned}$$

Thus,

$$v_1 = -\frac{1}{8}t^{-2}.$$

Next,

$$\begin{aligned} v_2' &= \frac{y_1 g(t)}{W} \\ &= \frac{t t^{-3}}{-4t^{-3}} \\ &= -\frac{1}{4}t. \end{aligned}$$

Thus,

$$v_2 = -\frac{1}{8}t^2.$$

Form

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= \left(-\frac{1}{8}t^{-2}\right)t + \left(-\frac{1}{8}t^2\right)t^{-3} \\ &= -\frac{1}{4t}. \end{aligned}$$

Thus, the general solution is

$$y(t) = C_1 t + \frac{C_2}{t^3} - \frac{1}{4t}.$$

14. The formulae given in the text depend upon the fact that the coefficient of y'' is 1. We start by dividing both sides of the equation

$$t^2 y''(t) + 3t y'(t) + y(t) = \frac{1}{t}$$

by t^2 .

$$y'' + \frac{3}{t}y' + \frac{1}{t^2}y = \frac{1}{t^3}.$$

First, check $y_1(t) = t^{-1}$ is a solution.

$$\begin{aligned} y'' + \frac{3}{t}y' + \frac{1}{t^2}y &= (2t^{-3}) + \frac{3}{t}(-t^{-2}) + \frac{1}{t^2}(t^{-1}) \\ &= 2t^{-3} - 3t^{-3} + t^{-3} \\ &= 0 \end{aligned}$$

Secondly, $y_2(t) = t^{-1} \ln t$ has derivatives

$$\begin{aligned} y_2'(t) &= t^{-2}(1 - \ln t) \\ y_2''(t) &= t^{-3}(-3 + 2 \ln t). \end{aligned}$$

Thus,

$$\begin{aligned} y'' + \frac{3}{t}y' + \frac{1}{t^2}y &= t^{-3}(-3 + 2 \ln t) \\ &\quad + 3t^{-3}(1 - \ln t) + t^{-3} \ln t \\ &= 0. \end{aligned}$$

Calculate the Wronskian.

$$\begin{aligned} W(t^{-1}, t^{-1} \ln t) &= \begin{vmatrix} t^{-1} & t^{-1} \ln t \\ -t^{-2} & t^{-2}(1 - \ln t) \end{vmatrix} \\ &= t^{-3}(1 - \ln t) + t^{-3} \ln t \\ &= t^{-3} \end{aligned}$$

Next,

$$\begin{aligned} v_1' &= \frac{-y_2 g(t)}{W} \\ &= -\frac{(t^{-1} \ln t)(t^{-3})}{t^{-3}} \\ &= -t^{-1} \ln t. \end{aligned}$$

Thus, $v_1 = -(1/2)(\ln t)^2$. Next,

$$\begin{aligned} v_2' &= \frac{y_1 g(t)}{W} \\ &= \frac{(t^{-1})(t^{-3})}{t^{-3}} \\ &= t^{-1}. \end{aligned}$$

Thus,

$$v_2 = \ln t.$$

Form

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= \left[-\frac{1}{2}(\ln t)^2 \right] (t^{-1}) + (\ln t)(t^{-1} \ln t) \\ &= \frac{1}{2} t^{-1} (\ln t)^2. \end{aligned}$$

Thus, the general solution is

$$y(t) = C_1 t^{-1} + C_2 t^{-1} \ln t + \frac{1}{2} t^{-1} (\ln t)^2,$$

provided, of course, that $t > 0$.

Section 4.7. Forced Harmonic Motion

1. (a) If
- $x_p = a \cos \omega t$
- , then

$$x_p' = -a\omega \sin \omega t$$

$$x_p'' = -a\omega^2 \cos \omega t.$$

Substituting these in $x'' + \omega_0^2 x = A \cos \omega t$,

$$-a\omega^2 \cos \omega t + \omega_0^2 a \cos \omega t = A \cos \omega t$$

$$a(\omega_0^2 - \omega^2) \cos \omega t = A \cos \omega t.$$

Comparing coefficients, $a = A/(\omega_0^2 - \omega^2)$ and

$$x_p = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

- (b) Substitute $z = ae^{i\omega t}$ into $z'' + \omega_0^2 z = Ae^{i\omega t}$ to get

$$ae^{i\omega t} = Ae^{i\omega t}$$

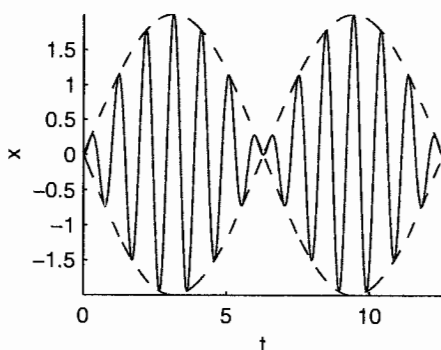
$$(\omega_0^2 - \omega^2)z = Ae^{i\omega t}$$

$$z = \frac{A}{\omega_0^2 - \omega^2} e^{i\omega t}.$$

The real part of this is the solution we seek.

$$x_p = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t$$

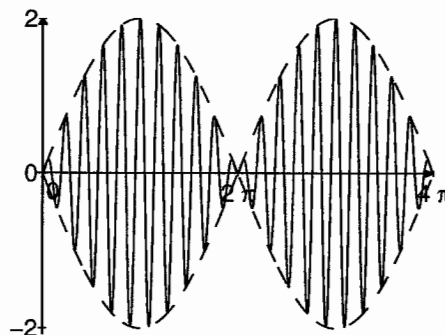
2. The envelope, $y = \pm 2 \sin(t/2)$, is shown as a dashed line in the figure.



3. The mean frequency is $\bar{\omega} = 21/2$ and the half difference is $\delta = 1/2$. Thus,

$$\begin{aligned} \cos 10t - \cos 11t &= \cos \left(\frac{21}{2} - \frac{1}{2} \right) t \\ &\quad - \cos \left(\frac{21}{2} + \frac{1}{2} \right) t, \\ &= 2 \sin \frac{1}{2} t \sin \frac{21}{2} t. \end{aligned}$$

The envelope is $y(t) = \pm 2 \sin(1/2)t$. The graph of the envelope is dashed in the following figure.

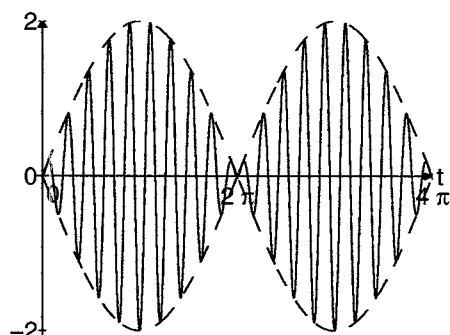


4. The mean frequency is $\bar{\omega} = 19/2$ and the half difference is $\delta = 1/2$. Thus,

$$\begin{aligned} \cos 9t - \cos 10t &= \cos \left(\frac{19}{2} - \frac{1}{2} \right) t \\ &\quad - \cos \left(\frac{19}{2} + \frac{1}{2} \right) t, \\ &= 2 \sin \frac{1}{2} t \sin \frac{19}{2} t. \end{aligned}$$

The envelope is $y(t) = \pm 2 \sin(1/2)t$. The graph of

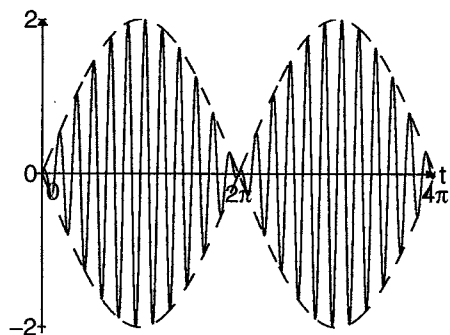
the envelope is dashed in the following figure.



5. The mean frequency is $\bar{\omega} = 23/2$ and the half difference is $\delta = 1/2$. Thus,

$$\begin{aligned}\sin 12t - \sin 11t &= \sin\left(\frac{23}{2} + \frac{1}{2}\right)t \\ &\quad - \sin\left(\frac{23}{2} - \frac{1}{2}\right)t, \\ &= 2 \sin \frac{1}{2}t \cos \frac{23}{2}t.\end{aligned}$$

The envelope is $y(t) = \pm 2 \sin(1/2)t$. The graph of the envelope is dashed in the following figure.

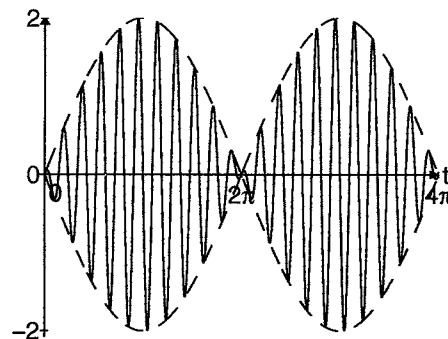


6. The mean frequency is $\bar{\omega} = 21/2$ and the half dif-

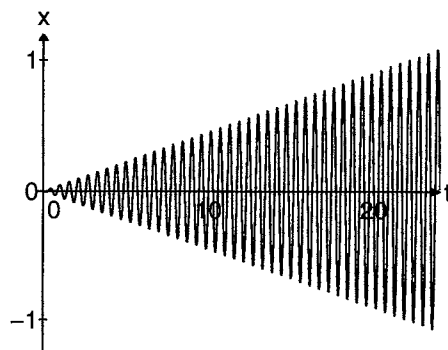
ference is $\delta = 1/2$. Thus,

$$\begin{aligned}\sin 11t - \sin 10t &= \sin\left(\frac{21}{2} + \frac{1}{2}\right)t \\ &\quad - \sin\left(\frac{21}{2} - \frac{1}{2}\right)t, \\ &= 2 \sin \frac{1}{2}t \cos \frac{21}{2}t.\end{aligned}$$

The envelope is $y(t) = \pm 2 \sin(1/2)t$. The graph of the envelope is dashed in the following figure.



7. The graph of $x(t) = \cos 11t - \cos \omega t$, with $\omega = 10.99$, is shown below.



At first glance, this appears to be resonance. However, with

$$\bar{\omega} = \frac{11 + \omega}{2} \quad \text{and} \quad \delta = \frac{11 - \omega}{2},$$

$x(t) = \cos 11t - \cos \omega t$ becomes

$$\begin{aligned} x(t) &= \cos(\bar{\omega} + \delta) - \cos(\bar{\omega} - \delta)t \\ &= -2 \sin \bar{\omega} t \sin \delta t. \end{aligned}$$

With $\omega = 10.99$,

$$\begin{aligned} \bar{\omega} &= \frac{11 + \omega}{2} = 10.995 \\ \delta &= \frac{11 - \omega}{2} = 0.005, \end{aligned}$$

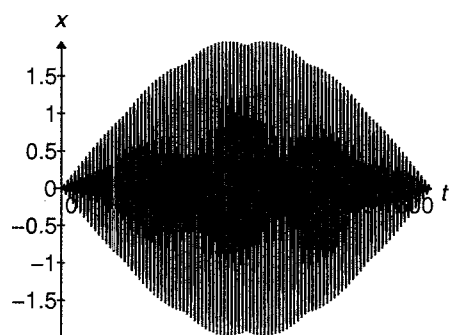
and

$$x(t) = -2 \sin 10.995t \sin 0.005t.$$

The period of the slow oscillation is

$$T = \frac{2\pi}{0.005} = 400\pi.$$

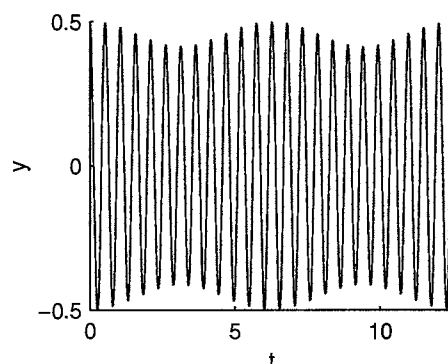
If we draw the graph of $x(t)$ on the interval $[0, 200\pi]$, we get the image below.



Note the aliasing in the image due to the frequency of the rapid oscillations. Still, note that “beats” are still present and this is non resonance. It’s just that the beats occur over a long time period.

8. The “slow” and “fast” frequencies are present in each case, although the slow frequency becomes less pronounced as y_0 increases. Below is the plot for

$$y_0 = 0.5.$$



9. (a) The displacement satisfies the differential equation $x'' + 4x = 4 \cos \omega t$, with initial conditions $x(0) = x'(0) = 0$. The solution is

$$x(t) = \frac{4}{4 - \omega^2} (\cos \omega t - \cos 2t).$$

- (b) If we set $\bar{\omega} = (2 + \omega)/2$, and $\delta = (2 - \omega)/2$, the solution becomes

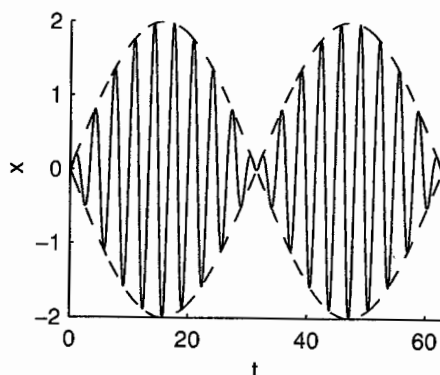
$$x(t) = \frac{2}{\bar{\omega}\delta} \sin \delta t \sin \bar{\omega} t.$$

We will take $\omega = 1.8$, which is near to $\omega_0 = 2$. Then $\bar{\omega} = 1.9$, and $\delta = 0.1$, so

$$x(t) = \frac{2}{0.19} \sin 0.1t \sin 1.9t.$$

The graph of x and its envelope is presented in

the following figure.



10. (a) As in the text we find the particular solution

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t = \frac{2}{5} t \sin 5t.$$

The solution of the homogeneous equation is $x_h(t) = C_1 \cos 5t + C_2 \sin 5t$, so our solution has the form

$$x(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{2}{5} t \sin 5t.$$

Our initial conditions are

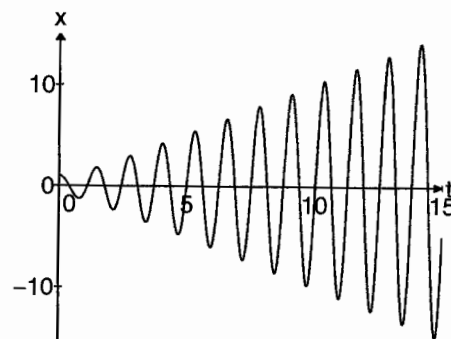
$$\begin{aligned} 1 &= x(0) = C_1 \\ 0 &= x'(0) = 5C_2, \end{aligned}$$

so the solution is

$$x(t) = \cos 5t + \frac{2}{5} t \sin 5t.$$

The particular solution $x_p(t)$ has a factor of t so its amplitude will grow, indicating a resonant solution.

(b)



11. (a) The equation for the current is $I'' + 4I = -12\omega \sin \omega t$. We find a particular solution of the form $I_p(t) = A \cos \omega t + B \sin \omega t$. Solving for A and B we find that

$$I_p(t) = \frac{12\omega}{\omega^2 - 4} \sin \omega t.$$

Since the general solution of the homogeneous equation is

$$I_h(t) = C_1 \cos 2t + C_2 \sin 2t,$$

we know that the current has the form

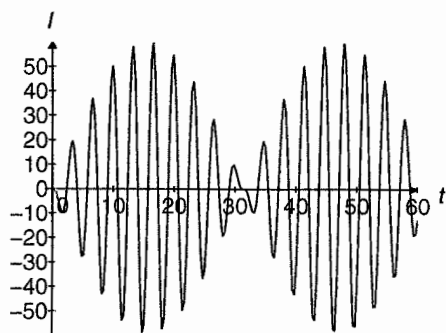
$$I(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{12\omega}{\omega^2 - 4} \sin \omega t.$$

The initial conditions $I(0) = 0$, and $I'(0) = 0$ imply that $C_1 = 0$, and $C_2 = -12\omega/(\omega^2 - 4)$. Hence the solution is

$$I(t) = \frac{6\omega}{\omega^2 - 4} (2 \sin \omega t - \omega \sin 2t).$$

With $\omega = 1.8$ we get the following graph of the

solution.



- (b) The equation for the current is $I'' + 4I = -24 \sin 2t$. We find a particular solution of the form $I_p(t) = t[A \cos 2t + B \sin 2t]$. Solving for A and B we find that

$$I_p(t) = 6t \cos 2t.$$

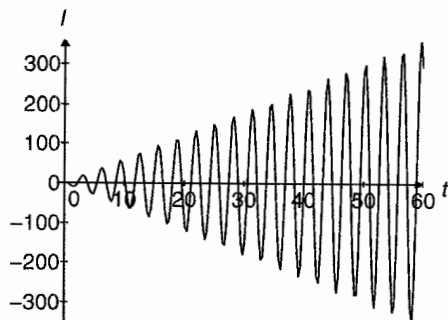
The general solution is

$$I(t) = 6t \cos 2t + C_1 \cos 2t + C_2 \sin 2t.$$

The initial conditions $I(0) = 0$, and $I'(0) = 0$ imply that $C_1 = 0$, and $C_2 = -3$. Hence the solution is

$$I(t) = 6t \cos 2t - 3 \sin 2t.$$

It is plotted in the following figure.



12. The characteristic polynomial is $P(\lambda) = \lambda^2 + 2c\lambda +$

$\omega_0^2 = \lambda^2 + \lambda + 4$. Hence $2c = 1$ and $\omega_0^2 = 4$. Hence

$$\begin{aligned} R(\omega) &= \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2} \\ &= \sqrt{(\omega^2 - 4)^2 + \omega^2} \\ \phi(\omega) &= \operatorname{arccot} \left(\frac{\omega_0^2 - \omega^2}{2c\omega} \right) \\ &= \operatorname{arccot} \left(\frac{4 - \omega^2}{\omega} \right). \end{aligned}$$

For the steady-state solution to $x'' + x' + 4x = 3 \cos 2t$ we use $\omega = 2 = \omega_0$. Hence $R(2) = 2$ and $\phi = \operatorname{arccot} 0 = \pi/2$. The solution is

$$\begin{aligned} x(t) &= A \frac{1}{R} \cos(\omega t - \phi) \\ &= 1.5 \cos(2t - \pi/2) = 1.5 \sin 2t. \end{aligned}$$

13. The characteristic polynomial is $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2 = \lambda^2 + 2\lambda + 2$. Hence $2c = 2$ and $\omega_0^2 = 2$. Therefore

$$\begin{aligned} R(\omega) &= \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2} \\ &= \sqrt{(\omega^2 - 2)^2 + 4\omega^2} \\ \phi(\omega) &= \operatorname{arccot} \left(\frac{\omega_0^2 - \omega^2}{2c\omega} \right) \\ &= \operatorname{arccot} \left(\frac{2 - \omega^2}{2\omega} \right). \end{aligned}$$

For the steady-state solution to $x'' + 2x' + 2x = 3 \sin 4t$, we look first for a complex solution to $z'' + 2z' + 2z = 3e^{4it}$. The solution will be $x(t) = \operatorname{Im} z(t)$. The complex solution is

$$z(t) = H(i\omega) \cdot 3e^{4it} = \frac{1}{R} \cdot 3e^{4it - i\phi}.$$

The real solution is then

$$x(t) = \operatorname{Im} z(t) = \frac{3}{R} \sin(4t - \phi).$$

We use $\omega = 4$. Hence $R(4) = \sqrt{260} \approx 15.8114$ and $\phi(4) = \operatorname{arccot}(-7/4) \approx 2.6224$. The solution is

$$\begin{aligned} x(t) &= \frac{3}{\sqrt{260}} \sin(4t - \phi) \\ &= \frac{3}{\sqrt{260}} \cos(4t - \phi - \pi/2). \end{aligned}$$

14. The characteristic polynomial is $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2 = \lambda^2 + 2\lambda + 4$. Hence $2c = 2$ and $\omega_0^2 = 4$, and

$$\begin{aligned} R(\omega) &= \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2} \\ &= \sqrt{(\omega^2 - 4)^2 + 4\omega^2} \\ \phi(\omega) &= \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right) \\ &= \operatorname{arccot}\left(\frac{4 - \omega^2}{2\omega}\right). \end{aligned}$$

For the steady-state solution to $x'' + 2x' + 4x = 2\sin 2\pi t$, we look first for a complex solution to $z'' + 2z' + 4z = 2e^{2\pi it}$. The solution will be $x(t) = \operatorname{Im} z(t)$. The complex solution is

$$z(t) = H(i\omega) \cdot 2e^{2\pi it} = \frac{1}{R} \cdot 2e^{2\pi it - i\phi}.$$

The real solution is then

$$x(t) = \operatorname{Im} z(t) = \frac{2}{R} \sin(2\pi t - \phi).$$

We use $\omega = 2\pi$. Hence $R(2\pi) = 37.6382$ and $\phi = 2.8012$. The solution is

$$x(t) = 0.0531 \sin(2\pi t - 2.8012).$$

15. We want to find the complex solution of $z'' + 4z' + 8z = 3e^{i2\pi t}$. Note that the frequency of the forcing term is $\omega = 2\pi$. The equation has characteristic polynomial $P(\lambda) = \lambda^2 + 4\lambda + 9$, so

$$P(i\omega) = (i\omega)^2 + 4(i\omega) + 8 = (8 - \omega^2) + 4i\omega,$$

which has magnitude and phase defined by

$$\begin{aligned} R(\omega) &= \sqrt{(8 - \omega^2)^2 + 16\omega^2} \\ \cot \phi(\omega) &= \frac{8 - \omega^2}{4\omega}. \end{aligned}$$

With $\omega = 2\pi$, $R(2\pi) \approx 40.2808$ and $\phi(2\pi) \approx 2.4678$, leading to the complex solution

$$\begin{aligned} z(t) &= H(i2\pi) \cdot 3e^{i2\pi t} \\ &= \frac{1}{R(2\pi)} e^{-i\phi(2\pi)} \cdot 3e^{i2\pi t} \\ &= \frac{1}{40.2808} e^{-i(2.4678)} \cdot 3e^{i2\pi t} \\ &= 0.0745 e^{i(2\pi t - 2.4678)}. \end{aligned}$$

The steady-state solution of $x'' + 4x' + 8x = 3\cos 2\pi t$ is the real part of this solution, namely

$$x(t) = 0.0745 \cos(2\pi t - 2.4678).$$

16. We will use the complex method, and look for a solution of the equation $z'' + 5z' + 4z = 2e^{2it}$ of the form $z(t) = ae^{2it}$. Then our particular solution will be $x_p = \operatorname{Im} z$. Differentiating we get

$$\begin{aligned} z'' + 5z' + 4z &= a((2i)^2 + 5(2i) + 4)e^{2it} \\ &= 10aie^{2it} = 2e^{2it}. \end{aligned}$$

Hence $a = -i/5$, $z(t) = -ie^{2it}/5$, and $x_p(t) = \operatorname{Im} z(t) = -\cos 2t/5$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4)$, which has roots -1 and -4 . Hence the general solution to the homogenous equation is $x_h(t) = C_1 e^{-t} + C_2 e^{-4t}$. The general solution to the inhomogeneous equation is

$$x(t) = -\frac{1}{5} \cos 2t + C_1 e^{-t} + C_2 e^{-4t}.$$

The initial conditions imply that

$$\begin{aligned} 1 &= x(0) = -\frac{1}{5} + C_1 + C_2 \\ 0 &= x'(0) = -C_1 - 4C_2 \end{aligned}$$

Hence we find C_1 and C_2 which solve the equations

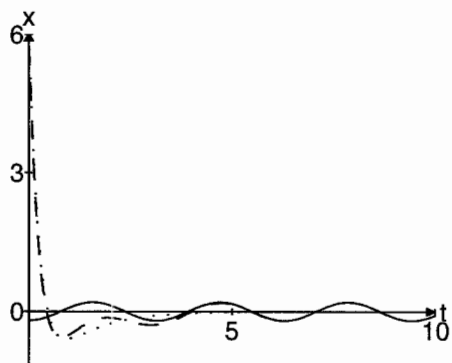
$$\begin{aligned} C_1 + C_2 &= \frac{6}{5}, \\ -C_1 - 4C_2 &= 0 \end{aligned}$$

The solutions are $C_1 = 8/5$ and $C_2 = -2/5$, so the solution to the initial value problem is

$$x(t) = -\frac{1}{5} \cos 2t + \frac{8}{5} e^{-t} - \frac{2}{5} e^{-4t}.$$

The steady-state response is the particular solution $x_p(t) = -\cos 2t/5$, and the transient response is $x_h(t) = [8e^{-t} - 2e^{-4t}]/5$. In the following plot the graph of the solution to the initial value problem is

the dashed curve, the transient response is dotted, and the steady-state solution is solid.



17. We will use the complex method, and look for a solution of the equation $z'' + 7z' + 10z = 3e^{3it}$ of the form $z(t) = ae^{3it}$. Then our particular solution will be $x_p = \text{Re } z$. Differentiating we get

$$\begin{aligned} z'' + 7z' + 10z &= a((3i)^2 + 7(3i) + 10)e^{3it} \\ &= a(1 + 21i)e^{3it} = 3e^{3it}. \end{aligned}$$

Hence

$$a = \frac{3}{1 + 21i} = 3 \cdot \frac{1 - 21i}{442},$$

so

$$\begin{aligned} z(t) &= 3 \cdot \frac{1 - 21i}{442} e^{3it} \\ &= \frac{3}{442} [1 - 21i][\cos 3t + i \sin 3t] \\ &= \frac{3}{442} [(\cos 3t + 21 \sin 3t) \\ &\quad + i(\sin 3t - 21 \cos 3t)], \end{aligned}$$

and $x_p(t) = \text{Re } z(t) = 3(\cos 3t + 21 \sin 3t)/442$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 7\lambda + 10 = (\lambda + 2)(\lambda + 5)$, which has roots -2 and -5 . Hence the general solution to the homogenous equation is $x_h(t) = C_1 e^{-2t} + C_2 e^{-5t}$. The general solution to the inhomogeneous equation is

$$x(t) = \frac{3}{442}(\cos 3t + 21 \sin 3t) + C_1 e^{-2t} + C_2 e^{-5t}.$$

The initial conditions imply that

$$\begin{aligned} -1 &= x(0) = \frac{3}{442} + C_1 + C_2 \\ 0 &= x'(0) = \frac{189}{442} - 2C_1 - 5C_2, \end{aligned}$$

or

$$\begin{aligned} C_1 + C_2 &= -\frac{445}{442} \\ 2C_1 + 5C_2 &= \frac{189}{442}. \end{aligned}$$

The solutions are $C_1 = -71/39$ and $C_2 = 83/102$, so the solution to the initial value problem is

$$x(t) = \frac{3}{442}(\cos 3t + 21 \sin 3t) - \frac{71}{39}e^{-2t} + \frac{83}{102}e^{-5t}.$$

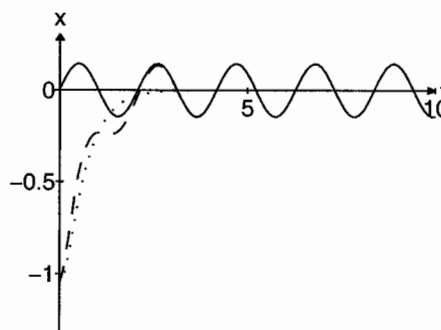
The steady-state solution is the particular solution

$$x_p(t) = \frac{3}{442}(\cos 3t + 21 \sin 3t),$$

and the transient response is

$$x_h(t) = -\frac{71}{39}e^{-2t} + \frac{83}{102}e^{-5t}.$$

In the following plot the graph of the solution to the initial value problem is the dashed curve, the transient response is dotted, and the steady-state solution is solid.



18. We will use the complex method, and look for a solution of the equation $z'' + 2z' + 2z = e^{2it}$ of the

form $z(t) = ae^{2it}$. Then our particular solution will be $x_p = \operatorname{Re} z$. Differentiating we get

$$\begin{aligned} z'' + 2z' + 2z &= a[(2i)^2 + 2(2i) + 2]e^{2it} \\ &= a[-2 + 4i]e^{2it} = e^{2it}. \end{aligned}$$

Hence

$$a = \frac{1}{-2 + 4i} = \frac{-2 - 4i}{20} = -\frac{1 + 2i}{10},$$

so

$$\begin{aligned} z(t) &= -\frac{1 + 2i}{10}e^{2it} \\ &= -\frac{1}{10}[1 + 2i][\cos 2t + i \sin 2t] \\ &= -\frac{1}{10}[(\cos 2t - 2 \sin 2t) \\ &\quad + i(2 \cos 2t + \sin 2t)], \end{aligned}$$

and $x_p(t) = \operatorname{Re} z(t) = (2 \sin 2t - \cos 2t)/10$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 2\lambda + 2$, which has the complex roots $\lambda = -1 \pm i$. Hence the general solution to the homogeneous equation is $x_h(t) = e^{-t}[C_1 \cos t + C_2 \sin t]$. The general solution to the inhomogeneous equation is

$$x(t) = \frac{1}{10}(2 \sin 2t - \cos 2t) + e^{-t}[C_1 \cos t + C_2 \sin t].$$

The initial conditions imply that

$$\begin{aligned} 0 &= x(0) = -\frac{1}{10} + C_1 \\ 2 &= x'(0) = \frac{2}{5} - C_1 + C_2. \end{aligned}$$

We solve these equations, finding $C_1 = 1/10$ and $C_2 = 17/10$, so the solution to the initial value problem is

$$\begin{aligned} x(t) &= \frac{1}{10}(2 \sin 2t - \cos 2t) \\ &\quad + e^{-t} \left[\frac{1}{10} \cos t + \frac{17}{10} \sin t \right]. \end{aligned}$$

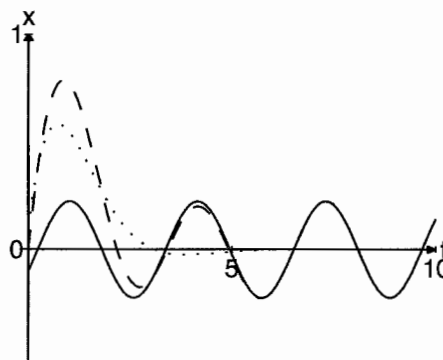
The steady-state solution is the particular solution

$$x_p(t) = (2 \sin 2t - \cos 2t)/10,$$

and the transient response is

$$x_h = e^{-t}[\cos t + 17 \sin t]/10.$$

In the following plot the graph of the solution to the initial value problem is the dashed curve, the transient response is dotted, and the steady-state solution is solid.



19. We will use the complex method, and look for a solution of the equation $z'' + 4z' + 5z = 3e^{it}$ of the form $z(t) = ae^{it}$. Then our particular solution will be $x_p = \operatorname{Im} z$. Differentiating we get

$$\begin{aligned} z'' + 4z' + 5z &= a[i^2 + 4i + 5]e^{it} \\ &= a[4 + 4i]e^{it} = 3e^{it}. \end{aligned}$$

Hence $a = 3/(4 + 4i) = 3(1 - i)/8$, so

$$\begin{aligned} z(t) &= \frac{3}{8}[1 - i]e^{it} \\ &= \frac{3}{8}[1 - i][\cos t + i \sin t] \\ &= \frac{3}{8}[(\cos t + \sin t) + i(\sin t - \cos t)], \end{aligned}$$

and $x_p(t) = \operatorname{Im} z(t) = 3(\sin t - \cos t)/8$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 4\lambda + 5$, which has the complex roots $\lambda = -2 \pm i$. Hence the general solution to the homogeneous equation is $x_h(t) = e^{-2t}[C_1 \cos t + C_2 \sin t]$. The general solution of the inhomogeneous equation is

$$x(t) = \frac{3}{8}(\sin t - \cos t) + e^{-2t}[C_1 \cos t + C_2 \sin t].$$

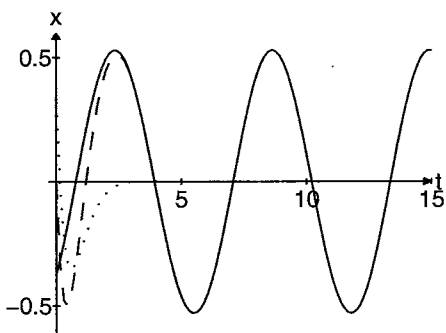
The initial conditions imply

$$\begin{aligned} 0 &= x(0) = -\frac{3}{8} + C_1 \\ -3 &= x'(0) = \frac{3}{8} - 2C_1 + C_2. \end{aligned}$$

We solve these equations, finding $C_1 = 3/8$ and $C_2 = -21/8$, so the solution to the initial value problem is

$$x(t) = \frac{3}{8}(\sin t - \cos t) + \frac{1}{8}e^{-2t}[3 \cos t - 21 \sin t].$$

The steady-state solution is the particular solution $x_p(t) = 3(\sin t - \cos t)/8$, and the transient response is $x_h = e^{-2t}[3 \cos t - 21 \sin t]/8$. In the following plot the graph of the solution to the initial value problem is the dashed curve, the transient response is dotted, and the steady-state solution is solid.

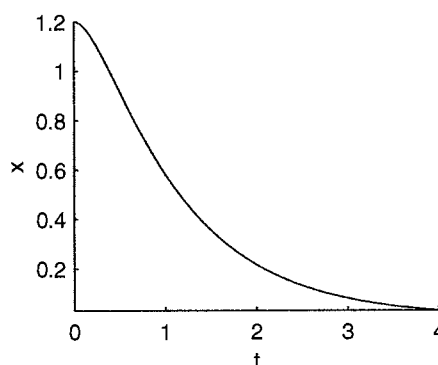


20. In Exercise 16, we found the transient solution

$$x_h(t) = \frac{8}{5}e^{-t} - \frac{2}{5}e^{-4t}.$$

Because there are two exponential terms, we will use the more slowly decaying term to determine a time constant. Thus, let $T_c = 1$. What follows is the plot of the transient solution on the interval

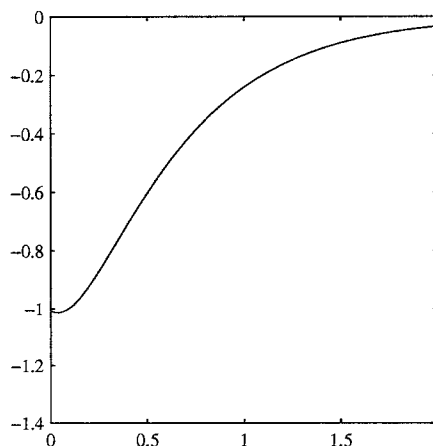
$$[0, 4T_c] = [0, 4].$$



21. In Exercise 17, we found the transient solution

$$x_h(t) = -\frac{71}{39}e^{-2t} + \frac{83}{102}e^{-5t}.$$

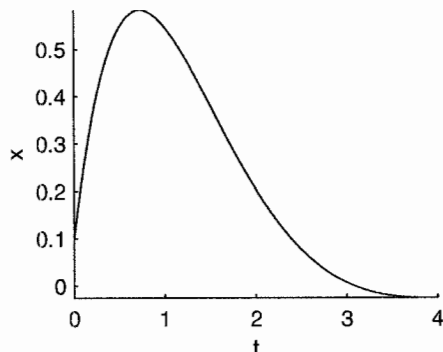
Because there are two exponential terms, we will use the more slowly decaying term to determine a time constant. Thus, let $T_c = 1/2$. What follows is the plot of the transient solution on the interval $[0, 4T_c] = [0, 2]$.



22. In Exercise 18, we found the transient solution

$$x_h(t) = \frac{1}{10}e^{-t}(\cos t + 17 \sin t).$$

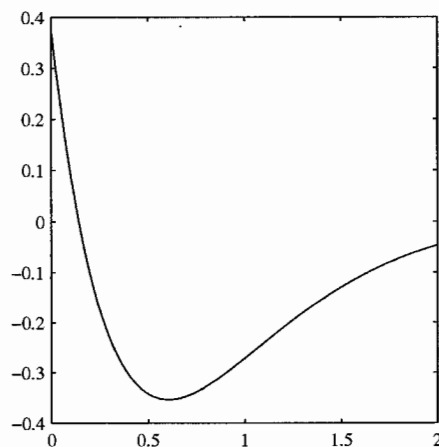
Thus, the time constant is $T_c = 1$. What follows is the plot of the transient solution on the interval $[0, 4T_c] = [0, 4]$.



23. In Exercise 19, we found the transient solution

$$x_h = e^{-2t}[3 \cos t - 21 \sin t]/8.$$

Thus, the time constant is $T_c = 1/2$. What follows is the plot of the transient solution on the interval $[0, 4T_c] = [0, 2]$.



24. We want to find a complex solution of $z'' + 2z' + 4z = 3e^{it}$. Note that the frequency of the forcing term is $\omega = 1$. The equation has characteristic polynomial

$$P(\lambda) = \lambda^2 + 2\lambda + 4, \text{ so}$$

$$P(i\omega) = (i\omega)^2 + 2(i\omega) + 4 = (4 - \omega^2) + 2i\omega,$$

which has magnitude and phase defined by

$$R(\omega) = \sqrt{(4 - \omega^2)^2 + 4\omega^2}$$

$$\cot \phi(\omega) = \frac{4 - \omega^2}{2\omega}$$

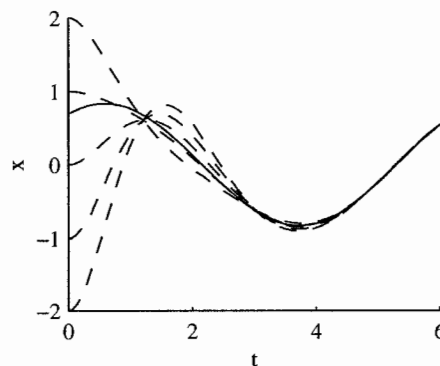
with $\omega = 1$, $R(1) = \sqrt{13} \approx 3.6056$ and $\phi(1) = \arccot(3/2) = 0.5880$, leading to the complex solution

$$z(t) = \frac{1}{\sqrt{13}} e^{-\phi i} \cdot 3e^{it} = \frac{3}{\sqrt{13}} e^{i(t-\phi)}.$$

The steady state solution of $x'' + 2x' + 4x = 3 \cos t$ is the real part of this solution, namely

$$x(t) = \frac{2}{\sqrt{13}} \cos(t - \phi).$$

Note, in the following figure, that the solution with the initial condition $(-2, 0)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, and $(2, 0)$, drawn as dashed lines, all converge to the steady-state solution (solid line).



25. We want to find the complex solution of $z'' + 2z' + 5z = 2e^{i3t}$. Note that the frequency of the forcing term is $\omega = 3$. The equation has characteristic polynomial $P(\lambda) = \lambda^2 + 2\lambda + 5$, so

$$P(i\omega) = (i\omega)^2 + 2(i\omega) + 5 = (5 - \omega^2) + 2i\omega,$$

which has magnitude and phase defined by

$$R(\omega) = \sqrt{(5 - \omega^2)^2 + 4\omega^2}$$

$$\cot \phi(\omega) = \frac{5 - \omega^2}{2\omega}.$$

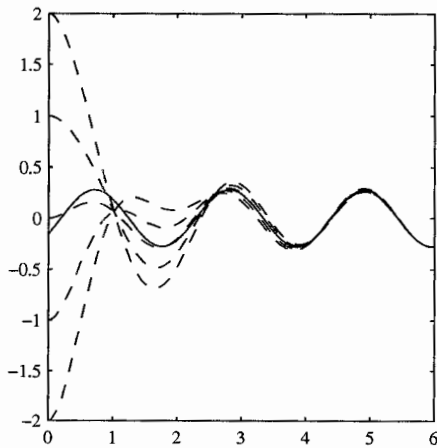
With $\omega = 3$, $R(3) = \sqrt{52} \approx 7.2111$ and $\phi(3) = \arccot(-2/3) \approx 2.1588$, leading to the complex solution

$$z(t) = \frac{1}{\sqrt{52}} e^{-\phi i} \cdot 2e^{3it} = \frac{1}{\sqrt{13}} e^{i(3t-\phi)}.$$

The steady-state solution of $x'' + 2x' + 5x = 2 \cos 3t$ is the real part of this solution, namely

$$x(t) = \frac{1}{\sqrt{13}} \cos(3t - \phi).$$

Note, in the following figure, that the solutions with the initial conditions $(-2, 0)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, and $(2, 0)$, drawn as dashed lines, all converge to the steady-state solution (solid line).



26. To find the complex solution of $z'' + 0.4z' + z = e^{it}$, note that the frequency of the forcing term is $\omega = 1$. The equation has characteristic polynomial $P(\lambda) = \lambda^2 + 0.4\lambda + 1$, so

$$P(i\omega) = (i\omega)^2 + 0.4(i\omega) + 1 = (1 - \omega^2) + 0.4i\omega,$$

which has magnitude and phase defined by

$$R(\omega) = \sqrt{(1 - \omega^2)^2 + 0.16\omega^2}$$

$$\cot \phi(\omega) = \frac{1 - \omega^2}{0.4\omega}.$$

Thus, the transfer function is

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R(\omega)e^{i\phi(\omega)}} = \frac{1}{R(\omega)}e^{-i\phi(\omega)}$$

$$= G(\omega)e^{-i\phi(\omega)},$$

where $G(\omega) = 1/R(\omega)$ is the gain. Thus,

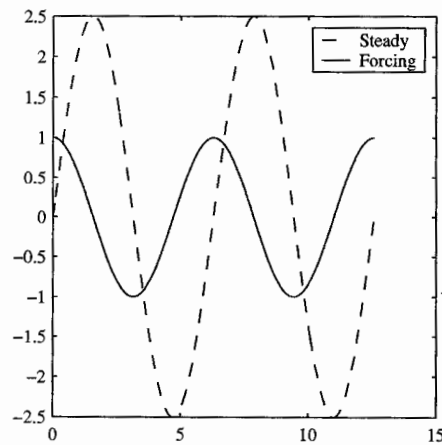
$$z(t) = H(i\omega)e^{it} = G(\omega)e^{-i\phi(\omega)}e^{it}$$

$$= G(\omega)e^{i(t-\phi(\omega))},$$

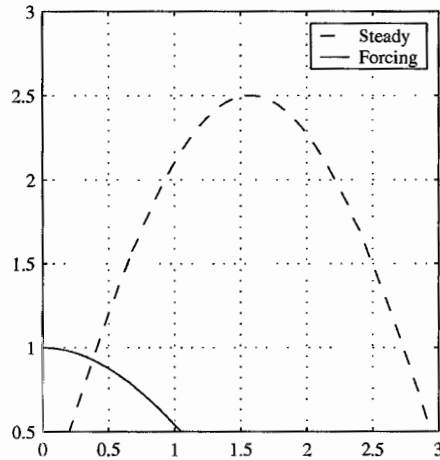
with $\omega = 1$, is the solution of the complex equation, and the real part, $x(t) = G(1) \cos(t - \phi(1))$, is the steady-state solution of $x'' + 0.4x' + x = \cos t$. Using the formulae above, $G(1) = 1/\sqrt{0.16} = 2.5$ and $\phi(1) = \arccot(0) = \pi/2$. Thus, the steady state solution is

$$x(t) = 2.5 \cos\left(t - \frac{\pi}{2}\right).$$

Note that the amplitude of the steady-state is the gain (not always the case), due to the fact that the amplitude of the forcing function, $\cos t$, is 1. Thus, the gain is easily read in the following graph, where one can estimate the gain by recording the amplitude of the steady-state response (plotted as a dashed line).



Further, note that the steady-state in the figure is shifted to the right of the forcing function. By zooming in on the upper left corner and adding a grid to the figure, one can estimate the phase shift.



Note that the steady-state response is shifted about 1.5 units ($\pi/2 \approx 1.5$) to the right of the forcing function. It is important to note that this reading of gain and phase is facilitated by the choice of forcing function. Because the forcing function $\cos t$ has amplitude 1, the gain is easily read as the amplitude of the steady-state response. Similarly, because the forcing function is $\cos t$, the phase is a simple shift. Things would be more complicated (but still doable) if we used $2 \cos 3t$ as the forcing function.

27. To find the complex solution of $z'' + 0.4z' + 2z = e^{it}$, note that the frequency of the forcing term is $\omega = 1$. The equation has characteristic polynomial $P(\lambda) = \lambda^2 + 0.4\lambda + 2$, so

$$P(i\omega) = (i\omega)^2 + 0.4(i\omega) + 2 = (2 - \omega^2) + 0.4i\omega,$$

which has magnitude and phase defined by

$$R(\omega) = \sqrt{(2 - \omega^2)^2 + 0.16\omega^2}$$

$$\cot \phi(\omega) = \frac{2 - \omega^2}{0.4\omega}.$$

Thus, the transfer function is

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R(\omega)e^{i\phi(\omega)}} = \frac{1}{R(\omega)}e^{-i\phi(\omega)}$$

$$= G(\omega)e^{-i\phi(\omega)},$$

where $G(\omega) = 1/R(\omega)$ is the gain. Thus,

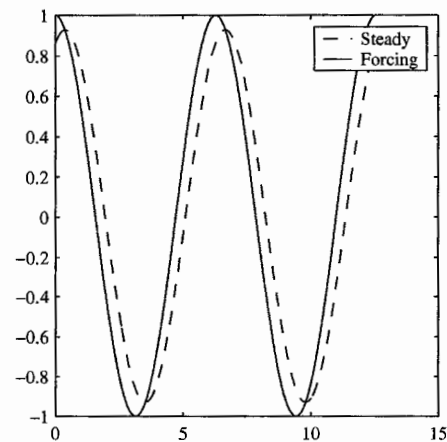
$$z(t) = H(i\omega)e^{it} = G(\omega)e^{-i\phi(\omega)}e^{it}$$

$$= G(\omega)e^{i(t-\phi(\omega))},$$

with $\omega = 1$, is the solution of the complex equation, and the real part, $x(t) = G(1) \cos(t - \phi(1))$, is the steady-state solution of $x'' + 0.4x' + 2x = \cos t$. Using the formulae above, $G(1) = 1/\sqrt{1.16} \approx 0.9285$ and $\phi(1) = \text{arccot}(2.5) \approx 0.3805$. Thus, the steady state solution is

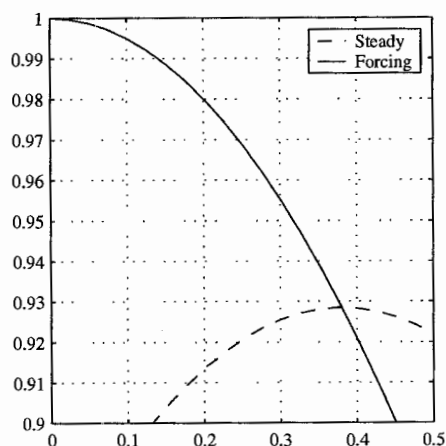
$$x(t) = \frac{1}{\sqrt{1.16}} \cos(t - \phi).$$

Note that the amplitude of the steady-state is the gain (not always the case), due to the fact that the amplitude of the forcing function, $\cos t$, is 1. Thus, the gain is easily read in the following graph, where one can estimate the gain by recording the amplitude of the steady-state response (plotted as a dashed line).



Further, note that the steady-state in the figure is shifted to the right of the forcing function. By zooming in on the upper left corner and adding a grid to

the figure, one can estimate the phase shift.

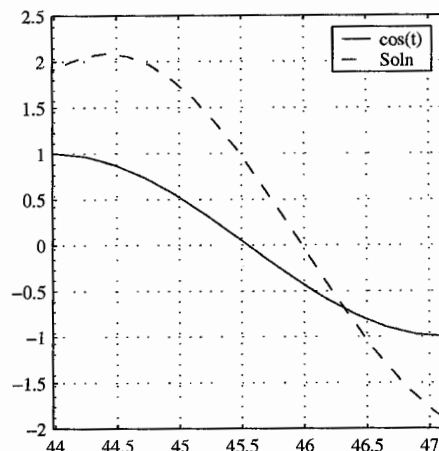
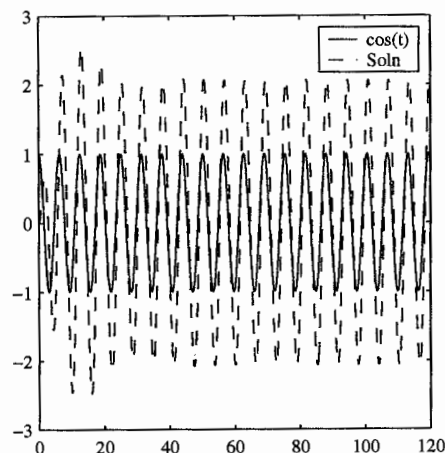


Note that the steady-state response is shifted about 0.38 units to the right of the forcing function. It is important to note that this reading of gain and phase is facilitated by the choice of forcing function. Because the forcing function $\cos t$ has amplitude 1, the gain is easily read as the amplitude of the steady-state response. Similarly, because $\omega = 1$, the forcing function is $\cos t$, the phase is a simple shift. Things would be more complicated (but still doable) if we used $2 \cos 3t$ as the forcing function.

28. The equation $x'' + 0.2x' + 1.44x = \cos t$ has characteristic polynomial $P(\lambda) = \lambda^2 + 0.2\lambda + 1.44$, having roots $\lambda = -(1/10) \pm \sqrt{143}/10$. Consequently, the time constant is

$$T_c = \frac{1}{c} = \frac{1}{1/10} = 10.$$

Thus, to avoid transients, we let at least $4T_c$ pass before examining the solution. Note that in the following figure that the solution settles down to its steady-state after about $[0, 4T_c] = [0, 40]$.



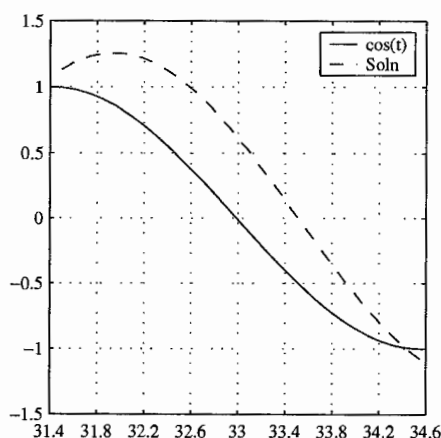
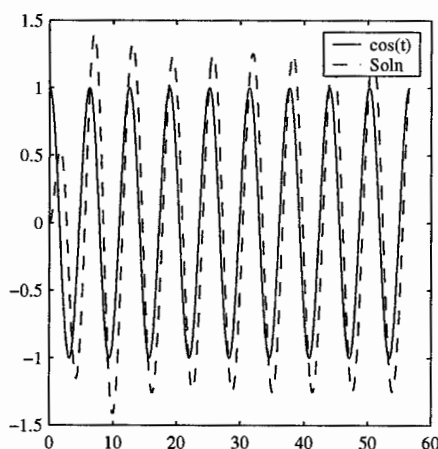
In the next image, we've zoomed in on the interval $[14\pi, 15\pi]$, which is well beyond $[0, 40]$. Note that the amplitude of the solution is about 2.1, which is the gain. Further, note that the cosine peaks at about 44, but the first peak of the solution occurs at approximately 44.4. Thus, the phase is $44.4 - 44$, or 0.4. The careful reader will want to compare these results with the actual gain and phase, 2.0690 and 0.4266.

29. The equation $x'' + 0.4x' + 1.69x = \cos t$ has charac-

teristic polynomial $P(\lambda) = \lambda^2 + 0.4\lambda + 1.69$, having roots $\lambda = -(1/5) \pm \sqrt{165}/10$. Consequently, the time constant is

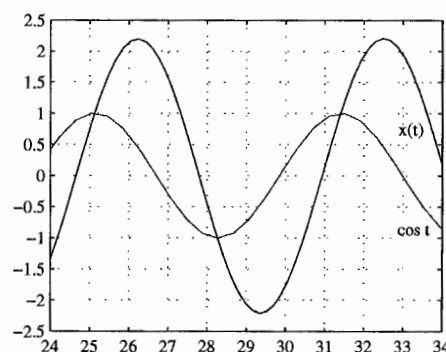
$$T_c = \frac{1}{c} = \frac{1}{1/5} = 5.$$

Thus, to avoid transients, we let at least $4T_c$ pass before examining the solution. Note that in the following figure that the solution settles down to its steady-state after about $[0, 4T_c] = [0, 20]$.



In the next image, we've zoomed in on the interval $[10\pi, 11\pi]$, which is well beyond $[0, 20]$. Note that the amplitude of the solution is about 1.25, which is the gain. Further, note that the cosine peaks at about 31.4, but the first peak of the solution occurs at approximately 32.0. Thus, the phase is $32.0 - 31.4$, or 0.6. The careful reader will want to compare these results with the actual gain and phase, 1.2538 and 0.5254.

30. From the image,



we can see that the solution $x(t)$ peaks at about 2.2, which is about 2.2 times the maximum of the forcing function $\cos t$. Thus, the gain is about 2.2. The first peak of $\cos t$ appears at about 25.2, while the first peak of the solution $x(t)$ occurs at about 26.3, making $\phi \approx 26.3 - 25.2 \approx 1.1$. The gain is given by

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

Because $\omega = 1$ and we've estimated the gain at $G(1) \approx 2.2$, we can write

$$2.2 = \frac{1}{\sqrt{(\omega_0^2 - 1)^2 + 4c^2}}.$$

The phase is given by

$$\cot \phi(\omega) = \frac{\omega_0^2 - \omega^2}{2c\omega}.$$

Again, $\omega = 1$ and we've estimated $\phi(1) = 1.1$. Thus,

$$\begin{aligned}\cot(\phi(1)) &= \frac{\omega_0^2 - (1)^2}{2c(1)} \\ 0.5090 &= \frac{\omega_0^2 - 1}{2c} \\ \omega_0^2 &= 1.0179c + 1.\end{aligned}$$

Substitute this result in the gain equation to get

$$\begin{aligned}2.2 &= \frac{1}{\sqrt{((1.0179c + 1) - 1)^2 + 4c^2}} \\ 2.2 &= \frac{1}{\sqrt{5.0362c^2}} \\ 4.84 &= \frac{1}{5.0362c^2} \\ c^2 &= \frac{1}{24.3752} \\ c &\approx 0.21.\end{aligned}$$

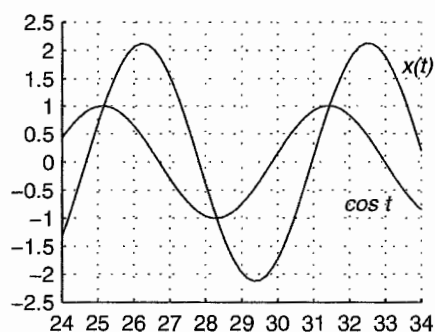
Therefore,

$$\omega_0^2 = 1.0179(0.21) + 1 \approx 1.21.$$

Thus, we guess that our differential equation is

$$\begin{aligned}x'' + 2cx' + \omega_0^2 x &= \cos t \\ x'' + 0.42x' + 1.21x &= \cos t.\end{aligned}$$

The plot of the solution of this equation, with initial conditions $x(0) = 3$ and $x'(0) = 0$ follows.



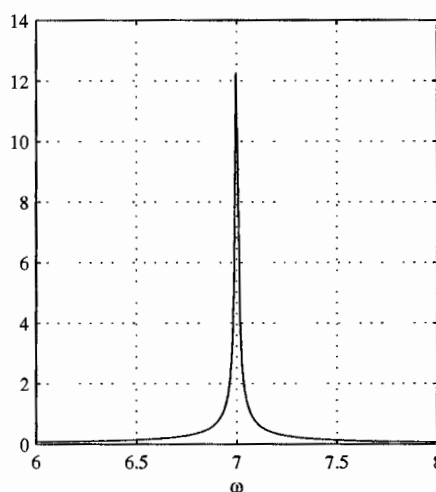
31. Note that the characteristic polynomial of $z'' + 0.01z' + 49z = Ae^{i\omega t}$ is $P(\lambda) = \lambda^2 + 0.01\lambda + 49$. Thus,

$$\begin{aligned}P(i\omega) &= (i\omega)^2 + 0.01(i\omega) + 49 \\ &= (49 - \omega^2) + 0.01i\omega.\end{aligned}$$

Thus, the gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(49 - \omega^2)^2 + 0.0001\omega^2}}.$$

Plotting the gain on the frequency interval $[6, 8]$, one easily sees that the maximum gain occurs at about $\omega \approx 7$.



32. Note that the characteristic polynomial of $z'' + 0.5z' + 4z = Ae^{i\omega t}$ is $P(\lambda) = \lambda^2 + 0.5\lambda + 4$. Thus,

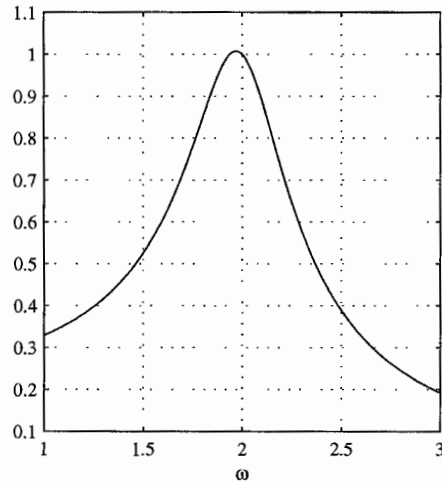
$$P(i\omega) = (i\omega)^2 + 0.5(i\omega) + 4 = (4 - \omega^2) + 0.5i\omega.$$

Thus, the gain is

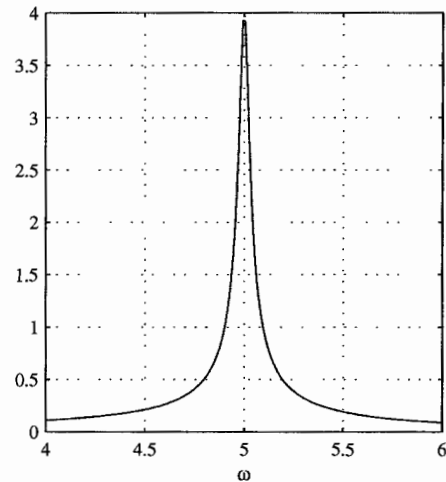
$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(4 - \omega^2)^2 + 0.25\omega^2}}.$$

Plotting the gain on the frequency interval $[6, 8]$, one easily sees that the maximum gain occurs at about

$w \approx 2.$



$w \approx 5.$



33. Note that the characteristic polynomial of $z'' + 0.05z' + 25z = Ae^{i\omega t}$ is $P(\lambda) = \lambda^2 + 0.05\lambda + 25$. Thus,

$$\begin{aligned} P(i\omega) &= (i\omega)^2 + 0.05(i\omega) + 25 \\ &= (25 - \omega^2) + 0.05i\omega. \end{aligned}$$

Thus, the gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(25 - \omega^2)^2 + 0.0025\omega^2}}.$$

Plotting the gain on the frequency interval $[4, 6]$, one easily sees that the maximum gain occurs at about

34. Note that the characteristic polynomial of $z'' + 0.25z' + z = Ae^{i\omega t}$ is $P(\lambda) = \lambda^2 + 0.25\lambda + 1$. Thus,

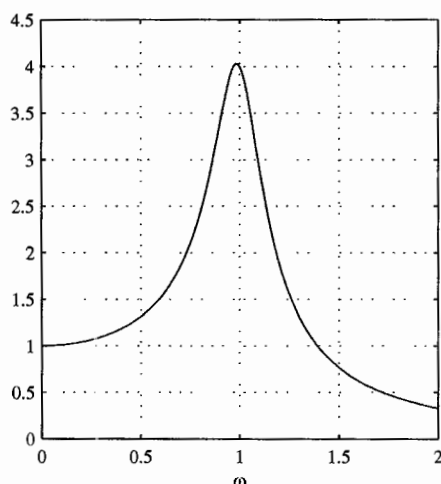
$$P(i\omega) = (i\omega)^2 + 0.25(i\omega) + 1 = (1 - \omega^2) + 0.25i\omega.$$

Thus, the gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(1 - \omega^2)^2 + 0.0625\omega^2}}.$$

Plotting the gain on the frequency interval $[4, 6]$, one easily sees that the maximum gain occurs at about

$\omega \approx 1$.



35. The characteristic polynomial of $z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$ is $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$. Thus,

$$\begin{aligned} P(i\omega) &= (i\omega)^2 + 2c(i\omega) + \omega_0^2 \\ &= (\omega_0^2 - \omega^2) + 2ci\omega. \end{aligned}$$

Thus, the gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

Taking the derivative,

$$\begin{aligned} G'(\omega) &= -\frac{1}{2}((\omega_0^2 - \omega^2)^2 + 4c^2\omega^2)^{-3/2} \\ &\quad \times (-4\omega(\omega_0^2 - \omega^2) + 8c^2\omega) \\ &= ((\omega_0^2 - \omega^2)^2 + 4c^2\omega^2)^{-3/2} \\ &\quad \times (2\omega(\omega_0^2 - \omega^2) - 4c^2\omega). \end{aligned}$$

Note that the first factor of G is always positive, so critical values are determined by setting the remaining factors equal to zero. Thus,

$$2\omega(\omega_0^2 - \omega^2 - 2c^2) = 0$$

leads to the critical values

$$\omega = 0 \quad \text{and} \quad \omega = \sqrt{\omega_0^2 - 2c^2},$$

provided, of course, that $\omega_0^2 > 2c^2$. Practically, we are not interested in a forcing function with zero frequency, so we concentrate on $\omega = \sqrt{\omega_0^2 - 2c^2}$. It is not difficult to show that the derivative of G is positive to the left of this critical value and negative to the right. Thus, we have a maximum at $\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2}$, which is the resonant frequency for the driven oscillator. Examining the equation $y'' + 0.01y' + 49y = A \cos \omega t$, the maximum gain occurs at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{7^2 - 2(0.005)^2},$$

which, to four decimal places, equals 7.0000, agreeing nicely with the estimate found in Exercise 31.

36. See the derivation of the formula in Exercise 35. Because $y'' + 0.5y' + 4y = A \sin \omega t$, the maximum gain occurs at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{2^2 - 2(0.25)^2} \approx 1.9685,$$

which agrees nicely with the result in Exercise 32.

37. See the derivation of the formula in Exercise 35. Because $y'' + 0.05y' + 25y = A \sin \omega t$, the maximum gain occurs at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{5^2 - 2(0.025)^2} \approx 4.9999,$$

which agrees nicely with the result in Exercise 33.

38. See the derivation of the formula in Exercise 35. Because $y'' + 0.25y' + y = A \cos \omega t$, the maximum gain occurs at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{1^2 - 2(0.125)^2} \approx 0.9843,$$

which agrees nicely with the result in Exercise 34.

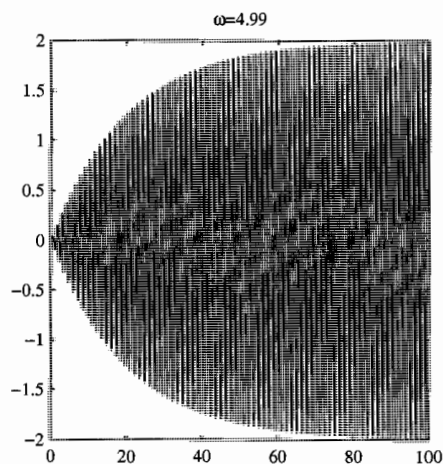
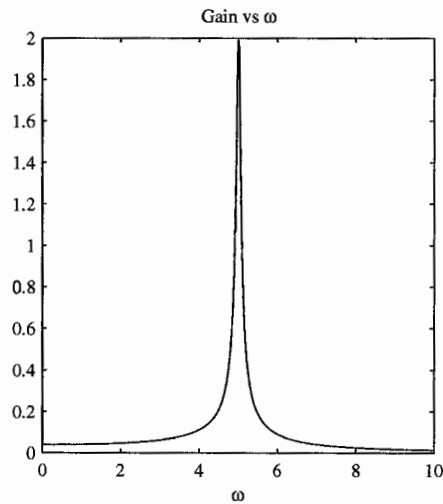
39. The equation $y'' + 0.1y' + 25y = \cos \omega t$ has characteristic polynomial $P(\lambda) = \lambda^2 + 0.1\lambda + 25$. Thus,

$$\begin{aligned} P(i\omega) &= (i\omega)^2 + 0.1(i\omega) + 25 \\ &= (25 - \omega^2) + 0.1i\omega. \end{aligned}$$

The gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(25 - \omega^2)^2 + 0.01\omega^2}}.$$

Next, the gain is plotted versus the frequency, indicating a maximum gain near $\omega = 5$.

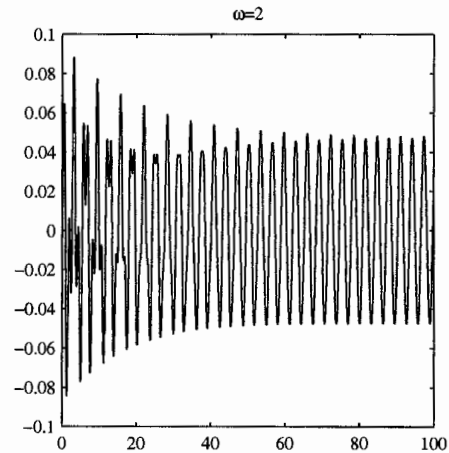


This estimate of the resonant frequency is verified with the calculation

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{5^2 - 2(0.05)^2} \approx 4.9995.$$

In the second figure above, careful attention was paid to eliminating transients. In this underdamped case, $T_c = 1/c = 1/0.05 = 20$. Thus, we must go beyond $4T_c = 80$ to eliminate most of the transient behavior.

Of course, this extended time interval allows truncation error to propagate, so we set the relative error tolerance of our Matlab solver to 1×10^{-8} to draw the solution in the second figure above, with $\omega = 4.99$ set near the resonant frequency. Note the gain in amplitude is about double that of the forcing function, $\cos 4.99t$. In the final figure, $\omega = 2$ was chosen at quite a distance from the resonant frequency.



Note the severe attenuation of the amplitude.

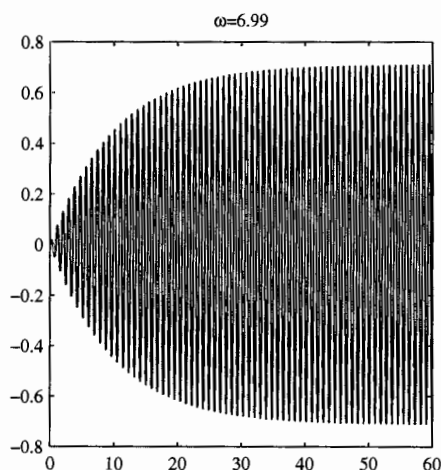
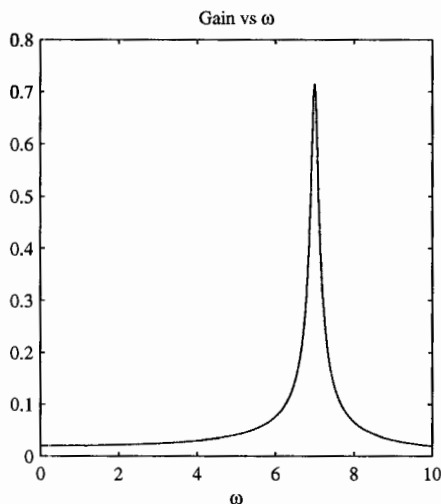
40. The equation $y'' + 0.2y' + 49y = \cos \omega t$ has characteristic polynomial $P(\lambda) = \lambda^2 + 0.2\lambda + 49$. Thus,

$$\begin{aligned} P(i\omega) &= (i\omega)^2 + 0.2(i\omega) + 49 \\ &= (49 - \omega^2) + 0.2i\omega. \end{aligned}$$

The gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(49 - \omega^2)^2 + 0.04\omega^2}}.$$

In the first figure, the gain is plotted versus the frequency, indicating a maximum gain near $\omega = 7$.

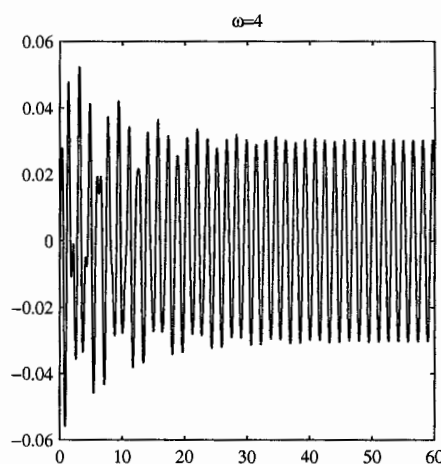


This estimate of the resonant frequency is verified with the calculation

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{7^2 - 2(0.1)^2} \approx 6.9986.$$

In the next figure, careful attention was paid to eliminating transients. In this underdamped case, $T_c = 1/c = 1/0.1 = 10$. Thus, we must go beyond $4T_c = 40$ to eliminate most of the transient behavior.

Of course, this extended time interval allows truncation error to propagate, so we set the relative error tolerance of our Matlab solver to 1×10^{-8} to draw the solution in the second figure above, with $\omega = 6.99$ set near the resonant frequency. Note the gain is a small attenuation of the amplitude of the forcing function, $\cos 6.99t$. This makes perfect sense in light of the graph of the gain at the left. Note that the maximum gain is about 0.72, so even near the resonant frequency, we can expect some small degradation of the amplitude of the forcing function. In the final figure, $\omega = 4$ was chosen at quite a distance from the resonant frequency.



Note the severe attenuation of the amplitude.

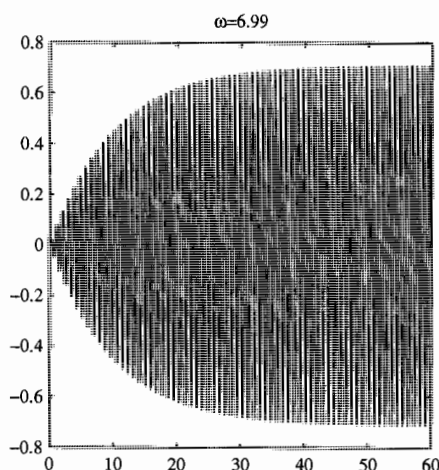
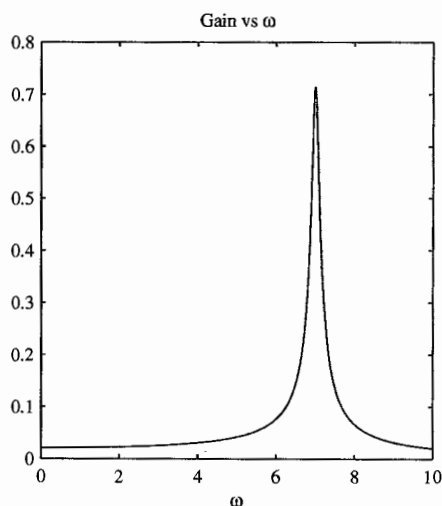
41. The equation $y'' + 0.2y' + 49y = \cos \omega t$ has characteristic polynomial $P(\lambda) = \lambda^2 + 0.2\lambda + 49$. Thus,

$$\begin{aligned} P(i\omega) &= (i\omega)^2 + 0.2(i\omega) + 49 \\ &= (49 - \omega^2) + 0.2i\omega. \end{aligned}$$

The gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(49 - \omega^2)^2 + 0.04\omega^2}}.$$

In the first figure, the gain is plotted versus the frequency, indicating a maximum gain near $\omega = 7$.

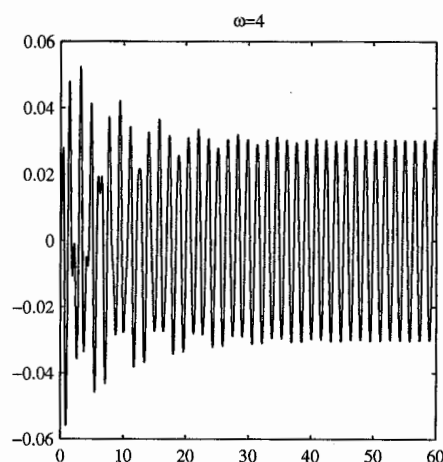


This estimate of the resonant frequency is verified with the calculation

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{7^2 - 2(0.1)^2} \approx 6.9986.$$

In the second figure above, careful attention was paid to eliminating transients. In this underdamped case, $T_c = 1/c = 1/0.1 = 10$. Thus, we must go beyond $4T_c = 40$ to eliminate most of the transient behavior.

Of course, this extended time interval allows truncation error to propagate, so we set the relative error tolerance of our Matlab solver to 1×10^{-8} to draw the solution in second figure above, with $\omega = 6.99$ set near the resonant frequency. Note the gain is a small attenuation of the amplitude of the forcing function, $\cos 6.99t$. This makes perfect sense in light of the graph of the gain in the first figure. Note that the maximum gain is about 0.72, so even near the resonant frequency, we can expect some small degradation of the amplitude of the forcing function. In the final figure, $\omega = 4$ was chosen at quite a distance from the resonant frequency.



Note the severe attenuation of the amplitude.

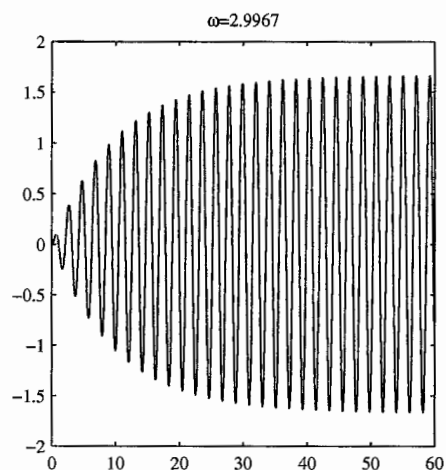
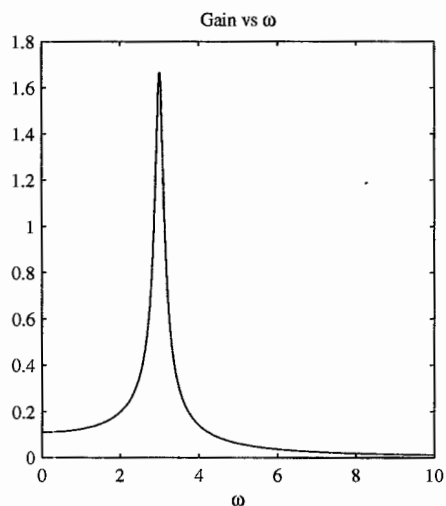
42. The equation $y'' + 0.2y' + 9y = \cos \omega t$ has characteristic polynomial $P(\lambda) = \lambda^2 + 0.2\lambda + 9$. Thus,

$$\begin{aligned} P(i\omega) &= (i\omega)^2 + 0.2(i\omega) + 9 \\ &= (9 - \omega^2) + 0.2i\omega. \end{aligned}$$

The gain is

$$G(\omega) = \frac{1}{R(\omega)} = \frac{1}{\sqrt{(9 - \omega^2)^2 + 0.04\omega^2}}.$$

In the first figure, the gain is plotted versus the frequency, indicating a maximum gain near $\omega = 3$.

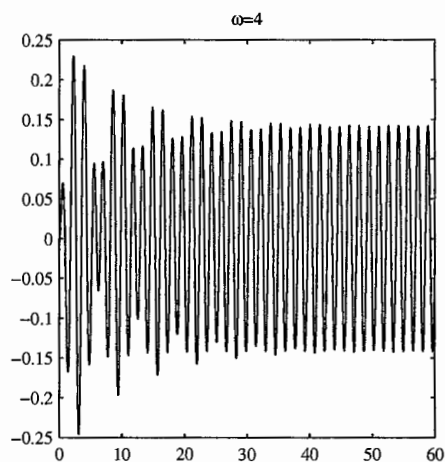


This estimate of the resonant frequency is verified with the calculation

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} = \sqrt{3^2 - 2(0.1)^2} \approx 2.9967.$$

In the second figure above, careful attention was paid to eliminating transients. In this underdamped case, $T_c = 1/c = 1/0.1 = 10$. Thus, we must go beyond

$4T_c = 40$ to eliminate most of the transient behavior. Of course, this extended time interval allows truncation error to propagate, so we set the relative error tolerance of our Matlab solver to 1×10^{-8} to draw the solution in the second figure above, with $\omega = 2.9967$ set near the resonant frequency. Note the gain is about 1.6 times the amplitude of the forcing function, $\cos 2.9967t$. This makes perfect sense in light of the graph of the gain in the first figure. In the figure that follows, $\omega = 4$ was chosen at quite a distance from the resonant frequency.



Note the severe attenuation of the amplitude.

43. (a) Because the current is the derivative of the charge ($I = Q'$), the equation $LI' + RI + (1/C)Q = A \cos \omega t$ becomes $LQ'' + RQ' + (1/C)Q = A \cos \omega t$, or upon dividing both sides by the inductance,

$$Q'' + \frac{R}{L}Q' + \frac{1}{LC}Q = \frac{A}{L} \cos \omega t.$$

The solution we seek is the real part of the complex solution of

$$z'' + \frac{R}{L}z' + \frac{1}{LC}z = \frac{A}{L}e^{i\omega t}.$$

Let $z = ae^{i\omega t}$ represent the steady-state solu-

tion. Substituting,

$$\begin{aligned} \left[(i\omega)^2 + \frac{R}{L}(i\omega) + \frac{1}{LC} \right] a e^{i\omega t} &= \frac{A}{L} e^{i\omega t} \\ \left[\left(\frac{1}{LC} - \omega^2 \right) + \frac{R}{L} i\omega \right] z &= \frac{A}{L} e^{i\omega t}. \end{aligned}$$

The complex coefficient of z can be written in polar form,

$$\sqrt{\left(\frac{1}{LC} - \omega^2 \right)^2 + \frac{R^2}{L^2} \omega^2} e^{i\phi(\omega)} z = \frac{A}{L} e^{i\omega t},$$

where

$$\cot \phi(\omega) = \frac{1/(LC) - \omega^2}{R\omega/L}.$$

Thus,

$$z = \frac{(A/L) e^{i(\omega t - \phi(\omega))}}{\sqrt{(1/(LC) - \omega^2)^2 + R^2 \omega^2 / L^2}}.$$

The real part of z ,

$$Q = \frac{(A/L) \cos(\omega t - \phi(\omega))}{\sqrt{(1/(LC) - \omega^2)^2 + R^2 \omega^2 / L^2}},$$

is the solution we seek. The charge will be maximized only if we can maximize

$$G(\omega) = \left[\left(\frac{1}{LC} - \omega^2 \right)^2 + \frac{R^2}{L^2} \omega^2 \right]^{-1/2}.$$

Taking the derivative,

$$\begin{aligned} G'(\omega) &= -\frac{1}{2} \left[\left(\frac{1}{LC} - \omega^2 \right)^2 + \frac{R^2}{L^2} \omega^2 \right]^{-3/2} \\ &\quad \times \left[2 \left(\frac{1}{LC} - \omega^2 \right) (-2\omega) + \frac{2R^2}{L^2} \omega \right] \\ &= \left[\left(\frac{1}{LC} - \omega^2 \right)^2 + \frac{R^2}{L^2} \omega^2 \right]^{-3/2} \\ &\quad \times \left[2\omega \left(\frac{1}{LC} - \omega^2 \right) - \frac{R^2}{L^2} \omega \right]. \end{aligned}$$

To find the critical value, set

$$2\omega \left(\frac{1}{LC} - \omega^2 \right) - \frac{R^2}{L^2} \omega = 0.$$

Assuming the forcing function has a nonzero frequency,

$$\begin{aligned} \frac{2}{LC} - 2\omega^2 - \frac{R^2}{L^2} &= 0 \\ \omega^2 &= \frac{1}{LC} - \frac{R^2}{2L^2} \\ \omega &= \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}. \end{aligned}$$

The careful reader will show that the oscillator is underdamped only if $1/(LC) > R^2/(4L^2)$. The resonant frequency occurs only if $1/(LC) > R^2/(2L^2)$, a bit more than underdamped.

- (b) Start by differentiating $LI' + RI + (1/C)Q = A \cos \omega t$, remembering that $Q' = I$.

$$LI'' + RI' + \frac{1}{C}I = -A\omega \sin \omega t,$$

or, upon dividing by the inductance,

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = -\frac{A\omega}{L} \sin \omega t.$$

Substitute $z = a e^{i\omega t}$ in

$$z'' + \frac{R}{L}z' + \frac{1}{LC}z = -\frac{A\omega}{L} e^{i\omega t}$$

to get

$$\begin{aligned} \left[(i\omega)^2 + \frac{R}{L}(i\omega) + \frac{1}{LC} \right] a e^{i\omega t} &= -\frac{A\omega}{L} e^{i\omega t} \\ \left[\left(\frac{1}{LC} - \omega^2 \right) + \frac{R}{L} i\omega \right] z &= -\frac{A\omega}{L} e^{i\omega t} \\ z &= \frac{-A\omega/L}{(1/(LC) - \omega^2) + Ri\omega/L} e^{i\omega t} \\ &= \frac{-A}{(1/(\omega C) - L\omega) + Ri} e^{i\omega t}. \end{aligned}$$

If we write the denominator of this last expression in polar form

$$\sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2} e^{i\phi(\omega)},$$

then

$$z = \frac{-A}{\sqrt{(1/(\omega C) - L\omega)^2 + R^2}} e^{i(\omega t - \phi(\omega))}.$$

The imaginary part of this is the steady-state solution for the current.

$$I = \frac{-A}{\sqrt{(1/(\omega C) - L\omega)^2 + R^2}} \sin(\omega t - \phi(\omega))$$

Again, the current is maximized by maximizing the amplitude,

$$\frac{-A}{\sqrt{(1/(\omega C) - L\omega)^2 + R^2}}.$$

We could proceed as before by taking derivatives, but in this case we can note that the amplitude is maximized when the denominator is minimized. The denominator's smallest possible value is R , which occurs when

$$\begin{aligned}\frac{1}{\omega C} - L\omega &= 0 \\ LC\omega^2 &= 1 \\ \omega^2 &= \frac{1}{LC} \\ \omega &= \sqrt{\frac{1}{LC}}.\end{aligned}$$

44. If $L = 0.1$ H, $R = 100 \Omega$, $C = 10^{-3}$ F, and $E(t) = 100 \sin 120\pi t$, then

$$LI' + RI + \frac{1}{C}Q = E(t)$$

becomes

$$0.1I' + 100I + 1000Q = 100 \sin 120\pi t.$$

Using the fact that $Q' = I$ and dividing through by 0.1, this becomes

$$Q'' + 1000Q' + 10000Q = 1000 \sin 120\pi t.$$

The characteristic equation is $\lambda^2 + 1000\lambda + 10000 = 0$, with roots $\lambda = -500 \pm 200\sqrt{6}$. Thus, the homogeneous solution is

$$Q_h = C_1 e^{(-500-200\sqrt{6})t} + C_2 e^{(-500+200\sqrt{6})t},$$

or approximately

$$Q_h \approx C_1 e^{-989.8t} + C_2 e^{-10.20t}.$$

Next,

$$\begin{aligned}R &= \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}, \\ R &= \sqrt{(10^4 - (120\pi)^2)^2 + 4(500)^2(120\pi)^2}, \\ R &\approx 3.9947 \times 10^5.\end{aligned}$$

Hence, the gain is

$$G = \frac{1}{R} \approx 2.5033 \times 10^{-6}.$$

The phase is given by

$$\begin{aligned}\phi &= \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right), \\ \phi &= \operatorname{arccot}\left(\frac{10^4 - (120\pi)^2}{2(500)(120\pi)}\right), \\ \phi &\approx 1.9079.\end{aligned}$$

Thus, the steady-state solution is the real part of $(1/R)(1000)e^{i(120\pi t - \phi)}$, or

$$Q_p = 2.5033 \times 10^{-3} \sin(120\pi t - 1.9079).$$

The charge Q , as a function of time, is given approximately by

$$\begin{aligned}Q &= Q_h + Q_p, \\ &= C_1 e^{-989.8t} + C_2 e^{-10.20t} \\ &\quad + 2.5033 \times 10^{-3} \sin(120\pi t - 1.9079).\end{aligned}$$

There is no initial charge, so $Q(0) = 0$ and

$$\begin{aligned}0 &= C_1 + C_2 - 2.5033 \times 10^{-3} \sin(1.9079), \\ 0 &= C_1 + C_2 - 0.002362.\end{aligned}$$

Differentiating,

$$I = -989.8C_1e^{-989.8t} - 10.20C_2e^{-10.20t} \\ + 2.5033 \times 10^{-3}(120\pi) \cos(120\pi t - 1.9079).$$

There is no initial current, so $I(0) = 0$ and

$$0 = -989.8C_1 - 10.20C_2 \\ + 2.5033 \times 10^{-3}(120\pi) \cos(1.9079), \\ 0 = -989.8C_1 - 10.20C_2 - 0.3121.$$

Thus, $C_1 \approx -3.432 \times 10^{-4}$ and $C_2 \approx 2.705 \times 10^{-3}$ and the current is given by

$$I \approx -989.8(-3.432 \times 10^{-4})e^{-989.8t} \\ - 10.20(2.705 \times 10^{-3})e^{-10.20t} \\ + 2.5033 \times 10^{-3}(120\pi) \cos(120\pi t - 1.9079), \\ I \approx 0.3395e^{-989.8t} - 0.02733e^{-10.20t} \\ + 0.9437 \cos(120\pi t - 1.9079).$$

45. First, $m = 50 \text{ g} = 0.05 \text{ kg}$ and $y = 10 \text{ cm} = 0.1 \text{ m}$. Use Hooke's law to compute the spring constant.

$$k = \frac{F}{y} = \frac{mg}{y} = \frac{(0.05)(9.8)}{0.1} = 4.9 \text{ N/m}.$$

We are given that the damping force is opposite the velocity, with magnitude $0.1v$, and the driving force is $F(t) = 5 \cos 4.4t$. Thus, $my'' + \mu y' + ky = F(t)$ becomes

$$0.05y'' + 0.01y' + 4.9y = 5 \cos 4.4t,$$

or

$$y'' + 0.2y' + 98y = 100 \cos 4.4t.$$

This has characteristic polynomial $P(\lambda) = \lambda^2 + 0.2\lambda + 98$ and zeros $\lambda = -0.1 \pm 9.8990i$. Thus, the homogeneous solution is

$$y_h(t) = e^{-0.1t}(C_1 \cos 9.8990t + C_2 \sin 9.8990t).$$

To find the steady-state solution, substitute $z = ae^{i4.4t}$ in $z'' + 0.2z' + 98z = 100e^{i4.4t}$.

$$ae^{i4.4t} = 100e^{i4.4t} \\ (78.64 + 0.88i)z = 100e^{i4.4t}.$$

The coefficient of z has magnitude 78.6449 and phase 0.0112, so

$$78.6449e^{0.0112i}z = 100e^{i4.4t} \\ z = 1.2716e^{i(4.4t - 0.0112)}.$$

The real part of this, $y_p(t) = 1.2716 \cos(4.4t - 0.0112)$, is the steady-state solution. Thus, the solution is

$$y(t) = y_h(t) + y_p(t) \\ = e^{-0.1t}(C_1 \cos 9.8990t + C_2 \sin 9.8990t) \\ + 1.2716 \cos(4.4t - 0.0112).$$

The initial condition $y(0) = 0$ leads to

$$0 = C_1 + 1.2716 \cos(-0.0112)$$

and $C_1 = -1.2715$. Differentiating,

$$y'(t) = 9.8990e^{-0.1t}(-C_1 \sin 9.8990t \\ + C_2 \cos 9.8990t) \\ - 0.1e^{-0.1t}(C_1 \cos 9.8990t \\ + C_2 \sin 9.8990t) \\ - 5.5950 \sin(4.4t - 0.0112).$$

The initial condition $y'(0) = 0$ leads to

$$0 = 9.8990C_2 - 0.1C_1 - 5.5950 \sin(-0.0112)$$

and $C_2 = -0.0192$. Thus, the solution is

$$y(t) = e^{-0.1t}(-1.2715 \cos 9.8990t \\ - 0.0192 \sin 9.8990t) \\ + 1.2716 \cos(4.4t - 0.0112).$$

Chapter 5. The Laplace Transform

Section 5.1. The Definition of the Laplace Transform

1. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^{\infty} 3e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T 3e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left. \frac{-3e^{-st}}{s} \right|_0^T \\ &= \lim_{T \rightarrow \infty} \left(\frac{-3e^{-sT}}{s} + \frac{3}{s} \right) \\ &= \frac{3}{s}, \end{aligned}$$

provided $s > 0$.

2. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^{\infty} -2e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T -2e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left. \frac{2e^{-st}}{s} \right|_0^T \\ &= \lim_{T \rightarrow \infty} \left[\frac{2e^{-sT}}{s} - \frac{2}{s} \right] \\ &= -\frac{2}{s}, \end{aligned}$$

provided, $s > 0$.

3. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-2t} e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-(s+2)t} dt \\ &= \lim_{T \rightarrow \infty} \left. \frac{-e^{-(s+2)t}}{s+2} \right|_0^T \\ &= \lim_{T \rightarrow \infty} \left[\frac{-e^{-(s+2)T}}{s+2} + \frac{1}{s+2} \right] \\ &= \frac{1}{s+2}, \end{aligned}$$

provided $s > -2$.

4. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{3t} e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-(s-3)t} dt \\ &= \lim_{T \rightarrow \infty} \left. \frac{e^{-(s-3)t}}{s-3} \right|_0^T \\ &= -\lim_{T \rightarrow \infty} \left[\frac{e^{-(s-3)T}}{s-3} - \frac{1}{s-3} \right] \\ &= \frac{1}{s-3}, \end{aligned}$$

provided $s > 3$.

5. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^{\infty} (\cos 2t) e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T (\cos 2t) e^{-st} dt. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int e^{-st} \cos 2t \, dt &= \frac{1}{2} \int e^{-st} d(\sin 2t) \\ &= \frac{1}{2} \left[e^{-st} \sin 2t - \int \sin 2t d(e^{-st}) \right] \\ &= \frac{1}{2} \left[e^{-st} \sin 2t + s \int e^{-st} \sin 2t \, dt \right]. \end{aligned}$$

Perform another integration by parts on the integral on the right.

$$\begin{aligned} \int e^{-st} \cos 2t \, dt &= \frac{1}{2} e^{-st} \sin 2t + \frac{s}{2} \left(-\frac{1}{2} \right) \int e^{-st} d(\cos 2t) \\ &= \frac{1}{2} e^{-st} \sin 2t - \frac{s}{4} \left[e^{-st} \cos 2t - \int \cos 2t d(e^{-st}) \right] \\ &= \frac{1}{2} e^{-st} \sin 2t - \frac{s}{4} \left[e^{-st} \cos 2t + s \int e^{-st} \cos 2t \, dt \right] \\ &= \frac{1}{2} e^{-st} \sin 2t - \frac{s}{4} e^{-st} \cos 2t - \frac{s^2}{4} \int e^{-st} \cos 2t \, dt. \end{aligned}$$

Transferring the last integral on the right to the left side of the equation,

$$\begin{aligned} \left(1 + \frac{s^2}{4}\right) \int e^{-st} \cos 2t &= \frac{1}{2} e^{-st} \sin 2t - \frac{s}{4} e^{-st} \cos 2t \\ \int e^{-st} \cos 2t \, dt &= \frac{2}{4 + s^2} e^{-st} \sin 2t - \frac{s}{4 + s^2} e^{-st} \cos 2t. \end{aligned}$$

Inserting the limits 0 and T , we obtain

$$\begin{aligned} F(s) &= \lim_{T \rightarrow \infty} \left(\frac{2}{4 + s^2} e^{-st} \sin 2t - \frac{s}{4 + s^2} e^{-st} \cos 2t \right) \Big|_{t=0}^T \\ &= \lim_{T \rightarrow \infty} \left(\frac{2}{4 + s^2} e^{-sT} \sin 2T - \frac{s}{4 + s^2} e^{-sT} \cos 2T + \frac{s}{4 + s^2} \right) \\ &= \frac{s}{4 + s^2}, \end{aligned}$$

provided $s > 0$.

6. Using Definition 1.1

$$\begin{aligned} F(s) &= \int_0^\infty (\sin 3t) e^{-st} \, dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-st} \sin 3t \, dt \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int e^{-st} \sin 3t \, dt &= -\frac{1}{s} e^{-st} \sin 3t + \frac{3}{s} \int e^{-st} \cos 3t \, dt. \end{aligned}$$

Perform another integration by parts on the integral on the right.

$$\begin{aligned} \int e^{-st} \sin 3t \, dt &= -\frac{1}{s} e^{-st} \sin 3t \\ &\quad + \frac{3}{s} \left[-\frac{1}{s} e^{-st} \cos 3t - \frac{3}{s} \int e^{-st} \sin 3t \, dt \right] \\ &= -\frac{1}{s} e^{-st} \sin 3t - \frac{3}{s^2} e^{-st} \cos 3t \\ &\quad - \frac{9}{s^2} \int e^{-st} \sin 3t \, dt \end{aligned}$$

Transferring the last integral to the left side of the

equation,

$$\begin{aligned} & \left(1 + \frac{9}{s^2}\right) \int e^{-st} \sin 3t \, dt \\ &= -\frac{1}{s} e^{-st} \sin 3t - \frac{3}{s^2} e^{-st} \cos 3t. \\ & \int e^{-st} \sin 3t \, dt \\ &= -\frac{s e^{-st}}{s^2 + 9} \sin 3t - \frac{3}{s^2 + 9} e^{-st} \cos 3t. \end{aligned}$$

Thus,

$$\begin{aligned} F(s) &= \lim_{T \rightarrow \infty} e^{-st} \left[-\frac{s}{s^2 + 9} \sin 3t \right. \\ &\quad \left. - \frac{3}{s^2 + 9} \cos 3t \right]_0^T \\ &= \frac{3}{s^2 + 9}, \end{aligned}$$

provided $s > 0$.

7. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^\infty t e^{2t} e^{-st} \, dt \\ &= \int_0^\infty t e^{-(s-2)t} \, dt \\ &= \lim_{T \rightarrow \infty} \int_0^T t e^{-(s-2)t} \, dt. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} & \int t e^{-(s-2)t} \, dt \\ &= \frac{-1}{s-2} \int t d(e^{-(s-2)t}) \\ &= -\frac{1}{s-2} \left[t e^{-(s-2)t} - \int e^{-(s-2)t} \, dt \right] \\ &= -\frac{1}{s-2} \left[t e^{-(s-2)t} + \frac{e^{-(s-2)t}}{s-2} \right] \\ &= \frac{-t e^{-(s-2)t}}{s-2} - \frac{e^{-(s-2)t}}{(s-2)^2} \end{aligned}$$

Inserting the limits 0 and T ,

$$\begin{aligned} F(s) &= \lim_{T \rightarrow \infty} \left[\frac{-T e^{-(s-2)T}}{s-2} - \frac{e^{-(s-2)T}}{(s-2)^2} \right. \\ &\quad \left. + \frac{1}{(s-2)^2} \right] \\ &= \frac{1}{(s-2)^2}, \end{aligned}$$

provided $s > 2$.

8. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^\infty t e^{-3t} e^{-st} \, dt \\ &= \lim_{T \rightarrow \infty} \int_0^T t e^{-(s+3)t} \, dt \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int t e^{-(s+3)t} \, dt &= -\frac{t}{s+3} e^{-(s+3)t} \\ &\quad - \frac{1}{(s+3)^2} e^{-(s+3)t}. \end{aligned}$$

Thus,

$$\begin{aligned} F(s) &= \lim_{T \rightarrow \infty} \left[-\frac{t}{s+3} e^{-(s+3)t} \right. \\ &\quad \left. - \frac{1}{(s+3)^2} e^{-(s+3)t} \right]_0^T \\ &= \frac{1}{(s+3)^2}, \end{aligned}$$

provided $s > -3$.

9. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^\infty (e^{2t} \cos 3t) e^{-st} \, dt \\ &= \int_0^\infty e^{-(s-2)t} \cos 3t \, dt. \end{aligned}$$

Integrating by parts,

$$\begin{aligned}
 & \int e^{-(s-2)t} \cos 3t \, dt \\
 &= \frac{1}{3} \int e^{-(s-2)t} d(\sin 3t) \\
 &= \frac{1}{3} \left[e^{-(s-2)t} \sin 3t - \int \sin 3t \, d(e^{-(s-2)t}) \right] \\
 &= \frac{1}{3} e^{-(s-2)t} \sin 3t + \frac{s-2}{3} \int e^{-(s-2)t} \sin 3t \, dt.
 \end{aligned}$$

We will compute the integral by parts.

$$\begin{aligned}
 & \int e^{-(s-2)t} \cos 3t \, dt \\
 &= \frac{1}{3} e^{-(s-2)t} \sin 3t \\
 &\quad + \frac{s-2}{3} \cdot \frac{-1}{3} \int e^{-(s-2)t} d(\cos 3t) \\
 &= \frac{1}{3} e^{-(s-2)t} \sin 3t \\
 &\quad - \frac{s-2}{9} \int e^{-(s-2)t} d(\cos 3t) \\
 &= \frac{1}{3} e^{-(s-2)t} \sin 3t - \frac{s-2}{9} e^{-(s-2)t} \cos 3t \\
 &\quad + \frac{s-2}{9} \int \cos 3t \, d(e^{-(s-2)t}) \\
 &= \frac{1}{3} e^{-(s-2)t} \sin 3t - \frac{s-2}{9} e^{-(s-2)t} \cos 3t \\
 &\quad - \frac{(s-2)^2}{9} \int e^{-(s-2)t} \cos 3t \, dt
 \end{aligned}$$

Combining the integral on the right with the one on the left,

$$\begin{aligned}
 & \left[1 + \frac{(s-2)^2}{9} \right] \int e^{-(s-2)t} \cos 3t \, dt \\
 &= \frac{1}{3} e^{-(s-2)t} \sin 3t - \frac{s-2}{9} e^{-(s-2)t} \cos 3t,
 \end{aligned}$$

so

$$\begin{aligned}
 & \int e^{-(s-2)t} \cos 3t \, dt \\
 &= \frac{9}{9 + (s-2)^2} \left[\frac{1}{3} e^{-(s-2)t} \sin 3t \right. \\
 &\quad \left. - \frac{s-2}{9} e^{-(s-2)t} \cos 3t \right].
 \end{aligned}$$

Inserting the limits 0 and T ,

$$\begin{aligned}
 F(s) &= \lim_{T \rightarrow \infty} \left[\frac{3}{9 + (s-2)^2} e^{-(s-2)T} \cos 3T \right. \\
 &\quad \left. + \frac{s-2}{9 + (s-2)^2} \right] \\
 &= \frac{s-2}{9 + (s-2)^2},
 \end{aligned}$$

provided $s > 2$.

10. Using Definition 1.1,

$$\begin{aligned}
 F(s) &= \int_0^\infty (e^{-3t} \sin 2t) e^{-st} \, dt \\
 &= \lim_{T \rightarrow \infty} \int_0^T e^{-(s+3)t} \sin 2t \, dt.
 \end{aligned}$$

Integrating by parts,

$$\begin{aligned}
 & \int e^{-(s+3)t} \sin 2t \, dt \\
 &= -\frac{1}{s+3} e^{-(s+3)t} \sin 2t \\
 &\quad + \frac{2}{s+3} \int e^{-(s+3)t} \cos 2t \, dt.
 \end{aligned}$$

Integrating by parts a second time,

$$\begin{aligned}
 & \int e^{-(s+3)t} \sin 2t \, dt \\
 &= -e^{-(s+3)t} \left[\frac{1}{s+3} \sin 2t + \frac{2}{(s+3)^2} \cos 2t \right] \\
 &\quad - \frac{4}{(s+3)^2} \int e^{-(s+3)t} \sin 2t \, dt
 \end{aligned}$$

Transferring the last integral on the right to the left side of the equation,

$$\begin{aligned} \left(1 + \frac{4}{(s+3)^2}\right) \int e^{-(s+3)t} \sin 2t \, dt \\ = -e^{-(s+3)t} \left[\frac{\sin 2t}{s+3} + \frac{2 \cos 2t}{(s+3)^2} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \int e^{-(s+3)t} \sin 2t \, dt \\ = -\frac{e^{-(s+3)t}}{(s+3)^2 + 4} [(s+3) \sin 2t + 2 \cos 2t]. \end{aligned}$$

Hence,

$$\begin{aligned} F(s) &= -\lim_{T \rightarrow \infty} \frac{e^{-(s+3)T}}{(s+3)^2 + 4} [(s+3) \sin 2T \\ &\quad + 2 \cos 2T] \Big|_0^T \\ &= \frac{2}{(s+3)^2 + 4}, \end{aligned}$$

provided $s > -3$.

11. (a) Using Definition 1.1,

$$\begin{aligned} G(s) &= \int_0^\infty t^2 e^{-st} \, dt \\ &= \lim_{T \rightarrow \infty} \int_0^T t^2 e^{-st} \, dt. \end{aligned}$$

Integrating by parts,

$$\int t^2 e^{-st} \, dt = -\frac{t^2 e^{-st}}{s} + \frac{2}{s} \int t e^{-st} \, dt.$$

Using integration by parts on the second integral,

$$\begin{aligned} \int t^2 e^{-st} \, dt &= -\frac{t^2 e^{-st}}{s} + \frac{2}{s} \left[-\frac{t e^{-st}}{s} + \frac{1}{s} \int e^{-st} \, dt \right] \\ &= -\frac{t^2 e^{-st}}{s} - \frac{2t e^{-st}}{s^2} + \frac{2}{s^2} \int e^{-st} \, dt \\ &= -\frac{t^2 e^{-st}}{s} - \frac{2t e^{-st}}{s^2} - \frac{2e^{-st}}{s^3}. \end{aligned}$$

Inserting the limits 0 and T ,

$$\begin{aligned} G(s) &= \lim_{T \rightarrow \infty} \left[-\frac{T^2 e^{-sT}}{s} - \frac{2T e^{-sT}}{s^2} \right. \\ &\quad \left. - \frac{2e^{-sT}}{s^3} + \frac{2}{s^3} \right] \\ &= \frac{2}{s^3} = \frac{2!}{s^3}. \end{aligned}$$

- (b) Using Definition 1.1,

$$\begin{aligned} H(s) &= \int_0^\infty t^3 e^{-st} \, dt \\ &= \lim_{T \rightarrow \infty} \int_0^T t^3 e^{-st} \, dt. \end{aligned}$$

Integrating by parts,

$$\int t^3 e^{-st} \, dt = -\frac{1}{s} t^3 e^{-st} + \frac{3}{s} \int t^2 e^{-st} \, dt.$$

In part (a), we evaluated the integral on the right.

$$\begin{aligned} \int t^3 e^{-st} \, dt &= -\frac{t^3 e^{-st}}{s} \\ &\quad + \frac{3}{s} \left[-\frac{t^2 e^{-st}}{s} - \frac{2t e^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right] \\ &= -e^{-st} \left[\frac{t^3}{s} + \frac{3t^2}{s^2} + \frac{6t}{s^3} + \frac{6}{s^4} \right]. \end{aligned}$$

Inserting the limits 0 and T ,

$$\begin{aligned} H(s) &= -\lim_{T \rightarrow \infty} e^{-sT} \left[\frac{T^3}{s} + \frac{3T^2}{s^2} + \frac{6T}{s^3} + \frac{6}{s^4} \right] \\ &\quad + \frac{6}{s^4} \\ &= \frac{6}{s^4} = \frac{3!}{s^4}. \end{aligned}$$

- (c) We've shown that $\mathcal{L}\{t\} = 1/s$. Assume that $\mathcal{L}\{t^k\} = k!/s^{k+1}$. Now,

$$\begin{aligned}\mathcal{L}\{t^{k+1}\}(s) &= \int_0^\infty t^{k+1} e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T t^{k+1} e^{-st} dt.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_0^T t^{k+1} e^{-st} dt &= -\frac{t^{k+1} e^{-st}}{s} \Big|_0^T + \frac{k+1}{s} \int_0^T t^k e^{-st} dt \\ &= \frac{-T^{k+1} e^{-sT}}{s} + \frac{k+1}{s} \int_0^T t^k e^{-st} dt.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{L}\{t^{k+1}\}(s) &= \lim_{T \rightarrow \infty} \left[\frac{-T^{k+1} e^{-sT}}{s} + \frac{k+1}{s} \int_0^T t^k e^{-st} dt \right] \\ &= \frac{k+1}{s} \int_0^\infty t^k e^{-st} dt.\end{aligned}$$

But this integral is the Laplace transform $\mathcal{L}\{t^k\}$, which by our induction hypothesis,

$$\begin{aligned}\mathcal{L}\{t^{k+1}\}(s) &= \frac{k+1}{s} \cdot \frac{k!}{s^{k+1}} \\ &= \frac{(k+1)!}{s^{k+2}}.\end{aligned}$$

Thus, by induction,

$$\mathcal{L}\{t^k\}(s) = \frac{k!}{s^{k+1}}$$

for all $k = 1, 2, 3, \dots$

12. Using Definition 1.1,

$$\begin{aligned}F(s) &= \int_0^\infty (\cos \omega t) e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-st} \cos \omega t dt.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int e^{-st} \cos \omega t dt &= -\frac{1}{s} e^{-st} \cos \omega t - \frac{\omega}{s} \int e^{-st} \sin \omega t dt.\end{aligned}$$

Integrating the second integral by parts,

$$\begin{aligned}\int e^{-st} \cos \omega t dt &= -\frac{1}{s} e^{-st} \cos \omega t \\ &\quad - \frac{\omega}{s} \left[-\frac{1}{s} e^{-st} \sin \omega t + \frac{\omega}{s} \int e^{-st} \cos \omega t dt \right] \\ &= -\frac{1}{s} e^{-st} \cos \omega t + \frac{\omega}{s^2} e^{-st} \sin \omega t \\ &\quad - \frac{\omega^2}{s^2} \int e^{-st} \cos \omega t dt.\end{aligned}$$

Transferring the last integral on the right to the left hand side of the equation

$$\begin{aligned}\left(1 + \frac{\omega^2}{s^2}\right) \int e^{-st} \cos \omega t dt &= -\frac{1}{s} e^{-st} \cos \omega t + \frac{\omega}{s^2} e^{-st} \sin \omega t\end{aligned}$$

Hence,

$$\begin{aligned}\int e^{-st} \cos \omega t dt &= \frac{e^{-st}}{s^2 + \omega^2} [-s \cos \omega t + \omega \sin \omega t]\end{aligned}$$

Thus,

$$\begin{aligned}F(s) &= \lim_{T \rightarrow \infty} \frac{e^{-st}}{s^2 + \omega^2} [-s \cos \omega t + \omega \sin \omega t] \Big|_0^T \\ &= \frac{s}{s^2 + \omega^2},\end{aligned}$$

provided $s > 0$.

13. Using Definition 1.1,

$$\begin{aligned}F(s) &= \int_0^\infty (e^{at} \cos \omega t) e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-(s-a)t} \cos \omega t dt.\end{aligned}$$

Integrating by parts,

$$\begin{aligned} & \int e^{-(s-a)t} \cos \omega t \, dt \\ &= \frac{e^{-(s-a)t} \sin \omega t}{\omega} \\ &+ \frac{s-a}{\omega} \int e^{-(s-a)t} \sin \omega t \, dt. \end{aligned}$$

Using integration by parts on the integral on the right,

$$\begin{aligned} & \int e^{-(s-a)t} \sin \omega t \, dt \\ &= \frac{-e^{-(s-a)t} \cos \omega t}{\omega} \\ &- \frac{s-a}{\omega} \int e^{-(s-a)t} \cos \omega t \, dt \end{aligned}$$

Substituting into the previous formula,

$$\begin{aligned} & \int e^{-(s-a)t} \cos \omega t \, dt \\ &= \frac{e^{-(s-a)t}}{\omega^2} [\omega \sin \omega t - (s-a) \cos \omega t] \\ &- \frac{(s-a)^2}{\omega^2} \int e^{-(s-a)t} \cos \omega t \, dt \end{aligned}$$

Transferring the last integral on the right to the left side of the equation,

$$\begin{aligned} & \frac{\omega^2 + (s-a)^2}{\omega^2} \int e^{-(s-a)t} \cos \omega t \, dt \\ &= \frac{e^{-(s-a)t}}{\omega^2} [\omega \sin \omega t - (s-a) \cos \omega t], \end{aligned}$$

Hence,

$$\begin{aligned} & \int e^{-(s-a)t} \cos \omega t \, dt \\ &= \frac{e^{-(s-a)t}}{\omega^2 + (s-a)^2} [\omega \sin \omega t - (s-a) \cos \omega t]. \end{aligned}$$

Inserting the limits 0 and T ,

$$\begin{aligned} F(s) &= \lim_{T \rightarrow \infty} \frac{e^{-(s-a)t}}{\omega^2 + (s-a)^2} \\ &\quad \times [\omega \sin \omega t - (s-a) \cos \omega t] \Big|_0^T \\ &= \frac{s-a}{\omega^2 + (s-a)^2}, \end{aligned}$$

provided $s > a$.

14. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^\infty (e^{at} \sin \omega t) e^{-st} \, dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-(s-a)t} \sin \omega t \, dt. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} & \int e^{-(s-a)t} \sin \omega t \, dt \\ &= -\frac{e^{-(s-a)t} \sin \omega t}{s-a} \\ &+ \frac{\omega}{s-a} \int e^{-(s-a)t} \cos \omega t \, dt. \end{aligned}$$

Integrating the last integral on the right by parts,

$$\begin{aligned} & \int e^{-(s-a)t} \cos \omega t \, dt \\ &= -\frac{e^{-(s-a)t} \cos \omega t}{s-a} \\ &- \frac{\omega}{s-a} \int e^{-(s-a)t} \sin \omega t \, dt. \end{aligned}$$

Substituting into the previous formula, we get

$$\begin{aligned} & \int e^{-(s-a)t} \sin \omega t \, dt \\ &= -\frac{e^{-(s-a)t}}{(s-a)^2} [(s-a) \sin \omega t + \omega \cos \omega t] \\ &- \frac{\omega^2}{(s-a)^2} \int e^{-(s-a)t} \sin \omega t \, dt. \end{aligned}$$

Transferring the last integral to the left side of the equation,

$$\begin{aligned} & \frac{(s-a)^2 + \omega^2}{(s-a)^2} \int e^{-(s-a)t} \sin \omega t \, dt \\ &= -\frac{e^{-(s-a)t}}{(s-a)^2} [(s-a) \sin \omega t + \omega \cos \omega t], \end{aligned}$$

so

$$\begin{aligned} & \int e^{-(s-a)t} \sin \omega t \, dt \\ &= -\frac{e^{-(s-a)t}}{(s-a)^2 + \omega^2} [(s-a) \sin \omega t + \omega \cos \omega t]. \end{aligned}$$

Thus,

$$\begin{aligned} F(s) &= \lim_{T \rightarrow \infty} \int_0^T e^{-(s-a)t} \sin \omega t \, dt \\ &= \frac{\omega}{(s-a)^2 + \omega^2}, \end{aligned}$$

provided $s > a$.

15. Using Table 1,

$$\begin{aligned} \mathcal{L}(3)(s) &= \int_0^\infty 3e^{-st} \, dt \\ &= 3 \int_0^\infty 1e^{-st} \, dt \\ &= 3 \mathcal{L}(1)(s) \\ &= 3 \cdot \frac{1}{s} \\ &= \frac{3}{s}, \end{aligned}$$

provided $s > 0$.

16. Using Table 1,

$$\begin{aligned} \mathcal{L}(-2)(s) &= \int_0^\infty -2e^{-st} \, dt \\ &= -2 \int_0^\infty 1e^{-st} \, dt \\ &= -2 \mathcal{L}(1)(s) \\ &= -2 \cdot \frac{1}{s} \\ &= -\frac{2}{s}, \end{aligned}$$

provided $s > 0$.

17. Using Table 1,

$$\mathcal{L}(e^{-2t})(s) = \frac{1}{s - (-2)} = \frac{1}{s + 2},$$

provided $s > -2$.

18. Using Table 1,

$$\mathcal{L}(e^{3t})(s) = \frac{1}{s - 3},$$

provided $s > 3$.

19. Using Table 1,

$$\mathcal{L}(\cos 2t)(s) = \frac{s}{s^2 + 2^2} = \frac{s}{s^2 + 4},$$

provided $s > 0$.

20. Using Table 1,

$$\mathcal{L}(\sin 3t) = \frac{3}{s^2 + 3^2} = \frac{3}{s^2 + 9},$$

provided $s > 0$.

21. Using Table 1,

$$\mathcal{L}(te^{2t})(s) = \frac{1!}{(s - 2)^2} = \frac{1}{(s - 2)^2},$$

provided $s > 2$.

22. Using Table 1,

$$\mathcal{L}(te^{-3t})(s) = \frac{1!}{(s - (-3))^2} = \frac{1}{(s + 3)^2},$$

provided $s > -3$.

23. Using Table 1,

$$\begin{aligned} \mathcal{L}(e^{2t} \cos 3t)(s) &= \frac{s - 2}{(s - 2)^2 + 3^2} \\ &= \frac{s - 2}{(s - 2)^2 + 9}, \end{aligned}$$

provided $s > 2$.

24. Using Table 1,

$$\begin{aligned} \mathcal{L}(e^{-3t} \sin 2t)(s) &= \frac{2}{(s - (-3))^2 + 2^2} \\ &= \frac{2}{(s + 3)^2 + 4}, \end{aligned}$$

provided $s > -3$.

25. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^{\infty} f(t)e^{-st} dt \\ &= \int_0^2 0e^{-st} dt + \int_2^{\infty} 2e^{-st} dt \\ &= 2 \lim_{T \rightarrow \infty} \left. \frac{-e^{-st}}{s} \right|_2^T \\ &= \frac{2e^{-2s}}{s}, \end{aligned}$$

provided $s > 0$.

26. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^{\infty} f(t)e^{-st} dt \\ &= \int_0^4 0e^{-st} dt + \int_4^{\infty} 5e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left. -\frac{5}{s}e^{-st} \right|_4^T \\ &= \frac{5}{s}e^{-4s}, \end{aligned}$$

provided $s > 0$.

27. Using Definition 1.1,

$$F(s) = \int_0^2 te^{-st} dt + \int_2^{\infty} 2e^{-st} dt.$$

Using integration by parts, we get

$$\begin{aligned} \int_0^2 te^{-st} dt &= \left. \frac{-te^{-st}}{s} \right|_0^2 + \frac{1}{s} \int_0^2 e^{-st} dt \\ &= \frac{-2e^{-2s}}{s} - \frac{1}{s^2} e^{-st} \Big|_0^2 \\ &= -\frac{2e^{-2s}}{s} - \frac{e^{-2s} - 1}{s^2} \end{aligned}$$

For the second integral we have

$$\begin{aligned} \int_2^{\infty} 2e^{-st} dt &= \lim_{T \rightarrow \infty} \int_2^T 2e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left. \frac{-2e^{-st}}{s} \right|_2^T \\ &= \frac{2e^{-2s}}{s} \end{aligned}$$

provided $s > 0$. Hence

$$F(s) = -\frac{e^{-2s} - 1}{s^2},$$

provided $s > 0$.

28. Using Definition 1.1,

$$\begin{aligned} F(s) &= \int_0^{\infty} f(t)e^{-st} dt \\ &= \int_0^3 te^{-st} dt + \int_3^{\infty} 3e^{-st} dt \end{aligned}$$

Use integration by parts on the first integral.

$$\begin{aligned} \int_0^3 te^{-st} dt &= \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} \right]_0^3 \\ &= -\frac{3}{s}e^{-3s} - \frac{1}{s^2}e^{-3s} + \frac{1}{s^2} \end{aligned}$$

The second integral is

$$\begin{aligned} \int_3^{\infty} 3e^{-st} dt &= \lim_{T \rightarrow \infty} \int_3^T 3e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left. -\frac{3}{s}e^{-st} \right|_3^T \\ &= \lim_{T \rightarrow \infty} \left[-\frac{3}{s}e^{-sT} + \frac{3}{s}e^{-3s} \right] \\ &= \frac{3}{s}e^{-3s}. \end{aligned}$$

Therefore,

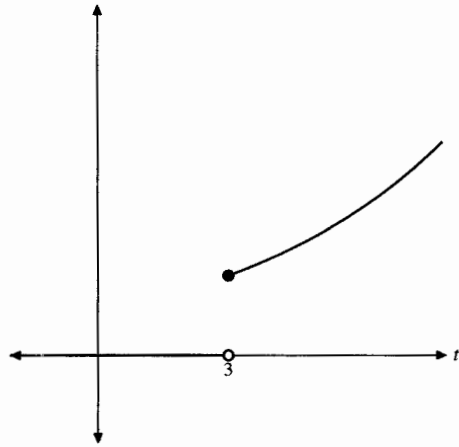
$$\begin{aligned} F(s) &= -\frac{3}{s}e^{-3s} - \frac{1}{s^2}e^{-3s} + \frac{1}{s^2} + \frac{3}{s}e^{-3s} \\ &= -\frac{1}{s^2}e^{-3s} + \frac{1}{s^2}, \end{aligned}$$

provided $s > 0$.

29. Because

$$y(t) = H(t-3)e^{0.2t} = \begin{cases} 0, & t < 3 \\ e^{0.2t}, & t \geq 3, \end{cases}$$

the graph is:



The Laplace transform is

$$\begin{aligned}
 Y(s) &= \int_0^{\infty} y(t)e^{-st} dt \\
 &= \int_0^3 0e^{-st} dt + \int_3^{\infty} e^{0.2t} e^{-st} dt \\
 &= \lim_{T \rightarrow \infty} \int_3^T e^{-(s-0.2)t} dt \\
 &= \lim_{T \rightarrow \infty} \left[\frac{-e^{-(s-0.2)t}}{s-0.2} \right]_3^T \\
 &= \lim_{T \rightarrow \infty} \left[\frac{-e^{-(s-0.2)T}}{s-0.2} + \frac{e^{-3(s-0.2)}}{s-0.2} \right] \\
 &= \frac{e^{-3(s-0.2)}}{s-0.2},
 \end{aligned}$$

provided $s > 0.2$.

30. (a) If there exists a C and an a such that $|f(t)| \leq Ce^{at}$ for all $t > 0$, then

$$\begin{aligned}
 |f(t)e^{-st}| &= |f(t)||e^{-st}| \\
 &\leq Ce^{at}e^{-st} \\
 &= Ce^{-(s-a)t},
 \end{aligned}$$

for all $t > 0$.

- (b) First,

$$\begin{aligned}
 &\int_0^{\infty} Ce^{-(s-a)t} dt \\
 &= \lim_{T \rightarrow \infty} \int_0^T Ce^{-(s-a)t} dt \\
 &= \lim_{T \rightarrow \infty} \left[\frac{-C}{s-a} e^{-(s-a)t} \right]_0^T \\
 &= \lim_{T \rightarrow \infty} \left[\frac{-C}{s-a} e^{-(s-a)T} + \frac{C}{s-a} \right] \\
 &= \frac{C}{s-a},
 \end{aligned}$$

provided $s > a$. Now, by part (a)

$$\begin{aligned}
 \left| \int_0^{\infty} f(t)e^{-st} dt \right| &\leq \int_0^{\infty} |f(t)e^{-st}| dt \\
 &\leq \int_0^{\infty} Ce^{-(s-a)t} dt \\
 &\leq \frac{C}{s-a}
 \end{aligned}$$

Because the absolute value of the integral is finite, so too is the regular integral.

31. (a) By a property of integrals,

$$\begin{aligned}
 |F(s)| &= \left| \int_0^{\infty} f(t)e^{-st} dt \right| \\
 &\leq \int_0^{\infty} |f(t)e^{-st}| dt \\
 &= \int_0^{\infty} |f(t)|e^{-st} dt.
 \end{aligned}$$

But, $|f(t)| \leq Ce^{at}$ for all $t > 0$, so

$$\begin{aligned}
 |F(s)| &\leq \int_0^{\infty} Ce^{at}e^{-st} dt \\
 &= C \int_0^{\infty} e^{at}e^{-st} dt \\
 &= C \mathcal{L}(e^{at})(s).
 \end{aligned}$$

But, in Table 1, $\mathcal{L}(e^{at})(s) = 1/(s-a)$, provided $s > 0$. Thus,

$$F(s) \leq \frac{C}{s-a},$$

provided $s > a$.

(b) By part (a),

$$|F(s)| \leq \frac{C}{s-a}.$$

Thus,

$$\lim_{s \rightarrow \infty} |F(s)| \leq \lim_{s \rightarrow \infty} \frac{C}{s-a} = 0.$$

Thus, $\lim_{s \rightarrow \infty} |F(s)| = 0$ and consequently,

$$\lim_{s \rightarrow \infty} F(s) = 0.$$

The function

$$F(s) = \frac{s+4}{s-2}$$

cannot be a Laplace transform of any function because

$$\begin{aligned} \lim_{s \rightarrow \infty} F(s) &= \lim_{s \rightarrow \infty} \frac{s+4}{s-2} \\ &= \lim_{s \rightarrow \infty} \frac{1 + \frac{4}{s}}{1 - \frac{2}{s}} \\ &= 1. \end{aligned}$$

32. For purposes of contradiction, assume that there are constants C and a so that

$$|e^{t^2}| \leq Ce^{at}$$

for all $t > 0$. Then

$$e^{t^2} \leq |e^{t^2}| \leq Ce^{at},$$

for all $t > 0$. Taking the logarithm of both sides of the inequality,

$$\begin{aligned} \ln e^{t^2} &\leq \ln Ce^{at} \\ t^2 &\leq \ln C + at. \end{aligned}$$

Thus,

$$t^2 - at - \ln C \leq 0$$

for all $t > 0$. But, this is impossible because $t^2 - at - \ln C$ is an upward opening parabola which is positive for all t -values greater than the value of its rightmost intercept. Therefore, e^{t^2} is *not* of exponential order.

—————x—————

Section 5.2. Basic Properties of the Laplace Transform

1. Using linearity and Table 1,

$$\begin{aligned} \mathcal{L}\{3t^2\}(s) &= 3 \mathcal{L}\{t^2\}(s) \\ &= 3 \cdot \frac{2!}{s^3} \\ &= \frac{6}{s^3}, \end{aligned}$$

provided $s > 0$.

2. Using Proposition 2.7 and Table 1,

$$\begin{aligned} \mathcal{L}(5e^{3t})(s) &= 5 \mathcal{L}(e^{3t})(s) \\ &= 5 \cdot \frac{1}{s-3} \\ &= \frac{5}{s-3}, \end{aligned}$$

provided $s > 3$.

3. Using linearity and Table 1,

$$\begin{aligned}
\mathcal{L}\{t^2 + 4t + 5\}(s) &= \mathcal{L}\{t^2\}(s) + 4\mathcal{L}\{t\}(s) + 5\mathcal{L}\{1\}(s) \\
&= \frac{2!}{s^3} + 4\left(\frac{1}{s^2}\right) + 5\left(\frac{1}{s}\right) \\
&= \frac{2}{s^3} + \frac{4}{s^2} + \frac{5}{s} \\
&= \frac{2 + 4s + 5s^2}{s^3},
\end{aligned}$$

provided $s > 0$.

4. Using Proposition 2.7 and Table 1,

$$\begin{aligned}
\mathcal{L}(3 - 5t - 11t^3)(s) &= 3\mathcal{L}(1)(s) - 5\mathcal{L}(t)(s) - 11\mathcal{L}(t^3)(s) \\
&= 3 \cdot \frac{1}{s} - 5 \cdot \frac{1!}{s^2} - 11 \cdot \frac{3!}{s^4} \\
&= \frac{3}{s} - \frac{5}{s^2} - \frac{66}{s^4},
\end{aligned}$$

provided $s > 0$.

5. Using linearity and Table 1,

$$\begin{aligned}
\mathcal{L}\{-2\cos t + 4\sin 3t\}(s) &= -2\mathcal{L}\{\cos t\}(s) + 4\mathcal{L}\{\sin 3t\}(s) \\
&= -2\left(\frac{s}{s^2 + 1}\right) + 4\left(\frac{3}{s^2 + 9}\right) \\
&= \frac{-2s(s^2 + 9) + 12(s^2 + 1)}{(s^2 + 1)(s^2 + 9)} \\
&= \frac{-2s^3 + 12s^2 - 18s + 12}{(s^2 + 1)(s^2 + 9)},
\end{aligned}$$

provided $s > 0$.

6. Using Proposition 2.7 and Table 1,

$$\begin{aligned}
\mathcal{L}(2\sin 3t + 3\cos 5t)(s) &= 2\mathcal{L}(\sin 3t)(s) + 3\mathcal{L}(\cos 5t)(s) \\
&= 2 \cdot \frac{3}{s^2 + 3^2} + 3 \cdot \frac{s}{s^2 + 5^2} \\
&= \frac{6}{s^2 + 9} + \frac{3s}{s^2 + 25},
\end{aligned}$$

provided $s > 0$.

7. Using linearity and Table 1,

$$\begin{aligned}
\mathcal{L}\{\cosh \omega t\}(s) &= \mathcal{L}\left\{\frac{e^{\omega t} + e^{-\omega t}}{2}\right\}(s) \\
&= \frac{1}{2}\mathcal{L}\{e^{\omega t}\}(s) + \frac{1}{2}\mathcal{L}\{e^{-\omega t}\}(s) \\
&= \frac{1}{2}\left(\frac{1}{s - \omega}\right) + \frac{1}{2}\left(\frac{1}{s + \omega}\right) \\
&= \frac{1}{2}\left[\frac{s + \omega + s - \omega}{(s - \omega)(s + \omega)}\right] \\
&= \frac{s}{s^2 - \omega^2}.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathcal{L}\{\sinh \omega t\}(s) &= \mathcal{L}\left\{\frac{e^{\omega t} - e^{-\omega t}}{2}\right\}(s) \\
&= \frac{1}{2}\mathcal{L}\{e^{\omega t}\}(s) - \frac{1}{2}\mathcal{L}\{e^{-\omega t}\}(s) \\
&= \frac{1}{2}\left(\frac{1}{s - \omega}\right) - \frac{1}{2}\left(\frac{1}{s + \omega}\right) \\
&= \frac{1}{2}\left[\frac{(s + \omega) - (s - \omega)}{(s - \omega)(s + \omega)}\right] \\
&= \frac{\omega}{s^2 - \omega^2}.
\end{aligned}$$

8. Using Proposition 2.7 and Table 1, if
- $y(t) = t^2$
- , then

$$\begin{aligned}
\mathcal{L}(y')(s) &= \mathcal{L}(2t)(s) \\
&= 2\mathcal{L}(t)(s) \\
&= 2 \cdot \frac{1!}{s^2} \\
&= \frac{2}{s^2},
\end{aligned}$$

provided $s > 0$. On the other hand,

$$\begin{aligned}
s\mathcal{L}(y)(s) - y(0) &= s\mathcal{L}(t^2)(s) - 0^2 \\
&= s \cdot \frac{2!}{s^3} \\
&= \frac{2}{s^2},
\end{aligned}$$

provided $s > 0$.

9. If
- $y(t) = t^3$
- , then

$$\begin{aligned}
 \mathcal{L}\{y'(t)\}(s) &= \mathcal{L}\{3t^2\}(s) \\
 &= 3 \mathcal{L}\{t^2\}(s) \\
 &= 3 \left(\frac{2!}{s^3} \right) \\
 &= \frac{6}{s^3}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 s \mathcal{L}\{y(t)\}(s) - y(0) &= s \mathcal{L}\{t^3\}(s) - 0 \\
 &= s \left(\frac{3!}{s^4} \right) \\
 &= \frac{6}{s^3}.
 \end{aligned}$$

10. Using Proposition 2.7 and Table 1, if
- $y(t) = e^{2t}$
- , then

$$\begin{aligned}
 \mathcal{L}(y')(s) &= \mathcal{L}(2e^{2t})(s) \\
 &= 2 \mathcal{L}(e^{2t})(s) \\
 &= 2 \cdot \frac{1}{s-2} \\
 &= \frac{2}{s-2},
 \end{aligned}$$

provided $s > 2$. On the other hand,

$$\begin{aligned}
 s \mathcal{L}(y)(s) - y(0) &= s \mathcal{L}(e^{2t}) - 1 \\
 &= s \cdot \frac{1}{s-2} - \frac{s-2}{s-2} \\
 &= \frac{2}{s-2},
 \end{aligned}$$

provided $s > 2$.

11. If
- $y(t) = e^{-3t}$
- , then

$$\begin{aligned}
 \mathcal{L}\{y'(t)\}(s) &= \mathcal{L}\{-3e^{-3t}\}(s) \\
 &= -3 \mathcal{L}\{e^{-3t}\}(s) \\
 &= -3 \left(\frac{1}{s+3} \right) \\
 &= \frac{-3}{s+3}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 s \mathcal{L}\{y(t)\}(s) - y(0) &= s \mathcal{L}\{e^{-3t}\}(s) - 1 \\
 &= s \left(\frac{1}{s+3} \right) - 1 \\
 &= \frac{s - (s+3)}{s+3} \\
 &= \frac{-3}{s+3}.
 \end{aligned}$$

12. Using Proposition 2.7 and Table 1, if
- $y(t) = \cos 3t$
- , then

$$\begin{aligned}
 \mathcal{L}(y')(s) &= \mathcal{L}(-3 \sin 3t)(s) \\
 &= -3 \mathcal{L}(\sin 3t)(s) \\
 &= -3 \cdot \frac{3}{s^2 + 3^2} \\
 &= -\frac{9}{s^2 + 9},
 \end{aligned}$$

provided $s > 0$. On the other hand,

$$\begin{aligned}
 s \mathcal{L}(y)(s) - y(0) &= s \mathcal{L}(\cos 3t) - 1 \\
 &= s \cdot \frac{s}{s^2 + 3^2} - \frac{s^2 + 3^2}{s^2 + 3^2} \\
 &= \frac{-9}{s^2 + 9},
 \end{aligned}$$

provided $s > 0$.

13. If
- $y(t) = \sin 5t$
- , then

$$\begin{aligned}
 \mathcal{L}\{y'(t)\}(s) &= \mathcal{L}\{5 \cos 5t\}(s) \\
 &= 5 \mathcal{L}\{\cos 5t\}(s) \\
 &= 5 \left(\frac{s}{s^2 + 25} \right) \\
 &= \frac{5s}{s^2 + 25}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 s \mathcal{L}\{y(t)\}(s) - y(0) &= s \mathcal{L}\{\sin 5t\}(s) - 0 \\
 &= s \left(\frac{5}{s^2 + 25} \right) \\
 &= \frac{5s}{s^2 + 25}.
 \end{aligned}$$

14. Using Proposition 2.7 and Table 1, if $y(t) = t^4$, then

$$\begin{aligned}\mathcal{L}(y'')(s) &= \mathcal{L}(12t^2)(s) \\ &= 12 \mathcal{L}(t^2)(s) \\ &= 12 \cdot \frac{2!}{s^3} \\ &= \frac{24}{s^3},\end{aligned}$$

provided $s > 0$. On the other hand,

$$\begin{aligned}s^2 \mathcal{L}(y)(s) - sy(0) - y'(0) &= s^2 \mathcal{L}(t^4)(s) - s(0) - 0 \\ &= s^2 \cdot \frac{4!}{s^5} \\ &= \frac{24}{s^3},\end{aligned}$$

provided $s > 0$.

15. If $y(t) = e^{-2t}$, then $y'(t) = -2e^{-2t}$ and $y''(t) = 4e^{-2t}$. On one hand,

$$\begin{aligned}\mathcal{L}\{y''(t)\}(s) &= \mathcal{L}\{4e^{-2t}\}(s) \\ &= 4 \mathcal{L}\{e^{-2t}\}(s) \\ &= 4 \left(\frac{1}{s+2} \right) \\ &= \frac{4}{s+2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}s^2 \mathcal{L}\{y(t)\}(s) - sy(0) - y'(0) &= s^2 \mathcal{L}\{e^{-2t}\}(s) - s(1) - (-2) \\ &= s^2 \left(\frac{1}{s+2} \right) - s + 2 \\ &= \frac{s^2 + (-s+2)(s+2)}{s+2} \\ &= \frac{s^2 - s^2 + 4}{s+2} \\ &= \frac{4}{s+2}.\end{aligned}$$

16. Using Proposition 2.7 and Table 1, if $y(t) = \sin 2t$,

then

$$\begin{aligned}\mathcal{L}(y'')(s) &= \mathcal{L}(-4 \sin 2t)(s) \\ &= -4 \mathcal{L}(\sin 2t)(s) \\ &= -4 \cdot \frac{2}{s^2 + 2^2} \\ &= -\frac{8}{s^2 + 4},\end{aligned}$$

provided $s > 0$. On the other hand,

$$\begin{aligned}s^2 \mathcal{L}(y)(s) - sy(0) - y'(0) &= s^2 \mathcal{L}(\sin 2t)(s) - s(0) - 2 \\ &= s^2 \cdot \frac{2}{s^2 + 2^2} - 2 \\ &= \frac{2s^2}{s^2 + 4} - \frac{2s^2 + 8}{s^2 + 4} \\ &= -\frac{8}{s^2 + 4},\end{aligned}$$

provided $s > 0$.

17. If $y(t) = t^2 + 3t + 5$, then $y'(t) = 2t + 3$ and $y''(t) = 2$. On the one hand,

$$\begin{aligned}\mathcal{L}\{y''(t)\}(s) &= \mathcal{L}\{2\}(s) \\ &= 2 \mathcal{L}\{1\}(s) \\ &= 2 \left(\frac{1}{s} \right) \\ &= \frac{2}{s}.\end{aligned}$$

On the other hand,

$$\begin{aligned}s^2 \mathcal{L}\{y(t)\}(s) - sy(0) - y'(0) &= s^2 \mathcal{L}\{t^2 + 3t + 5\}(s) - s(5) - 3 \\ &= s^2 (\mathcal{L}\{t^2\}(s) + 3 \mathcal{L}\{t\}(s) + 5 \mathcal{L}\{1\}(s)) - 5s - 3 \\ &= s^2 \left(\frac{2!}{s^3} + 3 \left(\frac{1}{s^2} \right) + 5 \left(\frac{1}{s} \right) \right) - 5s - 3 \\ &= s^2 \left(\frac{2 + 3s + 5s^2}{s^3} \right) - 5s - 3 \\ &= \frac{2 + 3s + 5s^2}{s} + \frac{-5s^2 - 3s}{s} \\ &= \frac{2}{s}.\end{aligned}$$

18. If

$$y' + 3y = t^2, \quad y(0) = -1,$$

then, letting $Y(s) = \mathcal{L}(y)(s)$,

$$s \mathcal{L}(y)(s) - y(0) + 3 \mathcal{L}(y)(s) = \mathcal{L}(t^2)(s)$$

$$sY(s) + 1 + 3Y(s) = \frac{2!}{s^3}.$$

Solving for $Y(s)$,

$$(s+3)Y(s) = \frac{2}{s^3} - 1$$

$$Y(s) = \frac{2-s^3}{s^3(s+3)}.$$

19. If $y' - 5y = e^{-2t}$, with $y(0) = 1$, then

$$\mathcal{L}\{y' - 5y\}(s) = \mathcal{L}\{e^{-2t}\}(s)$$

$$\mathcal{L}\{y'\}(s) - 5\mathcal{L}\{y\}(s) = \frac{1}{s+2}$$

$$s \mathcal{L}\{y\}(s) - y(0) - 5 \mathcal{L}\{y\}(s) = \frac{1}{s+2}.$$

If we let $Y(s) = \mathcal{L}\{y\}(s)$, then

$$sY(s) - 1 - 5Y(s) = \frac{1}{s+2}$$

$$(s-5)Y(s) = 1 + \frac{1}{s+2}$$

$$Y(s) = \frac{1}{s-5} + \frac{1}{(s-5)(s+2)}$$

$$Y(s) = \frac{(s+2)+1}{(s-5)(s+2)}$$

$$Y(s) = \frac{s+3}{(s-5)(s+2)}.$$

20. If $y' + 5y = t^2 + 2t + 3$, with $y(0) = 0$, then,

$$s \mathcal{L}(y)(s) - y(0) + 5 \mathcal{L}(y)(s)$$

$$= \mathcal{L}(t^2)(s) + 2 \mathcal{L}(t)(s) + 3 \mathcal{L}(1)(s)$$

Letting $Y(s) = \mathcal{L}(y)(s)$, this becomes

$$sY(s) + 5Y(s) = \frac{2!}{s^3} + 2 \cdot \frac{1!}{s^2} + 3 \cdot \frac{1}{s}$$

$$(s+5)Y(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}.$$

Solving for $Y(s)$,

$$(s+5)Y(s) = \frac{2+2s+3s^2}{s^3}$$

$$Y(s) = \frac{2+2s+3s^2}{s^3(s+5)}.$$

21. If $y' - 4y = \cos 2t$, with $y(0) = -2$, then

$$\mathcal{L}\{y' - 4y\}(s) = \mathcal{L}\{\cos 2t\}(s)$$

$$\mathcal{L}\{y'\}(s) - 4 \mathcal{L}\{y\}(s) = \frac{s}{s^2+4}$$

$$s \mathcal{L}\{y\}(s) - y(0) - 4 \mathcal{L}\{y\}(s) = \frac{s}{s^2+4}.$$

If we let $Y(s) = \mathcal{L}\{y\}(s)$, then

$$sY(s) + 2 - 4Y(s) = \frac{s}{s^2+4}$$

$$(s-4)Y(s) = -2 + \frac{s}{s^2+4}$$

$$Y(s) = -\frac{2}{s-4} + \frac{s}{(s-4)(s^2+4)}$$

$$Y(s) = \frac{-2(s^2+4)+s}{(s-4)(s^2+4)}$$

$$Y(s) = \frac{-2s^2+s-8}{(s-4)(s^2+4)}.$$

22. If

$$y'' + y = \sin 4t, \quad y(0) = 0, \quad y'(0) = 1,$$

then, letting $Y(s) = \mathcal{L}(y)(s)$,

$$s^2 \mathcal{L}(y)(s) - sy(0) - y'(0) + \mathcal{L}(y)(s) = \frac{4}{s^2+4^2}$$

$$s^2 Y(s) - 1 + Y(s) = \frac{4}{s^2+16}.$$

Solving for $Y(s)$,

$$(s^2+1)Y(s) = 1 + \frac{4}{s^2+16}$$

$$(s^2+1)Y(s) = \frac{s^2+20}{s^2+16}$$

$$Y(s) = \frac{s^2+20}{(s^2+1)(s^2+16)}.$$

23. If $y'' + 2y' + 2y = \cos 2t$, with $y(0) = 1$ and $y'(0) = 0$, then

$$\begin{aligned}\mathcal{L}\{y'' + 2y' + 2y\}(s) &= \mathcal{L}\{\cos 2t\}(s) \\ \mathcal{L}\{y''\}(s) + 2\mathcal{L}\{y'\}(s) + 2\mathcal{L}\{y\}(s) &= \frac{s}{s^2 + 4}\end{aligned}$$

If we let $Y(s) = \mathcal{L}\{y\}(s)$, this becomes

$$\begin{aligned}[s^2 Y(s) - s(1) - 0] + 2[sY(s) - 1] + 2Y(s) \\ = \frac{s}{s^2 + 4}.\end{aligned}$$

Collecting terms we get

$$(s^2 + 2s + 2)Y(s) - (s + 2) = \frac{s}{s^2 + 4},$$

which we solve for Y to get

$$\begin{aligned}Y(s) &= \frac{s + 2}{s^2 + 2s + 2} + \frac{s}{(s^2 + 4)(s^2 + 2s + 2)} \\ Y(s) &= \frac{s^3 + 2s^2 + 5s + 8}{(s^2 + 4)(s^2 + 2s + 2)}.\end{aligned}$$

24. If $y'' + y' + 2y = \cos 2t + \sin 3t$, with $y(0) = -1$, and $y'(0) = 1$, then,

$$\begin{aligned}[s^2 \mathcal{L}(y)(s) - sy(0) - y'(0)] \\ + [s \mathcal{L}(y)(s) - y(0)] + 2\mathcal{L}(y)(s) \\ = \mathcal{L}(\cos 2t) + \mathcal{L}(\sin 3t) \\ = \frac{s}{s^2 + 2^2} + \frac{3}{s^2 + 3^2}.\end{aligned}$$

Letting $Y(s) = \mathcal{L}(y)(s)$, and solving for $Y(s)$,

$$\begin{aligned}(s^2 + s + 2)Y(s) \\ = -s + \frac{s}{s^2 + 4} + \frac{3}{s^2 + 9} \\ = \frac{-s^5 - 13s^3 - 36s + s^3 + 9s + 3s^2 + 12}{(s^2 + 4)(s^2 + 9)}\end{aligned}$$

or

$$Y(s) = \frac{-s^5 - 12s^3 + 3s^2 - 27s + 12}{(s^2 + s + 2)(s^2 + 4)(s^2 + 9)}.$$

25. If $y'' + 3y' + 5y = t + e^{-t}$,

$$\begin{aligned}\mathcal{L}\{y'' + 3y' + 5y\}(s) \\ = \mathcal{L}\{y''\}(s) + 3\mathcal{L}\{y'\}(s) + 5\mathcal{L}\{y\}(s) \\ = [s^2 \mathcal{L}\{y\}(s) - sy(0) - y'(0)] \\ + 3[s \mathcal{L}\{y\}(s) - y(0)] + s \mathcal{L}\{y\}(s)\end{aligned}$$

and

$$\mathcal{L}\{t + e^{-t}\}(s) = \frac{1}{s^2} + \frac{1}{s + 1}.$$

If we equate these two, use the initial conditions $y(0) = -1$ and $y'(0) = 0$, and set $Y(s) = \mathcal{L}\{y\}(s)$, we get

$$s^2 Y(s) + s + 3sY(s) + 3 + 5Y(s) = \frac{s^2 + s + 1}{s^2(s + 1)}.$$

Solving for Y , we get

$$\begin{aligned}Y(s) &= \frac{-s - 3}{s^2 + 3s + 5} + \frac{s^2 + s + 1}{s^2(s + 1)(s^2 + 3s + 5)} \\ Y(s) &= \frac{(-s - 3)s^2(s + 1) + (s^2 + s + 1)}{s^2(s + 1)(s^2 + 3s + 5)} \\ Y(s) &= \frac{s^2(-s^2 - 4s - 3) + s^2 + s + 1}{s^2(s + 1)(s^2 + 3s + 5)} \\ Y(s) &= \frac{-s^4 - 4s^3 - 2s^2 + s + 1}{s^2(s + 1)(s^2 + 3s + 5)}.\end{aligned}$$

26. Consider the transform pair

$$y(t) = \sin 3t \leftrightarrow Y(s) = \frac{3}{s^2 + 9}.$$

By Proposition 2.12,

$$\begin{aligned}\mathcal{L}\{e^{-t} \sin 3t\} &= Y(s + 1) \\ &= \frac{3}{(s + 1)^2 + 9}.\end{aligned}$$

27. Because $f(t) = \cos 2t$ has transform $F(s) = s/(s^2 + 4)$, then $y(t) = e^{2t} \cos 2t$ will have transform

$$\begin{aligned}Y(s) &= F(s - 2) \\ &= \frac{s - 2}{(s - 2)^2 + 4} \\ &= \frac{s - 2}{s^2 - 4s + 8}.\end{aligned}$$

28. Consider the transform pair

$$y(t) = 2t + 3 \leftrightarrow Y(s) = 2 \cdot \frac{1}{s^2} + 3 \cdot \frac{1}{s} \\ = \frac{2 + 3s}{s^2}.$$

By Proposition 2.12,

$$\mathcal{L}\{e^{-2t}(2t + 3)\} = Y(s + 2) \\ = \frac{2 + 3(s + 2)}{(s + 2)^2} \\ = \frac{3s + 8}{(s + 2)^2}.$$

29. Because $f(t) = t^2 + 3t + 4$ has transform

$$F(s) = \frac{2!}{s^3} + \frac{3}{s^2} + \frac{4}{s} \\ = \frac{2 + 3s + 4s^2}{s^3},$$

then $y(t) = e^{-t}(t^2 + 3t + 4)$ will have transform

$$Y(s) = F(s + 1) \\ = \frac{2 + 3(s + 1) + 4(s + 1)^2}{(s + 1)^3} \\ = \frac{2 + 3s + 3 + 4s^2 + 8s + 4}{(s + 1)^3} \\ = \frac{4s^2 + 11s + 9}{(s + 1)^3}.$$

30. Consider the transform pair,

$$y(t) = \sin 3t \leftrightarrow Y(s) = \frac{3}{s^2 + 9}$$

By Proposition 2.14,

$$\mathcal{L}\{t \sin 3t\}(s) = -Y'(s) \\ = -3(-1)(s^2 + 9)^{-2}(2s) \\ = \frac{6s}{(s^2 + 9)^2}.$$

31. Because $f(t) = e^{-t}$ has transform $F(s) = 1/(s + 1)$,

then $y(t) = te^{-t}$ has transform

$$Y(s) = -F'(s) \\ = -\left[\frac{-1}{(s + 1)^2}\right] \\ = \frac{1}{(s + 1)^2}.$$

32. Consider the transform pair,

$$y(t) = \cos 2t \leftrightarrow Y(s) = \frac{s}{s^2 + 4}.$$

By Proposition 2.14,

$$\mathcal{L}\{t^2 \cos 2t\}(s) \\ = (-1)^2 Y''(s) \\ = \frac{d}{ds} \left[\frac{(s^2 + 4)(1) - s(2s)}{(s^2 + 4)^2} \right] \\ = \frac{d}{ds} \left[\frac{4 - s^2}{(s^2 + 4)^2} \right] \\ = \frac{(s^2 + 4)^2(-2s) - (4 - s^2)(2)(s^2 + 4)(2s)}{(s^2 + 4)^4} \\ = \frac{-2s(s^2 + 4)[(s^2 + 4) + 2(4 - s^2)]}{(s^2 + 4)^4} \\ = \frac{-2s(12 - s^2)}{(s^2 + 4)^3}.$$

33. Because $f(t) = e^{2t}$ has transform $F(s) = 1/(s - 2)$, then $y(t) = t^2 e^{2t}$ has transform

$$Y(s) = (-1)^2 F''(s) \\ = \frac{d}{ds} \left(\frac{-1}{(s - 2)^2} \right) \\ = \frac{2}{(s - 2)^3}.$$

34. If

$$y' + 2y = t \sin t, \quad y(0) = 1,$$

then, with $Y(s) = \mathcal{L}(y)(s)$,

$$\mathcal{L}(y' + 2y)(s) = s \mathcal{L}(y)(s) - y(0) + 2 \mathcal{L}(y)(s) \\ = sY(s) - 1 + 2Y(s) \\ = (s + 2)Y(s) - 1.$$

Consider the transform pair

$$f(t) = \sin t \leftrightarrow F(s) = \frac{1}{s^2 + 1}.$$

Using Proposition 2.14,

$$\begin{aligned}\mathcal{L}\{t \sin t\}(s) &= -F'(s) \\ &= -[-(s^2 + 1)^{-2}(2s)] \\ &= \frac{2s}{(s^2 + 1)^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}(s + 2)Y(s) - 1 &= \frac{2s}{(s^2 + 1)^2} \\ Y(s) &= \frac{1}{s + 2} + \frac{2s}{(s + 2)(s^2 + 1)^2}.\end{aligned}$$

35. If $y' - y = t^2 e^{-2t}$, with $y(0) = 0$, then with $Y(s) = \mathcal{L}\{y\}(s)$,

$$\begin{aligned}\mathcal{L}\{y' - y\}(s) &= s\mathcal{L}\{y\}(s) - y(0) - \mathcal{L}\{y\}(s) \\ &= sY(s) - Y(s) \\ &= (s - 1)Y(s).\end{aligned}$$

Because the transform of $f(t) = e^{-2t}$ is $F(s) = 1/(s + 2)$, the transform of $t^2 e^{-2t}$ is

$$\mathcal{L}\{t^2 e^{-2t}\}(s) = (-1)^2 F''(s) = \frac{2}{(s + 2)^3}.$$

Equating,

$$\begin{aligned}(s - 1)Y(s) &= \frac{2}{(s + 2)^3} \\ Y(s) &= \frac{2}{(s - 1)(s + 2)^3}.\end{aligned}$$

36. If

$$y' + y = e^t \sin 3t, \quad y(0) = 0,$$

then with $\mathcal{L}\{y\}(s) = Y(s)$,

$$\begin{aligned}\mathcal{L}\{y' + y\}(s) &= s\mathcal{L}\{y\}(s) - y(0) + \mathcal{L}\{y\}(s) \\ &= sY(s) + Y(s) \\ &= (s + 1)Y(s).\end{aligned}$$

Consider the transform pair,

$$f(t) = \sin 3t \leftrightarrow F(s) = \frac{3}{s^2 + 9}.$$

By Proposition 2.12,

$$\begin{aligned}\mathcal{L}\{e^{-t} \sin 3t\}(s) &= F(s + 1) \\ &= \frac{3}{(s + 1)^2 + 9}.\end{aligned}$$

Therefore,

$$\begin{aligned}(s + 1)Y(s) &= \frac{3}{(s + 1)^2 + 9} \\ Y(s) &= \frac{3}{(s + 1)(s^2 + 2s + 10)}.\end{aligned}$$

37. If $y' - 2y = e^{2t} \cos t$, with $y(0) = -2$, then with $Y(s) = \mathcal{L}\{y\}(s)$,

$$\begin{aligned}\mathcal{L}\{y' - 2y\}(s) &= s\mathcal{L}\{y\}(s) - y(0) - 2\mathcal{L}\{y\}(s) \\ &= sY(s) + 2 - 2Y(s) \\ &= (s - 2)Y(s) + 2.\end{aligned}$$

Because the transform of $f(t) = \cos t$ is $F(s) = s/(s^2 + 1)$, the transform of $e^{2t} \cos t$ is

$$F(s - 2) = \frac{s - 2}{(s - 2)^2 + 1} = \frac{s - 2}{s^2 - 4s + 5}.$$

Equating,

$$(s - 2)Y(s) + 2 = \frac{s - 2}{s^2 - 4s + 5}$$

and solving for Y , we get

$$\begin{aligned}Y(s) &= -\frac{2}{s - 2} + \frac{s - 2}{(s - 2)(s^2 - 4s + 5)} \\ &= \frac{-2(s^2 - 4s + 5) + s - 2}{(s - 2)(s^2 - 4s + 5)} \\ &= \frac{-2s^2 + 9s - 12}{(s - 2)(s^2 - 4s + 5)}.\end{aligned}$$

38. If

$$y'' + 4y = t^2 \sin 4t, \quad y(0) = 0, \quad y'(0) = -1,$$

then with $\mathcal{L}(y)(s) = Y(s)$,

$$\begin{aligned}\mathcal{L}(y'' + 4y) &= s^2 \mathcal{L}(y)(s) - sy(0) - y'(0) + 4\mathcal{L}(y)(s) \\ &= s^2 Y(s) + 1 + 4Y(s) \\ &= (s^2 + 4)Y(s) + 1.\end{aligned}$$

Consider the transform pair

$$f(t) = \sin 4t \leftrightarrow F(s) = \frac{4}{s^2 + 16}.$$

By Proposition 2.14,

$$\begin{aligned}\mathcal{L}\{t^2 \sin 4t\}(s) &= (-1)^2 F''(s) \\ &= \frac{d}{ds} [-4(s^2 + 16)^{-2}(2s)] \\ &= -\frac{d}{ds} \frac{8s}{(s^2 + 16)^2} \\ &= -\frac{(s^2 + 16)^2(8) - 8s[2(s^2 + 16)(2s)]}{(s^2 + 16)^4} \\ &= -\frac{8(s^2 + 16)[(s^2 + 16) - 4s^2]}{(s^2 + 16)^4} \\ &= -\frac{8(16 - 3s^2)}{(s^2 + 16)^3}.\end{aligned}$$

Therefore,

$$(s^2 + 4)Y(s) + 1 = -\frac{8(16 - 3s^2)}{(s^2 + 16)^3}.$$

Solving for Y ,

$$Y(s) = -\frac{1}{s^2 + 4} - \frac{8(16 - 3s^2)}{(s^2 + 4)(s^2 + 16)^3}.$$

39. If $y'' + y' + 2y = e^{-t} \cos 2t$, with $y(0) = 1$ and $y'(0) = -1$, then with $Y(s) = \mathcal{L}\{y\}(s)$,

$$\begin{aligned}\mathcal{L}\{y'' + y' + 2y\}(s) &= s^2 \mathcal{L}\{y\}(s) - sy(0) - y'(0) \\ &\quad + s \mathcal{L}\{y\}(s) - y(0) + 2 \mathcal{L}\{y\}(s) \\ &= s^2 Y(s) - s + 1 + sY(s) - 1 + 2Y(s) \\ &= (s^2 + s + 2)Y(s) - s.\end{aligned}$$

Because the transform of $f(t) = \cos 2t$ is $F(s) = s/(s^2 + 4)$, the transform of $e^{-t} \cos 2t$ is

$$F(s + 1) = \frac{s + 1}{(s + 1)^2 + 4} = \frac{s + 1}{s^2 + 2s + 5}.$$

Equating,

$$(s^2 + s + 2)Y(s) - s = \frac{s + 1}{s^2 + 2s + 5}.$$

Solving for Y

$$\begin{aligned}Y(s) &= \frac{s}{s^2 + s + 2} + \frac{s + 1}{(s^2 + s + 2)(s^2 + 2s + 5)} \\ &= \frac{s(s^2 + 2s + 5) + s + 1}{(s^2 + s + 2)(s^2 + 2s + 5)} \\ &= \frac{s^3 + 2s^2 + 6s + 1}{(s^2 + s + 2)(s^2 + 2s + 5)}.\end{aligned}$$

40. If

$$y'' + 2y' + 5y = t^2 e^{-t}, \quad y(0) = 1, \quad y'(0) = -2,$$

then with $\mathcal{L}(y)(s) = Y(s)$,

$$\begin{aligned}\mathcal{L}(y'' + 2y' + 5y) &= s^2 \mathcal{L}(y)(s) - sy(0) - y'(0) \\ &\quad + 2(s \mathcal{L}(y)(s) - y(0)) + 5 \mathcal{L}(y)(s) \\ &= s^2 Y(s) - s + 2 + 2(sY(s) - 1) + 5Y(s) \\ &= (s^2 + 2s + 5)Y(s) - s.\end{aligned}$$

Consider the transform pair,

$$f(t) = e^{-t} \leftrightarrow F(s) = \frac{1}{s + 1}.$$

By Proposition 2.14,

$$\begin{aligned}\mathcal{L}\{t^2 e^{-t}\}(s) &= (-1)^2 F''(s) \\ &= \frac{d}{ds} [-(s + 1)^{-2}] \\ &= \frac{2}{(s + 1)^3}.\end{aligned}$$

Hence,

$$(s^2 + 2s + 5)Y(s) - s = \frac{2}{(s + 1)^3}.$$

Solving for Y

$$Y(s) = \frac{s}{s^2 + 2s + 5} + \frac{2}{(s + 1)^3(s^2 + 2s + 5)}.$$

41. If $y'' + 5y = 3e^{-t} \cos 4t$, with $y(0) = -1$ and $y'(0) = 2$, then with $Y(s) = \mathcal{L}\{y\}(s)$,

$$\begin{aligned}\mathcal{L}\{y'' + 5y\}(s) &= s^2 \mathcal{L}\{y\}(s) - sy(0) - y'(0) + 5 \mathcal{L}\{y\}(s) \\ &= s^2 Y(s) + s - 2 + 5Y(s) \\ &= (s^2 + 5)Y(s) + s - 2.\end{aligned}$$

Because the transform of $f(t) = 3 \cos 4t$ is

$$F(s) = 3 \left(\frac{s}{s^2 + 16} \right) = \frac{3s}{s^2 + 16},$$

the transform of $3e^{-t} \cos 4t$ is

$$F(s+1) = \frac{3(s+1)}{(s+1)^2 + 16} = \frac{3s+3}{s^2+2s+17}.$$

Equating,

$$(s^2 + 5)Y(s) + s - 2 = \frac{3s+3}{s^2+2s+17}.$$

Solving for Y

$$\begin{aligned}Y(s) &= -\frac{s-2}{s^2+5} + \frac{3s+3}{(s^2+5)(s^2+2s+17)} \\ &= \frac{(2-s)(s^2+2s+17) + 3s+3}{(s^2+5)(s^2+2s+17)} \\ &= \frac{-s^3 - 10s + 37}{(s^2+5)(s^2+2s+17)}.\end{aligned}$$

42. (a) Using Definition 1.1,

$$\begin{aligned}\mathcal{L}\{\sin t\}(s) &= \int_0^\infty (\sin t)e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-st} \sin t dt.\end{aligned}$$

Two consecutive integration by parts provides

$$\begin{aligned}\int e^{-st} \sin t dt &= -\frac{1}{s}e^{-st} \sin t - \frac{1}{s^2}e^{-st} \cos t \\ &\quad - \frac{1}{s^2} \int e^{-st} \sin t dt.\end{aligned}$$

Thus,

$$\begin{aligned}\left(1 + \frac{1}{s^2}\right) \int e^{-st} \sin t dt &= -\frac{1}{s^2}e^{-st} [s \sin t + \cos t],\end{aligned}$$

and

$$\int e^{-st} \sin t dt = -\frac{1}{s^2+1}e^{-st} [s \sin t + \cos t].$$

Thus,

$$\begin{aligned}\mathcal{L}\{\sin t\}(s) &= -\frac{1}{s^2+1} \lim_{T \rightarrow \infty} e^{-st} [s \sin t + \cos t] \Big|_0^T \\ &= \frac{1}{s^2+1},\end{aligned}$$

provided $s > 0$.

- (b) Suppose that $\mathcal{L}\{f(t)\} = F(s)$. Consider

$$\mathcal{L}\{f(at)\}(s) = \int_0^\infty f(at)e^{-st} dt.$$

Let $u = at$ and $du = adt$. Then,

$$\begin{aligned}\mathcal{L}\{f(at)\}(s) &= \int_0^\infty f(u)e^{-s(u/a)} \frac{du}{a} \\ &= \frac{1}{a} \int_0^\infty f(u)e^{-(s/a)u} du \\ &= \frac{1}{a} F\left(\frac{s}{a}\right),\end{aligned}$$

provided $a > 0$. Since

$$f(t) = \sin t \leftrightarrow F(s) = \frac{1}{s^2+1}$$

forms a transform pair,

$$\begin{aligned}\mathcal{L}\{\sin \omega t\}(s) &= \mathcal{L}\{f(\omega t)\}(s) \\ &= \frac{1}{\omega} F\left(\frac{s}{\omega}\right) \\ &= \frac{1}{\omega} \cdot \frac{1}{\left(\frac{s}{\omega}\right)^2 + 1} \\ &= \frac{1}{\omega} \cdot \frac{\omega^2}{s^2 + \omega^2} \\ &= \frac{\omega}{s^2 + \omega^2}.\end{aligned}$$

(c) If $f(t) = \sin \omega t$, then $f'(t) = \omega \cos \omega t$. Thus,

$$\begin{aligned}\mathcal{L}\{\omega \cos \omega t\}(s) &= \mathcal{L}\{f'(t)\}(s) \\ &= s \mathcal{L}\{f(t)\}(s) - f(0) \\ &= s \mathcal{L}\{\sin \omega t\}(s) - \sin \omega(0) \\ &= s \cdot \frac{\omega}{s^2 + \omega^2} \\ &= \frac{s\omega}{s^2 + \omega^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}\{\cos \omega t\}(s) &= \frac{1}{\omega} \mathcal{L}\{\omega \cos \omega t\}(s) \\ &= \frac{1}{\omega} \cdot \frac{s\omega}{s^2 + \omega^2} \\ &= \frac{s}{s^2 + \omega^2}.\end{aligned}$$

43. (a) If $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$, then

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t} t^{1-1} dt \\ &= \int_0^\infty e^{-t} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-t} dt \\ &= \lim_{T \rightarrow \infty} -e^{-t} \Big|_0^T \\ &= -\lim_{T \rightarrow \infty} (e^{-T} - 1) \\ &= 1.\end{aligned}$$

(b) Furthermore,

$$\begin{aligned}\Gamma(\alpha + 1) &= \int_0^\infty e^{-t} t^{(\alpha+1)-1} dt \\ &= \int_0^\infty e^{-t} t^\alpha dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-t} t^\alpha dt.\end{aligned}$$

Integrating by parts,

$$\int e^{-t} t^\alpha dt = -e^{-t} t^\alpha + \alpha \int e^{-t} t^{\alpha-1} dt.$$

Inserting the limits 0 and T ,

$$\begin{aligned}\Gamma(\alpha + 1) &= \lim_{T \rightarrow \infty} \left[-e^{-t} t^\alpha \Big|_0^T + \alpha \int_0^T e^{-t} t^{\alpha-1} dt \right] \\ &= \lim_{T \rightarrow \infty} \left[-e^{-T} T^\alpha + \alpha \int_0^T e^{-t} t^{\alpha-1} dt \right] \\ &= 0 + \alpha \int_0^\infty e^{-t} t^{\alpha-1} dt \\ &= \alpha \Gamma(\alpha).\end{aligned}$$

Furthermore, if n is an integer, then

$$\begin{aligned}\Gamma(n + 1) &= n \Gamma(n) \\ &= n(n-1) \Gamma(n-1) \\ &= n(n-1)(n-2) \Gamma(n-2) \\ &= n(n-1)(n-2) \cdots (2) \Gamma(1) \\ &= n(n-1)(n-2) \cdots (2)(1) \\ &= n!.\end{aligned}$$

(c) First, by definition,

$$\mathcal{L}\{t^\alpha\}(s) = \int_0^\infty t^\alpha e^{-st} dt.$$

Let $u = st$, then $du = s dt$ and

$$\begin{aligned}\int_0^\infty t^\alpha e^{-st} dt &= \int_0^\infty \left(\frac{u}{s}\right)^\alpha e^{-u} \left(\frac{du}{s}\right) \\ &= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u} u^{(\alpha+1)-1} du \\ &= \frac{1}{s^{\alpha+1}} \Gamma(\alpha + 1) \\ &= \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}.\end{aligned}$$

In the case where α is a positive integer n , then

$$\mathcal{L}\{t^n\}(s) = \frac{\Gamma(n + 1)}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

44. (a) Suppose $f(t) \leftrightarrow F(s)$ is a transform pair. If we let

$$g(t) = \int_0^t f(\tau) d\tau,$$

then

$$\begin{aligned} s \mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} (s) &= s \mathcal{L}\{g(t)\}(s) - g(0) \\ &= \mathcal{L}\{g'(t)\}(s) \\ &= \mathcal{L}\{f(t)\}(s) \\ &= F(s). \end{aligned}$$

Therefore,

$$\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} (s) = \frac{F(s)}{s}.$$

(b)

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + 1)} \right\} (t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s^2 + 1} \right\} (t) \\ &= \int_0^t \sin \tau d\tau \\ &= -\cos \tau \Big|_0^t \\ &= -\cos t + \cos 0 \\ &= 1 - \cos t \end{aligned}$$

Section 5.3. The Inverse Laplace Transform

1. Factor.

$$Y(s) = \frac{1}{3s + 2} = \frac{1}{3} \cdot \frac{1}{s + 2/3}.$$

Now,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{3} \cdot \frac{1}{s + 2/3} \right\} \\ &= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s + 2/3} \right\} \\ &= \frac{1}{3} e^{-(2/3)t}. \end{aligned}$$

2. Adjust as follows.

$$Y(s) = \frac{2}{3 - 5s} = -\frac{2}{5} \cdot \frac{1}{s - 3/5}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ -\frac{2}{5} \cdot \frac{1}{s - 3/5} \right\} \\ &= -\frac{2}{5} \mathcal{L}^{-1} \left\{ \frac{1}{s - 3/5} \right\} \\ &= -\frac{2}{5} e^{(3/5)t} \end{aligned}$$

3. Adjust as follows:

$$Y(s) = \frac{1}{s^2 + 4} = \frac{1}{2} \cdot \frac{2}{s^2 + 4}.$$

Now,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{2} \cdot \frac{2}{s^2 + 4} \right\} \\ &= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} \\ &= \frac{1}{2} \sin 2t. \end{aligned}$$

4. Adjust as follows.

$$Y(s) = \frac{5s}{s^2 + 9} = 5 \cdot \frac{s}{s^2 + 9}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ 5 \cdot \frac{s}{s^2 + 9} \right\} \\ &= 5 \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 9} \right\} \\ &= 5 \cos 3t \end{aligned}$$

5. Adjust as follows:

$$Y(s) = \frac{3}{s^2} = 3 \cdot \frac{1}{s^2}.$$

Now,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ 3 \cdot \frac{1}{s^2} \right\} \\ &= 3 \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} \\ &= 3t. \end{aligned}$$

6. Adjust as follows.

$$Y(s) = \frac{2}{3s^4} = \frac{1}{9} \cdot \frac{3!}{s^4}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{9} \cdot \frac{3!}{s^4} \right\} \\ &= \frac{1}{9} \mathcal{L}^{-1} \left\{ \frac{3!}{s^4} \right\} \\ &= \frac{1}{9} t^3 \end{aligned}$$

7. Adjust as follows:

$$\begin{aligned} Y(s) &= \frac{3s+2}{s^2+25} \\ &= \frac{3s}{s^2+25} + \frac{2}{s^2+25} \\ &= 3 \cdot \frac{s}{s^2+25} + \frac{2}{5} \cdot \frac{5}{s^2+25}. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ 3 \cdot \frac{s}{s^2+25} + \frac{2}{5} \cdot \frac{5}{s^2+25} \right\} \\ &= 3 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+25} \right\} + \frac{2}{5} \mathcal{L}^{-1} \left\{ \frac{5}{s^2+25} \right\} \\ &= 3 \cos 5t + \frac{2}{5} \sin 5t. \end{aligned}$$

8. Adjust as follows.

$$\begin{aligned} Y(s) &= \frac{2-5s}{s^2+9} \\ &= \frac{2}{s^2+9} - \frac{5s}{s^2+9} \\ &= \frac{2}{3} \cdot \frac{3}{s^2+9} - 5 \cdot \frac{s}{s^2+9} \end{aligned}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{3}{s^2+9} \right\} - 5 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} \\ &= \frac{2}{3} \sin 3t - 5 \cos 3t. \end{aligned}$$

9. Adjust as follows:

$$\begin{aligned} Y(s) &= \frac{1}{3-4s} + \frac{3-2s}{s^2+49} \\ &= \frac{1}{-4} \cdot \frac{1}{s-3/4} + \frac{3}{s^2+49} - \frac{2s}{s^2+49} \\ &= -\frac{1}{4} \cdot \frac{1}{s-3/4} + \frac{3}{7} \cdot \frac{7}{s^2+49} \\ &\quad - 2 \cdot \frac{s}{s^2+49}. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{4} \cdot \frac{1}{s-3/4} + \frac{3}{7} \cdot \frac{7}{s^2+49} \right. \\ &\quad \left. - 2 \cdot \frac{s}{s^2+49} \right\} \\ &= -\frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s-3/4} \right\} + \frac{3}{7} \mathcal{L}^{-1} \left\{ \frac{7}{s^2+49} \right\} \\ &\quad - 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+49} \right\} \\ &= -\frac{1}{4} e^{(3/4)t} + \frac{3}{7} \sin 7t - 2 \cos 7t. \end{aligned}$$

10. Adjust as follows.

$$\begin{aligned} Y(s) &= \frac{1}{s^5} - \frac{5-9s}{s^2+100} \\ &= \frac{1}{4!} \cdot \frac{4!}{s^5} - \frac{5}{s^2+100} + \frac{9s}{s^2+100} \\ &= \frac{1}{24} \cdot \frac{4!}{s^5} - \frac{1}{2} \cdot \frac{10}{s^2+100} + 9 \cdot \frac{s}{s^2+100} \end{aligned}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \frac{1}{24} \mathcal{L}^{-1} \left\{ \frac{4!}{s^5} \right\} - \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{10}{s^2+100} \right\} \\ &\quad + 9 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+100} \right\} \\ &= \frac{1}{24} t^4 - \frac{1}{2} \sin 10t + 9 \cos 10t \end{aligned}$$

11. Note the transform pair:

$$t^2 \longleftrightarrow \frac{2}{s^3}$$

By Proposition 2.12,

$$e^{-2t} t^2 \longleftrightarrow \frac{2}{(s+2)^3}.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{5}{(s+2)^3} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{5}{2} \cdot \frac{2}{(s+2)^3} \right\} \\ &= \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{2}{(s+2)^3} \right\} \\ &= \frac{5}{2} e^{-2t} t^2. \end{aligned}$$

12. Note the transform pair

$$t^5 \longleftrightarrow \frac{5!}{s^6}.$$

By Proposition 2.12,

$$e^t t^5 \longleftrightarrow \frac{5!}{(s-1)^6}.$$

Hence,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^6} \right\} \\ &= \frac{1}{5!} \mathcal{L}^{-1} \left\{ \frac{5!}{(s-1)^6} \right\} \\ &= \frac{1}{120} e^t t^5. \end{aligned}$$

13. Note the transform pair:

$$\sin 5t \longleftrightarrow \frac{5}{s^2 + 25}$$

By Proposition 2.12,

$$e^{-2t} \sin 5t \longleftrightarrow \frac{5}{(s+2)^2 + 25}.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{3}{(s+2)^2 + 25} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{3}{5} \cdot \frac{5}{(s+2)^2 + 25} \right\} \\ &= \frac{3}{5} \mathcal{L}^{-1} \left\{ \frac{5}{(s+2)^2 + 25} \right\} \\ &= \frac{3}{5} e^{-2t} \sin 5t. \end{aligned}$$

14. Note the transform pair.

$$\cos 2t \longleftrightarrow \frac{s}{s^2 + 4}$$

By Proposition 2.12,

$$e^t \cos 2t \longleftrightarrow \frac{s-1}{(s-1)^2 + 4}.$$

Hence,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{4(s-1)}{(s-1)^2 + 4} \right\} \\ &= 4 \mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2 + 4} \right\} \\ &= 4e^t \cos 2t. \end{aligned}$$

15. Note the transform pairs:

$$\begin{aligned} \cos \sqrt{5}t &\longleftrightarrow \frac{s}{s^2 + 5} \\ \sin \sqrt{5}t &\longleftrightarrow \frac{\sqrt{5}}{s^2 + 5} \end{aligned}$$

By Proposition 2.12,

$$\begin{aligned} e^t \cos \sqrt{5}t &\longleftrightarrow \frac{s-1}{(s-1)^2 + 5} \\ e^t \sin \sqrt{5}t &\longleftrightarrow \frac{\sqrt{5}}{(s-1)^2 + 5}. \end{aligned}$$

Thus,

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1} \left\{ \frac{2s-3}{(s-1)^2+5} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{2s-2}{(s-1)^2+5} - \frac{1}{(s-1)^2+5} \right\} \\
 &= \mathcal{L}^{-1} \left\{ 2 \cdot \frac{s-1}{(s-1)^2+5} \right. \\
 &\quad \left. - \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{(s-1)^2+5} \right\} \\
 &= 2 \mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2+5} \right\} \\
 &\quad - \frac{1}{\sqrt{5}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{5}}{(s-1)^2+5} \right\} \\
 &= 2e^t \cos \sqrt{5}t - \frac{1}{\sqrt{5}} e^t \sin \sqrt{5}t \\
 &= e^t (2 \cos \sqrt{5}t - \frac{\sqrt{5}}{5} \sin \sqrt{5}t).
 \end{aligned}$$

16. Note the transform pairs.

$$\begin{aligned}
 \cos 2t &\leftrightarrow \frac{s}{s^2+4} \\
 \sin 2t &\leftrightarrow \frac{2}{s^2+4}
 \end{aligned}$$

By Proposition 2.12,

$$\begin{aligned}
 e^{-2t} \cos 2t &\leftrightarrow \frac{s+2}{(s+2)^2+4} \\
 e^{-2t} \sin 2t &\leftrightarrow \frac{2}{(s+2)^2+4}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1} \left\{ \frac{s}{(s+2)^2+4} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{s+2}{(s+2)^2+4} - \frac{2}{(s+2)^2+4} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{s+2}{(s+2)^2+4} \right\} \\
 &\quad - \mathcal{L}^{-1} \left\{ \frac{2}{(s+2)^2+4} \right\} \\
 &= e^{-2t} \cos 2t - e^{-2t} \sin 2t \\
 &= e^{-2t} (\cos 2t - \sin 2t).
 \end{aligned}$$

17. Complete the square.

$$Y(s) = \frac{3s+2}{s^2+4s+29} = \frac{3s+2}{(s+2)^2+25}$$

Note the transform pairs.

$$\begin{aligned}
 \cos 5t &\leftrightarrow \frac{s}{s^2+25} \\
 \sin 5t &\leftrightarrow \frac{5}{s^2+25}
 \end{aligned}$$

By Proposition 2.12,

$$\begin{aligned}
 e^{-2t} \cos 5t &\leftrightarrow \frac{s+2}{(s+2)^2+25} \\
 e^{-2t} \sin 5t &\leftrightarrow \frac{5}{(s+2)^2+25}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1} \left\{ \frac{3s+2}{(s+2)^2+25} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{3s+6}{(s+2)^2+25} - \frac{4}{(s+2)^2+25} \right\} \\
 &= \mathcal{L}^{-1} \left\{ 3 \cdot \frac{s+2}{(s+2)^2+25} \right. \\
 &\quad \left. - \frac{4}{5} \cdot \frac{5}{(s+2)^2+25} \right\} \\
 &= 3 \mathcal{L}^{-1} \left\{ \frac{s+2}{(s+2)^2+25} \right\} \\
 &\quad - \frac{4}{5} \mathcal{L}^{-1} \left\{ \frac{5}{(s+2)^2+25} \right\} \\
 &= 3e^{-2t} \cos 5t - \frac{4}{5} e^{-2t} \sin 5t \\
 &= e^{-2t} (3 \cos 5t - \frac{4}{5} \sin 5t).
 \end{aligned}$$

18. Complete the square.

$$Y(s) = \frac{5-2s}{s^2-2s+5} = \frac{5-2s}{(s-1)^2+4}$$

Note the transform pairs.

$$\begin{aligned}
 \cos 2t &\leftrightarrow \frac{s}{s^2+4} \\
 \sin 2t &\leftrightarrow \frac{2}{s^2+4}
 \end{aligned}$$

By Proposition 2.12,

$$\begin{aligned} e^t \cos 2t &= \frac{s-1}{(s-1)^2+4} \\ e^t \sin 2t &= \frac{2}{(s-1)^2+4}. \end{aligned}$$

Hence,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{5-2s}{(s-1)^2+4} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{3-2(s-1)}{(s-1)^2+4} \right\} \\ &= \frac{3}{2} \mathcal{L}^{-1} \left\{ \frac{2}{(s-1)^2+4} \right\} \\ &\quad - 2 \mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2+4} \right\} \\ &= \frac{3}{2} e^t \sin 2t - 2e^t \cos 2t \\ &= \frac{1}{2} e^t (3 \sin 2t - 4 \cos 2t). \end{aligned}$$

19. Find a partial fraction decomposition.

$$\begin{aligned} \frac{1}{(s+2)(s-1)} &= \frac{A}{s+2} + \frac{B}{s-1} \\ 1 &= A(s-1) + B(s+2) \end{aligned}$$

Now,

$$\begin{aligned} s=1 &\Rightarrow 1=3B \text{ or } B=\frac{1}{3} \\ s=-2 &\Rightarrow 1=-3A \text{ or } A=-\frac{1}{3}. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)(s-1)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{-1/3}{s+2} + \frac{1/3}{s-1} \right\} \\ &= -\frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} + \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} \\ &= -\frac{1}{3} e^{-2t} + \frac{1}{3} e^t. \end{aligned}$$

20. Find a partial fraction decomposition.

$$\begin{aligned} \frac{1}{(s+3)(s-4)} &= \frac{A}{s+3} + \frac{B}{s-4} \\ 1 &= A(s-4) + B(s+3) \end{aligned}$$

Now,

$$\begin{aligned} s=4 &\Rightarrow 1=7B \text{ or } B=1/7 \\ s=-3 &\Rightarrow 1=-7A \text{ or } A=-1/7. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{-1/7}{s+3} + \frac{1/7}{s-4} \right\} \\ &= -\frac{1}{7} \mathcal{L}^{-1} \left\{ \frac{1}{s+3} \right\} + \frac{1}{7} \mathcal{L}^{-1} \left\{ \frac{1}{s-4} \right\} \\ &= -\frac{1}{7} e^{-3t} + \frac{1}{7} e^{4t}. \end{aligned}$$

21. Find a partial fraction decomposition.

$$\begin{aligned} \frac{2s-1}{(s+1)(s-2)} &= \frac{A}{s+1} + \frac{B}{s-2} \\ 2s-1 &= A(s-2) + B(s+1) \end{aligned}$$

Now,

$$\begin{aligned} s=2 &\Rightarrow 3=3B \text{ or } B=1 \\ s=-1 &\Rightarrow -3=-3A \text{ or } A=1. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{2s-1}{(s+1)(s-2)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s+1} + \frac{1}{s-2} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} \\ &= e^{-t} + e^{2t}. \end{aligned}$$

22. Find a partial fraction decomposition.

$$\begin{aligned} \frac{2s-2}{(s-4)(s+2)} &= \frac{A}{s-4} + \frac{B}{s+2} \\ 2s-2 &= A(s+2) + B(s-4) \end{aligned}$$

Now,

$$\begin{aligned} s=-2 &\Rightarrow -6=-6B \text{ or } B=1 \\ s=4 &\Rightarrow 6=6A \text{ or } A=1. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s-4} + \frac{1}{s+2} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s-4} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} \\ &= e^{4t} + e^{-2t}. \end{aligned}$$

23. Find a partial fraction decomposition.

$$\begin{aligned} \frac{7s+13}{s^2+2s-3} &= \frac{A}{s+3} + \frac{B}{s-1} \\ 7s+13 &= A(s-1) + B(s+3) \end{aligned}$$

Now,

$$\begin{aligned} s=1 &\Rightarrow 20=4B \text{ or } B=5 \\ s=-3 &\Rightarrow -8=-4A \text{ or } A=2. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{7s+13}{s^2+2s-3} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2}{s+3} + \frac{5}{s-1} \right\} \\ &= 2\mathcal{L}^{-1} \left\{ \frac{1}{s+3} \right\} + 5\mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} \\ &= 2e^{-3t} + 5e^t. \end{aligned}$$

24. Find a partial fraction decomposition.

$$\begin{aligned} \frac{7-s}{s^2+s-2} &= \frac{A}{s+2} + \frac{B}{s-1} \\ 7-s &= A(s-1) + B(s+2) \end{aligned}$$

Now,

$$\begin{aligned} s=1 &\Rightarrow 6=3B \text{ or } B=2 \\ s=-2 &\Rightarrow 9=-3A \text{ or } A=-3. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{-3}{s+2} + \frac{2}{s-1} \right\} \\ &= -3\mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} + 2\mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} \\ &= -3e^{-2t} + 2e^t. \end{aligned}$$

25. Find a partial fraction decomposition.

$$\begin{aligned} \frac{13s-5}{2s^2-s} &= \frac{A}{s} + \frac{B}{2s-1} \\ 13s-5 &= A(2s-1) + Bs \end{aligned}$$

Now,

$$\begin{aligned} s=\frac{1}{2} &\Rightarrow \frac{3}{2} = \frac{1}{2}B \text{ or } B=3 \\ s=0 &\Rightarrow -5 = -A \text{ or } A=5. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{13s-5}{2s^2-s} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{5}{s} + \frac{3}{2s-1} \right\} \\ &= \mathcal{L}^{-1} \left\{ 5 \cdot \frac{1}{s} + \frac{3}{2} \cdot \frac{1}{s-1/2} \right\} \\ &= 5\mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + \frac{3}{2}\mathcal{L}^{-1} \left\{ \frac{1}{s-1/2} \right\} \\ &= 5 + \frac{3}{2}e^{(1/2)t}. \end{aligned}$$

26. Find a partial fraction decomposition.

$$\begin{aligned} \frac{4s+15}{2s^2+3s} &= \frac{A}{s} + \frac{B}{2s+3} \\ 4s+15 &= A(2s+3) + Bs \end{aligned}$$

Now,

$$\begin{aligned} s=-3/2 &\Rightarrow 9 = -3/2B \text{ or } B=-6 \\ s=0 &\Rightarrow 15 = 3A \text{ or } A=5. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{5}{s} - \frac{6}{2s+3} \right\} \\ &= 5\mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - \frac{6}{2}\mathcal{L}^{-1} \left\{ \frac{1}{s+3/2} \right\} \\ &= 5 - 3e^{-(3/2)t}. \end{aligned}$$

27. Find a partial fraction decomposition.

$$\frac{7s^2+3s+16}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

Equating numerators, we get

$$\begin{aligned} 7s^2 + 3s + 16 &= A(s^2 + 4) + (Bs + C)(s + 1) \\ &= (A + B)s^2 + (B + C)s + (4A + C) \end{aligned}$$

Thus,

$$\begin{aligned} A + B &= 7 \\ B + C &= 3 \\ 4A + C &= 16, \end{aligned}$$

leading to $A = 4$, $B = 3$ and $C = 0$. Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{7s^2 + 3s + 16}{(s + 1)(s^2 + 4)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{4}{s + 1} + \frac{3s}{s^2 + 4} \right\} \\ &= 4 \mathcal{L}^{-1} \left\{ \frac{1}{s + 1} \right\} + 3 \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\} \\ &= 4e^{-t} + 3 \cos 2t. \end{aligned}$$

28. Find a partial fraction decomposition.

$$\frac{3s^2 + s + 1}{(s - 2)(s^2 + 1)} = \frac{A}{s - 2} + \frac{Bs + C}{s^2 + 1}$$

Equating numerators we get

$$\begin{aligned} 3s^2 + s + 1 &= A(s^2 + 1) + (Bs + C)(s - 2) \\ &= (A + B)s^2 + (-2B + C)s + (A - 2C) \end{aligned}$$

Thus,

$$\begin{aligned} A + B &= 3 \\ -2B + C &= 1 \\ A - 2C &= 1, \end{aligned}$$

and $A = 3$, $B = 0$, and $C = 1$. Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{3}{s - 2} + \frac{1}{s^2 + 1} \right\} \\ &= 3e^{2t} + \sin t. \end{aligned}$$

29. Find a partial fraction decomposition.

$$\frac{2s^2 + 9s + 11}{(s + 1)(s^2 + 4s + 5)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 4s + 5}$$

$$\begin{aligned} 2s^2 + 9s + 11 &= A(s^2 + 4s + 5) + (Bs + C)(s + 1) \\ &= (A + B)s^2 + (4A + B + C)s + (5A + C) \end{aligned}$$

Thus,

$$\begin{aligned} A + B &= 2 \\ 4A + B + C &= 9 \\ 5A + C &= 11, \end{aligned}$$

leading to $A = 2$, $B = 0$, and $C = 1$. Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{2s^2 + 9s + 11}{(s + 1)(s^2 + 4s + 5)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2}{s + 1} + \frac{1}{s^2 + 4s + 5} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2}{s + 1} + \frac{1}{(s + 2)^2 + 1} \right\} \\ &= 2 \mathcal{L}^{-1} \left\{ \frac{1}{s + 1} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{(s + 2)^2 + 1} \right\} \\ &= 2e^{-t} + e^{-2t} \sin t. \end{aligned}$$

30. Find a partial fraction decomposition.

$$\frac{7s^2 + 20s + 53}{(s - 1)(s^2 + 2s + 5)} = \frac{A}{s - 1} + \frac{Bs + C}{s^2 + 2s + 5}$$

$$\begin{aligned} 7s^2 + 20s + 53 &= A(s^2 + 2s + 5) + (Bs + C)(s - 1) \\ &= (A + B)s^2 + (2A - B + C)s + (5A - C) \end{aligned}$$

Thus,

$$\begin{aligned} A + B &= 7 \\ 2A - B + C &= 20 \\ 5A - C &= 53, \end{aligned}$$

and $A = 10$, $B = -3$, and $C = -3$. Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{10}{s - 1} + \frac{-3s - 3}{s^2 + 2s + 5} \right\} \\ &= 10 \mathcal{L}^{-1} \left\{ \frac{1}{s - 1} \right\} - 3 \mathcal{L}^{-1} \left\{ \frac{s + 1}{(s + 1)^2 + 4} \right\} \end{aligned}$$

Note the transform pair.

$$\cos 2t \leftrightarrow \frac{s}{s^2 + 4}$$

By Proposition 2.12,

$$e^{-t} \cos 2t \leftrightarrow \frac{s+1}{(s+1)^2 + 4}.$$

Thus,

$$y(t) = 10e^t - 3e^{-t} \cos 2t.$$

31. Find a partial fraction decomposition.

$$\begin{aligned} \frac{1}{(s-2)^2(s+1)^3} &= \frac{A}{s-2} + \frac{B}{(s-2)^2} \\ &\quad + \frac{C}{s+1} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)^3} \end{aligned}$$

Equating numerators we get

$$\begin{aligned} 1 &= A(s-2)(s+1)^3 + B(s+1)^3 \\ &\quad + C(s-2)^2(s+1)^2 + D(s-2)^2(s+1) + E(s-2)^2 \\ &= (A+C)s^4 + (A+B-2C+D)s^3 \\ &\quad + (-3A+3B-3C-3D+E)s^2 \\ &\quad + (-5A+3B+4C-4E)s \\ &\quad + (-2A+B+4C+4D+4E) \end{aligned}$$

Thus,

$$\begin{aligned} A+C &= 0 \\ A+B-2C+D &= 0 \\ -3A+3B-3C-3D+E &= 0 \\ -5A+3B+4C-4E &= 0 \\ -2A+B+4C+4D+4E &= 1, \end{aligned}$$

leading to $A = -1/27$, $B = 1/27$, $C = 1/27$,

$D = 2/27$, and $E = 1/9$. Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{(s-2)^2(s+1)^3} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{-1/27}{s-2} + \frac{1/27}{(s-2)^2} + \frac{1/27}{s+1} \right. \\ &\quad \left. + \frac{2/27}{(s+1)^2} + \frac{1/9}{(s+1)^3} \right\} \\ &= -\frac{1}{27} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} + \frac{1}{27} \mathcal{L}^{-1} \left\{ \frac{1}{(s-2)^2} \right\} \\ &\quad + \frac{1}{27} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} + \frac{2}{27} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2} \right\} \\ &\quad + \frac{1}{9} \cdot \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{(s+1)^3} \right\} \\ &= -\frac{1}{27} e^{2t} + \frac{1}{27} e^{2t} t + \frac{1}{27} e^{-t} \\ &\quad + \frac{2}{27} e^{-t} t + \frac{1}{18} e^{-t} t^2 \\ &= \frac{1}{27} e^{2t} (t-1) + \frac{1}{27} e^{-t} (1+2t+\frac{3}{2}t^2). \end{aligned}$$

32. Find a partial fraction decomposition.

$$\begin{aligned} \frac{1}{(s-1)^2(s^2+4)} &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{(s^2+4)} \end{aligned}$$

Equating numerators we get

$$\begin{aligned} 1 &= A(s-1)(s^2+4) + B(s^2+4) \\ &\quad + (Cs+D)(s-1)^2 \\ &= (A+C)s^3 + (-A+B-2C+D)s^2 \\ &\quad + (4A+C-2D)s + (-4A+4B+D) \end{aligned}$$

Thus,

$$\begin{aligned} A+C &= 0 \\ -A+B-2C+D &= 0 \\ 4A+C-2D &= 0 \\ -4A+4B+D &= 1 \end{aligned}$$

and $A = -2/25$, $B = 1/5$, $C = 2/25$, and

$D = -3/25$. Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{-2/25}{s-1} + \frac{1/5}{(s-1)^2} \right. \\ &\quad \left. + \frac{(2/25)s}{s^2+4} + \frac{-3/25}{s^2+4} \right\} \\ &= -\frac{2}{25} \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} + \frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2} \right\} \\ &\quad + \frac{2}{25} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} \\ &\quad - \frac{3}{25} \cdot \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2+4} \right\} \end{aligned}$$

Note the transform pair.

$$t \leftrightarrow \frac{1}{s^2}$$

By Proposition 2.12,

$$e^t t \leftrightarrow \frac{1}{(s-1)^2}.$$

Thus,

$$y(t) = -\frac{2}{25}e^t + \frac{1}{5}te^t + \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t.$$

33. Find a partial fraction decomposition.

$$\begin{aligned} &\frac{1}{(s+2)^2(s^2+9)} \\ &= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+9} \end{aligned}$$

$$\begin{aligned} 1 &= A(s+2)(s^2+9) + B(s^2+9) \\ &\quad + (Cs+D)(s+2)^2 \\ &= (A+C)s^3 + (2A+B+4C+D)s^2 \\ &\quad + (9A+4C+4D)s + (18A+9B+4D) \end{aligned}$$

Thus,

$$\begin{aligned} A+C &= 0 \\ 2A+B+4C+D &= 0 \\ 9A+4C+4D &= 0 \\ 18A+9B+4D &= 1, \end{aligned}$$

leading to $A = 4/169$, $B = 1/13$, $C = -4/169$, and $D = -5/169$. Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2(s^2+9)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{4/169}{s+2} + \frac{1/13}{(s+2)^2} \right. \\ &\quad \left. + \frac{-(4/169)s - 5/169}{s^2+9} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{4}{169} \cdot \frac{1}{s+2} + \frac{1}{13} \cdot \frac{1}{(s+2)^2} \right. \\ &\quad \left. - \frac{4}{169} \cdot \frac{s}{s^2+9} - \frac{5}{169} \cdot \frac{1}{s^2+9} \right\} \\ &= \frac{4}{169}e^{-2t} + \frac{1}{13}te^{-2t} - \frac{4}{169}\cos 3t \\ &\quad - \frac{5}{169} \cdot \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3}{s^2+9} \right\} \\ &= \frac{4}{169}e^{-2t} + \frac{1}{13}te^{-2t} - \frac{4}{169}\cos 3t \\ &\quad - \frac{5}{507}\sin 3t. \end{aligned}$$

34. Find a partial fraction decomposition.

$$\begin{aligned} &\frac{1}{(s-1)^2(s^2-9)} \\ &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3} + \frac{D}{s-3} \end{aligned}$$

Equating numerators

$$\begin{aligned} 1 &= A(s-1)(s^2-9) + B(s^2-9) \\ &\quad + C(s-1)^2(s-3) + D(s-1)^2(s+3) \end{aligned}$$

Now,

$$\begin{aligned} s=1 &\Rightarrow 1 = -8B \text{ or } B = -1/8 \\ s=3 &\Rightarrow 1 = 24D \text{ or } D = 1/24 \\ s=-3 &\Rightarrow 1 = -96C \text{ or } C = -1/96. \end{aligned}$$

We need an equation involving A . It is not difficult

to see that the coefficient of s^3 is $A + C + D$. Thus,

$$\begin{aligned} A + C + D &= 0 \\ A - \frac{1}{96} + \frac{1}{24} &= 0 \\ A &= -\frac{1}{32}. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{-1/32}{s-1} + \frac{-1/8}{(s-1)^2} \right. \\ &\quad \left. + \frac{-1/96}{s+3} + \frac{1/24}{s-3} \right\} \\ &= -\frac{1}{32} \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} - \frac{1}{8} \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2} \right\} \\ &\quad - \frac{1}{96} \mathcal{L}^{-1} \left\{ \frac{1}{s+3} \right\} + \frac{1}{24} \mathcal{L}^{-1} \left\{ \frac{1}{s-3} \right\}. \end{aligned}$$

Note the transform pair

$$t \leftrightarrow \frac{1}{s^2}.$$

By Proposition 2.12,

$$e^t t \leftrightarrow \frac{1}{(s-1)^2}.$$

Thus,

$$y(t) = -\frac{1}{32}e^t - \frac{1}{8}te^t - \frac{1}{96}e^{-3t} + \frac{1}{24}e^{3t}.$$

35. Find a partial fraction decomposition.

$$\begin{aligned} &\frac{1}{(s+1)^2(s^2-4)} \\ &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2} + \frac{D}{s-2} \end{aligned}$$

Equating numerators

$$\begin{aligned} 1 &= A(s+1)(s^2-4) + B(s^2-4) \\ &\quad + C(s+1)^2(s-2) + D(s+1)^2(s+2) \end{aligned}$$

Now,

$$\begin{aligned} s = -1 &\Rightarrow 1 = -3B \text{ or } B = -1/3 \\ s = 2 &\Rightarrow 1 = 36D \text{ or } D = 1/36 \\ s = -2 &\Rightarrow 1 = -4C \text{ or } C = -1/4, \end{aligned}$$

and

$$\begin{aligned} s = 0 &\Rightarrow 1 = -4A - 4B - 2C + 2D \\ 1 &= -4A + \frac{4}{3} + \frac{1}{2} + \frac{1}{18} \\ A &= \frac{2}{9}. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2(s^2-4)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2/9}{s+1} + \frac{-1/3}{(s+1)^2} + \frac{-1/4}{s+2} + \frac{1/36}{s-2} \right\} \\ &= \frac{2}{9} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2} \right\} \\ &\quad - \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} + \frac{1}{36} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} \\ &= \frac{2}{9}e^{-t} - \frac{1}{3}e^{-t}t - \frac{1}{4}e^{-2t} + \frac{1}{36}e^{2t}. \end{aligned}$$

36. Find a partial fraction decomposition.

$$\begin{aligned} &\frac{s}{(s+2)^2(s^2+9)} \\ &= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+9} \end{aligned}$$

Equating numerators

$$\begin{aligned} s &= A(s+2)(s^2+9) + B(s^2+9) \\ &\quad + (Cs+D)(s+2)^2 \\ &= (A+C)s^3 + (2A+B+4C+D)s^2 \\ &\quad + (9A+4C+4D)s + (18A+9B+4D) \end{aligned}$$

Thus,

$$\begin{aligned} A + C &= 0 \\ 2A + B + 4C + D &= 0 \\ 9A + 4C + 4D &= 1 \\ 18A + 9B + 4D &= 0 \end{aligned}$$

and $A = 5/169$, $B = -2/13$, $C = -5/169$, and $D = 36/169$. Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{5/169}{s+2} + \frac{-2/13}{(s+2)^2} \right. \\ &\quad \left. + \frac{-(5/169)s}{s^2+9} + \frac{36/169}{s^2+9} \right\} \\ &= \frac{5}{169} \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} - \frac{2}{13} \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2} \right\} \\ &\quad - \frac{5}{169} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} \\ &\quad + \frac{36}{169} \cdot \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3}{s^2+9} \right\} \end{aligned}$$

Note the transform pair.

$$t \leftrightarrow \frac{1}{s^2}.$$

By Proposition 2.12,

$$e^{-2t}t \leftrightarrow \frac{1}{(s+2)^2}.$$

Thus,

$$y(t) = \frac{5}{169}e^{-2t} - \frac{2}{13}te^{-2t} - \frac{5}{169}\cos 3t + \frac{12}{169}\sin 3t.$$

Section 5.4. Using the Laplace Transform to Solve Differential Equations

1. Set $Y = \mathcal{L}(y)$. Then,

$$\begin{aligned} \mathcal{L}(y' + 3y) &= \mathcal{L}(e^{2t}) \\ \mathcal{L}(y') + 3\mathcal{L}(y) &= \frac{1}{s-2} \\ s \cdot Y(s) - y(0) + 3Y(s) &= \frac{1}{s-2}. \end{aligned}$$

But, $y(0) = -1$, so

$$\begin{aligned} (s+3)Y(s) + 1 &= \frac{1}{s-2} \\ Y(s) &= \frac{-1}{s+3} + \frac{1}{(s+3)(s-2)}. \end{aligned}$$

Find a partial fraction decomposition.

$$\begin{aligned} \frac{1}{(s+3)(s-2)} &= \frac{A}{s+3} + \frac{B}{s-2} \\ 1 &= A(s-2) + B(s+3). \end{aligned}$$

Now,

$$\begin{aligned} s=2 &\Rightarrow 1 = 5B \text{ or } B = 1/5 \\ s=-3 &\Rightarrow 1 = -5A \text{ or } A = -1/5. \end{aligned}$$

Thus,

$$\begin{aligned} Y(s) &= \frac{-1}{s+3} - \frac{1/5}{s+3} + \frac{1/5}{s-2} \\ &= \frac{-6/5}{s+3} + \frac{1/5}{s-2}, \end{aligned}$$

and

$$\begin{aligned} y(t) &= -\frac{6}{5} \mathcal{L}^{-1} \left\{ \frac{1}{s+3} \right\} + \frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} \\ &= -\frac{6}{5}e^{-3t} + \frac{1}{5}e^{2t}. \end{aligned}$$

2. Set $Y = \mathcal{L}(y)$. Then,

$$\begin{aligned} \mathcal{L}(y' + 9y) &= \mathcal{L}(e^{-t}) \\ \mathcal{L}(y') + 9\mathcal{L}(y) &= \frac{1}{s+1} \\ s \cdot Y(s) - y(0) + 9Y(s) &= \frac{1}{s+1}. \end{aligned}$$

But $y(0) = 0$, so

$$\begin{aligned} (s+9)Y(s) &= \frac{1}{s+1} \\ Y(s) &= \frac{1}{(s+1)(s+9)} \end{aligned}$$

Find a partial fraction decomposition.

$$\frac{1}{(s+1)(s+9)} = \frac{A}{s+1} + \frac{B}{s+9}$$

$$1 = A(s+9) + B(s+1)$$

Now,

$$s = -9 \Rightarrow 1 = -8B \text{ or } B = -\frac{1}{8}$$

$$s = -1 \Rightarrow 1 = 8A \text{ or } A = \frac{1}{8}.$$

Thus,

$$Y(s) = \frac{1/8}{s+1} - \frac{1/8}{s+9},$$

and

$$y(t) = \frac{1}{8} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - \frac{1}{8} \mathcal{L}^{-1} \left\{ \frac{1}{s+9} \right\}$$

$$y(t) = \frac{1}{8} e^{-t} - \frac{1}{8} e^{-9t}.$$

3. Set $Y = \mathcal{L}(y)$. Then,

$$\mathcal{L}(y' + 4y) = \mathcal{L}(\cos t)$$

$$\mathcal{L}(y') + 4\mathcal{L}(y) = \frac{s}{s^2 + 1}$$

$$s \cdot Y(s) - y(0) + 4Y(s) = \frac{s}{s^2 + 1}.$$

But $y(0) = 0$, so

$$(s+4)Y(s) = \frac{s}{s^2 + 1}$$

$$Y(s) = \frac{s}{(s+4)(s^2 + 1)}.$$

Find a partial fraction decomposition.

$$\frac{s}{(s+4)(s^2 + 1)} = \frac{A}{s+4} + \frac{Bs + C}{s^2 + 1}$$

Equating numerators

$$s = A(s^2 + 1) + (Bs + C)(s + 4)$$

$$= (A + B)s^2 + (4B + C)s + (A + 4C).$$

Thus,

$$A + B = 0$$

$$4B + C = 1$$

$$A + 4C = 0,$$

and $A = -4/17$, $B = 4/17$, and $C = 1/17$. Thus,

$$Y(s) = \frac{-4/17}{s+4} + \frac{(4/17)s + 1/17}{s^2 + 1}$$

$$= -\frac{4}{17} \cdot \frac{1}{s+4} + \frac{4}{17} \cdot \frac{s}{s^2 + 1}$$

$$+ \frac{1}{17} \cdot \frac{1}{s^2 + 1},$$

and

$$y(t) = -\frac{4}{17} e^{-4t} + \frac{4}{17} \cos t + \frac{1}{17} \sin t.$$

4. Set $Y = \mathcal{L}(y)$. Then

$$\mathcal{L}(y' + 16y) = \mathcal{L}(\sin 3t)$$

$$\mathcal{L}(y') + 16\mathcal{L}(y) = \frac{3}{s^2 + 9}$$

$$s \cdot Y(s) - y(0) + 16Y(s) = \frac{3}{s^2 + 9}.$$

But $y(0) = 1$, so

$$(s+16)Y(s) - 1 = \frac{3}{s^2 + 9}$$

$$Y(s) = \frac{1}{s+16} + \frac{3}{(s^2 + 9)(s+16)}.$$

Find a partial fraction decomposition.

$$\frac{3}{(s^2 + 9)(s+16)} = \frac{A}{s+16} + \frac{Bs + C}{s^2 + 9}$$

Equating numerators

$$3 = A(s^2 + 9) + (Bs + C)(s + 16)$$

$$3 = (A + B)s^2 + (16B + C)s + (9A + 16C)$$

Thus,

$$(A + B) = 0$$

$$(16B + C) = 0$$

$$(9A + 16C) = 3,$$

and $A = 3/265$, $B = -3/265$, and $C = 48/265$.
Then,

$$\begin{aligned} Y(s) &= \frac{1}{s+16} + \frac{3/265}{s+16} \\ &\quad + \frac{-(3/265)s + 48/265}{s^2+9} \\ &= \frac{268/265}{s+16} - \frac{(3/265)s}{s^2+9} + \frac{48/265}{s^2+9}, \end{aligned}$$

and

$$\begin{aligned} y(t) &= \frac{268}{265} \mathcal{L}^{-1} \left(\frac{1}{s+16} \right) \\ &\quad - \frac{3}{265} \mathcal{L}^{-1} \left(\frac{s}{s^2+9} \right) \\ &\quad + \frac{48}{265} \cdot \frac{1}{3} \mathcal{L}^{-1} \left(\frac{3}{s^2+9} \right) \\ &= \frac{268}{265} e^{-16t} - \frac{3}{265} \cos 3t + \frac{16}{265} \sin 3t. \end{aligned}$$

5. Set $Y = \mathcal{L}(y)$. Then,

$$\begin{aligned} \mathcal{L}(y' + 6y) &= \mathcal{L}(2t + 3) \\ \mathcal{L}(y') + 6\mathcal{L}(y) &= 2\mathcal{L}(t) + 3\mathcal{L}(1) \\ s \cdot Y(s) - y(0) + 6Y(s) &= \frac{2}{s^2} + \frac{3}{s}. \end{aligned}$$

But $y(0) = 1$, so

$$(s+6)Y(s) - 1 = \frac{2}{s^2} + \frac{3}{s}.$$

Solving for Y ,

$$\begin{aligned} Y(s) &= \frac{1}{s+6} + \frac{2}{s^2(s+6)} + \frac{3}{s(s+6)} \\ &= \frac{1}{s+6} + \frac{3s+2}{s^2(s+6)}. \end{aligned}$$

Find a partial fraction decomposition.

$$\frac{3s+2}{s^2(s+6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s+6}$$

Equating numerators,

$$\begin{aligned} 3s+2 &= As(s+6) + B(s+6) + (Cs+D)s^2 \\ 3s+2 &= Cs^3 + (A+D)s^2 + (6A+B)s + 6B. \end{aligned}$$

Thus,

$$\begin{aligned} C &= 0 \\ A+D &= 0 \\ 6A+B &= 3 \\ 6B &= 2, \end{aligned}$$

and $A = 4/9$, $B = 1/3$, $C = 0$, and $D = -4/9$.
Thus,

$$\begin{aligned} Y(s) &= \frac{1}{s+6} + \frac{4/9}{s} + \frac{1/3}{s^2} - \frac{4/9}{s+6} \\ &= \frac{5/9}{s+6} + \frac{4/9}{s} + \frac{1/3}{s^2}. \end{aligned}$$

Therefore,

$$y(t) = \frac{5}{9}e^{-6t} + \frac{4}{9} + \frac{1}{3}t.$$

6. Let $\mathcal{L}(y) = Y(s)$. Then,

$$\begin{aligned} \mathcal{L}(y' + 8y) &= \mathcal{L}(t^2) \\ \mathcal{L}(y') + 8\mathcal{L}(y) &= \frac{2!}{s^3} \\ s \cdot Y(s) - y(0) + 8Y(s) &= \frac{2}{s^3}. \end{aligned}$$

But, $y(0) = -1$.

$$(s+8)Y(s) + 1 = \frac{2}{s^3}$$

$$Y(s) = \frac{2}{s^3(s+8)} - \frac{1}{s+8}$$

Find a partial fraction decomposition.

$$\frac{2}{s^3(s+8)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+8}$$

Equating numerators,

$$\begin{aligned} 2 &= As^2(s+8) + Bs(s+8) \\ &\quad + C(s+8) + Ds^3 \\ &= (A+D)s^3 + (8A+B)s^2 \\ &\quad + (8B+C)s + 8C \end{aligned}$$

Then,

$$\begin{aligned} A+D &= 0 \\ 8A+B &= 0 \\ 8B+C &= 0 \\ 8C &= 2, \end{aligned}$$

and $A = 1/256$, $B = -1/32$, and $C = 1/4$, and $D = -1/256$. Thus,

$$\begin{aligned} Y(s) &= \frac{-1}{s+8} + \frac{1/256}{s} - \frac{1/32}{s^2} \\ &\quad + \frac{1/4}{s^3} - \frac{1/256}{s+8} \\ &= \frac{1}{256} \left(\frac{1}{s} \right) - \frac{1}{32} \left(\frac{1}{s^2} \right) + \frac{1}{4} \cdot \frac{1}{2} \left(\frac{2!}{s^3} \right) \\ &\quad + \left(\frac{-257}{256} \right) \left(\frac{1}{s+8} \right). \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{256} - \frac{1}{32}t + \frac{1}{8}t^2 - \frac{257}{256}e^{-8t}.$$

7. Let $\mathcal{L}(y) = Y$. Then,

$$\mathcal{L}(y' + 8y) = \mathcal{L}(e^{-2t} \sin t).$$

Note the transform pair:

$$\sin t \iff \frac{1}{s^2 + 1}$$

Thus, by Proposition 2.12,

$$e^{-2t} \sin t \iff \frac{1}{(s+2)^2 + 1}.$$

Thus,

$$\begin{aligned} \mathcal{L}(y') + 8\mathcal{L}(y) &= \frac{1}{(s+2)^2 + 1} \\ s \cdot Y(s) - y(0) + 8Y(s) &= \frac{1}{(s+2)^2 + 1}. \end{aligned}$$

But $y(0) = 0$, so

$$\begin{aligned} (s+8)Y(s) &= \frac{1}{(s+2)^2 + 1} \\ Y(s) &= \frac{1}{(s+8)((s+2)^2 + 1)}. \end{aligned}$$

Find a partial fraction decomposition.

$$\frac{1}{(s+8)((s+2)^2 + 1)} = \frac{A}{s+8} + \frac{Bs+C}{(s+2)^2 + 1}$$

$$1 = A((s+2)^2 + 1) + (Bs+C)(s+8)$$

$$1 = (A+B)s^2 + (4A+8B+C)s + (5A+8C).$$

Thus,

$$A+B=0$$

$$4A+8B+C=0$$

$$5A+8C=1,$$

and $A = 1/37$, $B = -1/37$, and $C = 4/37$. Thus,

$$Y(s) = \frac{1/37}{s+8} + \frac{-(1/37)s + 4/37}{(s+2)^2 + 1}.$$

Note the transform pairs.

$$\cos t \iff \frac{s}{s^2 + 1}$$

$$\sin t \iff \frac{1}{s^2 + 1}.$$

Proposition 2.12 provides

$$e^{-2t} \cos t = \frac{s+2}{(s+2)^2 + 1}$$

$$e^{-2t} \sin t = \frac{1}{(s+2)^2 + 1}.$$

Thus,

$$\begin{aligned} Y(s) &= \frac{1}{37} \cdot \frac{1}{s+8} - \frac{1}{37} \cdot \frac{s-4}{(s+2)^2 + 1} \\ &= \frac{1}{37} \cdot \frac{1}{s+8} \\ &\quad - \frac{1}{37} \left\{ \frac{s+2}{(s+2)^2 + 1} - \frac{6}{(s+2)^2 + 1} \right\} \\ &= \frac{1}{37} \cdot \frac{1}{s+8} - \frac{1}{37} \cdot \frac{s+2}{(s+2)^2 + 1} \\ &\quad + \frac{6}{37} \cdot \frac{1}{(s+2)^2 + 1}. \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{37}e^{-8t} - \frac{1}{37}e^{-2t} \cos t + \frac{6}{37}e^{-2t} \sin t.$$

8. Let $\mathcal{L}(y) = Y$. Then

$$\mathcal{L}(y' - 2y) = \mathcal{L}(e^{-t} \cos t).$$

Note the transform pair.

$$\cos t \iff \frac{s}{s^2 + 1}$$

Then by Proposition 2.12,

$$e^{-t} \cos t \leftrightarrow \frac{s+1}{(s+1)^2+1}.$$

Thus,

$$\begin{aligned}\mathcal{L}(y') - 2\mathcal{L}(y) &= \frac{s+1}{(s+1)^2+1} \\ s \cdot Y(s) - y(0) - 2Y(s) &= \frac{s+1}{(s+1)^2+1}.\end{aligned}$$

But $y(0) = -2$, so

$$\begin{aligned}(s-2)Y(s) + 2 &= \frac{s+1}{(s+1)^2+1} \\ Y(s) &= \frac{-2}{s-2} + \frac{s+1}{((s+1)^2+1)(s-2)}.\end{aligned}$$

Find a partial fraction decomposition.

$$\frac{s+1}{(s-2)((s+1)^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{(s+1)^2+1}$$

Equating numerators,

$$\begin{aligned}s+1 &= A((s+1)^2+1) + (Bs+C)(s-2) \\ &= (A+B)s^2 + (2A-2B+C)s \\ &\quad + (2A-2C)\end{aligned}$$

Thus,

$$\begin{aligned}A+B &= 0 \\ 2A-2B+C &= 1 \\ 2A-2C &= 1,\end{aligned}$$

and $A = 3/10$, $B = -3/10$, and $C = -1/5$. Thus,

$$\begin{aligned}Y(s) &= \frac{-2}{s-2} + \frac{3/10}{s-2} - \frac{(3/10)s-1/5}{(s+1)^2+1} \\ &= \frac{-17/10}{s-2} + \frac{-(3/10)s-1/5}{(s+1)^2+1}\end{aligned}$$

Note the transform pairs

$$\begin{aligned}\cos t &\leftrightarrow \frac{s}{s^2+1} \\ \sin t &\leftrightarrow \frac{1}{s^2+1}\end{aligned}$$

Proposition 2.12 provides

$$\begin{aligned}e^{-t} \cos t &\leftrightarrow \frac{s+1}{(s+1)^2+1} \\ e^{-t} \sin t &\leftrightarrow \frac{1}{(s+1)^2+1}.\end{aligned}$$

Thus,

$$\begin{aligned}Y(s) &= \frac{-17}{10} \left(\frac{1}{s-2} \right) - \frac{3}{10} \left(\frac{s+2/3}{(s+1)^2+1} \right) \\ &= \frac{-17}{10} \left(\frac{1}{s-2} \right) \\ &\quad - \frac{3}{10} \left(\frac{s+1}{(s+1)^2+1} - \frac{1/3}{(s+1)^2+1} \right) \\ &= \frac{-17}{10} \left(\frac{1}{s-2} \right) - \frac{3}{10} \left(\frac{s+1}{(s+1)^2+1} \right) \\ &\quad + \frac{1}{10} \left(\frac{1}{(s+1)^2+1} \right)\end{aligned}$$

Therefore,

$$y(t) = \frac{-17}{10} e^{2t} - \frac{3}{10} e^{-t} \cos t + \frac{1}{10} e^{-t} \sin t.$$

9. Let $\mathcal{L}(y) = Y$. Then,

$$\begin{aligned}\mathcal{L}(y' + y) &= \mathcal{L}(te^t) \\ \mathcal{L}(y') + \mathcal{L}(y) &= \frac{1}{(s-1)^2} \\ s \cdot Y(s) - y(0) + Y(s) &= \frac{1}{(s-1)^2}.\end{aligned}$$

But $y(0) = -2$, so

$$\begin{aligned}(s+1)Y(s) + 2 &= \frac{1}{(s-1)^2} \\ Y(s) &= -\frac{2}{s+1} + \frac{1}{(s+1)(s-1)^2}.\end{aligned}$$

Find a partial fraction decomposition.

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Equating numerators,

$$\begin{aligned}1 &= A(s-1)^2 + B(s+1)(s-1) + C(s+1) \\ &= (A+B)s^2 + (-2A-C)s + (A-B+C).\end{aligned}$$

Now,

$$s = 1 \Rightarrow 1 = 2C \text{ or } C = 1/2$$

$$s = -1 \Rightarrow 1 = 4A \text{ or } A = 1/4.$$

Finally, comparing coefficients of s^2 on each side,

$$0 = A + B = \frac{1}{4} + B \Rightarrow B = -\frac{1}{4}.$$

Thus,

$$\begin{aligned} Y(s) &= -\frac{2}{s+1} + \frac{1/4}{s+1} - \frac{1/4}{s-1} + \frac{1/2}{(s-1)^2} \\ &= -\frac{7/4}{s+1} - \frac{1/4}{s-1} + \frac{1/2}{(s-1)^2}. \end{aligned}$$

Therefore,

$$y(t) = -\frac{7}{4}e^{-t} - \frac{1}{4}e^t + \frac{1}{2}e^t t.$$

10. Let $\mathcal{L}(y) = Y(s)$. Then,

$$\mathcal{L}(y' - 4y) = \mathcal{L}(e^{-2t}t^2)$$

$$\mathcal{L}(y') - 4\mathcal{L}(y) = \frac{2!}{(s+2)^3}$$

$$s \cdot Y(s) - y(0) - 4Y(s) = \frac{2}{(s+2)^3}.$$

But $y(0) = 1$, so

$$(s-4)Y(s) - 1 = \frac{2}{(s+2)^3}$$

$$Y(s) = \frac{1}{s-4} + \frac{2}{(s-4)(s+2)^3}.$$

Find a partial fraction decomposition.

$$\begin{aligned} \frac{2}{(s-4)(s+2)^3} \\ = \frac{A}{s-4} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3} \end{aligned}$$

Equating numerators,

$$\begin{aligned} 2 &= A(s+2)^3 + B(s-4)(s+2)^2 \\ &\quad + C(s-4)(s+2) + D(s-4) \\ &= (A+B)s^3 + (6A+C)s^2 \\ &\quad + (12A-12B-2C+D)s \\ &\quad + (8A-16B-8C-4D) \end{aligned}$$

Thus,

$$A + B = 0$$

$$6A + C = 0$$

$$12A - 12B - 2C + D = 0$$

$$8A - 16B - 8C - 4D = 2,$$

and $A = 1/108$, $B = -1/108$, $C = -1/18$, and $D = -1/3$. Thus,

$$\begin{aligned} Y(s) &= \frac{1}{s-4} + \frac{1/108}{s-4} - \frac{1/108}{s+2} \\ &\quad - \frac{1/18}{(s+2)^2} - \frac{1/3}{(s+2)^3} \\ &= \frac{109}{108} \left(\frac{1}{s-4} \right) - \frac{1}{108} \left(\frac{1}{s+2} \right) \\ &\quad - \frac{1}{18} \left(\frac{1}{(s+2)^2} \right) - \frac{1}{3} \cdot \frac{1}{2} \left(\frac{2!}{(s+2)^2} \right) \end{aligned}$$

Therefore,

$$y(t) = \frac{109}{108}e^{4t} - \frac{1}{108}e^{-2t} - \frac{1}{18}te^{-2t} - \frac{1}{6}t^2e^{-2t}.$$

11. Let $\mathcal{L}(y) = Y$. Then,

$$\mathcal{L}(y'' - 4y) = \mathcal{L}(e^{-t})$$

$$\mathcal{L}(y'') - 4\mathcal{L}(y) = \frac{1}{s+1}$$

$$s^2Y(s) - s \cdot y(0) - y'(0) - 4Y(s) = \frac{1}{s+1}.$$

But $y(0) = -1$ and $y'(0) = 0$, so

$$(s^2-4)Y(s) + s = \frac{1}{s+1}$$

$$Y(s) = \frac{-s}{s^2-4} + \frac{1}{(s^2-4)(s+1)}.$$

Both terms on the right have partial fraction decompositions.

$$\begin{aligned} \frac{-s}{s^2-4} &= \frac{-1/2}{s-2} - \frac{1/2}{s+2} \\ \frac{1}{(s^2-4)(s+1)} &= \frac{1/12}{s-2} + \frac{1/4}{s+2} - \frac{1/3}{s+1}. \end{aligned}$$

Thus,

$$Y(s) = \frac{-5/12}{s-2} - \frac{1/4}{s+2} - \frac{1/3}{s+1},$$

and

$$y(t) = -\frac{5}{12}e^{2t} - \frac{1}{4}e^{-2t} - \frac{1}{3}e^{-t}.$$

12. Let $\mathcal{L}(y) = Y(s)$. Then

$$\begin{aligned}\mathcal{L}(y'' - 9y) &= \mathcal{L}(-2e^t) \\ \mathcal{L}(y'') - 9\mathcal{L}(y) &= \frac{-2}{s-1} \\ s^2Y(s) - s \cdot y(0) - y'(0) - 9Y(s) &= \frac{-2}{s-1}.\end{aligned}$$

But $y(0) = 0$ and $y'(0) = 1$, so

$$\begin{aligned}(s^2 - 9)Y(s) - 1 &= \frac{-2}{s-1} \\ Y(s) &= \frac{-2}{(s-1)(s^2-9)} + \frac{1}{s^2-9}.\end{aligned}$$

Both the terms on the right have a partial fraction decomposition.

$$\begin{aligned}\frac{1}{s^2-9} &= \frac{1/6}{s-3} - \frac{1/6}{s+3} \\ \frac{-2}{(s-1)(s^2-9)} &= \frac{1/4}{s-1} - \frac{1/6}{s-3} - \frac{1/12}{s+3}\end{aligned}$$

Thus

$$Y(s) = \frac{1/4}{s-1} - \frac{1/4}{s+3}.$$

Therefore,

$$y(t) = \frac{1}{4}e^t - \frac{1}{4}e^{-3t}.$$

13. Let $\mathcal{L}(y) = Y$. Then,

$$\begin{aligned}\mathcal{L}(y'' + 4y) &= \mathcal{L}(\cos t) \\ \mathcal{L}(y'') + 4\mathcal{L}(y) &= \frac{s}{s^2+1} \\ s^2Y(s) - s \cdot y(0) - y'(0) + 4Y(s) &= \frac{s}{s^2+1}.\end{aligned}$$

But $y(0) = 1$ and $y'(0) = 0$, so

$$\begin{aligned}(s^2 + 4)Y(s) - s &= \frac{s}{s^2+1} \\ Y(s) &= \frac{s}{s^2+4} + \frac{s}{(s^2+4)(s^2+1)}.\end{aligned}$$

But the term on the right has a decomposition

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{-(1/3)s}{s^2+4} + \frac{(1/3)s}{s^2+1}.$$

Thus,

$$Y(s) = \frac{(2/3)s}{s^2+4} + \frac{(1/3)s}{s^2+1},$$

and

$$\begin{aligned}y(t) &= \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\ &= \frac{2}{3}\cos 2t + \frac{1}{3}\cos t.\end{aligned}$$

14. Let $\mathcal{L}(y) = Y(s)$. Then

$$\begin{aligned}\mathcal{L}(y'' + 9y) &= \mathcal{L}(2\sin 2t) \\ \mathcal{L}(y'') + 9\mathcal{L}(y) &= \frac{4}{s^2+4} \\ s^2Y(s) - s \cdot y(0) - y'(0) + 9Y(s) &= \frac{4}{s^2+4}.\end{aligned}$$

But $y(0) = 0$ and $y'(0) = -1$.

$$\begin{aligned}(s^2 + 9)Y(s) + 1 &= \frac{4}{s^2+4} \\ Y(s) &= \frac{4}{(s^2+4)(s^2+9)} - \frac{1}{s^2+9}\end{aligned}$$

Find a partial fraction decomposition.

$$\frac{4}{(s^2+4)(s^2+9)} = \frac{4/5}{s^2+4} - \frac{4/5}{s^2+9}$$

Then

$$\begin{aligned}Y(s) &= \frac{4}{5}\left(\frac{1}{s^2+4}\right) - \frac{9}{5}\left(\frac{1}{s^2+9}\right) \\ &= \frac{4}{5} \cdot \frac{1}{2}\left(\frac{2}{s^2+4}\right) - \frac{3}{5}\left(\frac{3}{s^2+9}\right)\end{aligned}$$

and therefore,

$$y(t) = \frac{2}{5}\sin 2t - \frac{3}{5}\sin 3t.$$

15. Let
- $\mathcal{L}(y) = Y$
- . Then,

$$\begin{aligned}\mathcal{L}(y'' - y) &= \mathcal{L}(2t) \\ \mathcal{L}(y'') - \mathcal{L}(y) &= 2\mathcal{L}(t)\end{aligned}$$

$$s^2 Y(s) - s \cdot y(0) - y'(0) - Y(s) = \frac{2}{s^2}.$$

But $y(0) = 0$ and $y'(0) = -1$, so

$$\begin{aligned}(s^2 - 1)Y(s) + 1 &= \frac{2}{s^2} \\ Y(s) &= \frac{-1}{s^2 - 1} + \frac{2}{s^2(s^2 - 1)}.\end{aligned}$$

The terms on the right have the decompositions:

$$\begin{aligned}\frac{-1}{s^2 - 1} &= \frac{-1/2}{s - 1} + \frac{1/2}{s + 1} \\ \frac{2}{s^2(s^2 - 1)} &= \frac{2}{s^2} + \frac{1}{s - 1} - \frac{1}{s + 1}.\end{aligned}$$

Therefore,

$$Y(s) = -\frac{2}{s^2} + \frac{1/2}{s - 1} - \frac{1/2}{s + 1},$$

and

$$y(t) = -2t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}.$$

16. Let
- $\mathcal{L}(y) = Y(s)$
- . Since
- $y(0) = -1$
- and
- $y'(0) = 1$
- ,

$$\mathcal{L}(y'' + 3y') = s^2 Y(s) + s - 1 + 3s \cdot Y(s) + 3.$$

In addition

$$\mathcal{L}(-3t) = \frac{-3}{s^2}.$$

Solving

$$s^2 Y(s) + s - 1 + 3s Y(s) + 3 = \frac{-3}{s^2}$$

for Y , we get

$$Y(s) = \frac{-3}{s^3(s+3)} - \frac{s+2}{s(s+3)}$$

Find a partial fraction decomposition for both terms on the right.

$$\begin{aligned}\frac{-3}{s^3(s+3)} &= \frac{-1/9}{s} + \frac{1/3}{s^2} - \frac{1}{s^3} + \frac{1/9}{s+3} \\ \frac{s+2}{s(s+3)} &= \frac{2/3}{s} + \frac{1/3}{s+3}\end{aligned}$$

Therefore,

$$Y(s) = -\frac{7/9}{s} + \frac{1/3}{s^2} - \frac{1}{s^3} - \frac{2/9}{s+3}.$$

Further,

$$\begin{aligned}Y(s) &= -\frac{7}{9} \left(\frac{1}{s} \right) + \frac{1}{3} \left(\frac{1}{s^2} \right) \\ &\quad - \frac{1}{2} \left(\frac{2!}{s^3} \right) - \frac{2}{9} \left(\frac{1}{s+3} \right).\end{aligned}$$

Finally,

$$y(t) = -\frac{7}{9} + \frac{1}{3}t - \frac{1}{2}t^2 - \frac{2}{9}e^{-3t}.$$

17. Let
- $\mathcal{L}(y) = Y$
- . Since
- $y(0) = -2$
- and
- $y'(0) = 0$
- ,

$$\mathcal{L}(y'' + y') = s^2 Y(s) + 2s + sY(s) + 2.$$

In addition, Note the transform pair:

$$f(t) = e^{-t} \iff F(s) = \frac{1}{s+1}.$$

Thus, by Proposition 2.14,

$$te^{-t} \iff -F'(s) = \frac{1}{(s+1)^2}.$$

Thus, Y must satisfy

$$s^2 Y(s) + 2s + sY(s) + 2 = \frac{1}{(s+1)^2}.$$

Solving for Y , we get

$$Y(s) = -\frac{2}{s} + \frac{1}{s(s+1)^3}$$

Now,

$$s = 0 \Rightarrow 1 = A$$

$$s = -1 \Rightarrow 1 = -D \text{ or } D = -1.$$

The coefficient of s^3 is $A + B$, so

$$A + B = 0 \quad \text{and} \quad B = -A = -1.$$

Finally,

$$\begin{aligned} s = 1 &\Rightarrow 1 = 8A + 4B + 2C + D \\ 1 &= 8 - 4 + 2C - 1 \\ -2C &= 2 \\ C &= -1. \end{aligned}$$

Thus,

$$\begin{aligned} Y(s) &= -\frac{2}{s} + \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3} \\ &= -\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}, \end{aligned}$$

so

$$\begin{aligned} y(t) &= -1 - e^{-t} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\ &\quad - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^3}\right\}. \end{aligned}$$

By Proposition 2.12,

$$y(t) = -1 - e^{-t} - e^{-t}t - \frac{1}{2}e^{-t}t^2.$$

18. Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = 0$, and $y'(0) = -1$,
 $\mathcal{L}(y'' - y' - 2y) = s^2Y(s) + 1 - sY(s) - 2Y(s)$.

In addition,

$$\mathcal{L}(t^2e^{2t}) = \frac{2}{(s-2)^3}.$$

Solving

$$(s^2 - s - 2)Y(s) + 1 = \frac{2}{(s-2)^3}$$

for Y , we get

$$Y(s) = \frac{-1}{(s+1)(s-2)} + \frac{2}{(s+1)(s-2)^4}.$$

The terms on the right have decompositions:

$$\frac{-1}{(s+1)(s-2)} = \frac{1/3}{s+1} - \frac{1/3}{s-2}$$

and

$$\begin{aligned} \frac{2}{(s+1)(s-2)^4} &= \frac{2/81}{s+1} - \frac{2/81}{s-2} + \frac{2/27}{(s-2)^2} \\ &\quad - \frac{2/9}{(s-2)^3} + \frac{2/3}{(s-2)^4} \end{aligned}$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{29/81}{s+1} - \frac{29/81}{s-2} + \frac{2/27}{(s-2)^2} \\ &\quad - \frac{2/9}{(s-2)^3} + \frac{2/3}{(s-2)^4} \\ &= \frac{29}{81}\left(\frac{1}{s+1}\right) - \frac{29}{81}\left(\frac{1}{s-2}\right) \\ &\quad + \frac{2}{27}\left(\frac{1}{(s-2)^2}\right) - \frac{1}{9}\left(\frac{2!}{(s-2)^3}\right) \\ &\quad + \frac{1}{9}\left(\frac{3!}{(s-2)^4}\right) \end{aligned}$$

and

$$\begin{aligned} y(t) &= \frac{29}{81}e^{-t} - \frac{29}{81}e^{2t} + \frac{2}{27}te^{2t} \\ &\quad - \frac{1}{9}t^2e^{2t} + \frac{1}{9}t^3e^{2t}. \end{aligned}$$

19. Let $\mathcal{L}(y) = Y$. Since $y(0) = -1$ and $y'(0) = 0$,
 $\mathcal{L}(y'' - 4y' - 5y) = s^2Y(s) + s - 4[sY(s) + 1] - 5Y(s)$

In addition,

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2}.$$

Solving

$$(s^2 - 4s - 5)Y(s) + s - 4 = \frac{1}{s-2}$$

for Y , we get

$$Y(s) = \frac{-s+4}{s^2-4s-5} + \frac{1}{(s-2)(s^2-4s-5)}.$$

The terms on the right have decompositions:

$$\begin{aligned} \frac{-s+4}{s^2-4s-5} &= \frac{-5/6}{s+1} - \frac{1/6}{s-5} \\ \frac{1}{(s-2)(s^2-4s-5)} &= \frac{-1/9}{s-2} + \frac{1/18}{s+1} + \frac{1/18}{s-5}. \end{aligned}$$

Thus,

$$Y(s) = \frac{-7/9}{s+1} - \frac{1/9}{s-5} - \frac{1/9}{s-2},$$

and

$$y(t) = -\frac{7}{9}e^{-t} - \frac{1}{9}e^{5t} - \frac{1}{9}e^{2t}.$$

20. Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = 1$ and $y'(0) = -1$,

$$\begin{aligned}\mathcal{L}(y'' - 2y' - 3y) \\ = s^2 Y(s) + s - 1 - 2[s \cdot Y(s) - 1] - 3Y(s).\end{aligned}$$

In addition,

$$\mathcal{L}(e^{4t}) = \frac{1}{s-4}.$$

Solving for Y , we get

$$Y(s) = \frac{s-3}{(s+1)(s-3)} + \frac{1}{(s-4)(s+1)(s-3)}.$$

The terms on the right have decompositions

$$\begin{aligned}\frac{s-3}{(s+1)(s-3)} &= \frac{1}{s+1} \\ \frac{1}{(s-4)(s+1)(s-3)} &= \frac{1/5}{s-4} + \frac{1/20}{s+1} \\ &\quad + \frac{-1/4}{s-3}.\end{aligned}$$

Hence,

$$Y(s) = \frac{21/20}{s+1} - \frac{1/4}{s-3} + \frac{1/5}{s-4}.$$

and therefore

$$y(t) = \frac{21}{20}e^{-t} - \frac{1}{4}e^{3t} + \frac{1}{5}e^{4t}.$$

21. Let $\mathcal{L}(y) = Y$. Since $y(0) = -1$ and $y'(0) = 0$,

$$\begin{aligned}\mathcal{L}(y'' - y' - 2y) \\ = s^2 Y(s) + s - [sY(s) + 1] - 2Y(s).\end{aligned}$$

In addition,

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2}.$$

Solving

$$(s^2 - s - 2)Y(s) + s - 1 = \frac{1}{s-2}$$

for $Y(s)$, we get

$$Y(s) = \frac{-s+1}{s^2-s-2} + \frac{1}{(s-2)(s^2-s-2)}.$$

The terms on the right have decompositions.

$$\frac{-s+1}{s^2-s-2} = \frac{-2/3}{s+1} - \frac{1/3}{s-2}$$

and

$$\begin{aligned}\frac{1}{(s-2)(s^2-s-2)} \\ = \frac{-1/9}{s-2} + \frac{1/3}{(s-2)^2} + \frac{1/9}{s+1}.\end{aligned}$$

Thus,

$$Y(s) = \frac{-5/9}{s+1} - \frac{4/9}{s-2} + \frac{1/3}{(s-2)^2},$$

and

$$y(t) = -\frac{5}{9}e^{-t} - \frac{4}{9}e^{2t} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}.$$

Since

$$t \iff \frac{1}{s^2},$$

Proposition 2.14 tells us that

$$e^{2t}t \iff \frac{1}{(s-2)^2}.$$

Thus,

$$y(t) = -\frac{5}{9}e^{-t} - \frac{4}{9}e^{2t} + \frac{1}{3}te^{2t}.$$

22. Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = 0$ and $y'(0) = -1$,

$$\begin{aligned}\mathcal{L}(y'' - 3y' - 4y) \\ = s^2 Y(s) + 1 - 3sY(s) - 4Y(s)\end{aligned}$$

In addition,

$$\mathcal{L}(e^{-t}) = \frac{1}{s+1},$$

so

$$(s^2 - 3s - 4) + 1 = \frac{1}{s + 1}.$$

Solving for Y , we get

$$Y(s) = \frac{-1}{(s+1)(s-4)} + \frac{1}{(s+1)^2(s-4)}.$$

The terms on the right have decompositions

$$\begin{aligned}\frac{-1}{(s+1)(s-4)} &= \frac{1/5}{s+1} - \frac{1/5}{s-4} \\ \frac{1}{(s+1)^2(s-4)} &= \frac{-1/25}{s+1} - \frac{1/5}{(s+1)^2} + \frac{1/25}{s-4}\end{aligned}$$

Thus,

$$Y(s) = \frac{4/25}{s+1} - \frac{4/25}{s-4} - \frac{1/5}{(s+1)^2}$$

and

$$y(t) = \frac{4}{25}e^{-t} - \frac{4}{25}e^{4t} - \frac{1}{5}te^{-t}.$$

23. Let $\mathcal{L}(y) = Y$. Since $y(0) = -1$ and $y'(0) = 0$,

$$\begin{aligned}\mathcal{L}(y'' + 2y' + 2y) \\ = s^2Y(s) + s + 2[sY(s) + 1] + 2Y(s).\end{aligned}$$

In addition,

$$\mathcal{L}(\cos t) = \frac{s}{s^2 + 1},$$

so

$$(s^2 + 2s + 2)Y(s) + s + 2 = \frac{s}{s^2 + 1}.$$

Solving for Y , we get

$$Y(s) = \frac{-s-2}{s^2+2s+2} + \frac{s}{(s^2+1)(s^2+2s+2)}.$$

The second term on the right has decomposition

$$\begin{aligned}\frac{s}{(s^2+1)(s^2+2s+2)} &= \frac{(1/5)s+2/5}{s^2+1} \\ &\quad + \frac{-(1/5)s-4/5}{s^2+2s+2},\end{aligned}$$

so

$$\begin{aligned}Y(s) &= \frac{-(6/5)s-14/5}{s^2+2s+2} + \frac{(1/5)s+2/5}{s^2+1} \\ &= \frac{-(6/5)s-14/5}{(s+1)^2+1} + \frac{1}{5} \cdot \frac{s}{s^2+1} \\ &\quad + \frac{2}{5} \cdot \frac{1}{s^2+1}.\end{aligned}$$

Note the pairs

$$\begin{aligned}\cos t &\iff \frac{s}{s^2+1} \\ \sin t &\iff \frac{1}{s^2+1}.\end{aligned}$$

By Proposition 2.12,

$$\begin{aligned}e^{-t} \cos t &\iff \frac{s+1}{(s+1)^2+1} \\ e^{-t} \sin t &\iff \frac{1}{(s+1)^2+1}.\end{aligned}$$

Thus, write

$$\begin{aligned}&\frac{-(6/5)s-14/5}{(s+1)^2+1} \\ &= -\frac{6}{5} \cdot \frac{s+7/3}{(s+1)^2+1} \\ &= -\frac{6}{5} \left[\frac{s+1}{(s+1)^2+1} + \frac{4}{3} \cdot \frac{1}{(s+1)^2+1} \right] \\ &= -\frac{6}{5} \cdot \frac{s+1}{(s+1)^2+1} - \frac{8}{5} \cdot \frac{1}{(s+1)^2+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}Y(s) &= -\frac{6}{5} \cdot \frac{s+1}{(s+1)^2+1} - \frac{8}{5} \cdot \frac{1}{(s+1)^2+1} \\ &\quad + \frac{1}{5} \cdot \frac{s}{s^2+1} + \frac{2}{5} \cdot \frac{1}{s^2+1},\end{aligned}$$

and

$$y(t) = -\frac{6}{5}e^{-t} \cos t - \frac{8}{5}e^{-t} \sin t + \frac{1}{5} \cos t + \frac{2}{5} \sin t.$$

24. Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = 1$ and $y'(0) = -1$,

$$\begin{aligned}\mathcal{L}(y'' + 4y' + 5y) \\ = s^2Y(s) - s + 1 + 4[sY(s) - 1] + 5Y(s).\end{aligned}$$

In addition,

$$-3\mathcal{L}(\sin 2t) = \frac{-6}{s^2 + 4}, \quad (4.1)$$

so

$$(s^2 + 4s + 5)Y(s) - s - 3 = \frac{-6}{s^2 + 4}$$

and

$$Y(s) = \frac{s + 3}{(s + 2)^2 + 1} - \frac{6}{(s^2 + 4)((s + 2)^2 + 1)}.$$

However, the fraction on the right has decomposition

$$\begin{aligned} \frac{-6}{(s^2 + 4)((s + 2)^2 + 1)} &= \frac{1}{65} \cdot \frac{24s - 6}{s^2 + 4} \\ &\quad - \frac{1}{65} \cdot \frac{24s + 90}{(s + 2)^2 + 1}. \end{aligned}$$

Thus,

$$\begin{aligned} Y(s) &= \frac{1}{65} \cdot \frac{24s - 6}{s^2 + 4} + \frac{1}{65} \cdot \frac{41s + 105}{(s + 2)^2 + 1} \\ &= \frac{1}{65} \left\{ 24 \cdot \frac{s}{s^2 + 4} - 3 \cdot \frac{2}{s^2 + 4} \right\} \\ &\quad + \frac{1}{65} \left\{ 41 \cdot \frac{s + 2}{(s + 2)^2 + 1} \right. \\ &\quad \left. + 23 \cdot \frac{1}{(s + 2)^2 + 1} \right\} \end{aligned}$$

The transform pairs

$$\cos t \leftrightarrow \frac{s}{s^2 + 1}$$

$$\sin t \leftrightarrow \frac{1}{s^2 + 1}$$

and Proposition 2.12 yield

$$e^{-2t} \cos t \leftrightarrow \frac{s + 2}{(s + 2)^2 + 1},$$

$$e^{-2t} \sin t \leftrightarrow \frac{1}{(s + 2)^2 + 1}.$$

Therefore,

$$\begin{aligned} y(t) &= \frac{24}{65} \cos 2t - \frac{3}{65} \sin 2t \\ &\quad + \frac{41}{65} e^{-2t} \cos t + \frac{23}{65} e^{-2t} \sin t. \end{aligned}$$

25. Let $\mathcal{L}(y) = Y$. Since $y(0) = 1$ and $y'(0) = 0$,

$$\mathcal{L}(y'' + 4y) = s^2 Y(s) - s + 4Y(s).$$

On the other hand, we have the transform pair

$$\cos t \longleftrightarrow \frac{s}{s^2 + 1},$$

so by Proposition 2.12,

$$e^{-2t} \cos 2t \longleftrightarrow \frac{s + 2}{(s + 2)^2 + 1}.$$

Thus,

$$(s^2 + 4)Y(s) - s = \frac{s + 2}{(s + 2)^2 + 1}, \quad (4.2)$$

so

$$Y(s) = \frac{s}{s^2 + 4} + \frac{s + 2}{(s^2 + 4)((s + 2)^2 + 1)}. \quad (4.3)$$

The fraction on the right has decomposition

$$\begin{aligned} \frac{s + 2}{(s^2 + 4)((s + 2)^2 + 1)} &= -\frac{1}{65} \cdot \frac{7s - 18}{s^2 + 4} \\ &\quad + \frac{1}{65} \cdot \frac{7s + 10}{(s + 2)^2 + 1}. \end{aligned}$$

Thus,

$$\begin{aligned} Y(s) &= \frac{1}{65} \cdot \frac{58s + 18}{s^2 + 4} + \frac{1}{65} \cdot \frac{7s + 10}{(s + 2)^2 + 1} \\ &= \frac{1}{65} \left[58 \cdot \frac{s}{s^2 + 4} + 9 \cdot \frac{2}{s^2 + 4} \right] \\ &\quad + \frac{1}{65} \left[7 \cdot \frac{s + 2}{(s + 2)^2 + 1} - 4 \cdot \frac{1}{(s + 2)^2 + 1} \right] \end{aligned}$$

The transform pairs

$$\cos t \longleftrightarrow \frac{s}{s^2 + 1}$$

$$\sin t \longleftrightarrow \frac{1}{s^2 + 1}$$

and Proposition 2.12 yield

$$e^{-2t} \cos t \longleftrightarrow \frac{s + 2}{(s + 2)^2 + 1}$$

$$e^{-2t} \sin t \longleftrightarrow \frac{1}{(s + 2)^2 + 1}.$$

Therefore,

$$y(t) = \frac{58}{65} \cos 2t + \frac{9}{65} \sin 2t + \frac{7}{65} e^{-2t} \cos t - \frac{4}{65} e^{-2t} \sin t.$$

26. Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = -1$ and $y'(0) = 1$,

$$\mathcal{L}(y'' + 9y) = s^2 Y(s) + s - 1 + 9Y(s).$$

On the other hand, we have the transform pair

$$\sin t \leftrightarrow \frac{1}{s^2 + 1}.$$

So, by Proposition 2.12,

$$e^{-t} \sin t \leftrightarrow \frac{1}{(s+1)^2 + 1}.$$

Thus,

$$(s^2 + 9)Y(s) + (s - 1) = \frac{1}{(s+1)^2 + 1},$$

or

$$Y(s) = \frac{-s+1}{s^2+9} + \frac{1}{(s^2+9)((s+1)^2+1)}.$$

The fraction on the right has decomposition

$$\begin{aligned} & \frac{1}{(s^2+9)((s+1)^2+1)} \\ &= -\frac{1}{85} \cdot \frac{7+2s}{s^2+9} + \frac{1}{85} \cdot \frac{11+2s}{(s+1)^2+1}, \end{aligned}$$

so

$$\begin{aligned} Y(s) &= \frac{-s+1}{s^2+9} - \frac{1}{85} \cdot \frac{7+2s}{s^2+9} + \frac{1}{85} \cdot \frac{11+2s}{(s+1)^2+1} \\ &= \frac{78}{85} \cdot \frac{1}{3} \cdot \frac{3}{s^2+9} - \frac{87}{85} \cdot \frac{s}{s^2+9} \\ &\quad + \frac{2}{85} \cdot \frac{s+1}{(s+1)^2+1} + \frac{9}{85} \cdot \frac{1}{(s+1)^2+1} \end{aligned}$$

The transform pairs

$$\begin{aligned} \cos t &\leftrightarrow \frac{s}{s^2+1} \\ \sin t &\leftrightarrow \frac{1}{s^2+1} \end{aligned}$$

and Proposition 2.12 yield

$$\begin{aligned} e^{-t} \cos t &\leftrightarrow \frac{s+1}{(s+1)^2+1}, \\ e^{-t} \sin t &\leftrightarrow \frac{1}{(s+1)^2+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= \frac{26}{85} \sin 3t - \frac{87}{85} \cos 3t \\ &\quad + \frac{2}{85} e^{-t} \cos t + \frac{9}{85} e^{-t} \sin t. \end{aligned}$$

27. (a) Let $y = e^{rt}$. Then

$$\begin{aligned} y'' - 4y' + 3y &= 0 \\ (r^2 - 4r + 3)e^{rt} &= 0 \\ r^2 - 4r + 3 &= 0 \\ (r-3)(r-1) &= 0 \end{aligned}$$

Thus, $r = 3$ and $r = 1$ lead to independent solutions $y = e^{3t}$ and $y = e^t$. The general solution is

$$y = C_1 e^{3t} + C_2 e^t.$$

The initial condition $y(0) = 1$ gives us

$$1 = C_1 + C_2.$$

Differentiating,

$$y' = 3C_1 e^{3t} + C_2 e^t.$$

The initial condition $y'(0) = -1$ gives

$$-1 = 3C_1 + C_2.$$

Therefore, $C_1 = -1$ and $C_2 = 2$ and the solution is $y = -e^{3t} + 2e^t$.

- (b) Let $\mathcal{L}(y) = Y$. Since $y(0) = 1$ and $y'(0) = -1$,

$$\begin{aligned} 0 &= \mathcal{L}(y'' - 4y' + 3y) \\ &= (s^2 - 4s + 3)Y(s) - s + 1 + 4 = 0. \end{aligned}$$

Hence

$$Y(s) = \frac{s-5}{s^2-4s+3} = \frac{2}{s-1} - \frac{1}{s-3}.$$

Thus, $y(t) = 2e^t - e^{3t}$.

28. Let $y = e^{rt}$. Then

$$\begin{aligned}y'' - 3y' - 4y &= 0, \\r^2 e^{rt} - 3r e^{rt} - 4e^{rt} &= 0 \\r^2 - 3r - 4 &= 0 \\(r + 1)(r - 4) &= 0.\end{aligned}$$

Thus, $r = -1$ and $r = 4$ lead to independent solutions $y = e^{-t}$ and $y = e^{4t}$. The general solution and its derivative are

$$\begin{aligned}y &= C_1 e^{-t} + C_2 e^{4t}, \text{ and} \\y' &= -C_1 e^{-t} + 4C_2 e^{4t}.\end{aligned}$$

Thus,

$$\begin{aligned}y(0) = 1 &\Rightarrow C_1 + C_2 = 1, \\y'(0) = -1 &\Rightarrow -C_1 + 4C_2 = -1.\end{aligned}$$

Therefore, $C_1 = 1$ and $C_2 = 0$ and the solution is $y = e^{-t}$.

Now, let $\mathcal{L}(y) = Y(s)$. Since $y(0) = 1$ and $y'(0) = -1$,

$$\begin{aligned}0 &= \mathcal{L}(y'' - 3y' - 4y) \\&= (s^2 - 3s - 4)Y(s) - s + 1 + 3.\end{aligned}$$

Solving for Y , we get

$$Y(s) = \frac{s - 4}{(s + 1)(s - 4)} = \frac{1}{s + 1}.$$

Thus, $y(t) = e^{-t}$.

29. Let $y = e^{rt}$. Then

$$\begin{aligned}y'' - 3y' + 2y &= 0 \\(r^2 - 3r + 2)e^{rt} &= 0 \\r^2 - 3r + 2 &= 0 \\(r - 2)(r - 1) &= 0.\end{aligned}$$

Thus, $r = 2$ and $r = 1$ lead to independent solutions $y = e^{2t}$ and $y = e^t$. The general solution and its derivative are

$$\begin{aligned}y &= C_1 e^{2t} + C_2 e^t \\y' &= 2C_1 e^{2t} + C_2 e^t.\end{aligned}$$

Thus,

$$\begin{aligned}y(0) = 0 &\Rightarrow 0 = C_1 + C_2 \\y'(0) = -1 &\Rightarrow -1 = 2C_1 + C_2.\end{aligned}$$

Therefore, $C_1 = -1$, $C_2 = 1$, and

$$y = -e^{2t} + e^t.$$

Let $\mathcal{L}(y) = Y$. Since $y(0) = 0$ and $y'(0) = -1$,

$$\begin{aligned}0 &= \mathcal{L}(y'' - 3y' + 2y) \\&= (s^2 - 3s + 2)Y(s) + 1.\end{aligned}$$

Solving for Y , we get

$$Y(s) = \frac{-1}{s^2 - 3s + 2} = \frac{1}{s - 1} - \frac{1}{s - 2}.$$

Therefore,

$$y(t) = e^t - e^{2t}.$$

30. Let $y = e^{rt}$. Then

$$\begin{aligned}y'' + 4y &= 0 \\r^2 e^{rt} + 4e^{rt} &= 0 \\r^2 + 4 &= 0.\end{aligned}$$

Thus $r = \pm 2i$ leads to independent solutions $y = \cos 2t$ and $y = \sin 2t$. The general solution and its derivatives are

$$\begin{aligned}y &= C_1 \cos 2t + C_2 \sin 2t, \text{ and} \\y' &= -2C_1 \sin 2t + 2C_2 \cos 2t.\end{aligned}$$

Thus,

$$\begin{aligned}y(0) = 1 &\Rightarrow C_1 = 1, \\y'(0) = -1 &\Rightarrow 2C_2 = -1.\end{aligned}$$

Therefore $C_1 = 1$ and $C_2 = -1/2$ and the solution is

$$y = \cos 2t - \frac{1}{2} \sin 2t.$$

Let $\mathcal{L}(y) = Y(s)$. Then

$$\begin{aligned}\mathcal{L}(y'' + 4y) &= \mathcal{L}(0) \\s^2 Y(s) - s \cdot y(0) - y'(0) + 4Y(s) &= 0.\end{aligned}$$

The initial conditions $y(0) = 1$ and $y'(0) = -1$ give

$$(s^2 + 4)Y(s) - s + 1 = 0$$

$$\begin{aligned} Y(s) &= \frac{s-1}{s^2+4} \\ &= \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{2}{s^2+4} \end{aligned}$$

Thus,

$$y(t) = \cos 2t - \frac{1}{2} \sin 2t.$$

31. Let $y = e^{rt}$. Then

$$\begin{aligned} y'' + 2y' + 2y &= 0 \\ (r^2 + 2r + 2)e^{rt} &= 0 \\ r^2 + 2r + 2 &= 0. \end{aligned}$$

Using the quadratic formula, $r = -1 \pm i$. Thus,

$$y = e^{(-1+i)t} = e^{-t}e^{it} = e^{-t}(\cos t + i \sin t).$$

Therefore, $e^{-t} \cos t$ and $e^{-t} \sin t$ are independent solutions. The general solution and its derivative are

$$\begin{aligned} y &= C_1 e^{-t} \cos t + C_2 e^{-t} \sin t \\ y' &= -e^{-t} ((C_1 - C_2) \cos t + (C_1 + C_2) \sin t). \end{aligned}$$

But,

$$\begin{aligned} y(0) &= -1 \Rightarrow -1 = C_1 \\ y'(0) &= 1 \Rightarrow 1 = -(C_1 - C_2). \end{aligned}$$

Therefore, $C_1 = -1$ and $C_2 = 0$, and

$$y(t) = -e^{-t} \cos t.$$

Alternatively, let $\mathcal{L}(y) = Y$. Since $y(0) = -1$ and $y'(0) = 1$,

$$\begin{aligned} 0 &= \mathcal{L}(y'' + 2y' + 2y) \\ &= (s^2 + 2s + 2)Y(s) + s - 1 + 2. \end{aligned}$$

Solving for Y , we get

$$Y(s) = \frac{-s-1}{s^2+2s+2} = -\frac{s+1}{(s+1)^2+1}.$$

But, the transform pair

$$\cos t \iff \frac{s}{s^2+1},$$

and Proposition 2.12, give

$$e^{-t} \cos t \iff \frac{s+1}{(s+1)^2+1}.$$

Thus,

$$y(t) = -e^{-t} \cos t.$$

32. (a) Let $y = e^{rt}$. Then,

$$\begin{aligned} y'' + 4y &= 0 \\ r^2 e^{rt} + 4e^{rt} &= 0 \\ r^2 + 4 &= 0. \end{aligned}$$

Thus, $r = \pm 2i$ leads to the independent solutions $y = \cos 2t$ and $y = \sin 2t$. Therefore the general solution is

$$y_h = C_1 \cos 2t + C_2 \sin 2t.$$

(b) We examine $z'' + 4z = 2e^{i3t}$. Let $z_p = Ae^{i3t}$. Then $z' = 3Ai e^{i3t}$ and $z'' = -9Ae^{i3t}$ and

$$\begin{aligned} -9Ae^{i3t} + 4Ae^{i3t} &= 2e^{i3t} \\ -9A + 4A &= 2 \\ -5A &= 2 \\ A &= -\frac{2}{5}. \end{aligned}$$

Thus, $z_p = (-2/5)e^{i3t}$ and the real part of z_p is a solution of $y'' + 4y = 2 \cos 3t$. Thus,

$$y_p = -\frac{2}{5} \cos 3t.$$

(c) The general solution and its derivative are

$$\begin{aligned} y &= y_h + y_p \\ &= C_1 \cos 2t + C_2 \sin 2t - \frac{2}{5} \cos 3t \\ y' &= -2C_1 \sin 2t + 2C_2 \cos 2t + \frac{6}{5} \sin 3t. \end{aligned}$$

Thus

$$\begin{aligned} y(0) &= 1 \Rightarrow C_1 - \frac{2}{5} = 1, \\ y'(0) &= -1 \Rightarrow 2C_2 = -1. \end{aligned}$$

Therefore $C_1 = 7/5$ and $C_2 = -1/2$ and the solution is

$$y = \frac{7}{5} \cos 2t - \frac{1}{2} \sin 2t - \frac{2}{5} \cos 3t.$$

- (d) Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = 1$ and $y'(0) = -1$,

$$\mathcal{L}(y'' + 4y) = Y(s)(s^2 + 4) - s + 1.$$

In addition

$$\mathcal{L}(2 \cos 3t) = \frac{2s}{s^2 + 9},$$

so

$$Y(s)(s^2 + 4) - s + 1 = \frac{2s}{s^2 + 9}.$$

Solving for Y , we get

$$Y(s) = \frac{s-1}{s^2+4} + \frac{2s}{(s^2+4)(s^2+9)}.$$

The fraction on the right has decomposition

$$\frac{2s}{(s^2+4)(s^2+9)} = \frac{(2/5)s}{s^2+4} - \frac{(2/5)s}{s^2+9}.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{(7/5)s-1}{s^2+4} - \frac{2/5s}{s^2+9} \\ &= \frac{7}{5} \cdot \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{2}{s^2+4} \\ &\quad - \frac{2}{5} \cdot \frac{s}{s^2+9} \end{aligned}$$

and

$$y(t) = \frac{7}{5} \cos 2t - \frac{1}{2} \sin 2t - \frac{2}{5} \cos 3t.$$

33. Let $y = e^{rt}$. Then,

$$\begin{aligned} y'' + y &= 0 \\ (r^2 + 1)e^{rt} &= 0 \\ r^2 + 1 &= 0. \end{aligned}$$

Thus, $r = \pm i$ and

$$y = e^{it} = \cos t + i \sin t$$

produces a homogeneous solution

$$y_h = C_1 \cos t + C_2 \sin t.$$

We examine $z'' + z = -2e^{i2t}$. Let $z_p = Ae^{i2t}$. Then,

$$\begin{aligned} ((2i)^2 + 1) Ae^{i2t} &= -2e^{i2t} \\ -3A &= -2 \\ A &= \frac{2}{3}. \end{aligned}$$

Thus, $z_p = (2/3)e^{i2t}$ and the real part of z_p is a solution of $y'' + y = -2 \cos 2t$. Thus, $y_p = (2/3) \cos 2t$. The general solution is

$$y = y_h + y_p = C_1 \cos t + C_2 \sin t + \frac{2}{3} \cos 2t,$$

and

$$y(0) = 1 \Rightarrow 1 = C_1 + \frac{2}{3}.$$

The derivative is

$$y' = -C_1 \sin t + C_2 \cos t - \frac{4}{3} \sin 2t,$$

and

$$y'(0) = -1 \Rightarrow -1 = C_2.$$

Thus, $C_1 = 1/3$ and $C_2 = -1$ and

$$y = \frac{1}{3} \cos t - \sin t + \frac{2}{3} \cos 2t.$$

Alternatively, let $\mathcal{L}(y) = Y$. Then,

$$\begin{aligned} \mathcal{L}(y'' + y) &= \mathcal{L}(-2 \cos 2t) \\ s^2 Y(s) - s \cdot y(0) - y'(0) + Y(s) &= -2 \cdot \frac{s}{s^2 + 4}. \end{aligned}$$

But $y(0) = 1$ and $y'(0) = -1$, so

$$(s^2 + 1)Y(s) - s + 1 = -\frac{2s}{s^2 + 4},$$

or

$$Y(s) = \frac{s-1}{s^2+1} - \frac{2s}{(s^2+1)(s^2+4)}.$$

The fraction on the right has decomposition

$$\frac{-2s}{(s^2+1)(s^2+4)} = -\frac{2}{3} \cdot \frac{s}{s^2+1} + \frac{2}{3} \cdot \frac{s}{s^2+4}.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{(1/3)s - 1}{s^2 + 1} + \frac{2}{3} \cdot \frac{s}{s^2 + 4} \\ &= \frac{1}{3} \cdot \frac{s - 3}{s^2 + 1} + \frac{2}{3} \cdot \frac{s}{s^2 + 4} \\ &= \frac{1}{3} \left[\frac{s}{s^2 + 1} - 3 \cdot \frac{1}{s^2 + 1} \right] + \frac{2}{3} \cdot \frac{s}{s^2 + 4}. \end{aligned}$$

Thus,

$$y(t) = \frac{1}{3} \cos t - \sin t + \frac{2}{3} \cos 2t.$$

34. Let $y = e^{rt}$. Then

$$\begin{aligned} y'' + 9y &= 0 \\ r^2 e^{rt} + 9e^{rt} &= 0 \\ r^2 + 9 &= 0. \end{aligned}$$

Thus $r = \pm 3i$ leads to the independent solutions $y = \cos 3t$ and $y = \sin 3t$. Therefore the general solution is

$$y_h = C_1 \cos 3t + C_2 \sin 3t.$$

We examine $z'' + 9z = 3e^{i2t}$. Let $z_p = Ae^{i2t}$. Then $z' = 2Aie^{i2t}$ and $z'' = -4Ae^{i2t}$ and

$$\begin{aligned} -4Ae^{i2t} + 9Ae^{i2t} &= 3e^{i2t} \\ -4A + 9A &= 3 \\ 5A &= 3 \\ A &= \frac{3}{5}. \end{aligned}$$

Thus $z_p = (3/5)e^{i2t}$ and the imaginary part of z_p is a solution of $y'' + 9y = 3 \sin 2t$. Thus

$$y_p = \frac{3}{5} \sin 2t.$$

The general solution and its derivative are

$$\begin{aligned} y &= y_h + y_p \\ y &= C_1 \cos 3t + C_2 \sin 3t + \frac{3}{5} \sin 2t \\ y' &= -3C_1 \sin 3t + 3C_2 \cos 3t + \frac{6}{5} \cos 2t. \end{aligned}$$

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y'(0) = -1 \Rightarrow 3C_2 + \frac{6}{5} = -1$$

Therefore $C_1 = 0$ and $C_2 = -11/15$ and the solution is

$$y = -\frac{11}{15} \sin 3t + \frac{3}{5} \sin 2t.$$

Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = 0$ and $y'(0) = -1$,

$$\mathcal{L}(y'' + 9y) = Y(s)(s^2 + 9) + 1.$$

In addition,

$$\mathcal{L}(3 \sin 2t) = \frac{6}{s^2 + 4},$$

so

$$Y(s)(s^2 + 9) + 1 = \frac{6}{s^2 + 4}.$$

Solving for Y we get

$$Y(s) = \frac{-1}{s^2 + 9} + \frac{6}{(s^2 + 4)(s^2 + 9)}.$$

The fraction on the right has decomposition

$$\frac{6}{(s^2 + 4)(s^2 + 9)} = \frac{6/5}{s^2 + 4} - \frac{6/5}{s^2 + 9}.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{6/5}{s^2 + 4} - \frac{11/5}{s^2 + 9} \\ &= \frac{3}{5} \cdot \frac{2}{s^2 + 4} - \frac{11}{5} \cdot \frac{1}{3} \cdot \frac{3}{s^2 + 9} \end{aligned}$$

and $y(t) = (-11/15) \sin 3t + (3/5) \sin 2t$.

35. Let $y = e^{rt}$. Then,

$$\begin{aligned} y'' + 4y &= 0 \\ (r^2 + 4)e^{rt} &= 0 \\ r^2 + 4 &= 0. \end{aligned}$$

Thus, $r = \pm 2i$ and $y = e^{2it} = \cos 2t + i \sin 2t$ produces independent real solutions $\cos 2t$ and $\sin 2t$. The homogeneous solution is

$$y_h = C_1 \cos 2t + C_2 \sin 2t.$$

We examine $z'' + 4z = e^{i2t}$. Because $z = Ae^{i2t}$ is a solution of the homogeneous equation, we try $z_p = Ate^{i2t}$. Then,

$$\begin{aligned} z'_p &= 2iAte^{i2t} + Ae^{i2t} \\ z''_p &= -4Ate^{i2t} + 4iAe^{i2t}. \end{aligned}$$

Thus,

$$\begin{aligned} -4Ate^{i2t} + 4iAe^{i2t} + 4Ate^{i2t} &= e^{i2t} \\ 4iA &= 1 \\ A &= -\frac{1}{4}i. \end{aligned}$$

Thus,

$$z_p = -\frac{1}{4}ite^{i2t} = -\frac{1}{4}it(\cos 2t + i \sin 2t),$$

and the real part of z_p ,

$$y_p = \frac{1}{4}t \sin 2t,$$

is a particular solution of $y'' + 4y = \cos 2t$. The general solution is

$$y = y_h + y_p = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4}t \sin 2t,$$

and

$$y(0) = 1 \Rightarrow 1 = C_1.$$

Next, the derivative

$$\begin{aligned} y' &= -2C_1 \sin 2t + 2C_2 \cos 2t \\ &\quad + \frac{1}{4} \sin 2t + \frac{1}{2}t \cos 2t, \end{aligned}$$

and

$$y'(0) = -1 \Rightarrow -1 = 2C_2 \text{ or } C_2 = -1/2.$$

Thus, the solution is

$$y = \cos 2t - \frac{1}{2} \sin 2t + \frac{1}{4}t \sin 2t.$$

Alternatively, let $\mathcal{L}(y) = Y$. Since $y(0) = 1$ and $y'(0) = -1$,

$$\begin{aligned} \mathcal{L}(y'' + 4y) &= (s^2 + 4)Y(s) - s + 1 \\ \mathcal{L}(\cos 2t) &= \frac{s}{s^2 + 4}. \end{aligned}$$

Hence

$$(s^2 + 4)Y(s) - s + 1 = \frac{s}{s^2 + 4}.$$

Solving for Y we get

$$\begin{aligned} Y(s) &= \frac{s-1}{s^2+4} + \frac{s}{(s^2+4)^2} \\ &= \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{2}{s^2+4} + \frac{1}{4} \cdot \frac{4s}{(s^2+4)^2}. \end{aligned}$$

The last fraction is troublesome. However, the transform pair

$$\sin 2t \iff \frac{2}{s^2 + 4},$$

and Proposition 2.14,

$$\begin{aligned} t \sin 2t &\iff -\frac{d}{ds} \frac{2}{s^2 + 4} \\ t \sin 2t &\iff \frac{4s}{(s^2 + 4)^2}. \end{aligned}$$

Thus,

$$y(t) = \cos 2t - \frac{1}{2} \sin 2t + \frac{1}{4}t \sin 2t.$$

36. Let $y = e^{rt}$. Then,

$$\begin{aligned} y'' + y &= 0 \\ r^2 e^{rt} + e^{rt} &= 0 \\ r^2 + 1 &= 0. \end{aligned}$$

Thus, $r = \pm i$ leads to the independent solutions $y = \cos t$ and $y = \sin t$. Therefore the general solution is

$$y_h = C_1 \cos t + C_2 \sin t.$$

We examine $z'' + z = -2e^{it}$. Because $z = Ae^{it}$ is a solution of the homogeneous equation, let $z_p = Ate^{it}$. Then $z'_p = Aite^{it} + Ae^{it}$, and $z''_p = -Ate^{it} + 2iAe^{it}$. Thus,

$$\begin{aligned} -Ate^{it} + 2iAe^{it} + Ate^{it} &= -2e^{it} \\ 2iA &= -2 \\ A &= i. \end{aligned}$$

Therefore,

$$\begin{aligned} z_p &= ite^{it} \\ &= it(\cos t + i \sin t) \\ &= it \cos t - t \sin t \end{aligned}$$

and the imaginary part of z_p is a particular solution of $y'' + y = -2 \sin t$. Thus,

$$y_p = t \cos t.$$

The general solution ($y = y_h + y_p$) and its derivative are

$$\begin{aligned} y &= C_1 \cos t + C_2 \sin t + t \cos t \\ y' &= -C_1 \sin t + C_2 \cos t - t \sin t + \cos t. \end{aligned}$$

$$\begin{aligned} y(0) &= -1 \Rightarrow C_1 = -1 \\ y'(0) &= 1 \Rightarrow C_2 + 1 = 1 \end{aligned}$$

Therefore $C_1 = -1$ and $C_2 = 0$ and the solution is

$$y = -\cos t + t \cos t.$$

Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = -1$ and $y'(0) = 1$,

$$\begin{aligned} \mathcal{L}(y'' + y) &= Y(s)(s^2 + 1) + s - 1 \\ \mathcal{L}(-2 \sin t) &= \frac{-2}{s^2 + 1}. \end{aligned}$$

Equating the right-hand sides and solving for Y , we get

$$\begin{aligned} Y(s) &= \frac{-s + 1}{s^2 + 1} - \frac{2}{s^2 + 1} \\ &= \frac{-s}{s^2 + 1} + \frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2}. \end{aligned}$$

Note the transform pairs

$$\begin{aligned} \sin t &\leftrightarrow \frac{1}{s^2 + 1} \\ \cos t &\leftrightarrow \frac{s}{s^2 + 1}. \end{aligned}$$

Proposition 2.14 yields

$$\begin{aligned} \mathcal{L}(t \cos t) &= -\frac{d}{ds} \frac{s}{s^2 + 1} \\ &= -\frac{(s^2 + 1) - s(2s)}{(s^2 + 1)^2} \\ &= \frac{s^2 - 1}{(s^2 + 1)^2} \\ &= \frac{s^2 + 1}{(s^2 + 1)^2} - \frac{2}{(s^2 + 1)^2} \\ &= \frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2} \end{aligned}$$

Hence, by linearity,

$$\begin{aligned} &-\sin t + t \cos t \\ &\leftrightarrow -\frac{1}{s^2 + 1} + \left(\frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2} \right) \\ &\leftrightarrow -\frac{2}{(s^2 + 1)^2} \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= -\cos t + \sin t + [-\sin t + t \cos t] \\ &= -\cos t + t \cos t. \end{aligned}$$

37. (a) Let $\mathcal{L}(y) = Y$. Since $y(0) = y_0$ and $y'(0) = v_0$,

$$\begin{aligned} 0 &= \mathcal{L}(y'' + \omega_0^2 y) \\ &= (s^2 + \omega_0^2)Y(s) - sy_0 - v_0. \end{aligned}$$

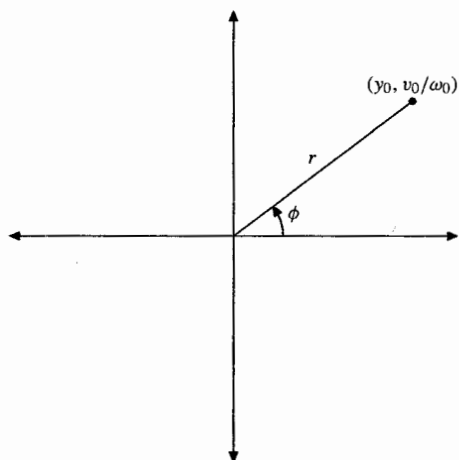
Solving for Y , we get

$$\begin{aligned} Y(s) &= \frac{s \cdot y_0 + v_0}{s^2 + \omega_0^2} \\ &= y_0 \cdot \frac{s}{s^2 + \omega_0^2} + \frac{v_0}{\omega_0} \cdot \frac{\omega_0}{s^2 + \omega_0^2}. \end{aligned}$$

Thus,

$$y(t) = y_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t.$$

(b) In the figure,



note that

$$r = \sqrt{y_0^2 + \frac{v_0^2}{\omega_0^2}} \quad \text{and} \quad \tan \phi = \frac{v_0}{y_0 \omega_0}.$$

Factor out the magnitude r .

$$\begin{aligned} y(t) &= y_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t \\ &= r \left[\frac{y_0}{r} \cos \omega_0 t + \frac{v_0/\omega_0}{r} \sin \omega_0 t \right] \\ &= r [\cos \phi \cos \omega_0 t + \sin \phi \sin \omega_0 t] \\ &= r \cos(\omega_0 t - \phi). \end{aligned}$$

38. Let $\mathcal{L}(y) = Y(s)$. Since $y(0) = y_0$ and $y'(0) = v_0$,

$$\begin{aligned} 0 &= \mathcal{L}(y'' + 2cy' + \omega_0^2 y) \\ &= Y(s)(s^2 + 2cs + \omega_0^2) - sy_0 - v_0 - 2cy_0. \end{aligned}$$

Solving for Y , we get

$$\begin{aligned} Y(s) &= \frac{sy_0 + v_0 + 2cy_0}{s^2 + 2cs + \omega_0^2} \\ &= \frac{y_0 s + v_0 + 2cy_0}{(s+c)^2 + (\omega_0^2 - c^2)}. \end{aligned}$$

39. Set $\omega = \sqrt{\omega_0^2 - c^2}$. This is possible because $\omega_0^2 > c^2$. In Exercise 38, we saw that $y'' + 2cy' + \omega_0^2 y = 0$,

$y(0) = y_0, y'(0) = v_0$, has transform

$$\begin{aligned} Y(s) &= \frac{y_0 s + v_0 + 2cy_0}{(s+c)^2 + (\omega_0^2 - c^2)} \\ &= \frac{y_0 s + v_0 + 2cy_0}{(s+c)^2 + \omega^2} \\ &= y_0 \cdot \frac{s+c}{(s+c)^2 + \omega^2} \\ &\quad + \frac{y_0 c + v_0}{\omega} \cdot \frac{\omega}{(s+c)^2 + \omega^2}. \end{aligned}$$

Thus,

$$y(t) = y_0 e^{-ct} \cos(\omega t) + \frac{y_0 c + v_0}{\omega} e^{-ct} \sin(\omega t).$$

40. In Exercise 38, we saw that $y'' + 2cy' + \omega_0^2 y = 0$, $y(0) = y_0, y'(0) = v_0$, has transform

$$Y(s) = \frac{y_0 s + v_0 + 2cy_0}{(s+c)^2 + (\omega_0^2 - c^2)}.$$

For the critically damped case, $\omega_0^2 - c^2 = 0$. Therefore

$$Y(s) = \frac{y_0 s + v_0 + 2cy_0}{(s+c)^2}.$$

Further,

$$\begin{aligned} Y(s) &= y_0 \cdot \frac{s+c}{(s+c)^2} + \frac{y_0 c + v_0}{(s+c)^2} \\ Y(s) &= y_0 \cdot \frac{1}{s+c} + (y_0 c + v_0) \cdot \frac{1}{(s+c)^2}, \end{aligned}$$

and

$$y(t) = y_0 e^{-ct} + (y_0 c + v_0) t e^{-ct}.$$

41. Set $a = \sqrt{c^2 - \omega_0^2}$. This is ok, since $c^2 - \omega_0^2 > 0$. In Exercise 38, we saw that $y'' + 2cy' + \omega_0^2 y = 0$, $y(0) = y_0, y'(0) = v_0$, has transform

$$\begin{aligned} Y(s) &= \frac{y_0 s + v_0 + 2cy_0}{(s+c)^2 + (\omega_0^2 - c^2)} \\ &= \frac{y_0 s + v_0 + 2cy_0}{(s+c)^2 - a^2} \end{aligned}$$

With the additional condition $y(0) = y_0 = 0$, this becomes

$$\begin{aligned} Y(s) &= \frac{v_0}{(s+c)^2 - a^2} \\ &= \frac{v_0}{(s+c+a)(s+c-a)} \\ &= \frac{v_0}{2a} \left[\frac{1}{s+c-a} - \frac{1}{s+c+a} \right]. \end{aligned}$$

Hence

$$y(t) = \frac{v_0}{2a} [e^{-(c-a)t} - e^{-(c+a)t}],$$

where $a = \sqrt{c^2 - \omega_0^2}$.

42. Let $\mathcal{L}(x) = X(s)$. Since $x(0) = x'(0) = 0$,

$$\begin{aligned} 0 &= \mathcal{L}(x'' + \omega_0^2 x) \\ &= X(s)(s^2 + \omega_0^2). \end{aligned}$$

Solving for X , we get

$$X(s) = \frac{F_0 \omega_0}{(s^2 + \omega_0^2)^2}.$$

Note the transform pairs

$$\begin{aligned} \sin \omega_0 t &\leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2} \\ \cos \omega_0 t &\leftrightarrow \frac{s}{s^2 + \omega_0^2}. \end{aligned}$$

Proposition 2.14 yields

$$\begin{aligned} t \cos \omega_0 t &\leftrightarrow -\frac{d}{ds} \frac{s}{s^2 + \omega_0^2} \\ &= -\frac{(s^2 + \omega_0^2) - s(2s)}{(s^2 + \omega_0^2)^2} \\ &= \frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2} \\ &= \frac{s^2 + \omega_0^2 - 2\omega_0^2}{(s^2 + \omega_0^2)^2} \\ &= \frac{1}{s^2 + \omega_0^2} - \frac{2\omega_0^2}{(s^2 + \omega_0^2)^2} \end{aligned}$$

Thus, by linearity,

$$\begin{aligned} \sin \omega_0 t - \omega_0 t \cos \omega_0 t &\leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2} - \frac{\omega_0}{s^2 + \omega_0^2} + \frac{2\omega_0^3}{(s^2 + \omega_0^2)^2} \\ &= \frac{2\omega_0^3}{(s^2 + \omega_0^2)^2}. \end{aligned}$$

Again, by linearity, if

$$X(s) = \frac{F_0 \omega_0}{(s^2 + \omega_0^2)^2} = \frac{F_0}{2\omega_0^2} \cdot \frac{2\omega_0^3}{(s^2 + \omega_0^2)^2},$$

then

$$x(t) = \frac{F_0}{2\omega_0^2} (\sin \omega_0 t - \omega_0 t \cos \omega_0 t).$$

Section 5.5. Discontinuous Forcing Terms

1. Since $f(t) = t$ has transform $F(s) = 1/s^2$, $g(t) = H(t-2)(t-2)$ has transform

$$\begin{aligned} G(s) &= e^{-2s} F(s) = e^{-2s} \cdot \frac{1}{s^2} \\ &= \frac{e^{-2s}}{s^2}. \end{aligned}$$

2. Since $f(t) = e^{2t}$ has transform $F(s) = 1/(s-2)$, $g(t) = H(t-1)e^{2(t-1)}$ has transform

$$\begin{aligned} G(s) &= e^{-s} F(s) = e^{-s} \frac{1}{s-2} \\ &= \frac{e^{-s}}{s-2}. \end{aligned}$$

3. Since $f(t) = \sin 3t$ has transform $F(s) = 3/(s^2 + 9)$, $g(t) = H(t - \pi/4) \sin 3(t - \pi/4)$ has transform

$$\begin{aligned} G(s) &= e^{-\pi s/4} F(s) \\ &= e^{-\pi s/4} \cdot \frac{3}{s^2 + 9} \\ &= \frac{3e^{-\pi s/4}}{s^2 + 9}. \end{aligned}$$

4. Since $f(t) = \sin t$ has transform

$$F(s) = \frac{1}{s^2 + 1},$$

$g(t) = H(t - \pi/6) \sin(t - \pi/6)$ has transform

$$\begin{aligned} G(s) &= e^{-\pi(s/6)} F(s) \\ &= e^{-\pi s/6} \frac{1}{s^2 + 1} \\ &= \frac{e^{-\pi s/6}}{s^2 + 1}. \end{aligned}$$

5. Since $(t - 1)^2 = t^2 - 2t + 1$, we can write

$$\begin{aligned} t^2 &= (t - 1)^2 + 2t - 1 \\ &= (t - 1)^2 + 2(t - 1) + 1. \end{aligned}$$

Thus,

$$\begin{aligned} H(t - 1)t^2 &= H(t - 1)(t - 1)^2 + 2H(t - 1)(t - 1) \\ &\quad + H(t - 1), \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}\{H(t - 1)t^2\}(s) &= \mathcal{L}\{H(t - 1)(t - 1)^2\}(s) \\ &\quad + 2\mathcal{L}\{H(t - 1)(t - 1)\}(s) \\ &\quad + \mathcal{L}\{H(t - 1)\}(s) \\ &= e^{-s} \cdot \frac{2!}{s^3} + 2e^{-s} \cdot \frac{1}{s^2} + e^{-s} \cdot \frac{1}{s} \\ &= \frac{e^{-s}(2 + 2s + s^2)}{s^3}. \end{aligned}$$

6. Since $-t = -(t - 2) - 2$,

$$\begin{aligned} H(t - 2)e^{-t} &= H(t - 2)e^{-(t-2)-2} \\ &= H(t - 2)e^{-(t-2)}e^{-2}. \end{aligned}$$

Thus $f(t) = e^{-2}e^{-t}$ has transform

$$F(s) = \frac{e^{-2}}{s + 1}$$

so $g(t) = H(t - 2)e^{-(t-2)}e^{-2}$ has transform

$$\begin{aligned} G(s) &= e^{-2s} F(s) \\ &= e^{-2s} \frac{e^{-2}}{s + 1} \\ &= \frac{e^{-2(s+1)}}{s + 1}. \end{aligned}$$

7. We need to express $\sin 2t$ in terms of $t - \pi/6$.

$$\begin{aligned} \sin 2t &= \sin\left(2t - \frac{\pi}{3} + \frac{\pi}{3}\right) \\ &= \sin\left(2t - \frac{\pi}{3}\right) \cos \frac{\pi}{3} \\ &\quad + \cos\left(2t - \frac{\pi}{3}\right) \sin \frac{\pi}{3} \\ &= \frac{1}{2} \sin 2\left(t - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} \cos 2\left(t - \frac{\pi}{6}\right). \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{L}\left\{H\left(t - \frac{\pi}{6}\right) \sin 2t\right\}(s) &= \frac{1}{2} \mathcal{L}\left\{H\left(t - \frac{\pi}{6}\right) \sin 2\left(t - \frac{\pi}{6}\right)\right\}(s) \\ &\quad + \frac{\sqrt{3}}{2} \mathcal{L}\left\{H\left(t - \frac{\pi}{6}\right) \cos 2\left(t - \frac{\pi}{6}\right)\right\}(s) \\ &= \frac{1}{2} e^{-\pi s/6} \mathcal{L}\{\sin 2t\}(s) \\ &\quad + \frac{\sqrt{3}}{2} e^{-\pi s/6} \mathcal{L}\{\cos 2t\}(s) \\ &= \frac{1}{2} e^{-\pi s/6} \cdot \frac{4}{s^2 + 4} + \frac{\sqrt{3}}{2} e^{-\pi s/6} \cdot \frac{s^2}{s^2 + 4} \\ &= \frac{1}{2} e^{-\pi s/6} \left[\frac{4}{s^2 + 4} + \frac{\sqrt{3}s^2}{s^2 + 4} \right] \\ &= \frac{1}{2} e^{-\pi s/6} \frac{4 + \sqrt{3}s^2}{s^2 + 4}. \end{aligned}$$

8. We have

$$\begin{aligned}
 \cos 3t &= \cos \left(2t - \frac{3\pi}{2} + \frac{3\pi}{2} \right) \\
 &= \cos \left(3t - \frac{3\pi}{2} \right) \cos \left(\frac{3\pi}{2} \right) \\
 &\quad - \sin \left(3t - \frac{3\pi}{2} \right) \sin \left(\frac{3\pi}{2} \right) \\
 &= \sin \left(3t - \frac{3\pi}{2} \right) \\
 &= \sin 3 \left(t - \frac{\pi}{2} \right).
 \end{aligned}$$

Therefore,

$$H \left(t - \frac{\pi}{2} \right) \cos 3t = H \left(t - \frac{\pi}{2} \right) \sin 3 \left(t - \frac{\pi}{2} \right).$$

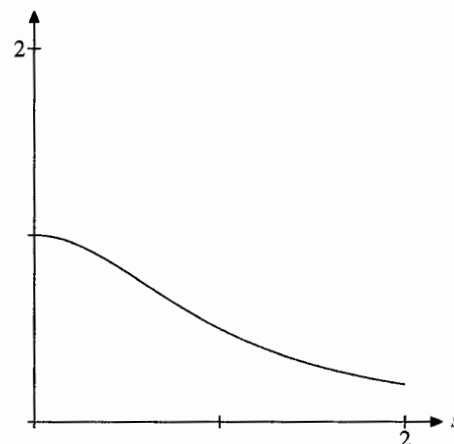
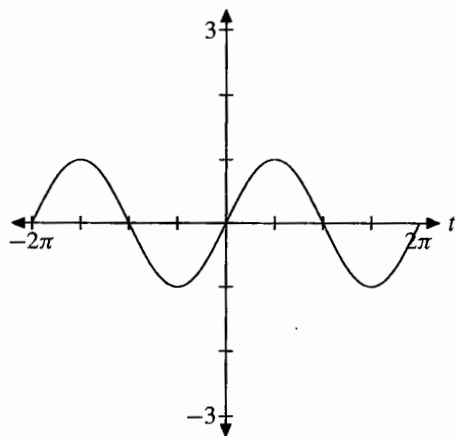
Because $f(t) = \sin 3t$ has transform

$$F(s) = \frac{3}{s^2 + 9},$$

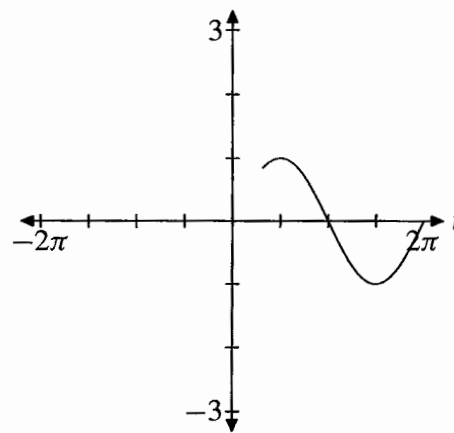
then $g(t) = H(t - \pi/2) \sin 3(t - \pi/2)$ has transform

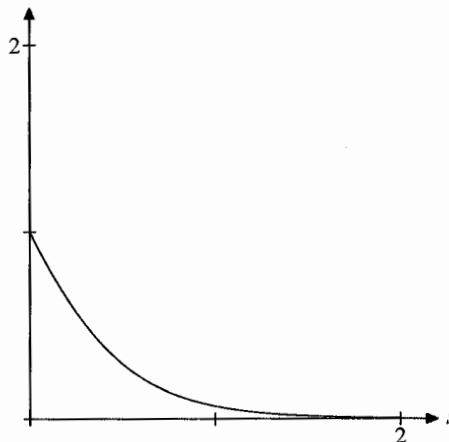
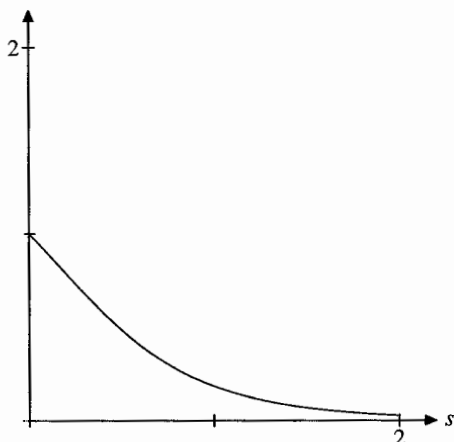
$$\begin{aligned}
 G(s) &= e^{-\pi s/2} F(s) \\
 &= e^{-\pi s/2} \frac{3}{s^2 + 9} \\
 &= \frac{3e^{-\pi s/2}}{s^2 + 9}.
 \end{aligned}$$

9. (a) If $f(t) = \sin t$, then $F(s) = 1/(s^2 + 1)$. The graph of f is pictured first, followed by the graph of F .



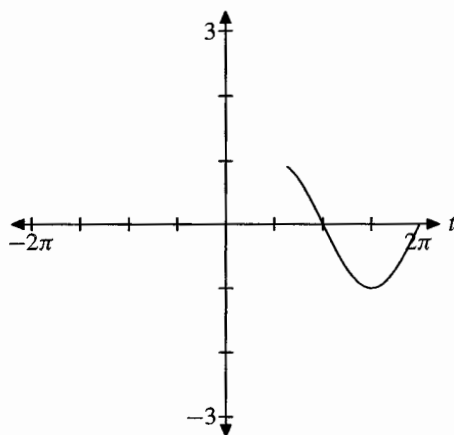
- (b) If $g(t) = H(t - 1) \sin(t - 1)$, then $G(s) = e^{-s}/(s^2 + 1)$. The graph of g is pictured first, followed by the graph of G .





In all cases, note that a shift in the time domain causes a scaling in the s -domain.

- (c) If $g(t) = H(t-2)\sin(t-2)$, then $G(s) = e^{-2s}/(s^2 + 1)$. The graph of g is pictured first, followed by the graph of G .



10. The function

$$f(t) = \begin{cases} 0, & \text{if } t < 0, \\ 3, & \text{if } t \geq 0 \end{cases}$$

can be written as $f(t) = 3H(t)$. Consequently,

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{3H(t)\}(s) \\ &= 3(\mathcal{L}\{H(t)\}(s)) \\ &= 3\frac{e^0}{s} \\ &= \frac{3}{s} \end{aligned}$$

11. The function

$$f(t) = \begin{cases} 5, & \text{if } 2 \leq t < 4, \\ 0, & \text{otherwise,} \end{cases}$$

can be written $f(t) = 5H_{24}(t)$. Consequently,

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{5H_{24}(t)\}(s) \\ &= 5(\mathcal{L}\{H_2(t)\}(s) - \mathcal{L}\{H_4(t)\}(s)) \\ &= 5\left(\frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}\right) \\ &= \frac{5}{s}(e^{-2s} - e^{-4s}). \end{aligned}$$

12. The function

$$f(t) = \begin{cases} 0, & \text{if } t < 0; \\ t, & \text{if } 0 \leq t < 3; \\ 3, & \text{if } t \geq 3 \end{cases}$$

can be written

$$\begin{aligned} f(t) &= tH_{03}(t) + 3H_3(t) \\ &= t(H(t) - H(t-3)) + 3H(t-3) \\ &= tH(t) - tH(t-3) + 3H(t-3) \\ &= tH(t) - (t-3)H(t-3). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{tH(t)\}(s) \\ &\quad - \mathcal{L}\{(t-3)H(t-3)\}(s) \\ &= \frac{1}{s^2} - \frac{e^{-3s}}{s^2} \\ &= \frac{1 - e^{-3s}}{s^2}. \end{aligned}$$

13. The function

$$f(t) = \begin{cases} 0, & \text{if } t < 0, \\ t^2, & \text{if } 0 \leq t < 2, \\ 4, & \text{if } t \geq 2, \end{cases}$$

can be written

$$\begin{aligned} f(t) &= t^2H_{02}(t) + 4H_2(t) \\ &= t^2(H(t) - H(t-2)) + 4H(t-2) \\ &= t^2H(t) - t^2H(t-2) + 4H(t-2) \\ &= t^2H(t) - (t-2+2)^2H(t-2) \\ &\quad - 4H(t-2) \\ &= t^2H(t) - (t-2)^2H(t-2) \\ &\quad - 4(t-2)H(t-2). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{t^2H(t)\}(s) \\ &\quad - \mathcal{L}\{(t-2)^2H(t-2)\}(s) \\ &\quad - 4\mathcal{L}\{(t-2)H(t-2)\}(s) \\ &= \frac{2!}{s^3} - e^{-2s} \cdot \frac{2!}{s^3} - 4e^{-2s} \cdot \frac{1}{s^2} \\ &= \frac{2}{s^3} - \frac{2e^{-2s}}{s^3} - \frac{4e^{-2s}}{s^2}. \end{aligned}$$

14. The function

$$f(t) = \begin{cases} 3, & \text{if } 0 \leq t < 1; \\ 2, & \text{if } 1 \leq t < 2; \\ 1, & \text{if } 2 \leq t < 3; \\ 0, & \text{otherwise} \end{cases}$$

can be written as

$$\begin{aligned} f(t) &= 3H_{01}(t) + 2H_{12}(t) + H_{23}(t) \\ &= 3H(t) - 3H(t-1) + 2H(t-1) \\ &\quad - 2H(t-2) + H(t-2) - H(t-3) \\ &= 3H(t) - H(t-1) - H(t-2) \\ &\quad - H(t-3) \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= 3\mathcal{L}\{H(t)\}(s) - \mathcal{L}\{H(t-1)\}(s) \\ &\quad - \mathcal{L}\{H(t-2)\}(s) \\ &\quad - \mathcal{L}\{H(t-3)\}(s) \\ &= \frac{3}{s} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}. \end{aligned}$$

15. The function

$$f(t) = \begin{cases} t, & \text{if } 0 \leq t < 1; \\ 2-t, & \text{if } 1 \leq t < 3; \\ t-4, & \text{if } 3 \leq t < 4; \\ 0, & \text{otherwise} \end{cases}$$

can be written

$$\begin{aligned} f(t) &= tH_{01}(t) + (2-t)H_{13}(t) + (t-4)H_{34}(t) \\ &= t(H(t) - H(t-1)) \\ &\quad + (2-t)(H(t-1) - H(t-3)) \\ &\quad + (t-4)(H(t-3) - H(t-4)) \\ &= tH(t) + (2-2t)H(t-1) \\ &\quad + (2t-6)H(t-3) - (t-4)H(t-4) \\ &= tH(t) - 2(t-1)H(t-1) \\ &\quad + 2(t-3)H(t-3) - (t-4)H(t-4). \end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{tH(t)\}(s) - 2\mathcal{L}\{(t-1)H(t-1)\}(s) \\
&\quad + 2\mathcal{L}\{(t-3)H(t-3)\}(s) \\
&\quad - \mathcal{L}\{(t-4)H(t-4)\}(s) \\
&= \frac{1}{s^2} - 2e^{-s} \cdot \frac{1}{s^2} + 2e^{-3s} \cdot \frac{1}{s^2} - e^{-4s} \cdot \frac{1}{s^2} \\
&= \frac{1}{s^2}(1 - 2e^{-s} + 2e^{-3s} - e^{-4s}).
\end{aligned}$$

16. Note the transform pair

$$f(t) = e^{-3t} \iff F(s) = \frac{1}{s+3}.$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s+3}\right\} &= \mathcal{L}^{-1}\{e^{-2s}F(s)\} \\
&= H(t-2)f(t-2) \\
&= H(t-2)e^{-3(t-2)} \\
&= \begin{cases} 0, & \text{if } 0 \leq t < 2, \\ e^{-3(t-2)}, & \text{for } 2 \leq t < \infty. \end{cases}
\end{aligned}$$

17. Note the transform pair

$$f(t) = e^{2t} \iff F(s) = \frac{1}{s-2}.$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{e^{-s} \frac{1}{s-2}\right\} &= \mathcal{L}^{-1}\{e^{-s}F(s)\} \\
&= H(t-1)f(t-1) \\
&= H(t-1)e^{2(t-1)} \\
&= \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ e^{2(t-1)}, & \text{if } 1 \leq t < \infty. \end{cases}
\end{aligned}$$

18. Note the transform pair

$$f(t) = t \iff F(s) = \frac{1}{s^2}.$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s^2}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) + \mathcal{L}^{-1}\left\{e^{-s} \frac{1}{s^2}\right\}(t) \\
&= \mathcal{L}^{-1}\{F(s)\}(t) + \mathcal{L}^{-1}\{e^{-s}F(s)\}(t) \\
&= t + H(t-1)f(t-1) \\
&= t + H(t-1)(t-1)
\end{aligned}$$

19. Note the transform pair

$$f(t) = t^2 \iff F(s) = \frac{2}{s^3}.$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2+e^{-2s}}{s^3}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t) + \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{2}{s^3}\right\}(t) \\
&= \mathcal{L}^{-1}\{F(s)\}(t) + \frac{1}{2}\mathcal{L}^{-1}\{e^{-2s}F(s)\}(t) \\
&= t^2 + \frac{1}{2}H(t-2)f(t-2) \\
&= t^2 + \frac{1}{2}H(t-2)(t-2)^2 \\
&= \begin{cases} t^2, & \text{if } 0 \leq t < 2, \\ t^2 + (1/2)(t-2)^2, & \text{if } 2 \leq t < \infty. \end{cases}
\end{aligned}$$

20. Note the transform pair

$$f(t) = \sin 2t \iff F(s) = \frac{2}{s^2+4}.$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+4}\right\}(t) &= \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{2}{s^2+4}\right\}(t) \\
&= \frac{1}{2}\mathcal{L}^{-1}\{e^{-s}F(s)\}(t) \\
&= \frac{1}{2}H(t-1)f(t-1) \\
&= \frac{1}{2}H(t-1)\sin 2(t-1)
\end{aligned}$$

21. Note the transform pair

$$f(t) = \cos 2t \iff F(s) = \frac{s}{s^2 + 4}.$$

Thus,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{se^{-2s}}{s^2 + 4} \right\}(t) &= \mathcal{L}^{-1} \{e^{-2s} F(s)\}(t) \\ &= H(t-2)f(t-2) \\ &= H(t-2)\cos 2(t-2) \\ &= \begin{cases} 0, & \text{if } 0 \leq t < 2, \\ \cos 2(t-2), & \text{if } 2 \leq t < \infty. \end{cases} \end{aligned}$$

22. A partial fraction decomposition.

$$\begin{aligned} \frac{1}{s(s+2)} &= \frac{A}{s} + \frac{B}{s+2} \\ 1 &= A(s+2) + Bs \end{aligned}$$

Then,

$$\begin{aligned} s = -2 &\Rightarrow B = -\frac{1}{2} \\ s = 0 &\Rightarrow A = \frac{1}{2} \end{aligned}$$

and

$$\frac{1 - e^{-s}}{s(s+2)} = \frac{1}{2s} - \frac{1}{2(s+2)} - \frac{e^{-s}}{2s} + \frac{e^{-s}}{2(s+2)}.$$

Note the transform pairs.

$$\begin{aligned} f(t) = 1 &\leftrightarrow F(s) = \frac{1}{s} \\ g(t) = e^{-2t} &\leftrightarrow G(s) = \frac{1}{s+2} \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1 - e^{-s}}{s(s+2)} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{2s} - \frac{1}{2(s+2)} \right. \\ &\quad \left. - e^{-s} \left[\frac{1}{2s} - \frac{1}{2(s+2)} \right] \right\}(t) \\ &= \frac{1}{2} \left[\mathcal{L}^{-1} \{F(s)\}(t) - \mathcal{L}^{-1} \{G(s)\}(t) \right. \\ &\quad \left. - \mathcal{L}^{-1} \{e^{-s} F(s)\}(t) + \mathcal{L}^{-1} \{e^{-s} G(s)\}(t) \right] \\ &= \frac{1}{2} \left[1 - e^{-2t} - H(t-1)f(t-1) \right. \\ &\quad \left. + H(t-1)g(t-1) \right] \\ &= \frac{1}{2} \left[1 - e^{-2t} - H(t-1) \right. \\ &\quad \left. + H(t-1)e^{-2(t-1)} \right] \end{aligned}$$

23. A partial fraction decomposition

$$\begin{aligned} \frac{1}{s^2 - 2s - 3} &= \frac{A}{s-3} + \frac{B}{s-1} \\ 1 &= A(s-1) + B(s-3). \end{aligned}$$

Then,

$$\begin{aligned} s = 1 &\Rightarrow B = -1/2, \\ s = 3 &\Rightarrow A = 1/2, \end{aligned}$$

and

$$\frac{1}{s^2 - 2s - 3} = \frac{1/2}{s-3} - \frac{1/2}{s-1}.$$

Note the transform pairs

$$\begin{aligned} f(t) = e^{3t} &\iff F(s) = \frac{1}{s-3}, \\ g(t) = e^{-t} &\iff G(s) = \frac{1}{s+1}. \end{aligned}$$

Thus,

$$\begin{aligned}
 \mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^2 - 2s - 3} \right\} (t) &= \mathcal{L}^{-1} \left\{ e^{-2s} \left[\frac{1/2}{s-3} - \frac{1/2}{s+1} \right] \right\} (t) \\
 &= \frac{1}{2} \mathcal{L}^{-1} \left\{ e^{-2s} \cdot \frac{1}{s-3} \right\} (t) \\
 &\quad - \frac{1}{2} \mathcal{L}^{-1} \left\{ e^{-2s} \cdot \frac{1}{s+1} \right\} (t) \\
 &= \frac{1}{2} \mathcal{L}^{-1} \{ e^{-2s} F(s) \} (t) \\
 &\quad - \frac{1}{2} \mathcal{L}^{-1} \{ e^{-2s} G(s) \} (t) \\
 &= \frac{1}{2} H(t-2) f(t-2) \\
 &\quad - \frac{1}{2} H(t-2) g(t-2) \\
 &= \frac{1}{2} H(t-2) [e^{3(t-2)} - e^{-(t-2)}] \\
 &= \begin{cases} 0, & \text{if } 0 \leq t < 2, \\ [e^{3(t-2)} - e^{-(t-2)}] / 2, & \text{if } 2 \leq t < \infty. \end{cases}
 \end{aligned}$$

24. A partial fraction decomposition.

$$\begin{aligned}
 \frac{1}{s(s-2)^2} &= \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2} \\
 1 &= A(s-2)^2 + Bs(s-2) + Cs
 \end{aligned}$$

Then,

$$\begin{aligned}
 s = 0 &\Rightarrow A = \frac{1}{4} \\
 s = 2 &\Rightarrow C = \frac{1}{2} \\
 s = 1 &\Rightarrow B = -\frac{1}{4}
 \end{aligned}$$

and

$$\frac{e^{-s}}{s(s-2)^2} = \frac{e^{-s}}{4s} - \frac{e^{-s}}{4(s-2)} + \frac{e^{-s}}{2(s-2)^2}.$$

Note the transform pairs:

$$\begin{aligned}
 f(t) = 1 &\leftrightarrow F(s) = \frac{1}{s} \\
 g(t) = e^{2t} &\leftrightarrow G(s) = \frac{1}{s-2} \\
 k(t) = te^{2t} &\leftrightarrow K(s) = \frac{1}{(s-2)^2}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s(s-2)^2} \right\} (t) &= \mathcal{L}^{-1} \left\{ e^{-s} \left[\frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2(s-2)^2} \right] \right\} (t) \\
 &= \frac{1}{4} \left[\mathcal{L}^{-1} \{ e^{-s} F(s) \} (t) \right. \\
 &\quad \left. - \mathcal{L}^{-1} \{ e^{-s} G(s) \} (t) + 2 \mathcal{L}^{-1} \{ e^{-s} K(s) \} (t) \right] \\
 &= \frac{1}{4} \left[H(t-1) f(t-1) - H(t-1) g(t-1) \right. \\
 &\quad \left. + 2 H(t-1) k(t-1) \right] \\
 &= \frac{1}{4} \left[H(t-1) - H(t-1) e^{2(t-1)} \right. \\
 &\quad \left. + 2 H(t-1) (t-1) e^{2(t-1)} \right] \\
 &= \frac{1}{4} H(t-1) - \frac{3}{4} H(t-1) e^{2(t-1)} \\
 &\quad + \frac{1}{2} H(t-1) t e^{2(t-1)}
 \end{aligned}$$

25. Complete the square.

$$F(s) = \frac{2 - e^{-2s}}{s^2 + 2s + 2} = \frac{2 - e^{-2s}}{(s+1)^2 + 1}.$$

Note the transform pair:

$$\sin t \longleftrightarrow \frac{1}{s^2 + 1}.$$

Thus, by Proposition 2.12, we have another transform pair:

$$e^{-t} \sin t \longleftrightarrow \frac{1}{(s+1)^2 + 1}.$$

Thus,

$$\begin{aligned} & \mathcal{L}^{-1} \left\{ \frac{2 - e^{-2s}}{(s+1)^2 + 1} \right\} (t) \\ &= 2 \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2 + 1} \right\} (t) \\ &\quad - \mathcal{L}^{-1} \left\{ e^{-2s} \cdot \frac{1}{(s+1)^2 + 1} \right\} (t) \\ &= 2e^{-t} \sin t - H(t-2)e^{-(t-2)} \sin(t-2), \end{aligned}$$

or, equivalently,

$$\begin{cases} 2e^{-t} \sin t, & 0 \leq t < 2, \\ 2e^{-t} \sin t - e^{-(t-2)} \sin(t-2), & 2 \leq t < \infty. \end{cases}$$

26. The forcing function f is described by

$$f(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0, \end{cases}$$

or equivalently

$$f(t) = H(t).$$

Hence,

$$F(s) = \mathcal{L}\{f(t)\}(s) = \frac{1}{s}.$$

Take the laplace transform of both sides of the given equation. Let $Y(s) = \mathcal{L}\{y(t)\}(s)$.

$$y'' + 4y = f(t)$$

$$\mathcal{L}(y'')(s) + 4\mathcal{L}(y)(s) = \mathcal{L}\{f(t)\}(s)$$

$$s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = F(s)$$

Use the initial conditions $y(0) = y'(0) = 0$ and the laplace transform of f found earlier.

$$s^2 Y(s) + 4Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s^2 + 4)}$$

A partial fraction decomposition is needed.

$$\begin{aligned} \frac{1}{s(s^2 + 4)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 4} \\ 1 &= A(s^2 + 4) + (Bs + C)s \\ 1 &= (A + B)s^2 + Cs + 4A \end{aligned}$$

Then,

$$s = 0 \Rightarrow A = \frac{1}{4}$$

$$A + B = 0 \Rightarrow B = -\frac{1}{4}$$

$$C = 0.$$

Therefore,

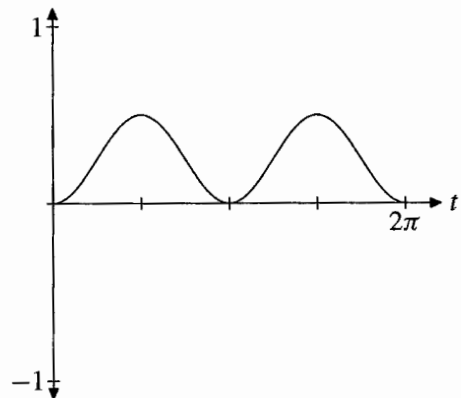
$$\frac{1}{s(s^2 + 4)} = \frac{1}{4s} - \frac{s}{4(s^2 + 4)}$$

which has transform

$$\frac{1}{4} - \frac{1}{4} \cos 2t.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + 4)} \right\} (t) \\ &= \frac{1}{4} - \frac{1}{4} \cos 2t. \end{aligned}$$



27. The forcing function f is described by

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ 0, & \text{otherwise,} \end{cases}$$

or equivalently,

$$\begin{aligned} f(t) &= H_{01}(t) \\ &= H_0(t) - H_1(t) \\ &= H(t) - H(t-1). \end{aligned}$$

Hence,

$$F(s) = \mathcal{L}\{f(t)\}(s) = \frac{1}{s} - \frac{e^{-s}}{s}.$$

Take the Laplace transform of both sides of the given equation. Let $Y(s) = \mathcal{L}\{y(t)\}(s)$.

$$\begin{aligned} y'' + 4y &= f(t) \\ \mathcal{L}(y'')(s) + 4\mathcal{L}(y)(s) &= \mathcal{L}\{f(t)\}(s) \\ s^2 Y(s) - sy(0) - y'(0) + 4Y(s) &= F(s). \end{aligned}$$

Use the initial conditions $y(0) = y'(0) = 0$ and the Laplace transform of f found earlier.

$$\begin{aligned} s^2 Y(s) + 4Y(s) &= \frac{1}{s} - \frac{e^{-s}}{s} \\ Y(s) &= \frac{1}{s^2 + 4} \left[\frac{1}{s} - \frac{e^{-s}}{s} \right] \\ Y(s) &= \frac{1}{s(s^2 + 4)} - \frac{e^{-s}}{s(s^2 + 4)}. \end{aligned}$$

A partial fraction decomposition is needed.

$$\begin{aligned} \frac{1}{s(s^2 + 4)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 4} \\ 1 &= A(s^2 + 4) + (Bs + C)s \end{aligned}$$

Now,

$$s = 0 \Rightarrow A = 1/4.$$

Furthermore,

$$\begin{aligned} 1 &= As^2 + 4A + Bs^2 + Cs \\ 1 &= (A + B)s^2 + Cs + 4A. \end{aligned}$$

Thus,

$$A + B = 0 \Rightarrow \frac{1}{4} + B = 0 \Rightarrow B = -1/4.$$

Also, $C = 0$. Therefore,

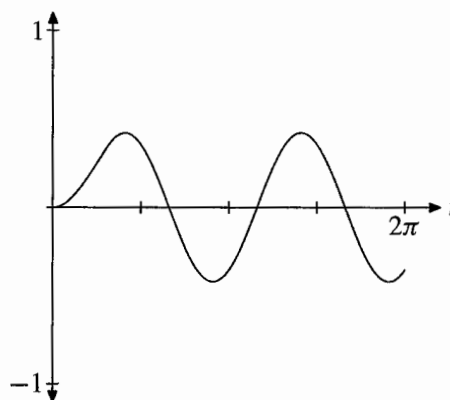
$$\frac{1}{s(s^2 + 4)} = \frac{1/4}{s} - \frac{(1/4)s}{s^2 + 4},$$

which has transform

$$\frac{1}{4} - \frac{1}{4} \cos 2t.$$

Finally,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + 4)} \right\}(t) \\ &\quad - \mathcal{L}^{-1} \left\{ e^{-s} \cdot \frac{1}{s(s^2 + 4)} \right\}(t) \\ &= \frac{1}{4} - \frac{1}{4} \cos 2t \\ &\quad - H(t - 1) \left[\frac{1}{4} - \frac{1}{4} \cos 2(t - 1) \right]. \end{aligned}$$



28. The forcing function f is described by

$$f(t) = \begin{cases} 1, & \text{if } 1 \leq t < 2, \\ 0, & \text{otherwise,} \end{cases}$$

or equivalently,

$$\begin{aligned} f(t) &= H_{12}(t) \\ &= H(t - 1) - H(t - 2). \end{aligned}$$

Hence,

$$F(s) = \mathcal{L}\{f(t)\}(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Take the Laplace transform of both sides of the given equation. Let $Y(s) = \mathcal{L}\{y(t)\}(s)$.

$$\begin{aligned} y'' + 4y &= f(t) \\ \mathcal{L}(y'')(s) + 4\mathcal{L}(y)(s) &= \mathcal{L}\{f(t)\}(s) \\ s^2 Y(s) - sy(0) - y'(0) + 4Y(s) &= F(s) \end{aligned}$$

Use the initial conditions $y(0) = y'(0) = 0$ and the Laplace transform of f found earlier.

$$\begin{aligned}s^2 Y(s) + 4Y(s) &= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\ Y(s) &= \frac{1}{s^2 + 4} \left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \right] \\ Y(s) &= \frac{e^{-s}}{s(s^2 + 4)} - \frac{e^{-2s}}{s(s^2 + 4)}\end{aligned}$$

A partial fraction decomposition is needed.

$$\begin{aligned}\frac{1}{s(s^2 + 4)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 4} \\ 1 &= A(s^2 + 4) + (Bs + C)s \\ 1 &= (A + B)s^2 + Cs + 4A\end{aligned}$$

Then,

$$\begin{aligned}s = 0 &\Rightarrow A = \frac{1}{4} \\ A + B = 0 &\Rightarrow B = -\frac{1}{4} \\ C &= 0.\end{aligned}$$

Therefore,

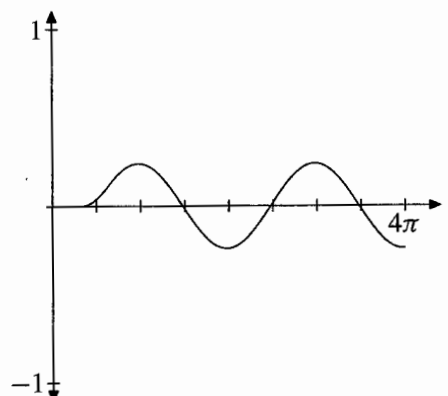
$$\frac{1}{s(s^2 + 4)} = \frac{1}{4s} - \frac{s}{4(s^2 + 4)},$$

which has transform

$$\frac{1}{4} - \frac{1}{4} \cos 2t.$$

Thus,

$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s(s^2 + 4)} \right\} (t) \\ &\quad - \mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s(s^2 + 4)} \right\} (t) \\ &= H(t - 1) \left[\frac{1}{4} - \frac{1}{4} \cos(t - 1) \right] \\ &\quad - H(t - 2) \left[\frac{1}{4} - \frac{1}{4} \cos(t - 2) \right].\end{aligned}$$



29. The forcing function f is described by

$$f(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 0, & \text{otherwise,} \end{cases}$$

or equivalently,

$$\begin{aligned}f(t) &= t H_{01}(t) \\ &= t(H(t) - H(t - 1)) \\ &= tH(t) - (t - 1)H(t - 1) - H(t - 1).\end{aligned}$$

Hence,

$$\begin{aligned}F(s) &= \frac{1}{s^2} - e^{-s} \cdot \frac{1}{s^2} - \frac{e^{-s}}{s} \\ &= \frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s}.\end{aligned}$$

Take the Laplace transform of each side of the equation, letting $\mathcal{L}(y)(s) = Y(s)$.

$$\begin{aligned}y'' + 4y &= f(t) \\ \mathcal{L}(y'')(s) + 4\mathcal{L}(y)(s) &= \mathcal{L}\{f(t)\}(s) \\ s^2 Y(s) - sy(0) - y'(0) + 4Y(s) &= F(s).\end{aligned}$$

Use the initial conditions $y(0) = y'(0) = 0$ and the Laplace transform of f found earlier.

$$\begin{aligned}s^2 Y(s) + 4Y(s) &= \frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s} \\ Y(s) &= \frac{1 - e^{-s}}{s^2(s^2 + 4)} - \frac{e^{-s}}{s(s^2 + 4)}\end{aligned}$$

Partial fraction decompositions are needed. We've seen (see Exercise 27)

$$\frac{1}{s(s^2 + 4)} = \frac{1/4}{s} - \frac{(1/4)s}{s^2 + 4},$$

which has inverse transform

$$\frac{1}{4} - \frac{1}{4} \cos 2t.$$

Next,

$$\frac{1}{s^2(s^2 + 4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4},$$

and

$$\begin{aligned} 1 &= As(s^2 + 4) + B(s^2 + 4) + s^2(Cs + D) \\ 1 &= As^3 + 4As + Bs^2 + 4B + Cs^3 + Ds^2 \\ 1 &= (A + C)s^3 + (B + D)s^2 + 4As + 4B. \end{aligned}$$

Thus,

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ 4A &= 0 \\ 4B &= 1, \end{aligned}$$

and $A = 0$, $B = 1/4$, $C = 0$, and $D = -1/4$. Therefore,

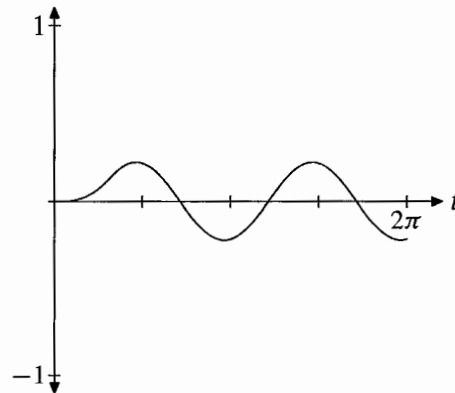
$$\begin{aligned} \frac{1}{s^2(s^2 + 4)} &= \frac{1/4}{s^2} - \frac{1/4}{s^2 + 4} \\ &= \frac{1}{4} \cdot \frac{1}{s^2} - \frac{1}{8} \cdot \frac{2}{s^2 + 4}, \end{aligned}$$

which has inverse transform

$$\frac{1}{4}t - \frac{1}{8} \sin 2t.$$

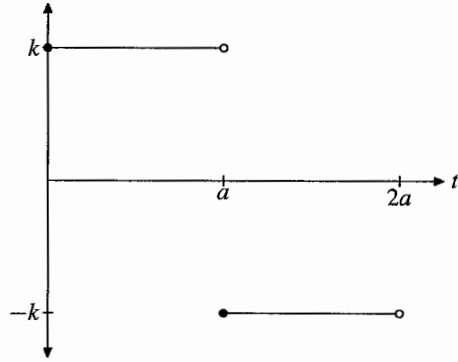
Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1 - e^{-s}}{s^2(s^2 + 4)} - \frac{e^{-s}}{s(s^2 + 4)} \right\}(t) \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2(s^2 + 4)} \right\}(t) \\ &\quad - \mathcal{L}^{-1} \left\{ e^{-s} \cdot \frac{1}{s^2(s^2 + 4)} \right\}(t) \\ &\quad - \mathcal{L}^{-1} \left\{ e^{-s} \cdot \frac{1}{s(s^2 + 4)} \right\}(t) \\ &= \frac{t}{4} - \frac{1}{8} \sin 2t \\ &\quad - H(t - 1) \left[\frac{1}{4}(t - 1) - \frac{1}{8} \sin 2(t - 1) \right] \\ &\quad - H(t - 1) \left[\frac{1}{4} - \frac{1}{4} \cos 2(t - 1) \right] \\ &= \frac{t}{4} - \frac{1}{8} \sin 2t \\ &\quad - H(t - 1) \left[\frac{1}{4}t - \frac{1}{8} \sin 2(t - 1) \right. \\ &\quad \left. - \frac{1}{4} \cos 2(t - 1) \right] \end{aligned}$$



30. To begin, the function f is periodic with period

$T = 2a$. Define a window,



or equivalently,

$$f_T(t) = \begin{cases} k, & \text{if } 0 \leq t < a, \\ -k, & \text{if } a \leq t < 2a, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} f_T(t) &= k[H(t) - H(t-a)] \\ &\quad - k[H(t-a) - H(t-2a)] \\ &= kH(t) - 2kH(t-a) + kH(t-2a), \end{aligned}$$

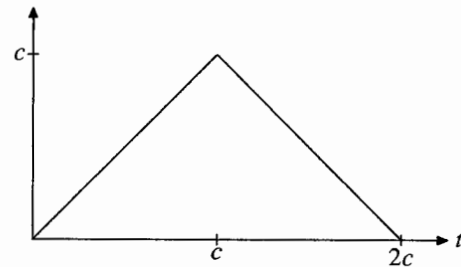
and

$$F_T(s) = \frac{k}{s} - \frac{2ke^{-as}}{s} + \frac{ke^{-2as}}{s}.$$

Therefore, by Proposition 5.19,

$$\begin{aligned} F(s) &= \frac{F_T(s)}{1 - e^{-Ts}} \\ &= \frac{\frac{k}{s} - \frac{2ke^{-as}}{s} + \frac{ke^{-2as}}{s}}{1 - e^{-2as}} \\ &= \frac{\frac{k}{s}(1 - 2e^{-as} + e^{-2as})}{1 - e^{-2as}} \\ &= \frac{k(1 - e^{-as})^2}{s(1 + e^{-as})(1 - e^{-as})} \\ &= \frac{k(1 - e^{-as})}{s(1 + e^{-as})} \\ &= \frac{k}{s} \cdot \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \\ &= \frac{k}{s} \tanh \frac{as}{2} \end{aligned}$$

31. To begin with, the function f is periodic with period $T = 2c$. Define a window,



or equivalently,

$$f_T(t) = \begin{cases} t, & \text{if } 0 \leq t < c, \\ 2c - t, & \text{if } c \leq t < 2c, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} f_T(t) &= t(H(t) - H(t-c)) \\ &\quad + (2c - t)(H(t-c) - H(t-2c)) \\ &= tH(t) + (2c - 2t)H(t-c) \\ &\quad - (2c - t)H(t-2c) \\ &= tH(t) - 2(t-c)H(t-c) \\ &\quad + (t-2c)H(t-2c). \end{aligned}$$

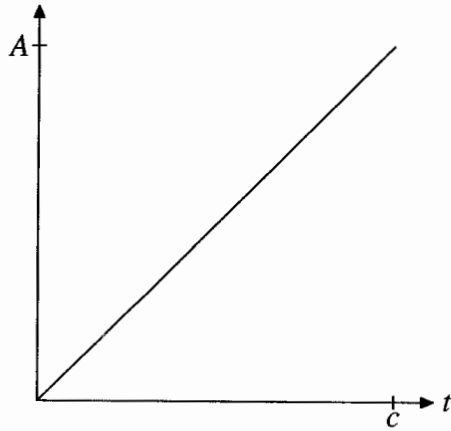
Thus,

$$F_T(s) = \frac{1}{s^2} - 2e^{-cs} \cdot \frac{1}{s^2} + e^{-2cs} \cdot \frac{1}{s^2}.$$

Therefore, by Proposition 5.17,

$$\begin{aligned} F(s) &= \frac{F_T(s)}{1 - e^{-Ts}} \\ &= \frac{\frac{1}{s^2} - \frac{2e^{-cs}}{s^2} + \frac{e^{-2cs}}{s^2}}{1 - e^{-2cs}} \\ &= \frac{1 - 2e^{-cs} + e^{-2cs}}{s^2(1 - e^{-2cs})} \\ &= \frac{(1 - e^{-cs})^2}{s^2(1 + e^{-cs})(1 - e^{-cs})} \\ &= \frac{1}{s^2} \cdot \frac{1 - e^{-cs}}{1 + e^{-cs}} \\ &= \frac{1}{s^2} \cdot \frac{e^{cs/2} - e^{-cs/2}}{e^{cs/2} + e^{-cs/2}} \\ &= \frac{1}{s^2} \tanh \frac{cs}{2}. \end{aligned}$$

32. The function f has period $T = c$. A window is defined



or equivalently,

$$f_T(t) = \begin{cases} At/c, & \text{if } 0 \leq t < c, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} f_T(t) &= \frac{A}{c} t(H(t) - H(t - c)) \\ &= \frac{A}{c} tH(t) - \frac{A}{c} (t - c)H(t - c) \\ &\quad - AH(t - c). \end{aligned}$$

Thus, with $T = c$,

$$F_T(s) = \frac{A}{cs^2} - \frac{A}{c} \cdot \frac{e^{-cs}}{s^2} - \frac{Ae^{-cs}}{s}.$$

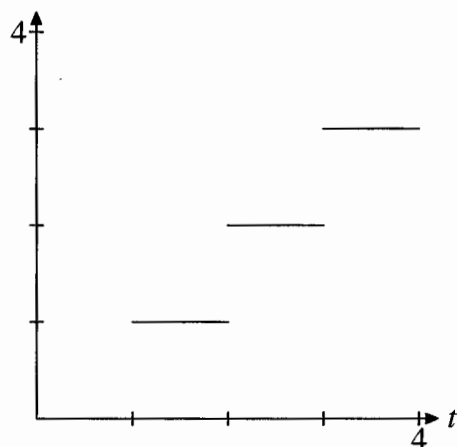
Therefore, with $T = c$,

$$\begin{aligned} F(s) &= \frac{F_T(s)}{1 - e^{-sT}} \\ &= \frac{\frac{A}{cs^2} - \frac{Ae^{-cs}}{cs^2} - \frac{Ae^{-cs}}{s}}{1 - e^{-cs}} \\ &= \frac{\frac{A}{cs^2}(1 - e^{-cs}) - \frac{Ae^{-cs}}{s}}{1 - e^{-cs}} \\ &= \frac{A}{cs^2} - \frac{Ae^{-cs}}{s(1 - e^{-cs})} \\ &= \frac{A}{cs^2} - \frac{A}{s(e^{cs} - 1)}. \end{aligned}$$

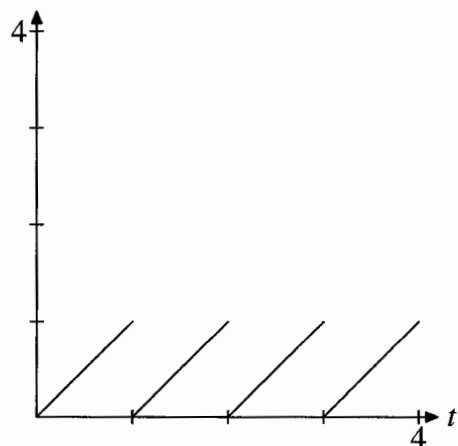
33. (a) A little thought provides

$$f(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ 1, & \text{if } 1 \leq t < 2, \\ 2, & \text{if } 2 \leq t < 3, \\ 3, & \text{if } 3 \leq t < 4, \\ \vdots, & \vdots \end{cases}$$

and the graph of f looks like a “staircase.”



(b) If h has graph



then

$$h(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ t-1, & \text{if } 1 \leq t < 2, \\ t-2, & \text{if } 2 \leq t < 3, \\ t-3, & \text{if } 3 \leq t < 4, \\ \vdots, & \vdots \end{cases}$$

Therefore,

$$t - h(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ 1, & \text{if } 1 \leq t < 2, \\ 2, & \text{if } 2 \leq t < 3, \\ 3, & \text{if } 3 \leq t < 4, \\ \vdots, & \vdots \end{cases}$$

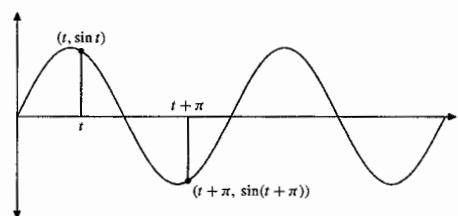
and $f(t) = t - h(t)$. By Exercise 32, with $A = 1$ and $c = 1$,

$$\mathcal{L}\{h(t)\}(s) = \frac{1}{s^2} - \frac{1}{s(e^s - 1)}.$$

Therefore,

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{t\}(s) - \mathcal{L}\{h(t)\}(s) \\ &= \frac{1}{s^2} - \left[\frac{1}{s^2} - \frac{1}{s(e^s - 1)} \right] \\ &= \frac{1}{s(e^s - 1)}. \end{aligned}$$

34. (a) The sine is an example of an antiperiodic function. This is clear from the following graph.



That is, if $f(t) = \sin t$, then

$$\begin{aligned} f\left(t + \frac{2\pi}{2}\right) &= f(t + \pi), \\ &= \sin(t + \pi), \\ &= -\sin t, \\ &= -f(t), \end{aligned}$$

for all t .

(b) If $f(t + T/2) = -f(t)$ for all t , then

$$\begin{aligned} f(t + T) &= f\left(\left(t + \frac{T}{2}\right) + \frac{T}{2}\right), \\ &= -f\left(t + \frac{T}{2}\right), \\ &= -(-f(t)), \\ &= f(t), \end{aligned}$$

for all t . Hence, f is periodic with period T .

(c) Let

$$f_{T/2} = \begin{cases} f(t), & 0 \leq t < T/2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, note that

$$G(s) = \int_0^{T/2} f(t)e^{-st} dt$$

is the Laplace transform of $f_{T/2}$. Next, note that

$$\begin{aligned} f(t) &= f_{T/2}(t)H(t) \\ &\quad - f_{T/2}(t - T/2)H(t - T/2) \\ &\quad + f_{T/2}(t - T)H(t - T) \\ &\quad - f_{T/2}(t - 3T/2)H(t - 3T/2) + \dots \end{aligned}$$

Taking the Laplace transform and using Proposition 5.6,

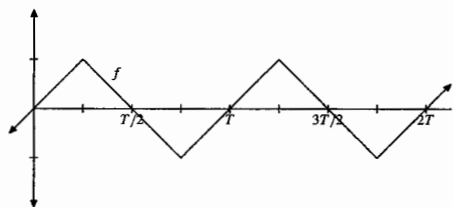
$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{f_{T/2}(t)H(t)\}(s) \\ &\quad - \mathcal{L}\{f_{T/2}(t - T/2)H(t - T/2)\}(s) \\ &\quad + \mathcal{L}\{f_{T/2}(t - T)H(t - T)\}(s) \\ &\quad - \mathcal{L}\{f_{T/2}(t - 3T/2)H(t - 3T/2)\}(s) \\ &\quad + \dots \\ &= G(s) - G(s)e^{-sT/2} + G(s)e^{-sT} \\ &\quad - G(s)e^{-3sT/2} + \dots \\ &= G(s)[1 - e^{-sT/2} + e^{-sT} - e^{-3sT/2} \\ &\quad + \dots] \end{aligned}$$

The series on the right is the geometric series

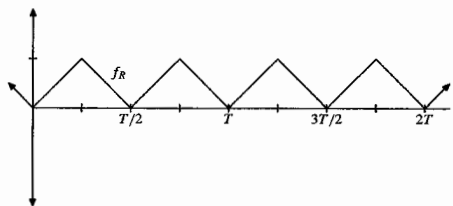
$a + ar + ar^2 + \dots$ with $r = -e^{-sT/2}$, so

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\}(s) \\ &= G(s) \frac{1}{1 - (-e^{-sT/2})} \\ &= \frac{G(s)}{1 + e^{-sT/2}}. \end{aligned}$$

(d) Consider the function with the following graph. It is implied that the function follows the indicated pattern indefinitely, both left and right.



Note that $f(t + T/2) = -f(t)$ for all t . Define $f_R(t)$ so that $f_R(t) = f(t)$ on $[0, T/2)$, and require that $f_R(t + T/2) = f_R(t)$ for all t . The graph of f_R is known as the “full wave” rectification of f and looks as follows.

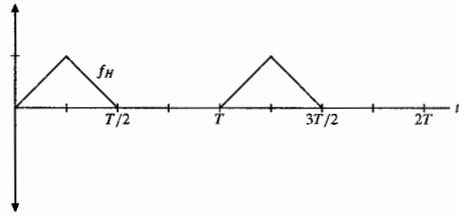


(e) We look at the “half wave” rectification $f_H(t)$ defined by

$$f_H(t) = \begin{cases} f_{T/2}(t), & 0 \leq t < T/2, \\ 0, & T/2 \leq t < T, \end{cases}$$

and $f_H(t + T) = f_H(t)$ for all t . The half wave rectification for the function pictured in part (d)

looks like the following.



The function f_H is periodic with period T , so

$$F_H(s) = \mathcal{L}\{f_H(t)\}(s) = \frac{\int_0^T f_H(t)e^{-st} dt}{1 - e^{-sT}}$$

by Proposition 5.17. But $f_H(t)$ is zero on $[T/2, T)$, so

$$\int_0^T f_H(t)e^{-st} dt = \int_0^{T/2} f_H(t)e^{-st} dt.$$

However, $f_H(t) = f(t)$ on $[0, T/2)$, so

$$\begin{aligned} \int_0^T f_H(t)e^{-st} dt &= \int_0^{T/2} f(t)e^{-st} dt \\ &= G(s). \end{aligned}$$

Therefore,

$$F_H(s) = \mathcal{L}\{f_H(t)\}(s) = \frac{G(s)}{1 - e^{-sT}}.$$

However, recall that in part (c) we show that

$$F(s) = \frac{G(s)}{1 + e^{-sT/2}}.$$

Thus,

$$G(s) = (1 + e^{-sT/2})F(s)$$

and $F_H(s)$ becomes

$$\begin{aligned} F_H(s) &= \frac{(1 + e^{-sT/2})F(s)}{1 - e^{-sT}} \\ &= \frac{(1 + e^{-sT/2})F(s)}{(1 + e^{-sT/2})(1 - e^{-sT/2})} \\ &= \frac{F(s)}{1 - e^{-sT/2}} \end{aligned}$$

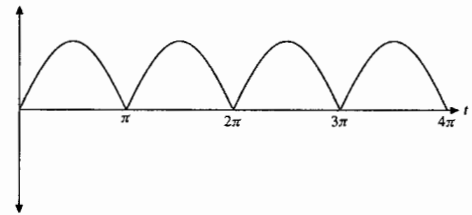
Finally, note that the full wave rectification f_R (2nd image in part (d)) is the sum of the half wave rectification pictured above and the same half wave rectification translated $T/2$ units to the right. Thus,

$$\begin{aligned} f_R(t) &= f_H(t)H(t) \\ &\quad + f_H\left(t - \frac{T}{2}\right)H\left(t - \frac{T}{2}\right) \end{aligned}$$

and

$$\begin{aligned} F_R(s) &= F_H(s) + F_H(s)e^{-sT/2} \\ &= (1 + e^{-sT/2})F_H(s) \\ &= (1 + e^{-sT/2})\frac{F(s)}{1 - e^{-sT/2}} \\ &= \frac{e^{sT/4} - e^{-sT/4}}{e^{sT/4} + e^{-sT/4}}F(s) \\ &= F(s) \coth \frac{sT}{4}. \end{aligned}$$

- (f) The function $g(t) = |\sin t|$ is a full wave rectification of the sine curve pictured in part (a).



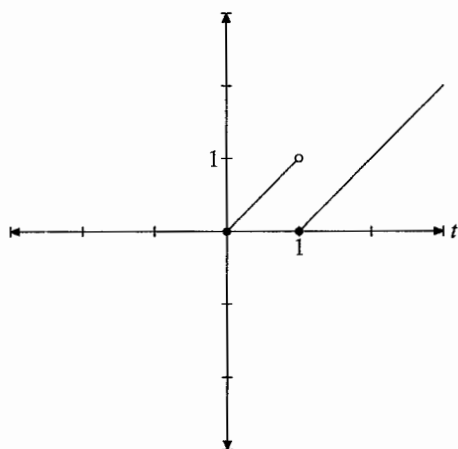
Thus, by part (e), noting that $T = 2\pi$,

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \mathcal{L}\{\sin t\}(s) \coth \frac{s(2\pi)}{4} \\ &= \frac{1}{s^2 + 1} \coth \frac{\pi s}{2}. \end{aligned}$$

35. (a) If

$$y(t) = \begin{cases} 0, & \text{if } t < 0, \\ t, & \text{if } 0 \leq t < 1, \\ t - 1, & \text{if } t \geq 1, \end{cases}$$

then y has graph



Moreover,

$$\begin{aligned} y(t) &= tH_0(t) + (t-1)H_1(t) \\ &= t(H(t) - H(t-1)) + (t-1)H(t-1) \\ &= tH(t) - H(t-1). \end{aligned}$$

Thus,

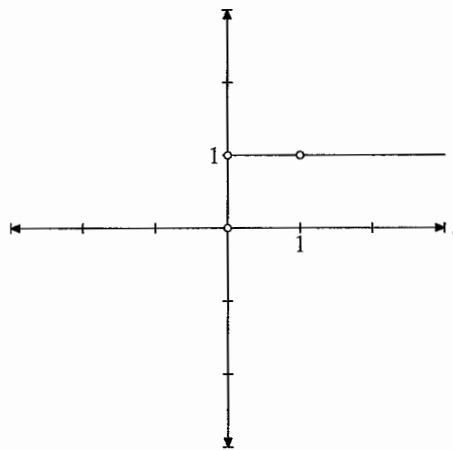
$$Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s}$$

and

$$\begin{aligned} s \mathcal{L}\{y(t)\}(s) - y(0) &= sY(s) - 0 \\ &= s \left[\frac{1}{s^2} - \frac{e^{-s}}{s} \right] \\ &= \frac{1}{s} - e^{-s}. \end{aligned}$$

- (b) Because $y'(t)$ is the slope of the line tangent to the graph of $y(t)$ at t , the derivative is easily

seen to be



or equivalently,

$$y'(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } 0 < t < 1, \\ 0, & \text{if } t > 1. \end{cases}$$

Of course, the derivative of y does not exist at $t = 0$ because there are different right and left-hand derivatives at this point (note the sharp cusp in the graph in part (a) at $t = 0$). Nor is y differentiable at $t = 1$, because y fails to be continuous at $t = 1$. Thus,

$$\begin{aligned} \mathcal{L}\{y'(t)\}(s) &= \int_0^{\infty} y'(t)e^{-st} dt \\ &= \int_0^1 1e^{-st} dt, \end{aligned}$$

as the removable discontinuity at $t = 1$ will not affect the overall integral. Note that this is identical to the Laplace transform of 1, which is $1/s$. Therefore,

$$\mathcal{L}\{y'(t)\}(s) = \frac{1}{s}.$$

- (c) There is a seeming contradiction, as parts (a) and (b) lead to different answers for $\mathcal{L}\{y'(t)\}(s)$. However, this does not contradict Proposition 2.1 as the hypotheses of Proposition 2.1 are not satisfied; namely, y is not piecewise differentiable (it isn't continuous).

(d) As we saw in part (b), $\mathcal{L}\{y'(t)\}(s) = 1/s$. Next,

$$\begin{aligned} s \mathcal{L}\{y(t)\}(s) - [y(1^+) - y(1^-)]e^{-s} - y(0) \\ = s \left[\frac{1}{s^2} - \frac{e^{-s}}{s} \right] - [0 - 1]e^{-s} - 0 \\ = \frac{1}{s} - e^{-s} + e^{-s} \\ = \frac{1}{s}. \end{aligned}$$

36. (a) Let $\mathcal{L}\{p\} = P(s)$. Since

$$p' = rp - h(H(t) - H(t-1)),$$

it follows that

$$\begin{aligned} \mathcal{L}\{p'\} &= r \mathcal{L}\{p\} - h \mathcal{L}\{H(t)\} \\ &\quad + h \mathcal{L}\{H(t-1)\}. \end{aligned}$$

Since $p(0) = p_0$, we have

$$sP(s) - p_0 = rP(s) - \frac{h}{s} + \frac{he^{-s}}{s}$$

We solve for P , getting

$$P(s) = \frac{p_0s - (1 - e^{-s})h}{s(s-r)}.$$

Since

$$\frac{1}{s(s-r)} = \frac{1}{r} \left[\frac{1}{s-r} - \frac{1}{s} \right],$$

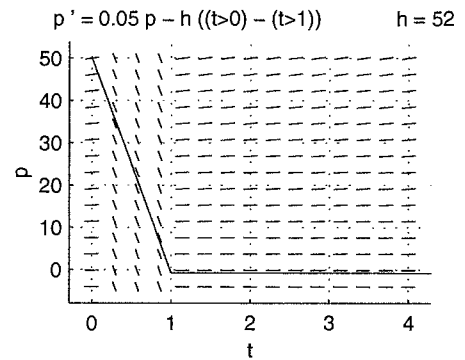
this becomes

$$P(s) = \frac{p_0}{s-r} + \frac{h(1 - e^{-s})}{r} \left[\frac{1}{s} - \frac{1}{s-r} \right].$$

Hence

$$\begin{aligned} p(t) &= p_0 e^{rt} + \frac{h}{r} [1 - e^{rt}] \\ &\quad - \frac{h}{r} H(t-1) [1 - e^{r(t-1)}] \end{aligned}$$

(b) $51 < h < 52$. See the Matlab figure.



(c) Using the result in part (a),

$$\begin{aligned} p(1) &= p_0 e^r + \frac{h}{r} [1 - e^r] \\ &\quad - \frac{h}{r} H(0) [1 - e^{r^0}] \\ &= p_0 e^r + \frac{h}{r} [1 - e^r] \end{aligned}$$

At extinction, $p(1) = 0$, so

$$\begin{aligned} h &= \frac{re^r}{e^r - 1} p_0 \\ &= \frac{0.05e^{0.05}}{e^{-0.05} - 1} \times 50 \\ &= 51.2604 \end{aligned}$$

After time $t = 1$, the term $h(H(t) - H(t-1))$ is zero and $dp/dt = rp$. If $p = 0$ at $t = 1$, then $dp/dt = 0$ for all time and the population remains extinct.

Section 5.6. The Delta Function

1. (a) If

$$\delta_p^\epsilon = \frac{1}{\epsilon}(H_p(t) - H_{p+\epsilon}(t)),$$

then

$$\begin{aligned}\mathcal{L}\{\delta_p^\epsilon(t)\}(s) &= \frac{1}{\epsilon} \mathcal{L}\{H_p(t)\}(s) \\ &\quad - \frac{1}{\epsilon} \mathcal{L}\{H_{p+\epsilon}(t)\}(s) \\ &= \frac{1}{\epsilon} \cdot \frac{e^{-ps}}{s} - \frac{1}{\epsilon} \cdot \frac{e^{-(p+\epsilon)s}}{s} \\ &= \frac{e^{-sp} - e^{-sp}e^{-s\epsilon}}{s\epsilon} \\ &= e^{-sp} \cdot \frac{1 - e^{-s\epsilon}}{s\epsilon}.\end{aligned}$$

- (b) Taking the limit,

$$\begin{aligned}\delta_p(t) &= \lim_{\epsilon \rightarrow 0} \delta_p^\epsilon(t) \\ &= \lim_{\epsilon \rightarrow 0} e^{-sp} \cdot \frac{1 - e^{-s\epsilon}}{s\epsilon} \\ &= e^{-sp} \lim_{\epsilon \rightarrow 0} \frac{1 - e^{-s\epsilon}}{s\epsilon}.\end{aligned}$$

Because both numerator and denominator approach zero as $\epsilon \rightarrow 0$, the indeterminate form $0/0$ allows the application of l'Hôpital's rule.

$$\begin{aligned}\delta_p(t) &= e^{-sp} \lim_{\epsilon \rightarrow 0} \frac{\frac{d}{d\epsilon}(1 - e^{-s\epsilon})}{\frac{d}{d\epsilon}(s\epsilon)} \\ &= e^{-sp} \lim_{\epsilon \rightarrow 0} \frac{se^{-s\epsilon}}{s} \\ &= e^{-sp} \lim_{\epsilon \rightarrow 0} e^{-s\epsilon} \\ &= e^{-sp}.\end{aligned}$$

Thus, $\mathcal{L}\{\delta_p(t)\}(s) = e^{-sp}$.

2. By Theorem 6.10, the unit impulse response of

$$y'' - 4y' + 3y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 - 4s + 3$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 - 4s + 3}.$$

Using a partial fraction decomposition,

$$E(s) = \frac{1}{2} \cdot \frac{1}{s-3} - \frac{1}{2} \cdot \frac{1}{s-1},$$

so the unit impulse response is

$$e(t) = \frac{1}{2}e^{3t} - \frac{1}{2}e^t.$$

3. By Theorem 6.10, the unit impulse response of

$$y'' - 4y' - 5y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 - 4s - 5$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 - 4s - 5}.$$

Using a partial fraction decomposition,

$$E(s) = \frac{1}{6} \cdot \frac{1}{s-5} - \frac{1}{6} \cdot \frac{1}{s+1},$$

so the unit impulse response is

$$e(t) = \frac{1}{6}e^{5t} - \frac{1}{6}e^{-t}.$$

4. By Theorem 6.10, the unit impulse response of

$$y'' + 4y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 + 4s$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 + 4s}.$$

Using a partial fraction decomposition,

$$E(s) = \frac{1}{4} \cdot \frac{1}{s} - \frac{1}{4} \cdot \frac{1}{s+4},$$

so the unit impulse response is

$$e(t) = \frac{1}{4} - \frac{1}{4}e^{-4t}.$$

5. By Theorem 6.10, the unit impulse response of

$$y'' - 9y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 - 9$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 - 9}.$$

Using a partial fraction decomposition,

$$E(s) = -\frac{1}{6} \cdot \frac{1}{s+3} + \frac{1}{6} \cdot \frac{1}{s-3},$$

so the unit impulse response is

$$e(t) = -\frac{1}{6}e^{-3t} + \frac{1}{6}e^{3t}.$$

6. By Theorem 6.10, the unit response of

$$y'' + 4y' + 5y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 + 4s + 5$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 + 4s + 5}.$$

Completing the square,

$$E(s) = \frac{1}{(s+2)^2 + 1},$$

so the unit impulse response is

$$e(t) = e^{-2t} \sin t.$$

7. By Theorem 6.10, the unit impulse response of

$$y'' + 2y' + 2y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 + 2s + 2$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 + 2s + 2}.$$

Completing the square,

$$E(s) = \frac{1}{(s+1)^2 + 1},$$

so the unit impulse response is

$$e(t) = e^{-t} \sin t.$$

8. (a)

$$\mathcal{L}\{\delta(t)\}(s) = 1$$

$$x'' = 2\delta(t)$$

$$\mathcal{L}\{x''\} = 2\mathcal{L}\{\delta(t)\}$$

$$s^2 X(s) - sx(0) - x'(0) = 2$$

But $x(0) = x'(0) = 0$, so

$$s^2 X(s) = 2$$

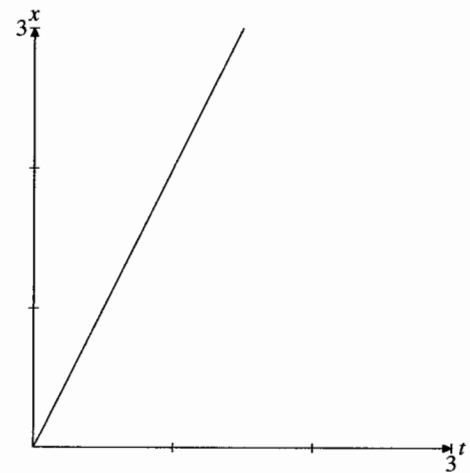
$$X(s) = \frac{2}{s^2}.$$

Thus,

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\},$$

and

$$x(t) = 2tH(t).$$



- (b) From Exercise 1

$$\mathcal{L}\{\delta_p^\epsilon(t)\} = e^{-sp} \frac{1 - e^{-s\epsilon}}{s\epsilon},$$

so,

$$\mathcal{L}\{\delta_0^\epsilon(t)\} = \frac{1 - e^{-s\epsilon}}{s\epsilon}.$$

Thus,

$$\begin{aligned}x'' &= 2\delta_0^\epsilon(t) \\ \mathcal{L}\{x''\} &= 2\mathcal{L}\{\delta_0^\epsilon(t)\} \\ s^2X(s) - sx(0) - x'(0) &= \frac{2(1 - e^{-s\epsilon})}{s\epsilon}\end{aligned}$$

But $x(0) = x'(0) = 0$.

$$\begin{aligned}s^2X(s) &= \frac{2(1 - e^{-s\epsilon})}{s\epsilon} \\ X(s) &= \frac{2(1 - e^{-s\epsilon})}{s^3\epsilon} \\ X(s) &= \frac{2}{s^3\epsilon} - \frac{2e^{-s\epsilon}}{s^3\epsilon}.\end{aligned}$$

Now, work backwards from the solution $x(t)$ and confirm it is consistent with $X(s)$.

$$x_\epsilon(t) = \begin{cases} t^2/\epsilon, & \text{if } 0 \leq t < \epsilon, \\ 2t - \epsilon, & \text{if } t \geq \epsilon \end{cases}$$

Thus,

$$\begin{aligned}x_\epsilon(t) &= \frac{1}{\epsilon}t^2H_{0\epsilon} + (2t - \epsilon)H_\epsilon \\ &= \frac{1}{\epsilon}t^2(H(t) - H(t - \epsilon)) \\ &\quad + (2t - \epsilon)H(t - \epsilon) \\ &= \frac{1}{\epsilon}t^2H(t) - \frac{1}{\epsilon}t^2H(t - \epsilon) \\ &\quad + (2t - \epsilon)H(t - \epsilon) \\ &= \frac{1}{\epsilon}t^2H(t) - \frac{1}{\epsilon}[(t - \epsilon)^2 \\ &\quad + 2\epsilon(t - \epsilon) + \epsilon^2]H(t - \epsilon) \\ &\quad + (2(t - \epsilon) + \epsilon)H(t - \epsilon).\end{aligned}$$

Therefore,

$$\begin{aligned}X(s) &= \frac{2}{\epsilon s^3} - \frac{1}{\epsilon} \cdot e^{-\epsilon s} \left[\frac{2}{s^3} + \frac{2\epsilon}{s^2} + \frac{\epsilon^2}{s} \right] \\ &\quad + e^{-\epsilon s} \left(\frac{2}{s^2} + \frac{\epsilon}{s} \right) \\ X(s) &= \frac{2}{\epsilon s^3} - \frac{2e^{-\epsilon s}}{\epsilon s^3}\end{aligned}$$

(c) As long as $t \geq 0$, the solution in part (b) will approach the solution in part (a).

9. (a) If

$$x'' + 2x' + 2x = \delta(t), \quad x(0) = x'(0) = 0,$$

then, by Theorem 6.10, the unit impulse response is

$$X(s) = \frac{1}{P(s)} = \frac{1}{s^2 + 2s + 2}.$$

Completing the square,

$$X(s) = \frac{1}{(s + 1)^2 + 1}.$$

Note the transform pair

$$\sin t \iff \frac{1}{s^2 + 1}.$$

Thus, by Proposition 2.12,

$$x(t) = e^{-t} \sin t.$$

(b) The Laplace transform of the right-hand side, δ_0^ϵ , is

$$\begin{aligned}\mathcal{L}\{\delta_0^\epsilon(t)\}(s) &= \mathcal{L}\left\{\frac{1}{\epsilon}(H(t) - H(t - \epsilon))\right\}(s) \\ &= \frac{1}{\epsilon} [\mathcal{L}\{H(t)\}(s) - \mathcal{L}\{H(t - \epsilon)\}(s)] \\ &= \frac{1}{\epsilon} \left[\frac{1}{s} - \frac{e^{-\epsilon s}}{s} \right] \\ &= \frac{1 - e^{-\epsilon s}}{\epsilon s}.\end{aligned}$$

Thus, if

$$\begin{aligned}x'' + 2x' + 2x &= \delta_0^\epsilon(t), \quad \text{and} \\ x(0) &= x'(0) = 0,\end{aligned}$$

then

$$\begin{aligned}\mathcal{L}(x'' + 2x' + 2x)(s) &= \mathcal{L}\{\delta_0^\epsilon(t)\}(s) \\ (s^2 + 2s + 2)X(s) &= \frac{1 - e^{-\epsilon s}}{\epsilon s} \\ X(s) &= \frac{1 - e^{-\epsilon s}}{\epsilon s(s^2 + 2s + 2)}.\end{aligned}$$

A partial fraction decomposition gives

$$\begin{aligned}
 & \frac{1}{s(s^2 + 2s + 2)} \\
 &= \frac{1/2}{s} + \frac{-(1/2)s - 1}{s^2 + 2s + 2} \\
 &= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s + 2}{(s + 1)^2 + 1} \\
 &= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s + 1}{(s + 1)^2 + 1} \\
 &\quad - \frac{1}{2} \cdot \frac{1}{(s + 1)^2 + 1} \\
 &= \frac{1}{2} \mathcal{L}\{f\}(s),
 \end{aligned}$$

where

$$f(t) = 1 - e^{-t}(\cos t + \sin t).$$

Thus,

$$\begin{aligned}
 x_\epsilon(t) &= \frac{1}{\epsilon} \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + 2s + 2)} \right\}(t) \\
 &\quad - \frac{1}{\epsilon} \mathcal{L}^{-1} \left\{ e^{-\epsilon s} \cdot \frac{1}{s(s^2 + 2s + 2)} \right\}(t) \\
 &= \frac{1}{2\epsilon} [f(t) - H(t - \epsilon)f(t - \epsilon)] \\
 &= \frac{1}{2\epsilon} \begin{cases} f(t), & 0 \leq t < \epsilon, \\ f(t) - f(t - \epsilon), & t \geq \epsilon. \end{cases}
 \end{aligned}$$

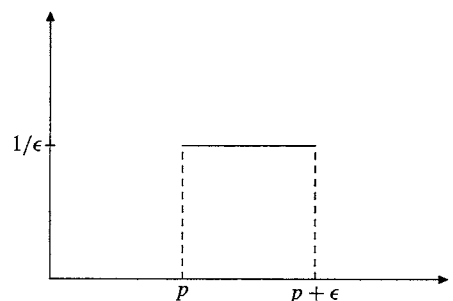
- (c) As $\epsilon \rightarrow 0$, the first interval ($0 \leq t < \epsilon$) in the piecewise definition in part (b) melts away. Let's concentrate on the second piece. By l'Hôpital's rule,

$$\begin{aligned}
 \frac{1}{2} \lim_{\epsilon \rightarrow 0} \frac{f(t) - f(t - \epsilon)}{\epsilon} &= \frac{1}{2} \lim_{\epsilon \rightarrow 0} f'(t - \epsilon) \\
 &= \frac{1}{2} f'(t) \\
 &= e^{-t} \sin t.
 \end{aligned}$$

10.

$$H_p^\epsilon(t) = H^\epsilon(t - p) = \begin{cases} 0, & \text{for } p + \epsilon < 0, \\ 1, & \text{for } p + \epsilon \geq 0 \end{cases}$$

The graph of the derivative of $H_p^\epsilon(t)$ is



Because the graph of $H_p^{\epsilon'}(t)$ is identical to the graph of $\delta_p^\epsilon(t)$, we can infer that $H_p'(t) = \delta_p(t)$.

11. If

$$x' = \delta_p(t), \quad x(0) = 0,$$

then

$$\begin{aligned}
 \mathcal{L}\{x'\}(s) &= \mathcal{L}\{\delta_p(t)\}(s) \\
 sX(s) - x(0) &= e^{-ps} \\
 X(s) &= \frac{1}{s} e^{-ps}.
 \end{aligned}$$

Thus,

$$x(t) = H_p(t).$$

Section 5.7. Convolutions

1. By definition,

$$\begin{aligned}
 f * (g + h)(t) &= \int_0^t f(u)(g + h)(t - u) du \\
 &= \int_0^t f(u)(g(t - u) + h(t - u)) du, \\
 &= \int_0^t f(u)g(t - u) du + \int_0^t f(u)h(t - u) du \\
 &= f * g(t) + f * h(t)
 \end{aligned}$$

Thus,

$$f * (g + h) = f * g + f * h.$$

2. Start with the left hand side of the equation and apply Definition 7.5 to the function
- $(f * g)(t)$
- and
- $h(t)$
- .

$$[(f * g) * h](t) = \int_0^t (f * g)(u)h(t - u) du$$

Next, again use Definition 7.5 to write the convolution of $f(u)$ and $g(u)$.

$$(f * g)(u) = \int_0^u f(v)g(u - v) dv.$$

Note that we use u as a limit of integration and introduce a new variable of integration v . Substitute the expression for $(f * g)(u)$ into the expression for $(f * g) * h$.

$$\begin{aligned}
 (f * g) * h(t) &= \int_0^t \left[\int_0^u f(v)g(u - v) dv \right] h(t - u) du \\
 &= \int_0^t \int_0^u f(v)g(u - v)h(t - u) dv du.
 \end{aligned}$$

Next we interchange the order of integration. The chain of inequalities $0 \leq v \leq u \leq t$ must be maintained when we change variables, so we get

$$\begin{aligned}
 (f * g) * h(t) &= \int_0^t f(v) \int_v^t g(u - v)h(t - u) du dv.
 \end{aligned}$$

When we introduce the new variable $w = u - v$ in the inner integral this becomes

$$\begin{aligned}
 (f * g) * h(t) &= \int_0^t f(v) \int_0^{t-v} g(w)h(t - v - w) dw dv.
 \end{aligned}$$

We recognize the inner integral as $g * h(t - v)$ so we finally have

$$\begin{aligned}
 (f * g) * h(t) &= \int_0^t f(v)g * h(t - v) dv \\
 &= f * (g * h)(t).
 \end{aligned}$$

3. By definition,

$$F * 0(t) = \int_0^t f(u)z(t - u) du,$$

where z is the zero function. Of course, $z(t - u) = 0$, so

$$\begin{aligned}
 f * 0(t) &= \int_0^t f(u) \cdot 0 du \\
 &= \int_0^t 0 du \\
 &= 0.
 \end{aligned}$$

Since this is true for all t ,

$$f * 0 = 0.$$

4. If
- $f(t) = e^{3t}$
- and
- $g(t) = e^{-2t}$
- , then

$$\begin{aligned}
 f * g(t) &= \int_0^t f(u)g(t - u) du \\
 &= \int_0^t e^{3u}e^{-2(t-u)} du \\
 &= \int_0^t e^{(-2t+5u)} du \\
 &= \frac{1}{5}e^{(-2t+5u)} \Big|_0^t \\
 &= \frac{1}{5}[e^{3t} - e^{-2t}]
 \end{aligned}$$

5. If $f(t) = e^t$ and $g(t) = e^{2t}$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t e^u e^{2(t-u)} du \\ &= \int_0^t e^{2t-u} du \\ &= -e^{2t-u} \Big|_0^t \\ &= -e^t - (-e^{2t}) \\ &= e^{2t} - e^t. \end{aligned}$$

6. If $f(t) = t - 1$ and $g(t) = t - 2$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t (u-1)(t-u-2) du \\ &= \int_0^t (ut - u^2 - u - t + 2) du \\ &= \frac{u^2 t}{2} - \frac{u^3}{3} - \frac{u^2}{2} - tu + 2u \Big|_0^t \\ &= \frac{t^3}{2} - \frac{t^3}{3} - \frac{t^2}{2} - t^2 + 2t \\ &= \frac{t^3}{6} - \frac{3t^2}{2} + 2t. \end{aligned}$$

7. If $f(t) = t$ and $g(t) = 3 - t$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t u(3 - (t-u)) du \\ &= \int_0^t (3u - ut + u^2) du \\ &= \frac{3}{2}u^2 - \frac{1}{2}ut^2 + \frac{1}{3}u^3 \Big|_0^t \\ &= \frac{3}{2}t^2 - \frac{1}{2}t^3 + \frac{1}{3}t^3 \\ &= \frac{3}{2}t^2 - \frac{1}{6}t^3. \end{aligned}$$

8. If $f(t) = t$ and $g(t) = e^t$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du, \\ &= \int_0^t u(e^{t-u}) du. \end{aligned}$$

Integration by parts provides

$$f * g(t) = -ue^{t-u} \Big|_0^t + \int_0^t e^{t-u} du.$$

A second integration yields

$$\begin{aligned} &= -ue^{t-u} \Big|_0^t - e^{t-u} \Big|_0^t, \\ &= -te^0 - e^0 + e^t, \\ &= -t - 1 + e^t. \end{aligned}$$

9. If $f(t) = t^2$ and $g(t) = e^{-t}$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t u^2 e^{-(t-u)} du \\ &= \int_0^t u^2 e^{u-t} du. \end{aligned}$$

Integration by parts provides

$$f * g(t) = u^2 e^{u-t} \Big|_0^t - \int_0^t 2ue^{u-t} du.$$

A second integration by parts provides

$$\begin{aligned} f * g(t) &= u^2 e^{u-t} \Big|_0^t - \left[2ue^{u-t} \Big|_0^t - \int_0^t 2e^{u-t} du \right] \\ &= u^2 e^{u-t} - 2ue^{u-t} + 2e^{u-t} \Big|_0^t \\ &= e^{u-t}(u^2 - 2u + 2) \Big|_0^t \\ &= (t^2 - 2t + 2) - e^{-t}(0 - 0 + 2) \\ &= t^2 - 2t + 2 - 2e^{-t}. \end{aligned}$$

10. (a) If $f(t) = e^t$ and $g(t) = e^{2t}$, then $F(s) = 1/(s-1)$ and $G(s) = 1/(s-2)$. Therefore

$$\begin{aligned} F(s)G(s) &= \left(\frac{1}{s-1}\right)\left(\frac{1}{s-2}\right) \\ &= \frac{1}{s^2 - 3s + 2}. \end{aligned}$$

(b)

$$\begin{aligned} \mathcal{L}\{f(t)g(t)\} &= \mathcal{L}\{e^t e^{2t}\}(s) \\ &= \mathcal{L}\{e^{3t}\}(s) \\ &= \frac{1}{s-3} \end{aligned}$$

Parts (a) and (b) emphasize the fact that the Laplace transform of a product of two functions is **not** the product of their Laplace transforms.

11. If $f(t) = \sin t$ and $g(t) = t$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t (\sin u)(t-u) du. \end{aligned}$$

Integration by parts provides

$$\begin{aligned} f * g(t) &= \int_0^t (u-t)d(\cos u) \\ &= (u-t)\cos u \Big|_0^t - \int_0^t \cos u du \\ &= (u-t)\cos u - \sin u \Big|_0^t \\ &= (0 - \sin t) - (-t - 0) \\ &= t - \sin t. \end{aligned}$$

However,

$$\begin{aligned} \mathcal{L}\{f * g(t)\}(s) &= \mathcal{L}\{t - \sin t\}(s) \\ &= \mathcal{L}\{t\}(s) - \mathcal{L}\{\sin t\}(s) \\ &= \frac{1}{s^2} - \frac{1}{s^2 + 1} \\ &= \frac{1}{s^2(s^2 + 1)}. \end{aligned}$$

Alternatively, we have transform pairs

$$\begin{aligned} f(t) = \sin t &\iff F(s) = \frac{1}{s^2 + 1} \\ g(t) = t &\iff G(s) = \frac{1}{s^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f * g(t)\}(s) &= F(s)G(s) \\ &= \frac{1}{s^2 + 1} \cdot \frac{1}{s^2} \\ &= \frac{1}{s^2(s^2 + 1)}. \end{aligned}$$

12. If $f(t) = \cos t$ and $g(t) = t^2$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t (\cos u)(t-u)^2 du. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int (\cos u)(t-u)^2 du &= (\sin u)(t-u)^2 \\ &\quad + 2 \int (\sin u)(t-u) du. \end{aligned}$$

Integrating by parts again

$$\begin{aligned} &= (\sin u)(t-u)^2 \\ &\quad + 2 \left[(-\cos u)(t-u) - \int \cos u du \right], \\ &= (\sin u)(t-u)^2 - 2(\cos u)(t-u) - 2 \sin u. \end{aligned}$$

Thus,

$$\begin{aligned} f * g(t) &= \left[(\sin u)(t-u)^2 - 2(\cos u)(t-u) \right. \\ &\quad \left. - 2 \sin u \right]_0^t \\ &= -2 \sin t + 2t. \end{aligned}$$

However,

$$\begin{aligned} \mathcal{L}\{f * g(t)\}(s) &= \mathcal{L}\{-2 \sin t + 2t\}(s), \\ &= \mathcal{L}\{-2 \sin t\}(s) + \mathcal{L}\{2t\}(s), \\ &= \frac{-2}{s^2 + 1} + \frac{2}{s^2}, \\ &= \frac{2}{s^2(s^2 + 1)}. \end{aligned}$$

Alternatively, we have the transform pairs

$$\begin{aligned} f(t) = \cos t &\iff F(s) = \frac{s}{s^2 + 1}, \\ g(t) = t^2 &\iff G(s) = \frac{2}{s^3}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f * g(t)\}(s) &= F(s)G(s), \\ &= \left(\frac{s}{s^2 + 1}\right)\left(\frac{2}{s^3}\right), \\ &= \frac{2}{s^2(s^2 + 1)}. \end{aligned}$$

13. If $f(t) = e^{-2t}$ and $g(t) = t$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t e^{-2u}(t-u) du \\ &= \int_0^t (t-u)d\left(-\frac{1}{2}e^{-2u}\right). \end{aligned}$$

Integrate by parts.

$$\begin{aligned} f * g(t) &= (t-u)\left(-\frac{1}{2}e^{-2u}\right)\Big|_0^t \\ &\quad - \int_0^t \left(-\frac{1}{2}e^{-2u}\right)(-du) \\ &= -\frac{1}{2}e^{-2u}(t-u)\Big|_0^t - \frac{1}{2} \int_0^t e^{-2u} du \\ &= -\frac{1}{2}e^{-2u}(t-u) + \frac{1}{4}e^{-2u}\Big|_0^t \\ &= \left(0 + \frac{1}{4}e^{-2t}\right) - \left(-\frac{1}{2}t + \frac{1}{4}\right) \\ &= \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}. \end{aligned}$$

However,

$$\begin{aligned} \mathcal{L}\{f * g(t)\}(s) &= \mathcal{L}\left\{\frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}\right\}(s) \\ &= \frac{1}{4} \cdot \frac{1}{s+2} + \frac{1}{2} \cdot \frac{1}{s^2} - \frac{1}{4} \cdot \frac{1}{s} \\ &= \frac{s^2 + 2(s+2) - s(s+2)}{4s^2(s+2)} \\ &= \frac{4}{4s^2(s+2)} \\ &= \frac{1}{s^2(s+2)}. \end{aligned}$$

Alternatively, we have the transform pairs

$$\begin{aligned} f(t) = e^{-2t} &\iff F(s) = \frac{1}{s+2} \\ g(t) = t &\iff G(s) = \frac{1}{s^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f * g(t)\}(s) &= F(s)G(s) \\ &= \frac{1}{s+2} \cdot \frac{1}{s^2} \\ &= \frac{1}{s^2(s+2)}. \end{aligned}$$

14. If $f(t) = e^{3t}$ and $g(t) = t^2$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t (e^{3u})(t-u)^2 du. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} &\int e^{3u}(t-u)^2 du \\ &= \frac{1}{3}e^{3u}(t-u)^2 - \int \left(\frac{1}{3}e^{3u}\right)(-2(t-u)) du, \\ &= \frac{1}{3}e^{3u}(t-u)^2 + \frac{2}{3} \int e^{3u}(t-u) du. \end{aligned}$$

Integrating by parts again,

$$\begin{aligned}
 &= \frac{1}{3}e^{3u}(t-u)^2 \\
 &\quad + \frac{2}{3} \left[\frac{1}{3}e^{3u}(t-u) - \int \left(\frac{1}{3}e^{3u} \right) (-1) du \right], \\
 &= \frac{1}{3}e^{3u}(t-u)^2 + \frac{2}{9}e^{3u}(t-u) + \frac{2}{27}e^{3u}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 f * g(t) &= \left[\frac{1}{3}e^{3u}(t-u)^2 + \frac{2}{9}e^{3u}(t-u) \right. \\
 &\quad \left. + \frac{2}{27}e^{3u} \right]_0^t, \\
 &= \frac{2}{27}e^{3t} - \frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27}.
 \end{aligned}$$

However,

$$\begin{aligned}
 \mathcal{L}\{f * g(t)\}(s) &= \mathcal{L}\left\{\frac{2e^{3t}}{27}\right\}(s) - \mathcal{L}\left\{\frac{2}{27}\right\}(s) \\
 &\quad - \mathcal{L}\left\{\frac{2t}{9}\right\}(s) - \mathcal{L}\left\{\frac{t^2}{3}\right\}(s), \\
 &= \frac{2}{27(s-3)} - \frac{2}{27s} \\
 &\quad - \frac{2}{9s^2} - \frac{2}{3s^3}, \\
 &= \frac{2}{(s-3)s^3}.
 \end{aligned}$$

Note the transform pairs

$$f(t) = e^{3t} \iff F(s) = \frac{1}{s-3},$$

$$g(t) = t^2 \iff G(s) = \frac{2}{s^3}.$$

Thus,

$$\begin{aligned}
 \mathcal{L}\{f * g(t)\}(s) &= F(s)G(s), \\
 &= \left(\frac{1}{s-3}\right)\left(\frac{2}{s^3}\right).
 \end{aligned}$$

15. If $f(t) = e^{-t}$ and $g(t) = \cos t$, then

$$\begin{aligned}
 f * g(t) &= \int_0^t f(u)g(t-u) du \\
 &= \int_0^t e^{-u} \cos(t-u) du.
 \end{aligned}$$

One integration by parts provides

$$\begin{aligned}
 &\int e^{-u} \cos(t-u) du \\
 &= \int \cos(t-u) d(-e^{-u}) \\
 &= -e^{-u} \cos(t-u) - \int (-e^{-u}) \sin(t-u) du \\
 &= -e^{-u} \cos(t-u) + \int e^{-u} \sin(t-u) du \\
 &= -e^{-u} \cos(t-u) + \int \sin(t-u) d(-e^{-u}).
 \end{aligned}$$

A second integration by parts leads to

$$\begin{aligned}
 &\int e^{-u} \cos(t-u) du \\
 &= -e^{-u} \cos(t-u) + (-e^{-u}) \sin(t-u) \\
 &\quad - \int (-e^{-u})(-\cos(t-u)) du \\
 &= -e^{-u} \cos(t-u) - e^{-u} \sin(t-u) \\
 &\quad - \int e^{-u} \cos(t-u) du.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 &2 \int e^{-u} \cos(t-u) du \\
 &= -e^{-u} \cos(t-u) - e^{-u} \sin(t-u).
 \end{aligned}$$

Hence

$$\begin{aligned}
 &\int e^{-u} \cos(t-u) du \\
 &= -\frac{1}{2}e^{-u} [\cos(t-u) + \sin(t-u)],
 \end{aligned}$$

and

$$\begin{aligned}
 f * g(t) &= -\frac{1}{2}e^{-t} [\cos(t-u) + \sin(t-u)] \Big|_0^t \\
 &= -\frac{1}{2}e^{-t} + \frac{1}{2}[\cos t + \sin t]
 \end{aligned}$$

We have

$$\begin{aligned}
 \mathcal{L}\{f * g(t)\}(s) &= \frac{1}{2} \mathcal{L}\{-e^{-t} + \cos t + \sin t\} \\
 &= \frac{1}{2} \left[\frac{-1}{s+1} + \frac{s}{s^2+1} + \frac{1}{s^2+1} \right] \\
 &= \frac{2s}{2(s+1)(s^2+1)} \\
 &= \frac{s}{(s+1)(s^2+1)}.
 \end{aligned}$$

Alternatively, we have transform pairs

$$\begin{aligned}
 f(t) = e^{-t} &\iff F(s) = \frac{1}{s+1} \\
 g(t) = \cos t &\iff G(s) = \frac{s}{s^2+1}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \mathcal{L}\{f * g(t)\}(s) &= F(s)G(s) \\
 &= \frac{1}{s+1} \cdot \frac{s}{s^2+1} \\
 &= \frac{s}{(s+1)(s^2+1)}.
 \end{aligned}$$

16. If $f(t) = e^{-2t}$ and $g(t) = \sin t$, then

$$\begin{aligned}
 f * g(t) &= \int_0^t f(u)g(t-u) du \\
 &= \int_0^t e^{-2u} \sin(t-u) du.
 \end{aligned}$$

Integration by parts provides

$$\begin{aligned}
 &\int e^{-2u} \sin(t-u) du \\
 &= -\frac{1}{2} \sin(t-u) e^{-2u} \\
 &\quad - \frac{1}{2} \int e^{-2u} \cos(t-u) du.
 \end{aligned}$$

Another integration gives

$$\begin{aligned}
 &= -\frac{1}{4} e^{-2u} [2 \sin(t-u) - \cos(t-u)] \\
 &\quad - \frac{1}{4} \int e^{-2u} \sin(t-u) du.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 &5 \int_0^t e^{-2u} \sin(t-u) du \\
 &= -e^{-2u} [2 \sin(t-u) - \cos(t-u)] \Big|_0^t
 \end{aligned}$$

from which we get

$$\begin{aligned}
 &\int_0^t e^{-2u} \sin(t-u) du \\
 &= -\frac{1}{5} e^{-2u} [2 \sin(t-u) - \cos(t-u)] \Big|_0^t \\
 &= \frac{1}{5} e^{-2t} + \frac{2}{5} \sin t - \frac{1}{5} \cos t.
 \end{aligned}$$

However,

$$\begin{aligned}
 \mathcal{L}\{f * g(t)\}(s) &= \mathcal{L}\left\{\frac{1}{5} e^{-2t}\right\}(s) + \mathcal{L}\left\{\frac{2}{5} \sin t\right\}(s) \\
 &\quad - \mathcal{L}\left\{\frac{1}{5} \cos t\right\}(s), \\
 &= \frac{1}{5(s+2)} + \frac{2}{5(s^2+1)} - \frac{s}{5(s^2+1)}, \\
 &= \frac{1}{(s+2)(s^2+1)}.
 \end{aligned}$$

Note the transform pairs

$$\begin{aligned}
 f(t) = e^{-2t} &\iff F(s) = \frac{1}{s+2}, \\
 g(t) = \sin t &\iff G(s) = \frac{1}{s^2+1}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \mathcal{L}\{f * g(t)\}(s) &= F(s)G(s), \\
 &= \frac{1}{s+2} \cdot \frac{1}{s^2+1}.
 \end{aligned}$$

17. The expression

$$\frac{1}{s(s+1)} = \frac{1}{s} \cdot \frac{1}{s+1} = F(s)G(s).$$

We have transform pairs

$$F(s) = \frac{1}{s} \iff f(t) = 1$$

$$G(s) = \frac{1}{s+1} \iff g(t) = e^{-t}.$$

Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t) \\ &= f * g(t) \\ &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t e^{-(t-u)} du \\ &= e^{-(t-u)} \Big|_0^t \\ &= 1 - e^{-t}.\end{aligned}$$

18. Note the expression

$$\begin{aligned}\frac{1}{s^2 - 3s} &= \frac{1}{s(s-3)} = \frac{1}{s} \cdot \frac{1}{s-3}, \\ &= F(s)G(s).\end{aligned}$$

We have the transform pairs

$$F(s) = \frac{1}{s} \iff f(t) = 1,$$

$$G(s) = \frac{1}{s-3} \iff g(t) = e^{3t}.$$

Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 3s}\right\}(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t), \\ &= f * g(t), \\ &= \int_0^t f(u)g(t-u) du, \\ &= \int_0^t e^{3(t-u)} du, \\ &= -\frac{1}{3}e^{3(t-u)} \Big|_0^t, \\ &= -\frac{1}{3} + \frac{1}{3}e^{3t}, \\ &= -\frac{1}{3}(1 - e^{3t}).\end{aligned}$$

19. The expression

$$\frac{1}{(s+1)(s-2)} = \frac{1}{s+1} \cdot \frac{1}{s-2} = F(s)G(s).$$

We have transform pairs

$$F(s) = \frac{1}{s+1} \iff f(t) = e^{-t}$$

$$G(s) = \frac{1}{s-2} \iff g(t) = e^{2t}.$$

Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t) \\ &= f * g(t) \\ &= \int_0^t e^{-u}e^{2(t-u)} du \\ &= -\frac{1}{3}e^{2t-3u} \Big|_0^t \\ &= \frac{1}{3}(e^{2t} - e^{-t}).\end{aligned}$$

20. The expression:

$$\begin{aligned}\frac{1}{s^2 - 2s - 3} &= \frac{1}{(s-3)(s+1)} \\ &= \frac{1}{s-3} \cdot \frac{1}{s+1} \\ &= F(s)G(s)\end{aligned}$$

We have transform pairs

$$F(s) = \frac{1}{s-3} \iff f(t) = e^{3t},$$

$$G(s) = \frac{1}{s+1} \iff g(t) = e^{-t}.$$

Thus,

$$\begin{aligned}
 \mathcal{L}^{-1}\left\{\frac{1}{s^2-2s-3}\right\}(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t), \\
 &= f * g(t), \\
 &= \int_0^t f(u)g(t-u) du, \\
 &= \int_0^t e^{3u}e^{-(t-u)} du, \\
 &= \int_0^t e^{4u-t} du, \\
 &= \frac{1}{4}e^{4u-t}\Big|_0^t, \\
 &= \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}, \\
 &= \frac{1}{4}(e^{3t} - e^{-t}).
 \end{aligned}$$

21. The expression

$$\frac{s}{(s-1)(s^2+1)} = \frac{1}{s-1} \cdot \frac{s}{s^2+1} = F(s)G(s).$$

We have transform pairs

$$\begin{aligned}
 F(s) &= \frac{1}{s-1} \iff f(t) = e^t \\
 G(s) &= \frac{s}{s^2+1} \iff g(t) = \cos t.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2+1)}\right\}(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t) \\
 &= f * g(t) \\
 &= \int_0^t e^u \cos(t-u) du.
 \end{aligned}$$

Two applications of integration by parts can be used to show that

$$\int e^u \cos(t-u) du = \frac{1}{2}e^u [\cos(t-u) - \sin(t-u)].$$

Continuing,

$$\begin{aligned}
 \mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2+1)}\right\}(t) &= \frac{1}{2}e^u [\cos(t-u) - \sin(t-u)]\Big|_0^t \\
 &= \frac{1}{2}e^t - \left(\frac{1}{2}\cos t - \frac{1}{2}\sin t\right) \\
 &= \frac{1}{2}e^t - \frac{1}{2}\cos t + \frac{1}{2}\sin t.
 \end{aligned}$$

22. The expression

$$\begin{aligned}
 \frac{1}{(s+1)(s^2+4)} &= \frac{1}{s+1} \cdot \frac{1}{s^2+4}, \\
 &= F(s)G(s).
 \end{aligned}$$

We have transform pairs

$$\begin{aligned}
 F(s) &= \frac{1}{s+1} \iff f(t) = e^{-t}, \\
 G(s) &= \frac{1}{2} \cdot \frac{2}{s^2+4} \iff g(t) = \frac{1}{2}\sin 2t.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\} &= \mathcal{L}^{-1}\{F(s)G(s)\}(t), \\
 &= f * g(t), \\
 &= \int_0^t f(u)g(t-u) du, \\
 &= \frac{1}{2}\int_0^t e^{-u} \sin 2(t-u) du.
 \end{aligned}$$

Two integrations by parts show that

$$\begin{aligned}
 \int e^{-u} \sin(2t-2u) du &= \frac{1}{5}e^{-u} [2\cos(2t-2u) - \sin(2t-2u)].
 \end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\}(t) &= \frac{1}{10}e^{-u}[2\cos(2t-2u) - \sin(2t-2u)]\Big|_0^t \\ &= \frac{1}{5}e^{-t} + \frac{1}{10}\sin 2t - \frac{1}{5}\cos 2t.\end{aligned}$$

23. The expression

$$\frac{1}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1} = F(s)G(s).$$

We have transform pairs

$$F(s) = \frac{1}{s^2+1} \iff f(t) = \sin t$$

$$G(s) = \frac{1}{s^2+1} \iff g(t) = \sin t.$$

Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t) \\ &= f * g(t) \\ &= \int_0^t f(u)g(t-u) du \\ &= \int_0^t \sin u \sin(t-u) du.\end{aligned}$$

A trig product-to-sum identity is useful.

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)].$$

Continuing,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}(t) &= \frac{1}{2} \int_0^t (\cos(2u-t) - \cos t) du \\ &= \frac{1}{2} \left[\frac{1}{2} \sin(2u-t) - u \cos t \right] \Big|_0^t \\ &= \frac{1}{4} \sin(2u-t) - \frac{1}{2} u \cos t \Big|_0^t \\ &= \left[\frac{1}{4} \sin t - \frac{1}{2} t \cos t \right] - \left[\frac{1}{4} \sin(-t) \right] \\ &= \frac{1}{2} \sin t - \frac{1}{2} t \cos t.\end{aligned}$$

24. The expression

$$\begin{aligned}\frac{s^2}{(s^2+9)^2} &= \frac{s}{s^2+9} \cdot \frac{s}{s^2+9}, \\ &= F(s)^2.\end{aligned}$$

We have transform pair

$$F(s) = \frac{s}{s^2+9} \iff f(t) = \cos 3t.$$

Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+9)^2}\right\}(t) &= \mathcal{L}^{-1}\{F(s)^2\}(t), \\ &= f * f(t), \\ &= \int_0^t f(u)f(t-u) du, \\ &= \int_0^t \cos 3u \cos 3(t-u) du.\end{aligned}$$

Using the trig product to sum identity,

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)],$$

gives

$$\begin{aligned}&= \int_0^t \frac{1}{2} [\cos(6u-3t) + \cos(3t)] du, \\ &= \frac{1}{12} \sin(6u-3t) + \frac{1}{2} u \cos 3t \Big|_0^t, \\ &= \frac{1}{6} \sin 3t + \frac{1}{2} t \cos 3t.\end{aligned}$$

25. (a) The expression

$$\begin{aligned}\frac{s\omega}{(s^2+\omega_0^2)(s^2+\omega^2)} &= \frac{s}{s^2+\omega_0^2} \cdot \frac{\omega}{s^2+\omega^2} \\ &= F(s)G(s).\end{aligned}$$

We have transform pairs

$$F(s) = \frac{s}{s^2+\omega_0^2} \iff f(t) = \cos \omega_0 t$$

$$G(s) = \frac{\omega}{s^2+\omega^2} \iff g(t) = \sin \omega t.$$

Thus,

$$\begin{aligned}
 & \mathcal{L}^{-1} \left\{ \frac{s\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)} \right\} (t) \\
 &= \mathcal{L}^{-1} \{ F(s)G(s) \} (t) \\
 &= f * g(t) \\
 &= \int_0^t f(u)g(t-u) du \\
 &= \int_0^t \cos \omega_0 u \sin \omega(t-u) du \\
 &= \int_0^t \sin \omega u \cos \omega_0(t-u) du.
 \end{aligned}$$

The product-to-sum identity is useful.

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)].$$

Continuing,

$$\begin{aligned}
 & \mathcal{L}^{-1} \left\{ \frac{s\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)} \right\} (t) \\
 &= \frac{1}{2} \int_0^t \sin((\omega - \omega_0)u + \omega_0 t) du \\
 &\quad + \frac{1}{2} \int_0^t \sin((\omega + \omega_0)u - \omega_0 t) du
 \end{aligned}$$

If $\omega = \omega_0$, then this is equal to

$$\begin{aligned}
 &= \frac{1}{2} \int_0^t \sin \omega_0 t du \\
 &\quad + \frac{1}{2} \int_0^t \sin(2\omega_0 u - \omega_0 t) du \\
 &= \frac{1}{2} t \sin \omega_0 t - \frac{1}{4\omega_0} (\cos \omega_0 t - \cos(-\omega_0 t)) \\
 &= \frac{1}{2} t \sin \omega_0 t.
 \end{aligned}$$

(b) If $\omega \neq \omega_0$, then

$$\begin{aligned}
 & f * g(t) \\
 &= \frac{-1}{2(\omega - \omega_0)} \cos((\omega - \omega_0)u + \omega_0 t) \Big|_0^t \\
 &\quad - \frac{1}{2(\omega + \omega_0)} \cos((\omega + \omega_0)u - \omega_0 t) \Big|_0^t \\
 &= \frac{-1}{2(\omega - \omega_0)} [\cos \omega t - \cos \omega_0 t] \\
 &\quad - \frac{1}{2(\omega + \omega_0)} [\cos \omega t - \cos(-\omega_0 t)] \\
 &= -\frac{1}{2} \left[\frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right] \\
 &\quad \times (\cos \omega t - \cos \omega_0 t) \\
 &= -\frac{1}{2} \cdot \frac{2\omega}{\omega^2 - \omega_0^2} (\cos \omega t - \cos \omega_0 t) \\
 &= \frac{\omega}{\omega^2 - \omega_0^2} (\cos \omega_0 t - \cos \omega t).
 \end{aligned}$$

26. We start by computing the impulse response function $e(t)$.

$$y'' + 4y = \delta(t), \quad y(0) = y'(0) = 0$$

By Theorem 6.10,

$$E(s) = \frac{1}{P(s)} = \frac{1}{s^2 + 4}.$$

Thus,

$$\begin{aligned}
 e(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\} (t), \\
 &= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} (t), \\
 &= \frac{1}{2} \sin 2t.
 \end{aligned}$$

Computing the derivative we get

$$e'(t) = \cos 2t.$$

From Theorem 7.16,

$$y'' + 4y = g(t), \quad y(0) = y'(0) = 1$$

and

$$\begin{aligned}
 y(t) &= e * g(t) + ay_0 e'(t) \\
 &\quad + (ay_1 + by_0)e(t), \\
 &= e * g(t) + (1)(1)e'(t) \\
 &\quad + [(1)(1) + (0)(1)]e(t), \\
 &= e * g(t) + e'(t) + e(t), \\
 &= \int_0^t \frac{1}{2} \sin 2(t-u)g(u) du \\
 &\quad + \cos 2t + \frac{1}{2} \sin 2t.
 \end{aligned}$$

27. We start by computing the impulse response function $e(t)$.

$$e'' + 9e = \delta(t), \quad e(0) = e'(0) = 0.$$

By Theorem 6.10, the Laplace transform of $e(t)$ is

$$E(s) = \frac{1}{P(s)} = \frac{1}{s^2 + 9}.$$

Thus,

$$\begin{aligned}
 e(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 9} \right\}(t) \\
 &= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 9} \right\}(t) \\
 &= \frac{1}{3} \sin 3t.
 \end{aligned}$$

Compute the derivative.

$$e'(t) = \cos 3t$$

From Theorem 7.16, we know that the solution of

$$y'' + 9y = g(t), \quad y(0) = -1, y'(0) = 2$$

is

$$\begin{aligned}
 y(t) &= e * g(t) + ay_0 e'(t) \\
 &\quad + (ay_1 + by_0)e(t) \\
 &= e * g(t) + (1)(-1)e'(t) \\
 &\quad + ((1)(2) + (0)(-1))e(t) \\
 &= e * g(t) - e'(t) + 2e(t) \\
 &= \frac{1}{3} \int_0^t (\sin 3u)g(t-u) du \\
 &\quad - \cos 3t + \frac{2}{3} \sin 3t.
 \end{aligned}$$

28. We start by computing the impulse response function $e(t)$.

$$y'' + 5y' + 4y = \delta(t), \quad y(0) = y'(0) = 0$$

By Theorem 6.10,

$$\begin{aligned}
 E(s) &= \frac{1}{P(s)} = \frac{1}{s^2 + 5s + 4} = \frac{1}{(s+4)(s+1)} \\
 &= \frac{-1/3}{s+4} + \frac{1/3}{s+1}.
 \end{aligned}$$

Thus,

$$e(t) = -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t},$$

and the derivative is

$$e'(t) = \frac{4}{3}e^{-4t} - \frac{1}{3}e^{-t}.$$

From Theorem 7.16, the solution to

$$y'' + 5y' + 4y = g(t), \quad y(0) = 1, \quad y'(0) = 0$$

is

$$\begin{aligned}
 y(t) &= e * g(t) + ay_0 e'(t) \\
 &\quad + (ay_1 + by_0)e(t), \\
 &= e * g(t) + (1)(1)e'(t) \\
 &\quad + [(1)(0) + (5)(1)]e(t), \\
 &= e * g(t) + e'(t) + 5e(t), \\
 &= \int_0^t \left(-\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} \right) g(t-u) du \\
 &\quad + \frac{4}{3}e^{-4t} - \frac{1}{3}e^{-t} - \frac{5}{3}e^{-4t} + \frac{5}{3}e^{-t}, \\
 &= \int_0^t \left(-\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} \right) g(t-u) du \\
 &\quad - \frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t}.
 \end{aligned}$$

29. We start by computing the impulse response function $e(t)$.

$$e'' + 4e' + 3e = \delta(t), \quad e(0) = e'(0) = 0.$$

By Theorem 6.10, the Laplace transform of $e(t)$ is

$$\begin{aligned} E(s) &= \frac{1}{P(s)} \\ &= \frac{1}{s^2 + 4s + 3} \\ &= \frac{1}{(s+1)(s+3)} \\ &= \frac{1/2}{s+1} - \frac{1/2}{s+3}. \end{aligned}$$

Thus,

$$e(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

Compute the derivative.

$$e'(t) = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}$$

From Theorem 7.16, we know that the solution of

$$y'' + 4y' + 3y = g(t), \quad y(0) = -1, y'(0) = 1$$

is

$$\begin{aligned} y(t) &= e * g(t) + ay_0e'(t) + (ay_1 + by_0)e(t) \\ &= e * g(t) + (1)(-1)e'(t) \\ &\quad + ((1)(1) + (4)(-1))e(t) \\ &= e * g(t) - e'(t) - 3e(t) \\ &= \int_0^t \left(\frac{1}{2}e^{-u} - \frac{1}{2}e^{-3u} \right) g(t-u) du \\ &\quad + \frac{1}{2}e^{-t} - \frac{3}{2}e^{-3t} - \frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} \\ &= \int_0^t \left(\frac{1}{2}e^{-u} - \frac{1}{2}e^{-3u} \right) g(t-u) du \\ &\quad - e^{-t}. \end{aligned}$$

30. We start by computing the impulse response function $e(t)$.

$$y'' + 2y' + 5y = \delta(t) \quad y(0) = y'(0) = 0$$

By Theorem 6.10,

$$E(s) = \frac{1}{P(s)} = \frac{1}{s^2 + 2s + 5} = \frac{1}{(s+2)^2 + 1}.$$

Thus,

$$e(t) = e^{-2t} \sin t,$$

and

$$e'(t) = e^{-2t} \cos t - 2e^{-2t} \sin t.$$

From Theorem 7.16, the solution to

$$y'' + 2y' + 5y = g(t), \quad y(0) = 1, \quad y'(0) = -1,$$

is

$$\begin{aligned} y(t) &= e * g(t) + ay_0e'(t) + (ay_1 + by_0)e(t), \\ &= e * g(t) + (1)(1)e'(t) \\ &\quad + [(1)(-1) + (2)(1)]e(t), \\ &= e * g(t) + e'(t) + e(t), \\ &= \int_0^t e^{-2(t-u)} \sin(t-u) g(u) du \\ &\quad + e^{-2t} \cos t - 2e^{-2t} \sin t + e^{-2t} \sin t, \\ &= \int_0^t e^{-2(t-u)} \sin(t-u) g(u) du \\ &\quad + e^{-2t} \cos t - e^{-2t} \sin t. \end{aligned}$$

31. We start by computing the impulse response function $e(t)$.

$$e'' + 4e' + 29e = \delta(t), \quad e(0) = e'(0) = 0$$

by Theorem 6.10, the Laplace transform of $e(t)$ is

$$\begin{aligned} E(s) &= \frac{1}{P(s)} \\ &= \frac{1}{s^2 + 4s + 29} \\ &= \frac{1}{(s+2)^2 + 25} \\ &= \frac{1}{5} \cdot \frac{5}{(s+2)^2 + 25}. \end{aligned}$$

Thus,

$$e(t) = \frac{1}{5}e^{-2t} \sin 5t.$$

Differentiate.

$$\begin{aligned} e'(t) &= \frac{1}{5}e^{-2t}(5 \cos 5t) - \frac{2}{5}e^{-2t} \sin 5t \\ &= e^{-2t} \cos 5t - \frac{2}{5}e^{-2t} \sin 5t. \end{aligned}$$

From Theorem 7.16, we know that the solution of is

$$\begin{aligned}
 y(t) &= e * g(t) + ay_0 e'(t) + (ay_1 + by_0)e(t) \\
 &= e * g(t) + (1)(-1)e'(t) + ((1)(2) + (4)(-1))e(t) \\
 &= e * g(t) - e'(t) - 2e(t) \\
 &= e * g(t) - e^{-2t} \cos 5t + \frac{2}{5}e^{-2t} \sin 5t - \frac{2}{5}e^{-2t} \sin 5t \\
 &= \frac{1}{5} \int_0^t e^{-2u} (\sin 5u) g(t-u) du - e^{-2t} \cos 5t.
 \end{aligned}$$

$$y'' + 4y' + 29y = g(t), \quad y(0) = -1, y'(0) = 2$$

Chapter 6. Numerical Methods

Section 6.1. Euler's Method

1. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = t + y$. Thus, the first step of Euler's method is completed as follows.

$$y_1 = y_0 + (t_0 + y_0)h = 1 + (0 + 1) \times 0.1 = 1.1$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$y_2 = y_1 + (t_1 + y_1)h = 1.1 + (0.1 + 1.1) \times 0.1 = 1.22$$
$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	t_k	y_k	$f(t_k, y_k) = y_k$	h	$f(t_k, y_k)h$
0	0.0	1.0000	1.0000	0.1	0.1000
1	0.1	1.1000	1.2000	0.1	0.1200
2	0.2	1.2200	1.4200	0.1	0.1420
3	0.3	1.3620	1.6620	0.1	0.1662
4	0.4	1.5282	1.9282	0.1	0.1928
5	0.5	1.7210	2.2210	0.1	0.2221

2. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = y$. Thus, the first step of Euler's method is completed as follows.

$$y_1 = y_0 + y_0 h = 1 + 1 \times 0.1 = 1.1$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$y_2 = y_1 + y_1 h = 1.1 + 1.1 \times 0.1 = 1.21$$
$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	t_k	y_k	$f(t_k, y_k) = y_k$	h	$f(t_k, y_k)h$
0	0.0	1.0000	1.0000	0.1	0.1000
1	0.1	1.1000	1.1000	0.1	0.1100
2	0.2	1.2100	1.2100	0.1	0.1210
3	0.3	1.3310	1.3310	0.1	0.1331
4	0.4	1.4641	1.4641	0.1	0.1464
5	0.5	1.6105	1.6105	0.1	0.1611

3. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = ty$. Thus, the first step of Euler's method is completed as follows.

$$y_1 = y_0 + t_0 y_0 h = 1 + 0 \times 1 \times 0.1 = 1.0$$

$$t_1 = t_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$y_2 = y_1 + t_1 y_1 h = 1.0 + 0.1 \times 1 \times 0.1 = 1.01$$

$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	t_k	y_k	$f(t_k, y_k) = t_k y_k$	h	$f(t_k, y_k)h$
0	0.0	1.0000	0.0000	0.1	0.0000
1	0.1	1.0000	0.1000	0.1	0.0100
2	0.2	1.0100	0.2020	0.1	0.0202
3	0.3	1.0302	0.3091	0.1	0.0309
4	0.4	1.0611	0.4244	0.1	0.0424
5	0.5	1.1036	0.5518	0.1	0.0552

4. We have $x_0 = 0$, $z_0 = 0$ and $f(x, z) = 5 - z$. Thus, the first step of Euler's method is completed as follows.

$$z_1 = z_0 + (5 - z_0)h = 0 + (5 - 0) \times 0.1 = 0.5$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$z_2 = z_1 + (5 - z_1)h = 0.5 + (5 - 0.5) \times 0.1 = 0.95$$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	x_k	z_k	$f(x_k, z_k) = 5 - z_k$	h	$f(x_k, z_k)h$
0	0.0	0.0000	5.0000	0.1	0.5000
1	0.1	0.5000	4.5000	0.1	0.4500
2	0.2	0.9500	4.0500	0.1	0.4050
3	0.3	1.3550	3.6450	0.1	0.3645
4	0.4	1.7195	3.2805	0.1	0.3281
5	0.5	2.0476	2.9524	0.1	0.2952

5. We have $x_0 = 0$, $z_0 = 1$ and $f(x, z) = x - 2z$. Thus, the first step of Euler's method is completed as follows.

$$z_1 = z_0 + (x_0 - 2z_0)h = 1 + (0 - 2 \times 1) \times 0.1 = 0.8$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$z_2 = z_1 + (x_1 - 2z_1)h = 0.8 + (0.1 - 2 \times 0.8) \times 0.1 = 0.65$$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	x_k	z_k	$f(x_k, z_k) = x_k - 2z_k$	h	$f(x_k, z_k)h$
0	0.0	1.0000	-2.0000	0.1	-0.2000
1	0.1	0.8000	-1.5000	0.1	-0.1500
2	0.2	0.6500	-1.1000	0.1	-0.1100
3	0.3	0.5400	-0.7800	0.1	-0.0780
4	0.4	0.4620	-0.5240	0.1	-0.0524
5	0.5	0.4096	-0.3192	0.1	-0.0319

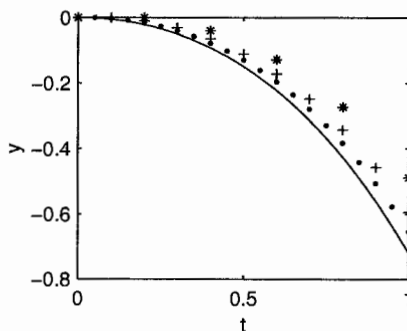
6. An integrating factor for $y' = y - t$ is e^{-t} .

$$\begin{aligned}y' - y &= -t \\e^{-t}y' - e^{-t}y &= -te^{-t} \\(e^{-t}y)' &= -te^{-t}\end{aligned}$$

Integrating by parts,

$$\begin{aligned}e^{-t}y &= te^{-t} + e^{-t} + C \\y &= t + 1 + Ce^t.\end{aligned}$$

The initial condition $y(0) = 0$ produces $C = -1$ and $y = t + 1 - e^t$. In the figure, three numerical solutions and the exact solution are pictured. The numerical solutions were calculated using Euler's method and step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$.



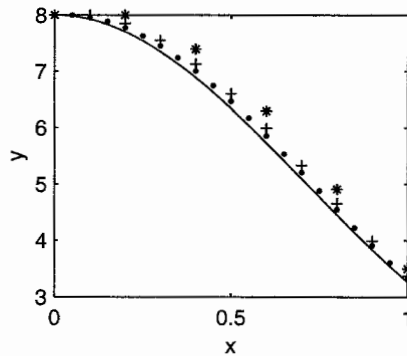
7. An integrating factor for $y' + 2xy = x$ is e^{x^2} .

$$\begin{aligned}e^{x^2}y' + 2xe^{x^2}y &= xe^{x^2} \\(e^{x^2}y)' &= xe^{x^2}\end{aligned}$$

Integrate.

$$\begin{aligned}e^{x^2}y &= \frac{1}{2}e^{x^2} + C \\y &= \frac{1}{2} + Ce^{-x^2}\end{aligned}$$

The initial condition $y(0) = 8$ produces $C = 15/2$ and $y = 1/2 + (15/2)e^{-x^2}$. In the figure, three numerical solutions and the exact solution are pictured. The numerical solutions were calculated using Euler's method and step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$.



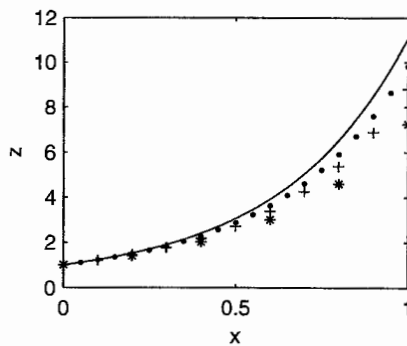
8. An integrating factor for $z - 2z = xe^{2x}$ is e^{-2x} .

$$\begin{aligned} e^{-2x} z' - 2e^{-2x} z &= x \\ (e^{-2x} z)' &= x \end{aligned}$$

Integrate.

$$\begin{aligned} e^{-2x} z &= \frac{1}{2} x^2 + c \\ z &= \frac{1}{2} x^2 e^{2x} + C e^{2x} \end{aligned}$$

The initial condition $z(0) = 1$ produces $C = 1$ and $z = (1/2)x^2 e^{2x} + e^{2x}$. In the figure, three numerical solutions and the exact solution are pictured. The numerical solutions were calculated using Euler's method and step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$.



9. The equation $z' = (1 + t)z$ is separable.

$$\frac{dz}{z} = (1 + t) dt$$

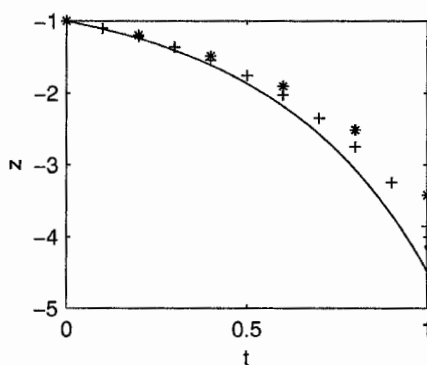
$$\ln |z| = t + \frac{1}{2}t^2 + C$$

The initial condition $z(0) = -1$ produces $C = 0$ and $\ln |z| = t + (1/2)t^2$. Of course, we choose the negative branch.

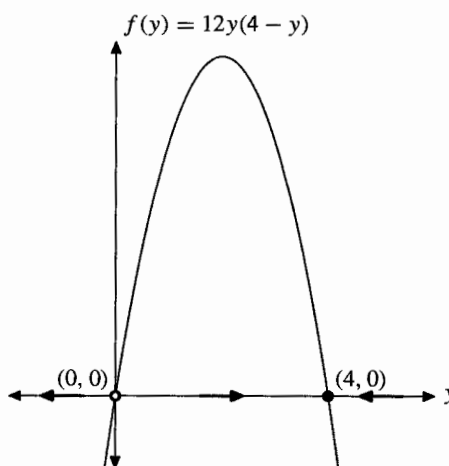
$$|z| = e^{t+(1/2)t^2}$$

$$z = -e^{t+(1/2)t^2}$$

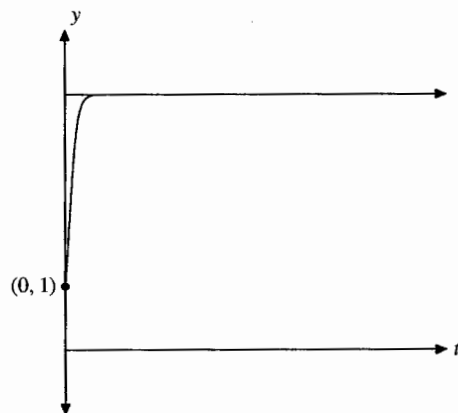
In the figure, three numerical solutions and the exact solution are pictured. The numerical solutions were calculated using Euler's method and step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$.



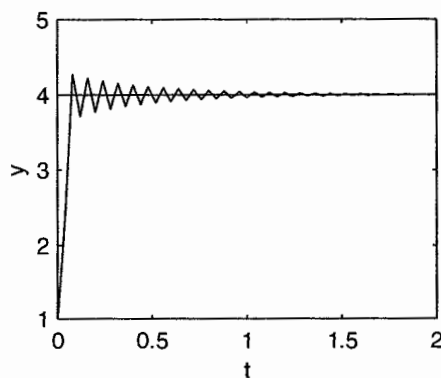
10. (a) Phase plane analysis shows a stable equilibrium point at $(4, 0)$.



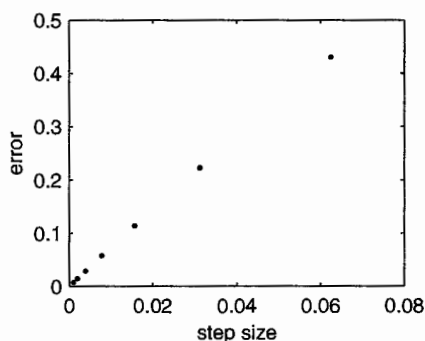
Thus, a solution starting at $(0, 1)$ should approach the stable solution $y = 4$.



- (b) The following figure displays a numerical solution of $y' = 12y(4 - y)$, $y(0) = 1$, using Euler's method with step size $h = 0.04$. Note the obvious contradiction of uniqueness as the numerical solution repeatedly crosses the stable solution $y = 4$.



11. (a) Note that a plot of the error versus the step size signifies a linear relationship. Indeed, a line through the origin with the appropriate slope should pass through or close to each data point.



- (b) We can estimate the proportionality constant by picking two points from the figure and calculating the slope of the line through the chosen points. Let's use the first and last points.

$$\lambda = \frac{0.0072076631 - 0.4303893417}{0.0009765625 - 0.0625000000} \approx 6.8784$$

We can use $E = \lambda h$ to calculate the step size.

$$\begin{aligned} E &= \lambda h \\ h &= \frac{E}{\lambda} \\ h &= \frac{0.001}{6.8784} \\ h &= 1.454 \times 10^{-4} \end{aligned}$$

Since $h = (b - a)/N$,

$$\begin{aligned} N &= \frac{b - a}{h} \\ N &= \frac{2 - 0}{1.454 \times 10^{-4}} \\ N &= 13757. \end{aligned}$$

It will take about 13,757 iterations to achieve the required accuracy.

- (c) We ran our Euler routine on a 300 MHz PC using the step size from part (b). The run took approximately 56 seconds and reported an error from the true value of 0.00107417614304.

12. A integrating factor for $y' = t - y$ is e^t . Thus,

$$\begin{aligned} e^t y' + e^t y &= t \\ (e^t y)' &= t e^t. \end{aligned}$$

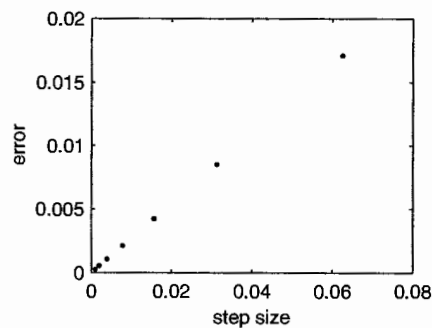
Integrate, using integration by parts.

$$\begin{aligned} e^t y &= t e^t - e^t + C \\ y &= t - 1 + C e^{-t} \end{aligned}$$

The initial condition $y(0) = 1$ provides $C = 2$ and the exact solution $y = t - 1 + 2e^{-t}$. Thus, $y(2) \approx 1.27067$. This value, plus our Euler routine, generated the data in the following table.

Step size h	Euler approx.	True value	Error E_h
0.06250	1.25358	1.27067	0.01709
0.03125	1.26217	1.27067	0.00850
0.01563	1.26643	1.27067	0.00424
0.00781	1.26855	1.27067	0.00212
0.00391	1.26961	1.27067	0.00106
0.00195	1.27014	1.27067	0.00053
0.00098	1.27041	1.27067	0.00026

A plot of the error versus the step size reveals a linear relationship.



Choose two points and use the slope formula to estimate λ .

$$\lambda \approx \frac{0.00026 - 0.01709}{0.00098 - 0.06250}$$

$$\lambda \approx 0.27357$$

Calculate the step size required to maintain error less than 0.01 with the calculation

$$h = \frac{E}{\lambda}$$

$$h \approx \frac{0.01}{0.27357}$$

$$h \approx 0.03655.$$

Calculate the number of iterations with

$$N = \frac{b - a}{h}$$

$$N \approx \frac{2 - 0}{0.03655}$$

$$N \approx 55.$$

Using a step size $h \approx 0.03655$, our Euler routine on a 300 MHz PC produced an error of 0.00991658799458 in about 0.05 seconds.

13. A integrating factor for $y' = y + \sin t$ is e^{-t} . Thus,

$$\begin{aligned} e^{-t}y' - e^{-t}y &= e^{-t} \sin t \\ (e^{-t}y)' &= e^{-t} \sin t. \end{aligned}$$

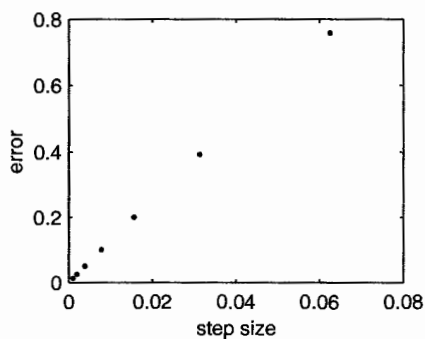
Integrate, using integration by parts.

$$\begin{aligned} e^{-t}y &= -\frac{1}{2}e^{-t} \sin t - \frac{1}{2}e^{-t} \cos t + C \\ y &= -\frac{1}{2} \sin t - \frac{1}{2} \cos t + Ce^t \end{aligned}$$

The initial condition $y(0) = 1$ provides $C = 3/2$ and the exact solution $y = -(1/2) \sin t - (1/2) \cos t + (3/2)e^t$. Thus, $y(2) \approx 10.83701$. This value, plus our Euler routine, generated the data in the following table.

Step size h	Euler approx.	True value	Error E_h
0.06250	10.07789	10.83701	0.75912
0.03125	10.44404	10.83701	0.39297
0.01563	10.63700	10.83701	0.20001
0.00781	10.73610	10.83701	0.10091
0.00391	10.78632	10.83701	0.05069
0.00195	10.81161	10.83701	0.02540
0.00098	10.82429	10.83701	0.01271

A plot of the error versus the step size reveals a linear relationship.



Choose two points and use the slope formula to estimate λ .

$$\begin{aligned} \lambda &\approx \frac{0.01271 - 0.75912}{0.00098 - 0.06250} \\ \lambda &\approx 12.1321 \end{aligned}$$

Calculate the step size required to maintain error less than 0.01 with the calculation

$$h = \frac{E}{\lambda}$$

$$h \approx \frac{0.01}{12.1321}$$

$$h \approx 8.2426 \times 10^{-4}.$$

Calculate the number of iterations with

$$N = \frac{b-a}{h}$$

$$N \approx \frac{2-0}{8.2426 \times 10^{-4}}$$

$$N \approx 2426.$$

Using a step size $h \approx 8.2426 \times 10^{-4}$, our Euler routine running on a 300 MHz PC produced an error of 0.01073284032773 in about 3.95 seconds.

14. The equation $y' = ty$ is separable. Thus,

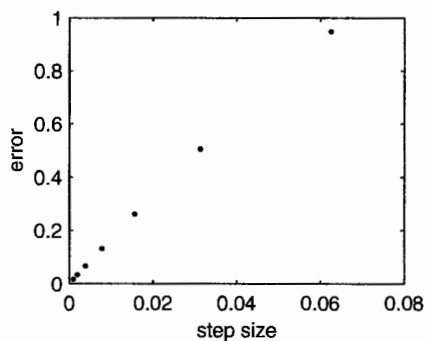
$$\frac{dy}{y} = t \, dt$$

$$\ln y = \frac{1}{2}t^2 + C.$$

The initial condition $y(0) = 1$ provides $C = 0$ and the exact solution $y = e^{(1/2)t^2}$. Thus, $y(2) \approx 7.38906$. This value, plus our Euler routine, generated the data in the following table.

Step size h	Euler approx.	True value	Error E_h
0.06250	6.44037	7.38906	0.94868
0.03125	6.88449	7.38906	0.50456
0.01563	7.12849	7.38906	0.26056
0.00781	7.25660	7.38906	0.13245
0.00391	7.32227	7.38906	0.06678
0.00195	7.35552	7.38906	0.03353
0.00098	7.37225	7.38906	0.01680

A plot of the error versus the step size reveals a linear relationship.



Choose two points and use the slope formula to estimate λ .

$$\lambda \approx \frac{0.01680 - 0.94868}{0.00098 - 0.06250}$$

$$\lambda \approx 15.1467$$

Calculate the step size required to maintain error less than 0.01 with the calculation

$$h = \frac{E}{\lambda}$$

$$h \approx \frac{0.01}{15.1467}$$

$$h \approx 6.60207 \times 10^{-4}.$$

Calculate the number of iterations with

$$N = \frac{b - a}{h}$$

$$N \approx \frac{2 - 0}{6.60207 \times 10^{-4}}$$

$$N \approx 3029.$$

Using a step size $h \approx 6.60207 \times 10^{-4}$, our Euler routine running on a 300 MHz PC produced an error of 0.01136462652632 in about 5.39 seconds.

15. The equation $y' = -(1/3)y^2$ is separable. Thus,

$$\frac{dy}{y^2} = -\frac{1}{3} dt$$

$$-y^{-1} = -\frac{1}{3}t + C$$

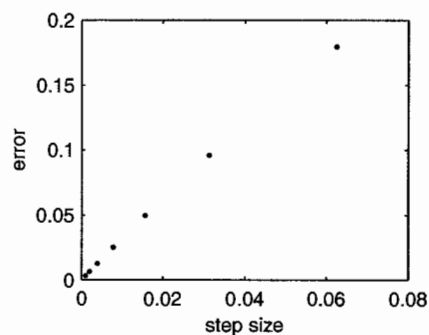
$$y = \frac{1}{(1/3)t + C}$$

$$y = \frac{3}{t + 3C}.$$

The initial condition $y(0) = 1$ provides $C = -1$ and the exact solution $y = 3/(t - 3)$. Thus, $y(2) = -3$. This value, plus our Euler routine, generated the data in the following table.

Step size h	Euler approx.	True value	Error E_h
0.06250	-2.82040	-3.00000	0.17960
0.03125	-2.90412	-3.00000	0.09588
0.01563	-2.95035	-3.00000	0.04965
0.00781	-2.97472	-3.00000	0.02528
0.00391	-2.98725	-3.00000	0.01275
0.00195	-2.99359	-3.00000	0.00641
0.00098	-2.99679	-3.00000	0.00321

A plot of the error versus the step size reveals a linear relationship.



Choose two points and use the slope formula to estimate λ .

$$\lambda \approx \frac{0.00321 - 0.17960}{0.00098 - 0.06250}$$

$$\lambda \approx 2.8670$$

Calculate the step size required to maintain error less than 0.01 with the calculation

$$h = \frac{E}{\lambda}$$

$$h \approx \frac{0.01}{2.8670}$$

$$h \approx 0.003487.$$

Calculate the number of iterations with

$$N = \frac{b - a}{h}$$

$$N \approx \frac{2 - 0}{0.003487}$$

$$N \approx 573.$$

Using a step size $h \approx 0.003487$, our Euler routine on a 300 MHz PC produced an error of 0.01139175100243 in about 0.66 seconds.

16. Because $E_h \approx \lambda h$, halving the step size should halve the error. A simple calculation will show this.

$$E \approx \lambda \left(\frac{1}{2} h \right) \approx \frac{1}{2} (\lambda h) \approx \frac{1}{2} E_h$$

Because the number of iterations with step size h is $N_h = (b - a)/h$, halving the step size should double the number of iterations. A simple calculation shows this.

$$N \approx \frac{b-a}{\frac{1}{2}h} \approx 2 \left(\frac{b-a}{h} \right) \approx 2N_h$$

17. We have $t_0 = 0$, $x_0 = 1$, and $y_0 = 0$. We also have $f(t, x, y) = y$ and $g(t, x, y) = -x$. The first step of Euler's method is completed as follows.

$$\begin{aligned} x_1 &= x_0 + y_0 h = 1 + 0 \times 0.1 = 1 \\ y_1 &= y_0 - x_0 h = 0 - 1 \times 0.1 = -0.1 \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1 \end{aligned}$$

We iterate a second time as follows.

$$\begin{aligned} x_2 &= x_1 + y_1 h = 1 - 1 \times 0.1 = 0.99 \\ y_2 &= y_1 - x_1 h = -0.1 - 1 \times 0.1 = -0.2 \\ t_2 &= t_1 + h = 0.1 + 0.1 = 0.2 \end{aligned}$$

Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = y_k h$	$g(t_k, x_k, y_k)h = -x_k h$
0.0	1.0000	0.0000	0.0000	-0.1000
0.1	1.0000	-0.1000	-0.0100	-0.1000
0.2	0.9900	-0.2000	-0.0200	-0.0990
0.3	0.9700	-0.2990	-0.0299	-0.0970
0.4	0.9401	-0.3960	-0.0396	-0.0940
0.5	0.9005	-0.4900	-0.0490	-0.0901

18. We have $t_0 = 0$, $x_0 = 1$, and $y_0 = -1$. We also have $f(t, x, y) = y$ and $g(t, x, y) = -x$. The first step of Euler's method is completed as follows.

$$\begin{aligned} x_1 &= x_0 + y_0 h = 1 - 1 \times 0.1 = 0.9 \\ y_1 &= y_0 - x_0 h = -1 - 1 \times 0.1 = -1.1 \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1 \end{aligned}$$

We iterate a second time as follows.

$$\begin{aligned} x_2 &= x_1 + y_1 h = 0.9 - 1.1 \times 0.1 = 0.79 \\ y_2 &= y_1 - x_1 h = -1.1 - 0.9 \times 0.1 = -1.19 \\ t_2 &= t_1 + h = 0.1 + 0.1 = 0.2 \end{aligned}$$

Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = y_k h$	$g(t_k, x_k, y_k)h = -x_k h$
0.0	1.0000	-1.0000	-0.1000	-0.1000
0.1	0.9000	-1.1000	-0.1100	-0.0900
0.2	0.7900	-1.1900	-0.1190	-0.0790
0.3	0.6710	-1.2690	-0.1269	-0.0671
0.4	0.5441	-1.3361	-0.1336	-0.0544
0.5	0.4105	-1.3905	-0.1391	-0.0410

19. We have $t_0 = 0$, $x_0 = 0$, and $y_0 = -1$. We also have $f(t, x, y) = -2y$ and $g(t, x, y) = x$. The first step of Euler's method is completed as follows.

$$\begin{aligned}x_1 &= x_0 - 2y_0h = 0 - 2 \times -1 \times 0.1 = 0.2 \\y_1 &= y_0 + x_0h = -1 + 0 \times 0.1 = -1 \\t_1 &= t_0 + h = 0 + 0.1 = 0.1\end{aligned}$$

We iterate a second time as follows.

$$\begin{aligned}x_2 &= x_1 - 2y_1h = 0.2 - 2 \times -1 \times 0.1 = 0.4 \\y_2 &= y_1 + x_1h = -1 + 0.2 \times 0.1 = -0.98 \\t_2 &= t_1 + h = 0.1 + 0.1 = 0.2\end{aligned}$$

Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = -2y_k h$	$g(t_k, x_k, y_k)h = x_k h$
0.0	0.0000	-1.0000	-0.1000	0.0000
0.1	0.2000	-1.0000	-0.1000	-0.0200
0.2	0.4000	-0.9800	-0.0980	-0.0400
0.3	0.5960	-0.9400	-0.0940	-0.0596
0.4	0.7840	-0.8804	-0.0880	-0.0784
0.5	0.9601	-0.8020	-0.0802	-0.0960

20. We have $t_0 = 0$, $x_0 = 1$, and $y_0 = 1$. We also have $f(t, x, y) = x + y$ and $g(t, x, y) = x - y$. The first step of Euler's method is completed as follows.

$$\begin{aligned}x_1 &= x_0 + (x_0 + y_0)h = 1 + (1 + 1) \times 0.1 = 1.2 \\y_1 &= y_0 + (x_0 - y_0)h = 1 + (1 - 1) \times 0.1 = 1 \\t_1 &= t_0 + h = 0 + 0.1 = 0.1\end{aligned}$$

We iterate a second time as follows.

$$\begin{aligned}x_2 &= x_1 + (x_1 + y_1)h = 1.2 + (1.2 + 1) \times 0.1 = 1.42 \\y_2 &= y_1 + (x_1 - y_1)h = 1 + (1.2 - 1) \times 0.1 = 1.02 \\t_2 &= t_1 + h = 0.1 + 0.1 = 0.2\end{aligned}$$

Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = (x_k + y_k)h$	$g(t_k, x_k, y_k)h = (x_k - y_k)h$
0.0	1.0000	1.0000	0.2000	0.0000
0.1	1.2000	1.0000	0.2200	0.0200
0.2	1.4200	1.0200	0.2440	0.0400
0.3	1.6640	1.0600	0.2724	0.0604
0.4	1.9364	1.1204	0.3057	0.0816
0.5	2.2421	1.2020	0.3444	0.1040

21. We have $t_0 = 0$, $x_0 = 1$, and $y_0 = -1$. We also have $f(t, x, y) = -y$ and $g(t, x, y) = x + y$. The first step of Euler's method is completed as follows.

$$x_1 = x_0 - y_0 h = 1 + 1 \times 0.1 = 1.1$$

$$y_1 = y_0 + (x_0 + y_0)h = -1 + (1 - 1) \times 0.1 = -1$$

$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

We iterate a second time as follows.

$$x_2 = x_1 - y_1 h = 1.1 + 1 \times 0.1 = 1.2$$

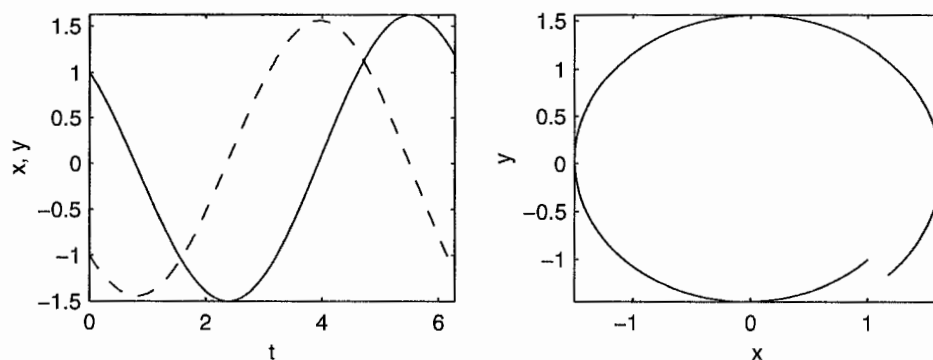
$$y_2 = y_1 + (x_1 + y_1)h = -1 + (1.1 - 1) \times 0.1 = -0.99$$

$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2$$

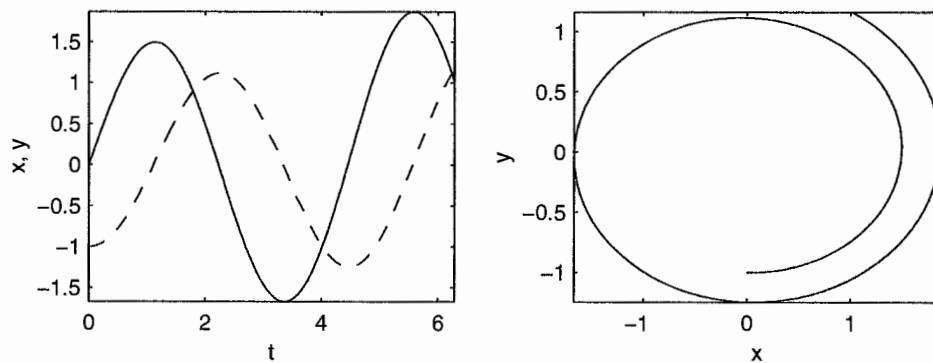
Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = -y_k h$	$g(t_k, x_k, y_k)h = (x_k + y_k)h$
0.0	1.0000	-1.0000	0.1000	0.0000
0.1	1.1000	-1.0000	0.1000	0.0100
0.2	1.2000	-0.9900	0.0990	0.0210
0.3	1.2990	-0.9690	0.0969	0.0330
0.4	1.3959	-0.9360	0.0936	0.0460
0.5	1.4895	-0.8900	0.0890	0.0599

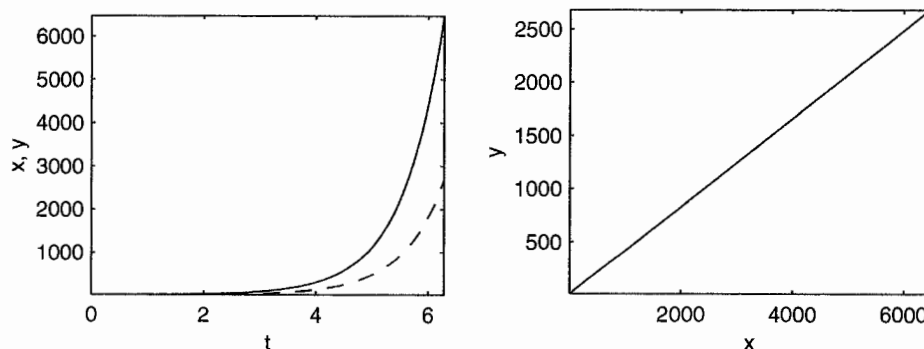
22. Pictured below are two plots. The first shows the plots of x versus t (solid line) and y versus t (dashed line) on the time interval $[0, 2\pi]$. The second shows the plot of y versus x . The plots were generated with Euler's method, using a step size $h = 0.05$.



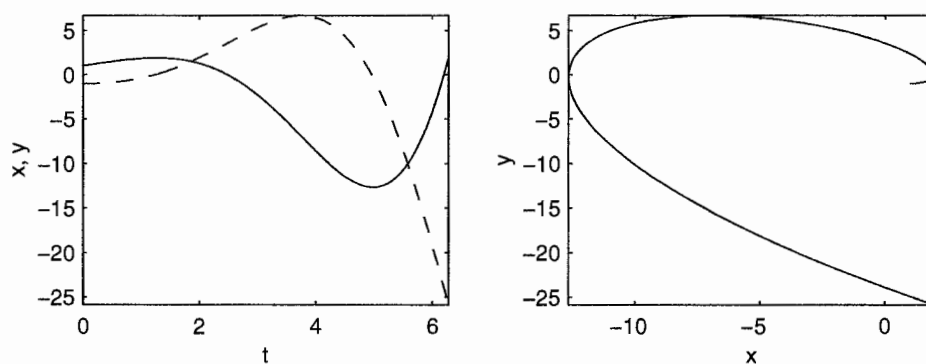
23. Pictured below are two plots. The first shows the plots of x versus t (solid line) and y versus t (dashed line) on the time interval $[0, 2\pi]$. The second shows the plot of y versus x . The plots were generated with Euler's method, using a step size $h = 0.05$.



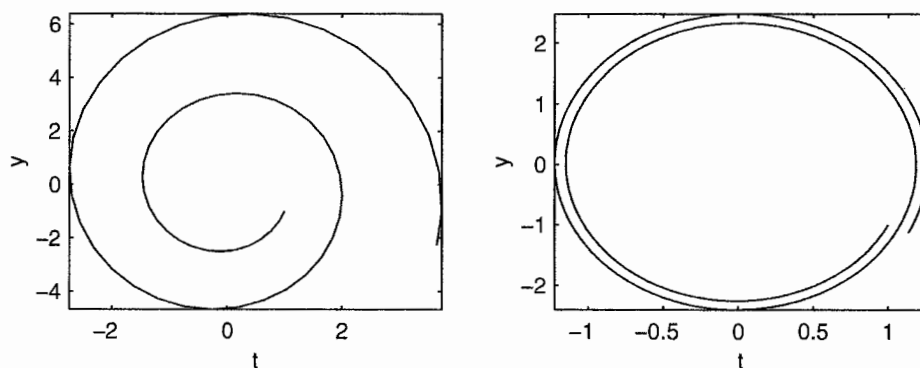
24. Pictured below are two plots. The first shows the plots of x versus t (solid line) and y versus t (dashed line) on the time interval $[0, 2\pi]$. The second shows the plot of y versus x . The plots were generated with Euler's method, using a step size $h = 0.05$.



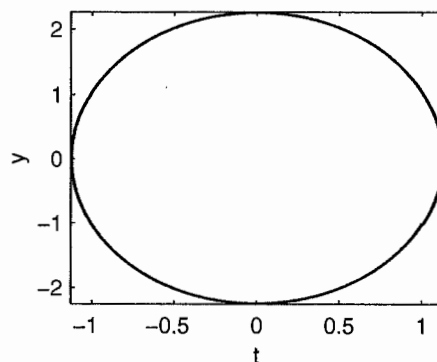
25. Pictured below are two plots. The first shows the plots of x versus t (solid line) and y versus t (dashed line) on the time interval $[0, 2\pi]$. The second shows the plot of y versus x . The plots were generated with Euler's method, using a step size $h = 0.05$.



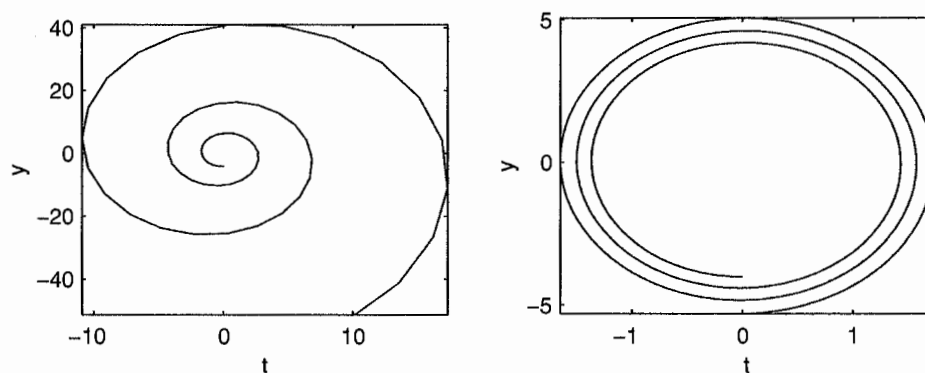
26. The first plot uses step size $h = 0.1$, the second uses $h = 0.01$



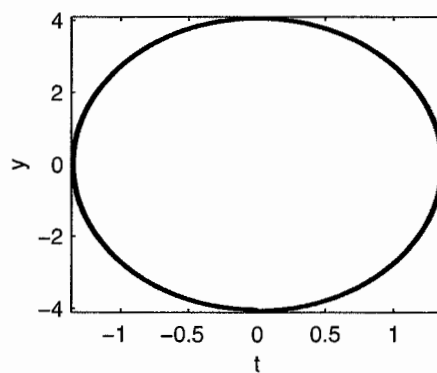
The third plot uses step size $h = 0.001$.



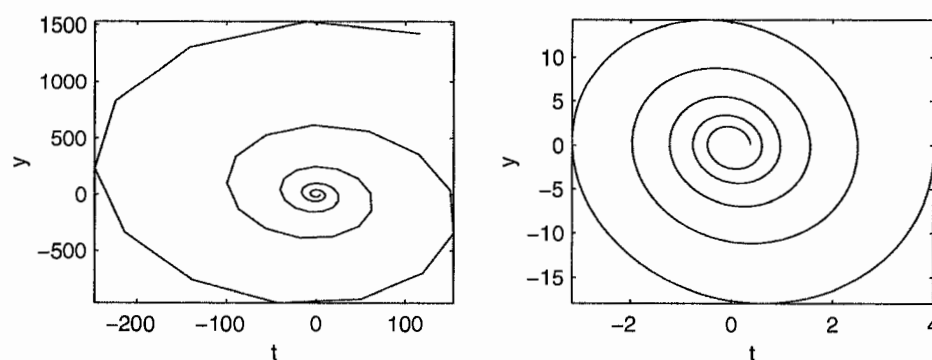
27. The first plot uses step size $h = 0.1$, the second uses $h = 0.01$



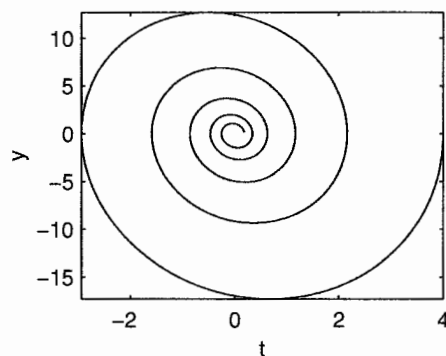
The third plot uses step size $h = 0.001$.



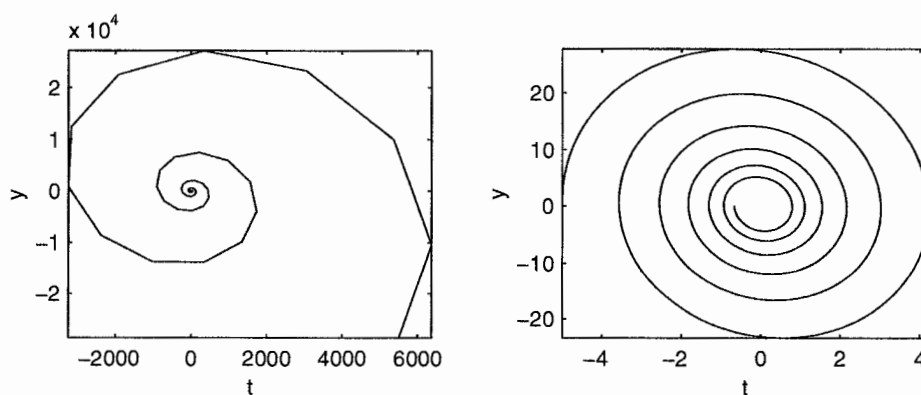
28. The first plot uses step size $h = 0.1$, the second uses $h = 0.01$



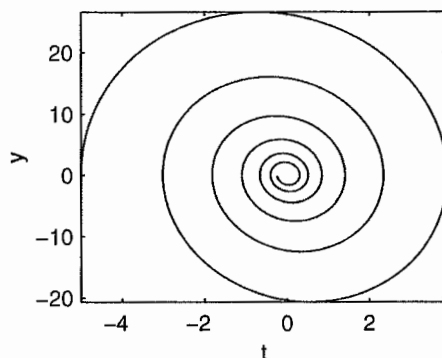
The third plot uses step size $h = 0.001$.



29. The first plot uses step size $h = 0.1$, the second uses $h = 0.01$



The third plot uses step size $h = 0.001$.



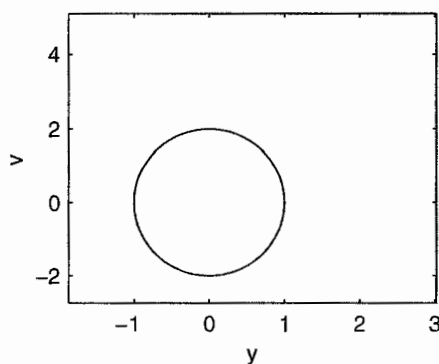
30. (a) Substitute $y(t) = \cos 2t$ and $v(t) = -2 \sin 2t$ into the first equation, $y' = v$.

$$\begin{aligned} y' &= v \\ (\cos 2t)' &= -2 \sin 2t \\ -2 \sin 2t &= -2 \sin 2t \end{aligned}$$

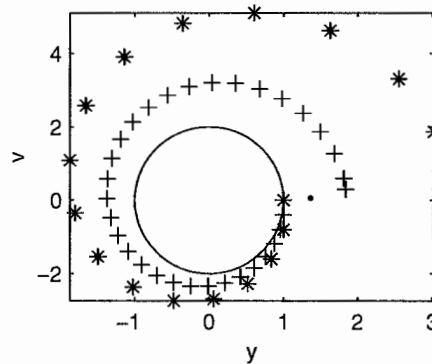
Thus, the first equation is satisfied. Substitute the proposed solutions into the second equation.

$$\begin{aligned} v' &= -4y \\ (-2 \sin 2t)' &= -4 \cos 2t \\ -4 \cos 2t &= -4 \cos 2t \end{aligned}$$

The second equation is also satisfied. Therefore, $y(t) = \cos 2t$, $v(t) = -2 \sin 2t$ is a solution of the system. A plot of v versus y on the interval $[0, \pi]$ follows.



- (b) In the following plot, we've superimposed Euler solutions on the exact solution from part (a). Note the improvement in the numerical solution as the step size decreases from $h = 0.2$ to $h = 0.1$ to $h = 0.05$.



×

Section 6.2. Runge-Kutta Methods

1. We start with initial condition $t_0 = 0$ and $y_0 = 1$. Also, $f(t, y) = t + y$. The first step of the Runge-Kutta 2 algorithm follows. First we compute the slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 0 + 1 = 1$$

$$s_2 = f(t_0 + h, y_0 + hs_1) = f(0.1, 1.1) = 0.1 + 1.1 = 1.2$$

You can now update y and t .

$$y_1 = y_0 + h \frac{s_1 + s_2}{2} = 1 + 0.1 \frac{1 + 1.2}{2} = 1.1$$

$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

The second iteration begins with computing the slopes.

$$s_1 = f(t_1, y_1) = f(0.1, 1.1) = 0.1 + 1.1 = 1.21$$

$$s_2 = f(t_1 + h, y_1 + hs_1) = f(0.2, 1.231) = 0.2 + 1.231 = 1.431$$

You can now update y and t .

$$y_2 = y_1 + h \frac{s_1 + s_2}{2} = 1.1 + 0.1 \frac{1.21 + 1.431}{2} = 1.24205$$

$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2$$

Continuing in this manner, we can complete the table.

k	t_k	y_k	s_1	s_2	h	$h(s_1 + s_2)/2$
0	0.0	1.0000	1.0000	1.2000	0.1	0.1100
1	0.1	1.1100	1.2100	1.4310	0.1	0.1321
2	0.2	1.2421	1.4421	1.6863	0.1	0.1564
3	0.3	1.3985	1.6985	1.9683	0.1	0.1833
4	0.4	1.5818	1.9818	2.2800	0.1	0.2131
5	0.5	1.7949	2.2949	2.6244	0.1	0.2460

2. We begin with $t_0 = 0$, $y_0 = 1$ and $f(t, y) = y$. First, the slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 1$$

$$s_2 = f(t_0 + h, y_0 + hs_1) = f(0.1, 1.1) = 1.1$$

Update t and y .

$$y_1 = y_0 + h \frac{s_1 + s_2}{2} = 1 + 0.1 \frac{1 + 1.1}{2} = 1.105$$

$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner, we arrive at the following table.

k	t_k	y_k	s_1	s_2	h	$h(s_1 + s_2)/2$
0	0.0	1.0000	1.0000	1.1000	0.1	0.1050
1	0.1	1.1050	1.1050	1.2155	0.1	0.1160
2	0.2	1.2210	1.2210	1.3431	0.1	0.1282
3	0.3	1.3492	1.3492	1.4842	0.1	0.1417
4	0.4	1.4909	1.4909	1.6400	0.1	0.1565
5	0.5	1.6474	1.6474	1.8122	0.1	0.1730

3. We begin with $t_0 = 0$, $y_0 = 1$ and $f(t, y) = ty$. First, the slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 0 \times 1 = 0$$

$$s_2 = f(t_0 + h, y_0 + hs_1) = f(0.1, 1) = 0.1 \times 1 = 0.1$$

Update t and y .

$$y_1 = y_0 + h \frac{s_1 + s_2}{2} = 1 + 0.1 \frac{0 + 0.1}{2} = 1.005$$

$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner, we arrive at the following table.

k	t_k	y_k	s_1	s_2	h	$h(s_1 + s_2)/2$
0	0.0	1.0000	0.0000	0.1000	0.1	0.0050
1	0.1	1.0050	0.1005	0.2030	0.1	0.0152
2	0.2	1.0202	0.2040	0.3122	0.1	0.0258
3	0.3	1.0460	0.3138	0.4309	0.1	0.0372
4	0.4	1.0832	0.4333	0.5633	0.1	0.0498
5	0.5	1.1331	0.5665	0.7138	0.1	0.0640

4. We begin with $x_0 = 0$, $z_0 = 0$ and $f(x, z) = 5 - z$. First, the slopes.

$$s_1 = f(x_0, z_0) = f(0, 0) = 5 - 0 = 5$$

$$s_2 = f(x_0 + h, z_0 + hs_1) = f(0.1, 0.5) = 5 - 0.5 = 4.5$$

Update x and z .

$$z_1 = z_0 + h \frac{s_1 + s_2}{2} = 0 + 0.1 \frac{5 + 4.5}{2} = 0.475$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner, we arrive at the following table.

k	x_k	z_k	s_1	s_2	h	$h(s_1 + s_2)/2$
0	0.0	0.0000	5.0000	4.5000	0.1	0.4750
1	0.1	0.4750	4.5250	4.0725	0.1	0.4299
2	0.2	0.9049	4.0951	3.6856	0.1	0.3890
3	0.3	1.2939	3.7061	3.3355	0.1	0.3521
4	0.4	1.6460	3.3540	3.0186	0.1	0.3186
5	0.5	1.9646	3.0354	2.7318	0.1	0.2884

5. We begin with $x_0 = 0$, $z_0 = 1$ and $f(x, z) = x - 2z$. First, the slopes.

$$s_1 = f(x_0, z_0) = f(0, 1) = 0 - 2(1) = -2$$

$$s_2 = f(x_0 + h, z_0 + hs_1) = f(0.1, 0.8) = 0.1 - 2(0.8) = -1.5$$

Update x and z .

$$z_1 = z_0 + h \frac{s_1 + s_2}{2} = 1 + 0.1 \frac{-2 - 1.5}{2} = 0.825$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner, we arrive at the following table.

k	x_k	z_k	s_1	s_2	h	$h(s_1 + s_2)/2$
0	0.0	1.0000	-2.0000	-1.5000	0.1	-0.1750
1	0.1	0.8250	-1.5500	-1.1400	0.1	-0.1345
2	0.2	0.6905	-1.1810	-0.8448	0.1	-0.1013
3	0.3	0.5892	-0.8784	-0.6027	0.1	-0.0741
4	0.4	0.5152	-0.6303	-0.4042	0.1	-0.0517
5	0.5	0.4634	-0.4268	-0.2415	0.1	-0.0334

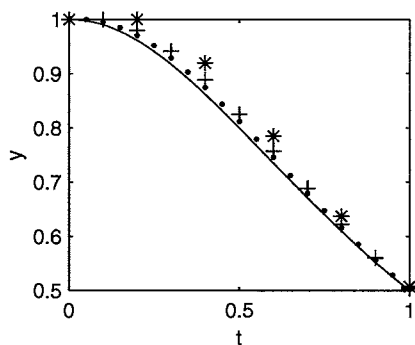
6. The equation is separable.

$$\frac{dy}{y^2} = -2t \, dt$$

$$-y^{-1} = -t^2 + C$$

$$y = \frac{1}{t^2 + C}$$

The initial condition $y(0) = 1$ provides $C = -1$ and the exact solution $y = 1/(t^2 + 1)$. In the image that follows, the exact solution is plotted on the interval $[0, 1]$. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.



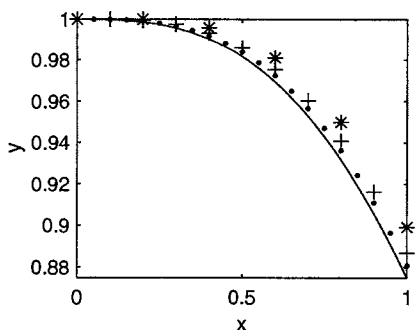
7. An integrating factor is e^x .

$$\begin{aligned} e^x z' + e^x z &= e^x \cos x \\ (e^x z)' &= e^x \cos x \end{aligned}$$

Integration is by parts.

$$\begin{aligned} e^x z &= \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C \\ z &= \frac{1}{2} \sin x + \frac{1}{2} \cos x + C e^{-x} \end{aligned}$$

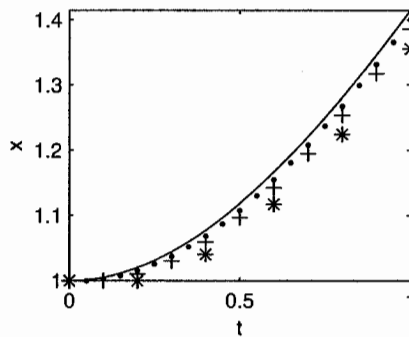
The initial condition $z(0) = 1$ provides $C = 1/2$ and the exact solution $z = (1/2) \sin x + (1/2) \cos x + (1/2)e^{-x}$. In the image that follows, the exact solution is plotted on the interval $[0, 1]$. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.



8. The equation is separable.

$$\begin{aligned}x dx &= t dt \\ \frac{1}{2}x^2 &= \frac{1}{2}t^2 + C \\ x^2 &= t^2 + 2C\end{aligned}$$

The initial condition $x(0) = 1$ provides $C = 1/2$ and the implicit solution $x^2 = t^2 + 1$. Of course, because we want $x(0) = 1$, we take the right-hand branch, $x = \sqrt{t^2 + 1}$. In the image that follows, the exact solution is plotted on the interval $[0, 1]$. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.



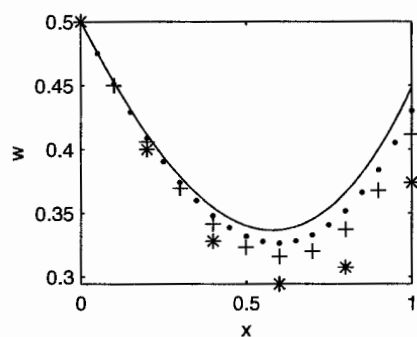
9. An integrating factor is e^x .

$$\begin{aligned}e^x w' + e^x w &= x^2 e^x \\ (e^x w)' &= x^2 e^x\end{aligned}$$

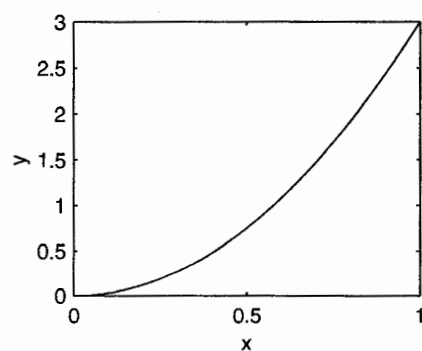
Integrate by parts.

$$\begin{aligned}e^x w &= x^2 e^x - 2x e^x + 2e^x + C \\ w &= x^2 - 2x + 2 + C e^{-x}\end{aligned}$$

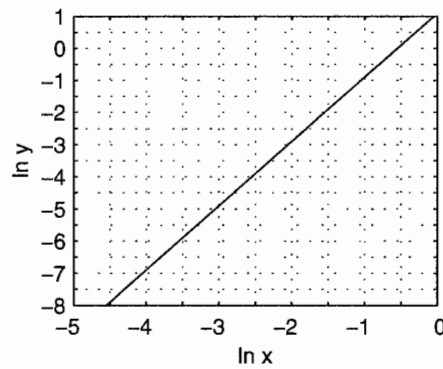
The initial condition $w(0) = 1/2$ provides $C = -3/2$ and the exact solution $w = x^2 - 2x + 2 - (3/2)e^{-x}$. In the image that follows, the exact solution is plotted on the interval $[0, 1]$. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.



10. The plot of the power function $y = 3x^2$ on $[0, 1]$.



The difficulty most encounter is in circumventing the logarithm of zero. We actually made our plot on $[0.01, 1]$ (close enough).

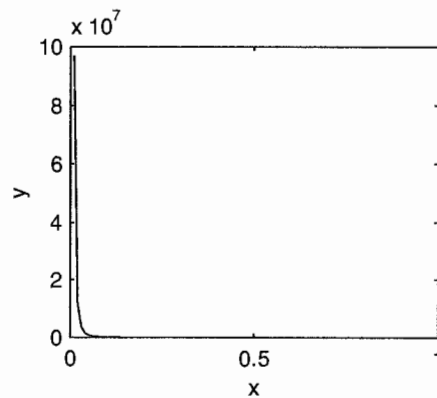


Pick any two points on the line and use the slope formula. We estimate $(-4, -6.8)$ and $(-1, -0.8)$ from the graph, giving a slope of

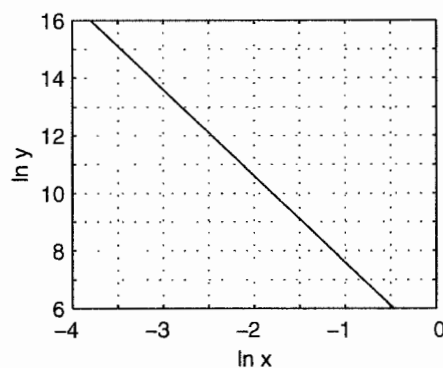
$$m \approx \frac{-6.8 - (-4)}{-0.8 - (-1)} \approx 2.$$

Note that the slope of the line is identical to the exponent of the power function $y = 3x^2$.

11. The plot of the power function $y = 100x^{-3}$ on $[0, 1]$.



The difficulty most encounter is in circumventing the logarithm of zero. We actually made our plot on $[0.01, 1]$ (close enough).

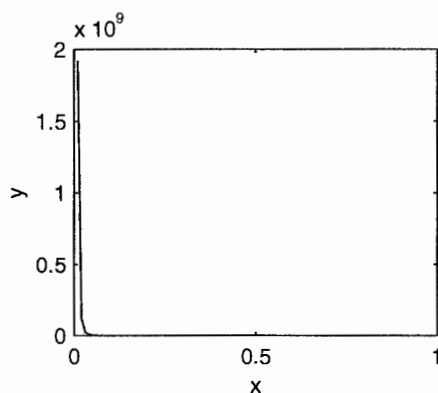


Pick any two points on the line and use the slope formula. We estimate $(-3, 13.6)$ and $(-1, 7.6)$ from the graph, giving a slope of

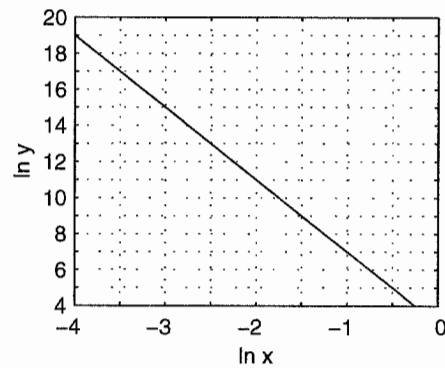
$$m \approx \frac{7.6 - 13.6}{-1 - (-3)} \approx -3.$$

Note that the slope of the line is identical to the exponent of the power function $y = 100x^{-3}$.

12. The plot of the power function $y = 20x^{-4}$ on $[0, 1]$.



The difficulty most encounter is in circumventing the logarithm of zero. We actually made our plot on $[0.01, 1]$ (close enough).

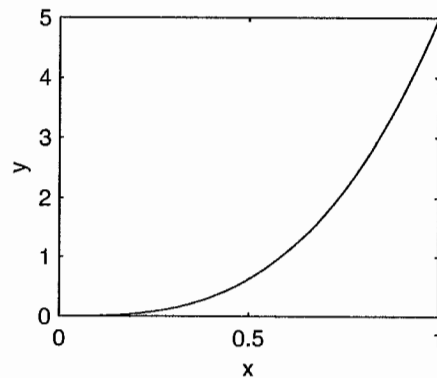


Pick any two points on the line and use the slope formula. We estimate $(-3, 15)$ and $(-1, 7)$ from the graph, giving a slope of

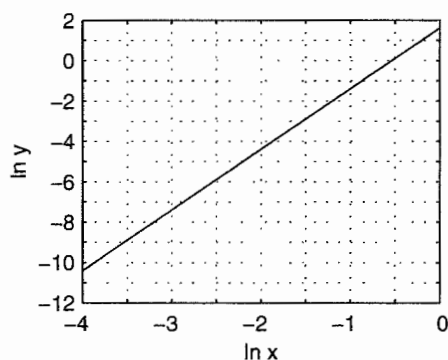
$$m \approx \frac{7 - 15}{-1 - (-3)} \approx -4.$$

Note that the slope of the line is identical to the exponent of the power function $y = 20x^{-4}$.

13. The plot of the power function $y = 5x^3$ on $[0, 1]$.



The difficulty most encounter is in circumventing the logarithm of zero. We actually made our plot on $[0.01, 1]$ (close enough).

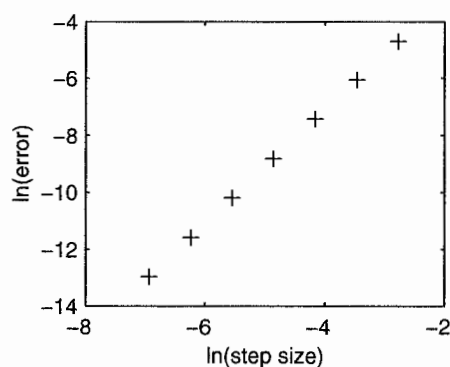


Pick any two points on the line and use the slope formula. We estimate $(-3, -7.4)$ and $(-1, -1.4)$ from the graph, giving a slope of

$$m \approx \frac{-1.4 - (-7.4)}{-1 - (-3)} \approx 3.$$

Note that the slope of the line is identical to the exponent of the power function $y = 5x^3$.

14. (a) Prepare a plot of $\ln(E_h)$ (column 4) versus the logarithm of the step size (column 1).



Note the linear relationship between $\ln(E_h)$ and $\ln h$.

- (b) Using a calculator, we fit a linear model to the data, having form

$$\ln E_h = 1.9901 \ln h + 0.8397.$$

- (c) Solve the equation in part (b) for E_h . Exponentiate both sides.

$$E_h = e^{1.9901 \ln h + 0.8397}$$

$$E_h = e^{0.8397} e^{\ln h^{1.9901}}$$

$$E_h = 2.316h^{1.9901}$$

Note that the exponent is near 2. This agrees with our claim that RK2 is a second order method.

(d) Let $E_h = 0.001$ in the formula developed in part (c).

$$\begin{aligned} .001 &= 2.316h^{1.9901} \\ h &= \left(\frac{0.001}{2.316} \right)^{1/1.9901} \\ h &\approx 0.0203 \end{aligned}$$

The number of iterations is calculated with

$$\begin{aligned} N &= \frac{b-a}{h} \\ N &= \frac{2}{0.0203} \\ N &\approx 99 \end{aligned}$$

It will take approximately 99 iterations of the RK2 method to achieve the desired accuracy.

(e) We ran the RK2 algorithm on a computer with step size $h = 0.0203$. A 300 MHz PC reported an error of 0.0009957 in an elapsed time of about 0.17 seconds.

15. An integrating factor is e^t .

$$\begin{aligned} e^t y' + e^t y &= e^t \sin t \\ (e^t y)' &= e^t \sin t \end{aligned}$$

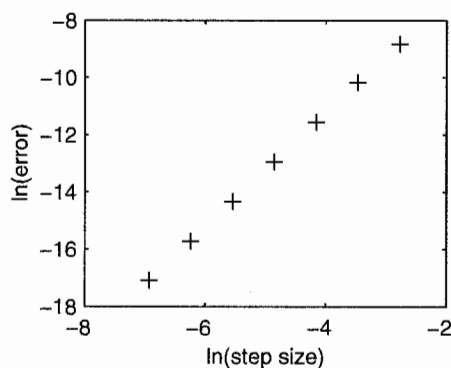
Integrate by parts.

$$\begin{aligned} e^t y &= \frac{1}{2} e^t \sin t - \frac{1}{2} e^t \cos t + C \\ y &= \frac{1}{2} \sin t - \frac{1}{2} \cos t + C e^{-t} \end{aligned}$$

The initial condition $y(0) = 1$ gives us $C = 3/2$ and the exact solution $y = (1/2) \sin t - (1/2) \cos t + (3/2)e^{-t}$. Thus, the true solution at $t = 2$ is $y(2) \approx .8657250565$. This value allows us to use RK2 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.06250000	0.86557660	0.86572506	0.00014846
0.03125000	0.86568735	0.86572506	0.00003771
0.01562500	0.86571556	0.86572506	0.00000950
0.00781250	0.86572267	0.86572506	0.00000238
0.00390625	0.86572446	0.86572506	0.00000060
0.00195313	0.86572491	0.86572506	0.00000015
0.00097656	0.86572502	0.86572506	0.00000004

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 1.9938 \ln h - 3.278$$

$$E_h = e^{-3.278} e^{\ln h^{1.9938}}$$

$$E_h = 0.03770h^{1.9938}$$

Note that the exponent is approximately 2, which is in keeping with the fact that RK2 is a second order method.

16. The equation is separable.

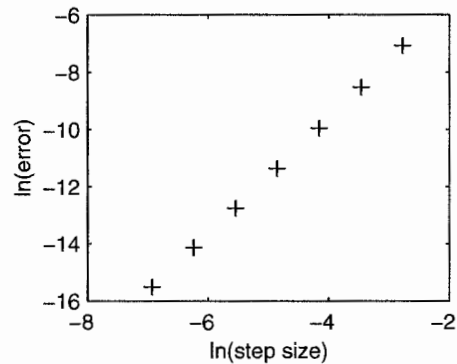
$$\frac{dy}{y} = -t^2 dt$$

$$\ln y = -\frac{1}{3}t^3 + C$$

The initial condition $y(0) = 1$ gives $C = 0$. Thus, $y = e^{-(1/3)t^3}$ and $y(2) \approx 0.06948345$. This value allows us to use RK2 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.06250000	0.07032847	0.06948345	0.00084502
0.03125000	0.06968051	0.06948345	0.00019706
0.01562500	0.06953107	0.06948345	0.00004761
0.00781250	0.06949516	0.06948345	0.00001170
0.00390625	0.06948635	0.06948345	0.00000290
0.00195313	0.06948417	0.06948345	0.00000072
0.00097656	0.06948363	0.06948345	0.00000018

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 2.029 \ln h - 1.490$$

$$E_h = e^{-1.490} e^{\ln h^{2.029}}$$

$$E_h = 0.2254h^{2.029}$$

Note that the exponent is approximately 2, which is in keeping with the fact that RK2 is a second order method.

17. The equation is separable.

$$y^2 dy = t dt$$

$$\frac{1}{3}y^3 = \frac{1}{2}t^2 + C$$

The initial condition $y(0) = 1$ gives $C = 1/3$. Thus,

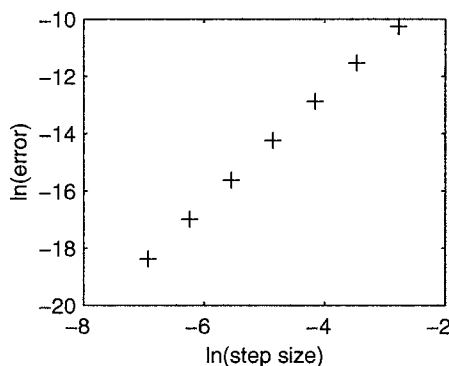
$$\frac{1}{3}y^3 = \frac{1}{2}t^2 + \frac{1}{3}$$

$$y = \sqrt[3]{(3/2)t^2 + 1}$$

Thus, $y(2) \approx 1.91293118$. This value allows us to use RK2 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.06250000	1.91289601	1.91293118	0.00003517
0.03125000	1.91292139	1.91293118	0.00000979
0.01562500	1.91292861	1.91293118	0.00000257
0.00781250	1.91293053	1.91293118	0.00000066
0.00390625	1.91293102	1.91293118	0.00000017
0.00195313	1.91293114	1.91293118	0.00000004
0.00097656	1.91293117	1.91293118	0.00000001

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 1.958 \ln h - 4.765$$

$$E_h = e^{-4.765} e^{\ln h^{1.958}}$$

$$E_h = 0.008522h^{1.958}$$

Note that the exponent is approximately 2, which is in keeping with the fact that RK2 is a second order method.

18. An integrating factor is e^{3t} .

$$e^{3t} y' + 3e^{3t} y = te^{3t}$$

$$(e^{3t} y)' = te^{3t}$$

Integration by parts.

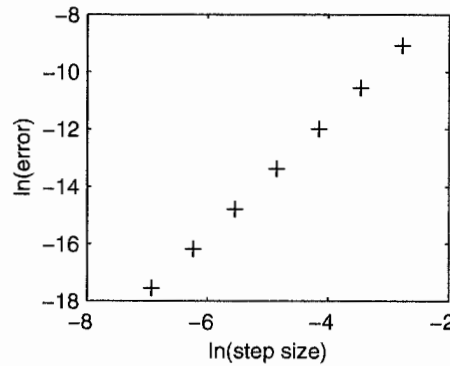
$$e^{3t} y = \frac{1}{3} te^{3t} - \frac{1}{9} e^{3t} + C$$

$$y = \frac{1}{3} t - \frac{1}{9} + Ce^{-3t}$$

The initial condition $y(0) = 1$ gives us $C = 10/9$ and the exact solution $y = (1/3)t - 1/9 + (10/9)e^{-3t}$. Thus, $y(2) \approx 0.55830972$. This value allows us to use RK2 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.06250000	0.55842346	0.55830972	0.00011373
0.03125000	0.55833582	0.55830972	0.00002610
0.01562500	0.55831600	0.55830972	0.00000628
0.00781250	0.55831126	0.55830972	0.00000154
0.00390625	0.55831011	0.55830972	0.00000038
0.00195313	0.55830982	0.55830972	0.00000009
0.00097656	0.55830975	0.55830972	0.00000002

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 2.033 \ln h - 3.492$$

$$E_h = e^{-3.492} e^{\ln h^{2.033}}$$

$$E_h = 0.03043 h^{2.033}$$

Note that the exponent is approximately 2, which is in keeping with the fact that RK2 is a second order method.

19. We've seen that the error made by RK2 is related to the step size via $E_h = \lambda h^2$. So, if h leads to an error of E_h , then the error associated with step size $h/\sqrt{2}$ is

$$E = \lambda \left(\frac{h}{\sqrt{2}} \right)^2$$

$$E = \lambda \frac{h^2}{2}$$

$$E = \frac{1}{2} \lambda h^2$$

$$E = \frac{1}{2} E_h.$$

Thus, dividing the step size by the square root of 2 halves the error. Of course, if N_h is the number of iterations associated with step size h , then

$$N_h = \frac{b-a}{h}.$$

If we reduce the step size to $h/\sqrt{2}$, then the number of iterations is

$$N = \frac{b-a}{h/\sqrt{2}}$$

$$N = \sqrt{2} \frac{b-a}{h}$$

$$N = \sqrt{2} N_h.$$

Thus, the number of iterations is increased by a factor of $\sqrt{2}$.

This exercise may be better understood in reverse. That is, what happens when we halve the step size? If E_h is the error associated with step size h , then the error associated with step size $h/2$ is

$$E = \lambda \left(\frac{h}{2} \right)^2$$

$$E = \frac{1}{4} \lambda h^2$$

$$E = \frac{1}{4} E_h.$$

Thus, halving the step size quarters the error, which is considerably better than the performance given by Euler's method.

20. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = y$. First compute four slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 1$$

$$s_2 = f(t_0 + h/2, y_0 + hs_1/2) = f(0.05, 1.05) = 1.05$$

$$s_3 = f(t_0 + h/2, y_0 + hs_2/2) = f(0.05, 1.0525) = 1.0525$$

$$s_4 = f(t_0 + h, y_0 + hs_3) = f(0.1, 1.10525) = 1.10525$$

Update t and y .

$$y_1 = y_0 + h \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

$$= 1 + 0.1 \frac{1 + 2(1.05) + 2(1.0525) + 1.10525}{6} = 1.05170833$$

$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner produces the following table.

t_k	y_k	s_1	s_2	s_3	s_4	$h \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$
0.0	1.0000	1.0000	1.0500	1.0525	1.1053	0.1052
0.1	1.1052	1.1052	1.1604	1.1632	1.2215	0.1162
0.2	1.2214	1.2214	1.2825	1.2855	1.3500	0.1285
0.3	1.3499					

21. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = t + y$. First compute four slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 1$$

$$s_2 = f(t_0 + h/2, y_0 + hs_1/2) = f(0.05, 1.05) = 1.10$$

$$s_3 = f(t_0 + h/2, y_0 + hs_2/2) = f(0.05, 1.055) = 1.105$$

$$s_4 = f(t_0 + h, y_0 + hs_3) = f(0.1, 1.1105) = 1.2105$$

Update t and y .

$$y_1 = y_0 + h \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

$$= 1 + 0.1 \frac{1 + 2(1.10) + 2(1.105) + 1.2105}{6} = 1.110341667$$

$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner produces the following table.

t_k	y_k	s_1	s_2	s_3	s_4	$h \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$
0.0	1.0000	1.0000	1.1000	1.1050	1.2105	0.1103
0.1	1.1103	1.2103	1.3209	1.3264	1.4430	0.1325
0.2	1.2428	1.4428	1.5649	1.5711	1.6999	0.1569
0.3	1.3997					

22. We have $x_0 = 0$, $z_0 = 0$ and $f(x, z) = 5 - z$. First compute four slopes.

$$\begin{aligned} s_1 &= f(x_0, z_0) = f(0, 0) = 5 \\ s_2 &= f(x_0 + h/2, z_0 + hs_1/2) = f(0.05, 0.25) = 4.75 \\ s_3 &= f(x_0 + h/2, z_0 + hs_2/2) = f(0.05, 0.2375) = 4.7625 \\ s_4 &= f(x_0 + h, z_0 + hs_3) = f(0.1, 0.47625) = 4.52375 \end{aligned}$$

Update x and z .

$$\begin{aligned} z_1 &= z_0 + h \frac{s_1 + 2s_2 + 2s_3 + s_4}{6} \\ &= 0 + 0.1 \frac{5 + 2(4.75) + 2(4.7625) + 4.52375}{6} = 0.4758125 \\ x_1 &= x_0 + h = 0 + 0.1 = 0.1 \end{aligned}$$

Continuing in this manner produces the following table.

$s \ x_k$	z_k	s_1	s_2	s_3	s_4	$h \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$
0.0	0.0000	5.0000	4.7500	4.7625	4.5237	0.4758
0.1	0.4758	4.5242	4.2980	4.3093	4.0933	0.4305
0.2	0.9063	4.0937	3.8890	3.8992	3.7037	0.3896
0.3	1.2959					

23. We have $x_0 = 0$, $z_0 = 1$ and $f(x, z) = x - 2z$. First compute four slopes.

$$\begin{aligned} s_1 &= f(x_0, z_0) = f(0, 1) = -2 \\ s_2 &= f(x_0 + h/2, z_0 + hs_1/2) = f(0.05, 0.9) = -1.75 \\ s_3 &= f(x_0 + h/2, z_0 + hs_2/2) = f(0.05, 0.9125) = -1.775 \\ s_4 &= f(x_0 + h, z_0 + hs_3) = f(0.1, 0.8225) = -1.545 \end{aligned}$$

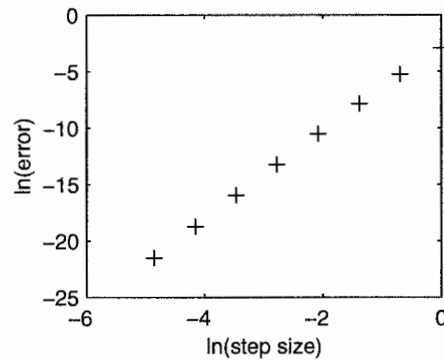
Update x and z .

$$\begin{aligned} z_1 &= z_0 + h \frac{s_1 + 2s_2 + 2s_3 + s_4}{6} \\ &= 1 + 0.1 \frac{-2 + 2(-1.75) + 2(-1.775) + (-1.545)}{6} = 0.8234166667 \\ x_1 &= x_0 + h = 0 + 0.1 = 0.1 \end{aligned}$$

Continuing in this manner produces the following table.

s	x_k	z_k	s_1	s_2	s_3	s_4	$h \frac{s_1+2s_2+2s_3+s_4}{6}$
0.0	1.0000	-2.0000	-1.7500	-1.7500	-1.7750	-1.5450	-0.1766
0.1	0.8234	-1.5468	-1.3422	-1.3422	-1.3626	-1.1743	-0.1355
0.2	0.6879	-1.1758	-1.0082	-1.0082	-1.0250	-0.8708	-0.1019
0.3	0.5860						

24. (a) Prepare a plot of $\ln(E_h)$ (column 4) versus the logarithm of the step size (column 1).



Note the linear relationship between $\ln(E_h)$ and $\ln h$.

- (b) Using a calculator, we fit a linear model to the data, having form

$$\ln E_h = 3.8569 \ln h - 2.647.$$

- (c) Solve the equation in part (b) for E_h . Exponentiate both sides.

$$E_h = e^{3.8569 \ln h - 2.647}$$

$$E_h = e^{-2.647} e^{\ln h^{3.8569}}$$

$$E_h = 0.07086h^{3.8569}$$

Note that the exponent is near 4. This agrees with our claim that RK4 is a fourth order method.

- (d) Let $E_h = 0.000001$ in the formula developed in part (c).

$$.000001 = 0.07086h^{3.8569}$$

$$h = \left(\frac{0.000001}{0.07086} \right)^{1/3.8569}$$

$$h \approx 0.0552$$

The number of iterations is calculated with

$$N = \frac{b-a}{h}$$

$$N = \frac{2}{0.0552}$$

$$N \approx 36$$

It will take approximately 36 iterations of the RK4 method to achieve the desired accuracy.

- (e) We ran the RK4 algorithm on a computer with step size $h = 0.0552$. A 300 MHz PC reported an error of 1.085×10^{-6} in an elapsed time of about 0.11 seconds.

25. An integrating factor is e^t .

$$\begin{aligned} e^t y' + e^t y &= e^t \sin t \\ (e^t y)' &= e^t \sin t \end{aligned}$$

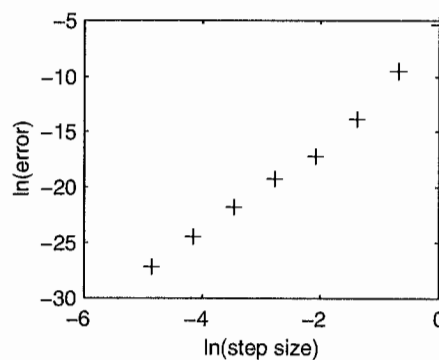
Integrate by parts.

$$\begin{aligned} e^t y &= \frac{1}{2} e^t \sin t - \frac{1}{2} e^t \cos t + C \\ y &= \frac{1}{2} \sin t - \frac{1}{2} \cos t + C e^{-t} \end{aligned}$$

The initial condition $y(0) = 1$ gives us $C = 3/2$ and the exact solution $y = (1/2) \sin t - (1/2) \cos t + (3/2)e^{-t}$. Thus, the true solution at $t = 2$ is $y(2) \approx .8657250565$. This value allows us to use RK4 and a computer to construct the following table.

Step size h	RK4 approx.	True value	Error E_h
1.000000	0.87036095021	0.86572505654	0.00463589367
0.500000	0.86580167967	0.86572505654	0.00007662313
0.250000	0.86572599485	0.86572505654	0.00000093831
0.125000	0.86572502504	0.86572505654	0.00000003150
0.062500	0.86572505215	0.86572505654	0.00000000439
0.031250	0.86572505620	0.86572505654	0.00000000034
0.015625	0.86572505652	0.86572505654	0.00000000002
0.0078125	0.86572505654	0.86572505654	0.00000000000

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 4.352 \ln h - 6.778$$

$$E_h = e^{-6.778} e^{\ln h^{4.352}}$$

$$E_h = 0.001138h^{4.352}$$

Note that the exponent is approximately 4, which is in keeping with the fact that RK4 is a fourth order method.

26. The equation is separable.

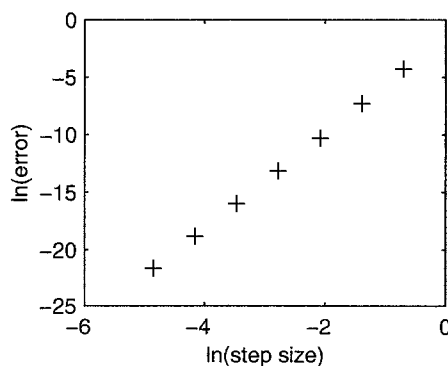
$$\frac{dy}{y} = -t^2 dt$$

$$\ln y = -\frac{1}{3}t^3 + C$$

The initial condition $y(0) = 1$ gives $C = 0$. Thus, $y = e^{-(1/3)t^3}$ and $y(2) \approx 0.06948345$. This value allows us to use RK4 and a computer to construct the following table.

Step size h	RK4 approx.	True value	Error E_h
0.50000000	0.08308652	0.06948345	0.01360307
0.25000000	0.07018576	0.06948345	0.00070231
0.12500000	0.06951834	0.06948345	0.00003489
0.06250000	0.06948535	0.06948345	0.00000190
0.03125000	0.06948356	0.06948345	0.00000011
0.01562500	0.06948346	0.06948345	0.00000001
0.00781250	0.06948345	0.06948345	0.00000000

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 4.165 \ln h - 1.516$$

$$E_h = e^{-1.516} e^{\ln h^{4.165}}$$

$$E_h = 0.2195h^{4.165}$$

Note that the exponent is approximately 4, which is in keeping with the fact that RK4 is a fourth order method.

27. The equation is separable.

$$y^2 dy = t dt$$

$$\frac{1}{3}y^3 = \frac{1}{2}t^2 + C$$

The initial condition $y(0) = 1$ gives $C = 1/3$. Thus,

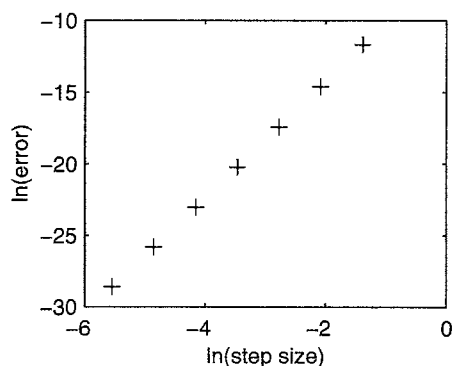
$$\frac{1}{3}y^3 = \frac{1}{2}t^2 + \frac{1}{3}$$

$$y = \sqrt[3]{(3/2)t^2 + 1}$$

Thus, $y(2) \approx 1.91293118$. This value allows us to use RK4 and a computer to construct the following table.

Step size h	RK4 approx.	True value	Error E_h
0.25000000	1.91293958	1.91293118	0.00000840
0.12500000	1.91293164	1.91293118	0.00000046
0.06250000	1.91293121	1.91293118	0.00000003
0.03125000	1.91293118	1.91293118	0.00000000
0.01562500	1.91293118	1.91293118	0.00000000
0.00781250	1.91293118	1.91293118	0.00000000
0.00390625	1.91293118	1.91293118	0.00000000

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 4.055 \ln h - 6.139$$

$$E_h = e^{-6.139} e^{\ln h^{4.055}}$$

$$E_h = 0.002157h^{4.055}$$

Note that the exponent is approximately 4, which is in keeping with the fact that RK4 is a fourth order method.

28. An integrating factor is e^{3t} .

$$e^{3t}y' + 3e^{3t}y = te^{3t}$$

$$(e^{3t}y)' = te^{3t}$$

Integration by parts.

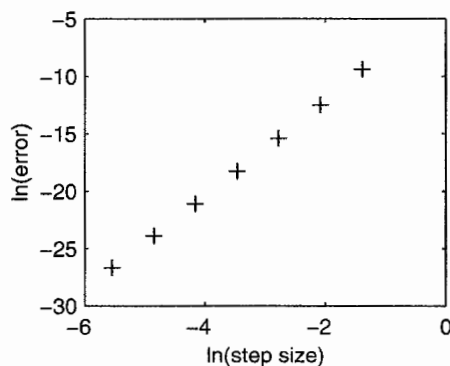
$$e^{3t}y = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + C$$

$$y = \frac{1}{3}t - \frac{1}{9} + Ce^{-3t}$$

The initial condition $y(0) = 1$ gives us $C = 10/9$ and the exact solution $y = (1/3)t - 1/9 + (10/9)e^{-3t}$. Thus, $y(2) \approx 0.55830972$. This value allows us to use RK4 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.25000000	0.55839264	0.55830972	0.00008291
0.12500000	0.55831345	0.55830972	0.00000373
0.06250000	0.55830992	0.55830972	0.00000020
0.03125000	0.55830974	0.55830972	0.00000001
0.01562500	0.55830973	0.55830972	0.00000000
0.00781250	0.55830972	0.55830972	0.00000000
0.00390625	0.55830972	0.55830972	0.00000000

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 4.134 \ln h - 3.850$$

$$E_h = e^{-3.850} e^{\ln h^{4.134}}$$

$$E_h = 0.02127h^{4.134}$$

Note that the exponent is approximately 4, which is in keeping with the fact that RK4 is a fourth order method.

29. Let $E_h = \lambda h^4$ be the error associated with the step size h . Now, what will be the error if we halve the step size?

$$E = \lambda \left(\frac{h}{2}\right)^4$$

$$E = \frac{1}{16} \lambda h^4$$

$$E = \frac{1}{16} E_h$$

Thus, halving the step size reduces the error to 1/16 of its former size.

————— × —————

Section 6.3. Numerical Error Comparisons

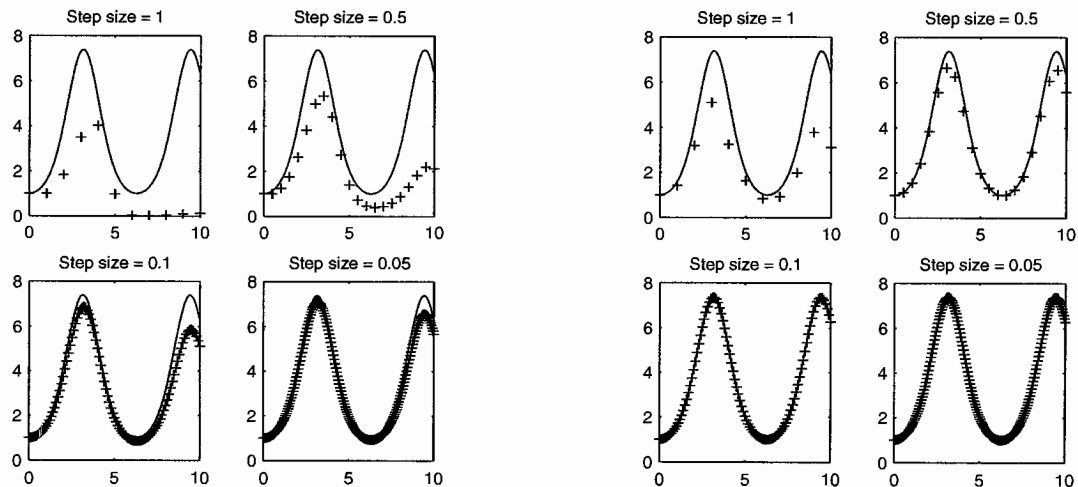
1. The equation is separable.

RK2 does a little better.

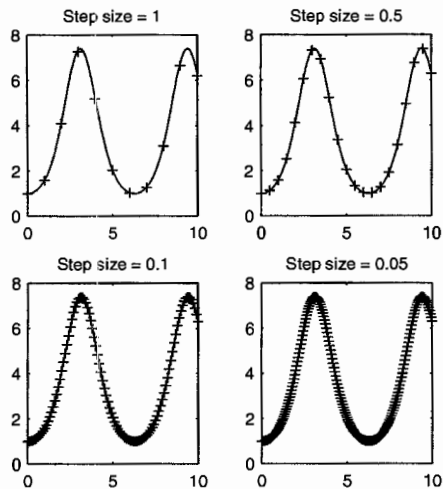
$$\frac{dx}{x} = \sin t \, dt$$

$$\ln x = -\cos t + C$$

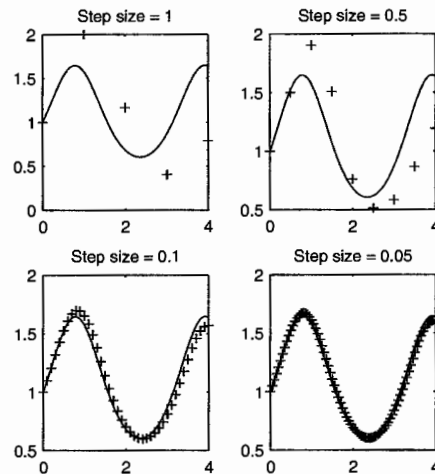
The initial condition $x(0) = 1$ gives us $C = 1$ and $\ln x = 1 - \cos t$. Solving for x , $x = e^{1-\cos t}$. Euler's method, with step sizes $h = 1, 0.5, 0.1$, and 0.05 , produces the following result.



RK4 is the most accurate.



0.05, produces the following result.



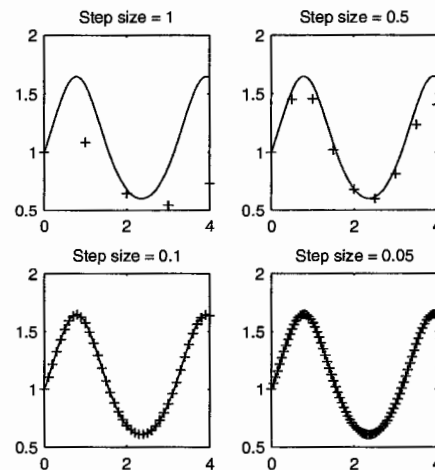
RK2 does a little better.

2. The equation is separable.

$$\frac{dx}{x} = \cos 2t \, dt$$

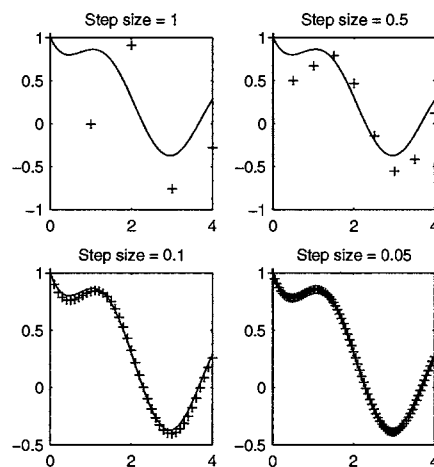
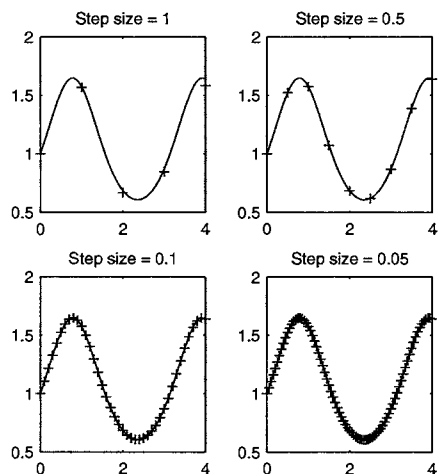
$$\ln x = \frac{1}{2} \sin 2t + C$$

The initial condition $x(0) = 1$ gives us $C = 0$ and $\ln x = (1/2) \sin 2t$. Solving for x , $x = e^{(1/2) \sin 2t}$. Euler's method, with step sizes $h = 1, 0.5, 0.1$, and



RK4 is the most accurate.

produces the following result.



3. An integrating factor is e^t .

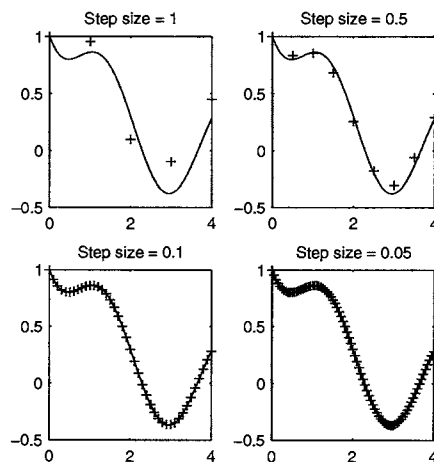
RK2 does a little better.

$$\begin{aligned} e^t x' + e^t x &= e^t \sin 2t \\ (e^t x)' &= e^t \sin 2t \end{aligned}$$

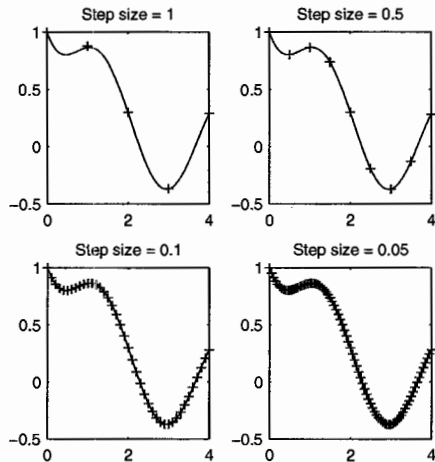
Integration is by parts.

$$\begin{aligned} e^t x &= \frac{1}{5} e^t \sin 2t - \frac{2}{5} e^t \cos 2t + C \\ x &= \frac{1}{5} \sin 2t - \frac{2}{5} \cos 2t + ce^{-t} \end{aligned}$$

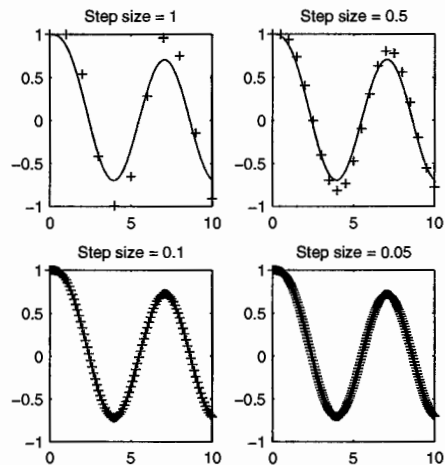
The initial condition $x(0) = 1$ gives us $C = 7/5$ and $x = (1/5) \sin 2t - (2/5) \cos 2t + (7/5)e^{-t}$. Euler's method, with step sizes $h = 1, 0.5, 0.1$, and 0.05 ,



RK4 is the most accurate.



produces the following result.



4. An integrating factor is e^t .

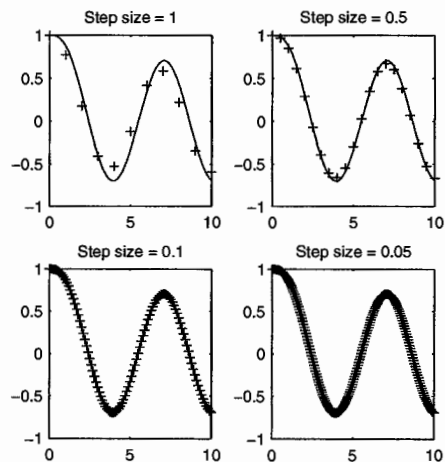
$$\begin{aligned} e^t x' + e^t x &= e^t \cos t \\ (e^t x)' &= e^t \cos t \end{aligned}$$

Integration is by parts.

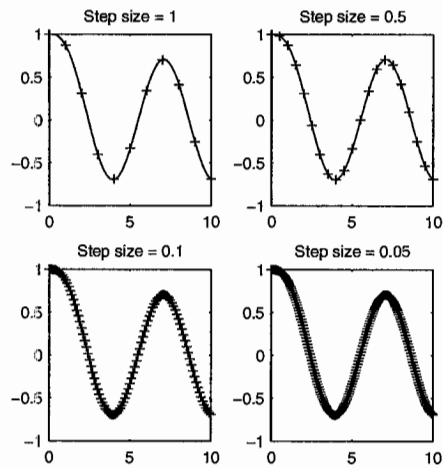
$$\begin{aligned} e^t x &= \frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t + C \\ x &= \frac{1}{2} \cos t + \frac{1}{2} \sin t + C e^{-t} \end{aligned}$$

The initial condition $x(0) = 1$ gives us $C = 1/2$ and $x = (1/2) \cos t + (1/2) \sin t + (1/2)e^{-t}$. Euler's method, with step sizes $h = 1, 0.5, 0.1$, and 0.05 ,

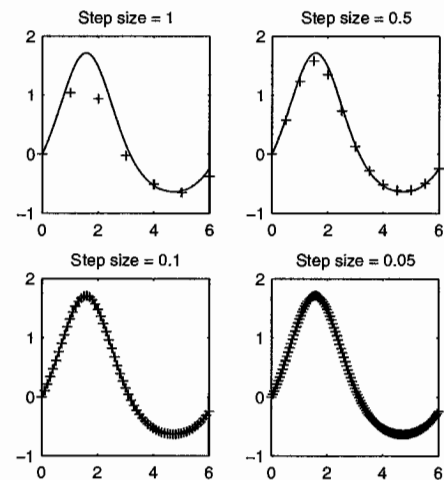
RK2 does a little better.



RK4 is the most accurate.



RK2 does a little better.



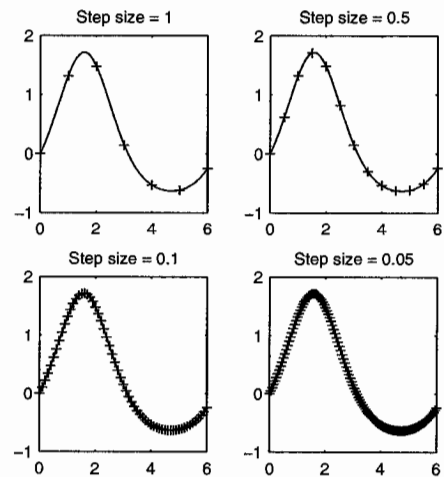
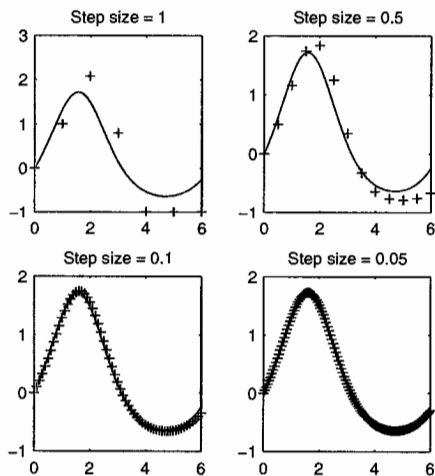
5. The equation is separable.

$$\frac{dx}{1+x} = \cos t \, dt$$

$$\ln(1+x) = \sin t + C$$

The initial condition $x(0) = 0$ gives us $C = 0$ and $\ln(1+x) = \sin t$. Solving for x , $x = -1 + e^{\sin t}$. Euler's method, with step sizes $h = 1, 0.5, 0.1$, and 0.05 , produces the following result.

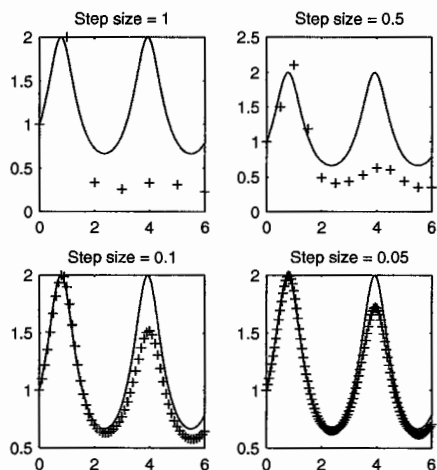
RK4 is the most accurate.



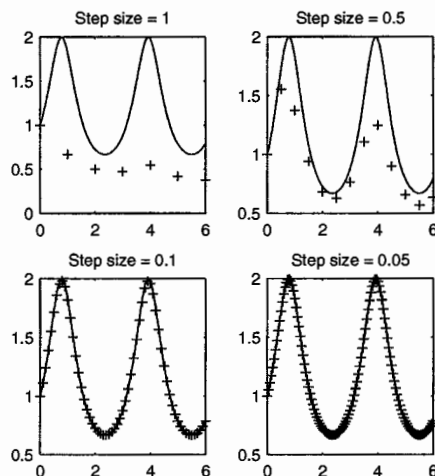
6. The equation is separable.

$$\begin{aligned}\frac{dx}{x^2} &= \cos 2t \, dt \\ -\frac{1}{x} &= \frac{1}{2} \sin 2t + C \\ x &= \frac{-2}{\sin 2t + 2C}\end{aligned}$$

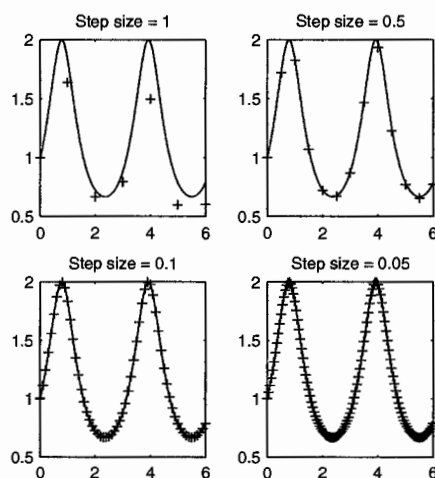
The initial condition $x(0) = 1$ gives us $C = -1$ and $x = 2/(2 - \sin 2t)$. Euler's method, with step sizes $h = 1, 0.5, 0.1$, and 0.05 , produces the following result.



RK2 does a little better.



RK4 is the most accurate.

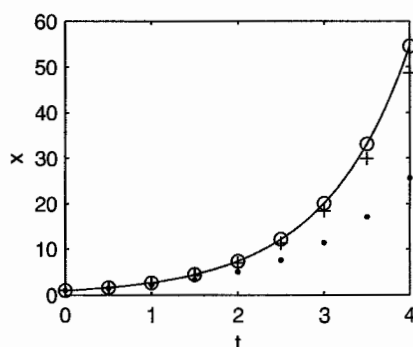


7. The equation is separable.

$$\begin{aligned}\frac{dx}{x} &= dt \\ \ln x &= t + C\end{aligned}$$

The initial condition $x(0) = 1$ gives $C = 0$ and $x = e^t$. The figure that follows shows three nu-

merical solutions on the interval $[0, 4]$ with step size $h = 0.5$. The exact solution is the solid line, the Euler solution is plotted with discrete dots, the RK2 solution is plotted with discrete plus signs, and the RK4 solution is plotted with discrete circles.

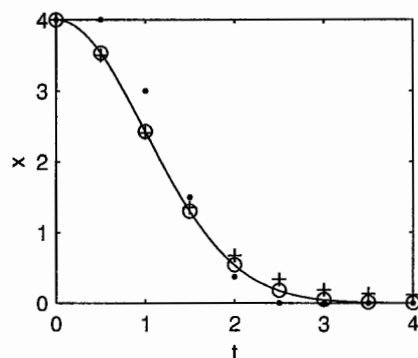


8. The equation is separable. Thus,

$$\begin{aligned}\frac{dx}{x} &= -t \, dt \\ \ln x &= -\frac{1}{2}t^2 + C \\ x &= De^{-(1/2)t^2},\end{aligned}$$

where $D = e^C$. The initial condition $x(0) = 4$ gives $D = 4$ and $x = 4e^{-(1/2)t^2}$. The figure that follows shows three numerical solutions on the interval $[0, 4]$ with step size $h = 0.5$. The exact solution is the solid line, the Euler solution is plotted with discrete dots, the RK2 solution is plotted with discrete plus signs,

and the RK4 solution is plotted with discrete circles.



9. The equation is separable.

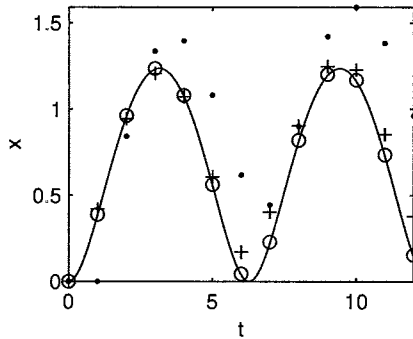
$$\begin{aligned}(x+1) \, dx &= \sin t \, dt \\ \frac{1}{2}x^2 + x &= -\cos t + C\end{aligned}$$

The initial condition $x(0) = 0$ gives $C = 1$ and

$$\begin{aligned}\frac{1}{2}x^2 + x &= 1 - \cos t \\ x^2 + 2x &= 2 - 2\cos t \\ (x+1)^2 &= 3 - 2\cos t \\ x &= -1 \pm \sqrt{3 - 2\cos t}\end{aligned}$$

Because $x(0) = 0$, choose $x = -1 + \sqrt{3 - 2\cos t}$. The figure that follows shows three numerical solutions on the interval $[0, 12]$ with step size $h = 1$. The exact solution is the solid line, the Euler solution is plotted with discrete dots, the RK2 solution is plotted with discrete plus signs, and the RK4 solution is

plotted with discrete circles.



10. The equation is separable.

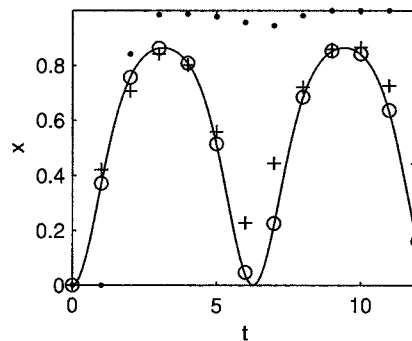
$$\begin{aligned}\frac{dx}{1-x} &= \sin t \, dt \\ -\ln|1-x| &= -\cos t + C \\ \ln|1-x| &= \cos t - C\end{aligned}$$

The initial condition $x(0) = 0$ gives $C = 1$ and

$$\begin{aligned}\ln|1-x| &= \cos t - 1 \\ |1-x| &= e^{-1+\cos t} \\ 1-x &= \pm e^{-1+\cos t} \\ x &= 1 \pm e^{-1+\cos t}.\end{aligned}$$

Because $x(0) = 0$, choose $x = 1 - e^{-1+\cos t}$. The figure that follows shows three numerical solutions on the interval $[0, 12]$ with step size $h = 1$. The exact solution is the solid line, the Euler solution is plotted with discrete dots, the RK2 solution is plotted with discrete plus signs, and the RK4 solution is

plotted with discrete circles.



11. (a) First, we complete the entries in the table using a computer.

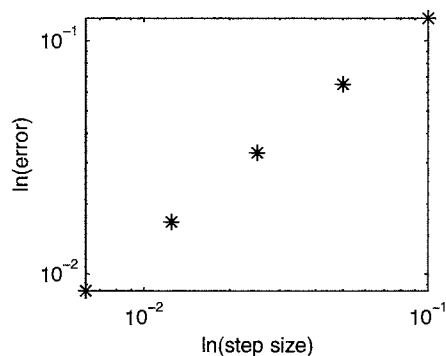
h	Euler	RK2	RK4
0.10000	2.59374	2.71408	2.71828
0.05000	2.65330	2.71719	2.71828
0.02500	2.68506	2.71800	2.71828
0.01250	2.70148	2.71821	2.71828
0.00625	2.70984	2.71826	2.71828

Next, since we will be plotting the magnitude of the error, we craft a new table with the errors at each step size. The error is calculated by taking the magnitude of the difference of $x(1) = e$ and the solution at $t = 1$ given by the numerical routine.

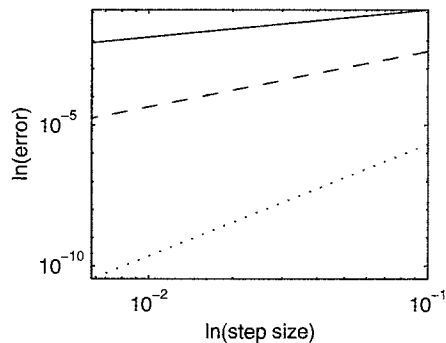
h	Eul error	RK2 error	RK4 error
0.10000	0.12454	0.00420	0.00000
0.05000	0.06498	0.00109	0.00000
0.02500	0.03322	0.00028	0.00000
0.01250	0.01680	0.00007	0.00000
0.00625	0.00845	0.00002	0.00000

We will work with the internal precision of the numbers in the computer, not the numbers displayed in the above tables. Here is a loglog graph of the Euler error versus the step size.

Note the logarithmic scale on each axis.



- (b) What follows is a plot of the logarithm of the error versus the logarithm of the step size for each numerical method. Note the logarithmic scale on each axis. Euler (solid line), RK2 (dashed), RK4 (dotted).



- (c) We used a computer to find the slope of each line in part (b). For Euler's method, the slope was reported as 0.9716, which is close to 1, consistent with the fact that the Euler routine is a first order algorithm. The slope of the second line came in at 1.9755, which is close to 2, consistent with the fact that RK2 is a second order method. Finally, the slope of the third line was estimated as 3.9730, which is close to 4, consistent with the fact that RK4 is a fourth order method.

12. The equation is separable.

$$\begin{aligned}\frac{dx}{x} &= -tx \\ \ln x &= -\frac{1}{2}t^2 + C\end{aligned}$$

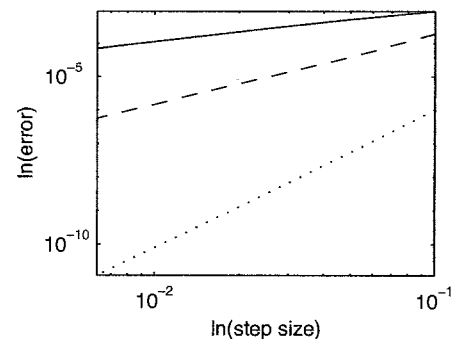
The initial condition $x(0) = 4$ gives $C = \ln 4$. Thus,

$$\begin{aligned}\ln x &= -\frac{1}{2}t^2 + \ln 4 \\ x &= e^{-(1/2)t^2 + \ln 4} \\ x &= e^{\ln 4} e^{-(1/2)t^2} \\ x &= 4e^{-(1/2)t^2}.\end{aligned}$$

A table is constructed to evaluate the error made at each step size. The error is the magnitude of the difference between $x(4) = 4e^{-8}$ and the approximation predicted by the particular method (Eul, RK2, RK4) at $t = 4$. The step size used in constructing the table is $h = 0.1$.

h	Eul error	RK2 error	RK4 error
0.10000	0.00089	0.00019	0.00000
0.05000	0.00051	0.00004	0.00000
0.02500	0.00027	0.00001	0.00000
0.01250	0.00014	0.00000	0.00000
0.00625	0.00007	0.00000	0.00000

Again, we use the internal precision of each number to continue with the analysis. A loglog plot of the error versus step size follows for each numerical method. Euler (solid line), RK2 (dashed), RK4 (dotted).



Finally, we used a computer routine to find the slope of each line. Euler: 0.9145, RK2: 2.0978, RK4: 4.0919. Note that each of these numbers is consistent with the order of the particular method.

13. The equation is separable.

$$\begin{aligned}\frac{dx}{x^2} &= -t \, dt \\ -\frac{1}{x} &= -\frac{1}{2}t^2 + C \\ \frac{1}{x} &= \frac{1}{2}t^2 - C\end{aligned}$$

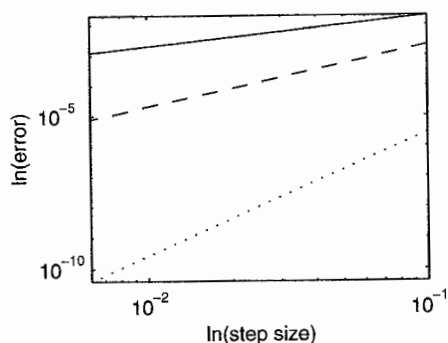
The initial condition $x(0) = 3$ gives $C = -(1/3)$. Thus,

$$\begin{aligned}\frac{1}{x} &= \frac{1}{2}t^2 + \frac{1}{3} \\ x &= \frac{6}{3t^2 + 2}.\end{aligned}$$

A table is constructed to evaluate the error made at each step size. The error is the magnitude of the difference between $x(2) = 3/7$ and the approximation predicted by the particular method (Eul, RK2, RK4) at $t = 2$. The step size used in constructing the table is $h = 0.1$.

h	Eul error	RK2 error	RK4 error
0.10000	0.02045	0.00220	0.00000
0.05000	0.01001	0.00052	0.00000
0.02500	0.00495	0.00013	0.00000
0.01250	0.00246	0.00003	0.00000
0.00625	0.00123	0.00001	0.00000

Again, we use the internal precision of each number to continue with the analysis. A loglog plot of the error versus step size follows for each numerical method. Euler (solid line), RK2 (dashed), RK4 (dotted).



Finally, we used a computer routine to find the slope of each line. Euler: 1.0135, RK2: 2.0303, RK4: 4.0256. Note that each of these numbers is consistent with the order of the particular method.

14. (a) Substitute $x = \cos t$ and $y = -\sin t$ into the first equation.

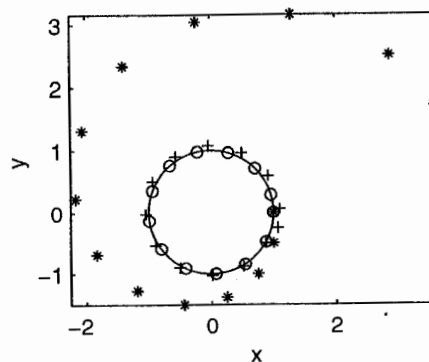
$$\begin{aligned}x' &= y \\ (\cos t)' &= -\sin t \\ -\sin t &= -\sin t\end{aligned}$$

Therefore, the first equation is satisfied. Substitute into the second equation.

$$\begin{aligned}y' &= -x \\ (-\sin t)' &= -\cos t \\ -\cos t &= -\cos t\end{aligned}$$

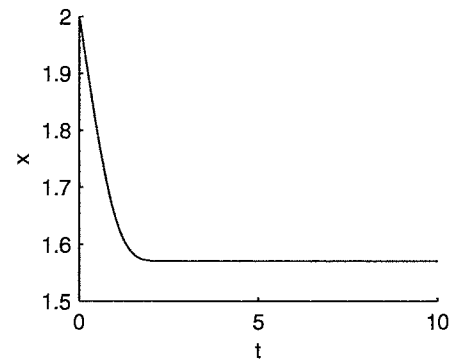
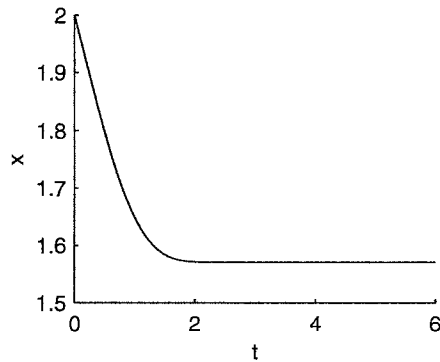
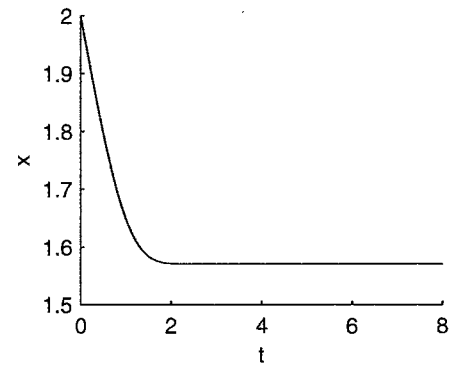
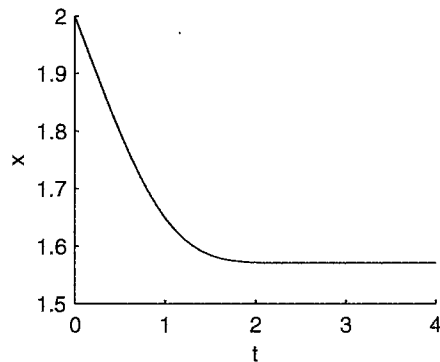
Therefore, the equations satisfy the second equation. Thus, $x = \cos t$ and $y = -\sin t$ is a solution of the system. Furthermore, $x(0) = \cos 0 = 1$ and $y(0) = -\sin 0 = 0$, so the initial conditions are also satisfied.

- (b) The exact solution plus three numerical solutions are shown in the next figure. In each case, a step size of $h = 0.5$ was used. Note that the RK4 is again the most accurate. Exact (solid), Euler (asterisk), RK2 (plus), RK4 (circles).



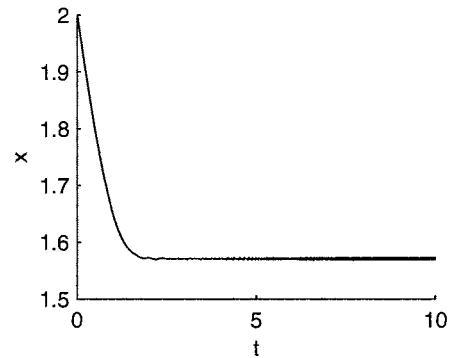
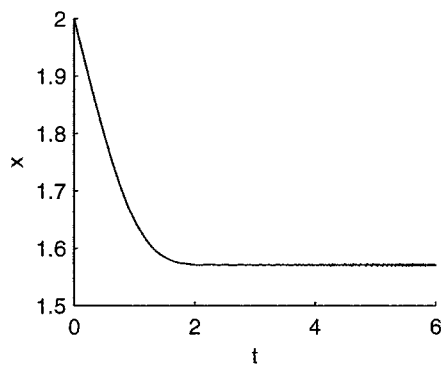
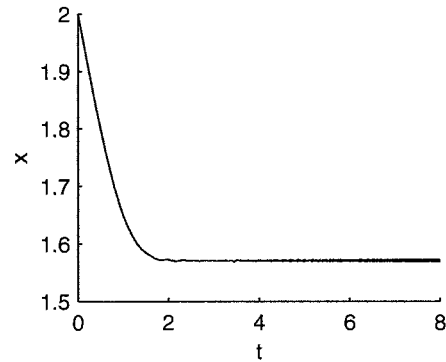
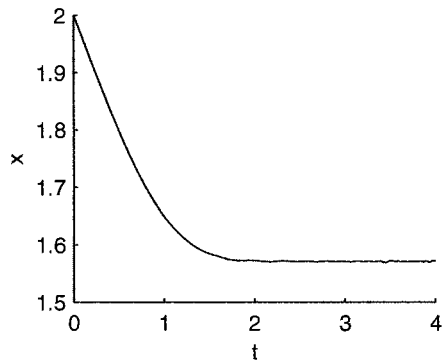
Section 6.4. Practical Use of Solvers

- The following figure shows the solution on $[0, 4]$, with step-size $h = 0.1$, and elapsed time $t \approx 0.11$ s. The next figure shows the solution on $[0, 6]$, with step-size $h = 0.01$, and elapsed time $t \approx 1.43$ s. The run was done on a P166 Pentium machine. Answers will vary on different systems.



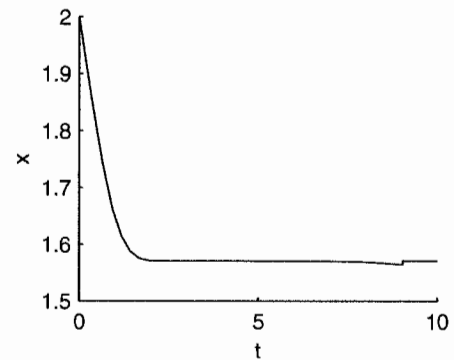
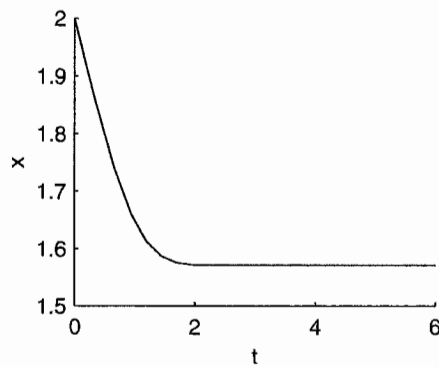
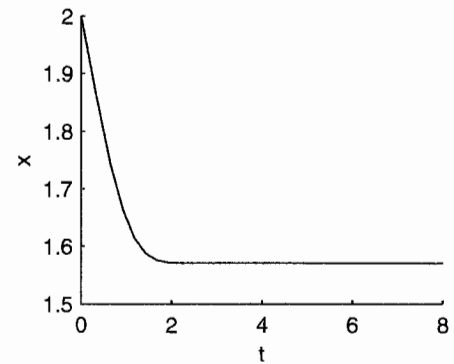
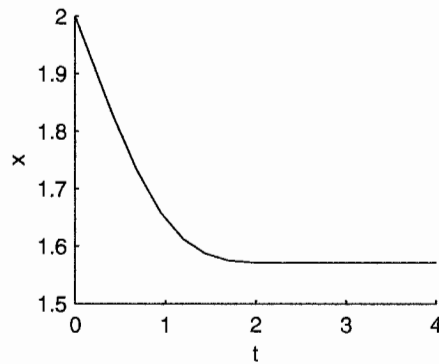
The following figure shows the solution on $[0, 8]$, with step-size $h = 0.001$, and elapsed time $t \approx 19.12$ s. The next figure shows the solution on $[0, 10]$, with step-size $h = 0.0001$, and elapsed time $t \approx 238.6$ s. The run was also done on a P166 machine. Answers will vary on different systems.

- The following figure shows the solution on $[0, 4]$. The elapsed time was $t \approx 0.55$ s, the minimum step-size was 0.006, and the maximum step-size was 0.1. The next figure shows the solution on $[0, 6]$. The elapsed time was $t \approx 1.27$ s, the minimum step-size was 0.0018, and the maximum step size was 0.15.



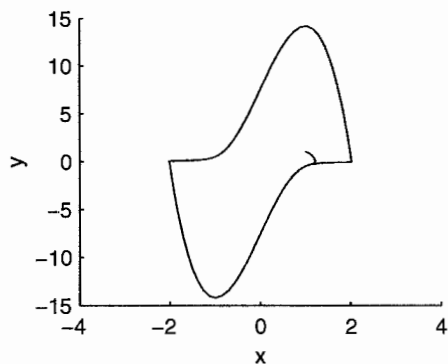
The following figure shows the solution on $[0, 8]$. The elapsed time was $t \approx 8.46$ s, the minimum step-size was 1.6×10^{-4} , and the maximum step-size was 0.2. The next figure shows the solution on $[0, 10]$. The elapsed times was $t \approx 66.19$ s, the minimum step-size was 2.1×10^{-5} , and the maximum step-size was 0.24. All runs were on a 166 MHz machine. Answers will vary on other systems.

3. The following figure shows the solution on the interval $[0, 4]$. The elapsed time was $t \approx 0.72$ s, the minimum step-size was 0.2420, and the maximum step-size was 0.4232. The next figure shows the solution on $[0, 6]$. The elapsed times was $t \approx 0.22$ s, the minimum step-size was 0.2422, and the maximum step-size was 0.6304.

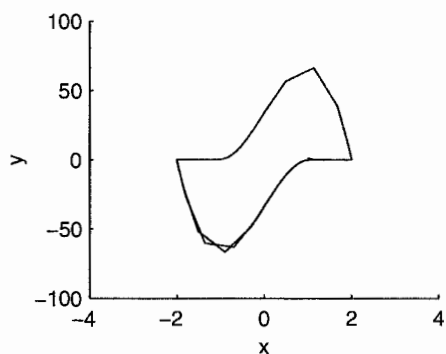
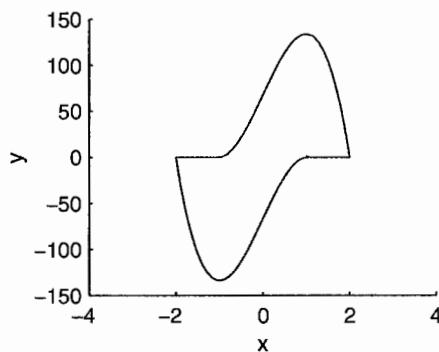


The following figure shows the solution on the interval $[0, 8]$. The elapsed time was $t \approx 0.28$ s. The minimum step-size was 0.1571, and the maximum step-size was 0.8. The next figure shows the solution on $[0, 10]$. The elapsed time was $t \approx 0.44$ s, the minimum step-size was 4.3×10^{-4} , and the maximum step-size was 1. Results were run on a 166 MHz system. Answers will vary on other systems.

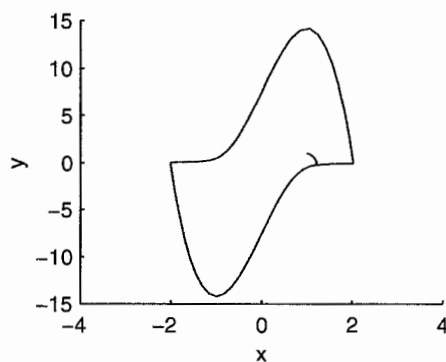
4. The following figure shows the solution for $\mu = 10$ on the interval $I = [0, 20]$. The elapsed time was $t \approx 8.02$ s, using step size $h = 0.01$. The next figure shows the solution for $\mu = 50$ on the interval $I = [0, 100]$. The elapsed time was $t \approx 43.89$, using step-size $h = 0.01$.



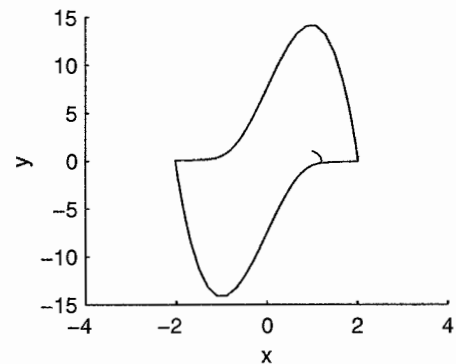
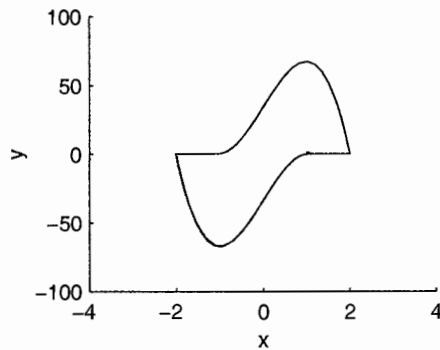
vary on different systems.



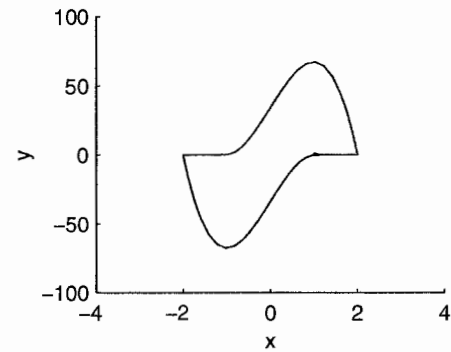
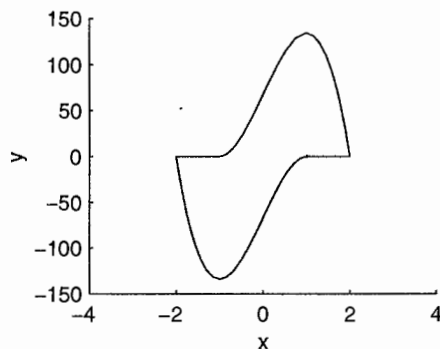
5. The following figure shows the solution for $\mu = 10$ on $I = [0, 20]$. The elapsed time was $t \approx 2.03$ s, the minimum step-size was 0.0051, and the maximum step-size was 0.0889. The next figure shows the solution for $\mu = 50$ on $I = [0, 100]$. The elapsed time was $t \approx 39.27$ s, the minimum step-size was 0.0010, and the maximum step-size was 0.0585.



The following figure shows the solution for $\mu = 100$ on the interval $I = [0, 200]$. The elapsed time was $t \approx 844.53$ s, using step-size $h = 0.001$. The runs were made on a 166 MHz machine. Answers will

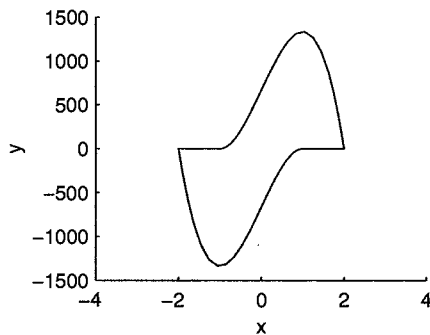
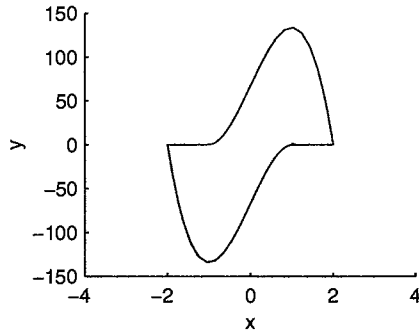


The following figure shows the solution for $\mu = 50$ on $I = [0, 200]$. The elapsed time was $t \approx 187.9$ s, the minimum step-size was 5.3×10^{-4} , and the maximum step-size was 0.0442. Answers will vary on different systems.



6. The following figure shows the solution for $\mu = 10$ on $I = [0, 20]$. The elapsed times was $t \approx 3.79$ s, the minimum step-size was 0.0035, and the maximum step-size was 0.4918. The next figure shows the solution for $\mu = 50$ on $I = [0, 100]$. The elapsed time was $t \approx 7.69$ s, the minimum step-size was 7.4×10^{-4} , and the maximum step-size was 2.1179.

The following figure shows the solution for $\mu = 100$ on $I = [0, 200]$. The elapsed time was $t \approx 9.67$ s, the minimum step-size was 4.2×10^{-4} , and the maximum step-size was 4.1960. The next figure shows the solution for $\mu = 1000$ on the interval $I = [0, 2000]$. The elapsed time was $t \approx 11.42$ s.



7. (a) If we substitute $x(t) = 2e^{-t} + \sin t$ into the left-hand side of the first equation,

$$x' = (2e^{-t} + \sin t)' = -2e^{-t} + \cos t.$$

If we substitute $x(t)$ and $y(t) = 2e^{-t} + \cos t$ into the right-hand side of the first equation,

$$\begin{aligned} -2x + y + 2 \sin t &= -2(2e^{-t} + \sin t) \\ &\quad + (2e^{-t} + \cos t) + 2 \sin t \\ &= -2e^{-t} + \cos t. \end{aligned}$$

Thus, the solution satisfies the first equation. Substituting into the left-hand side of the second equation,

$$y' = (2e^{-t} + \cos t)' = -2e^{-t} - \sin t.$$

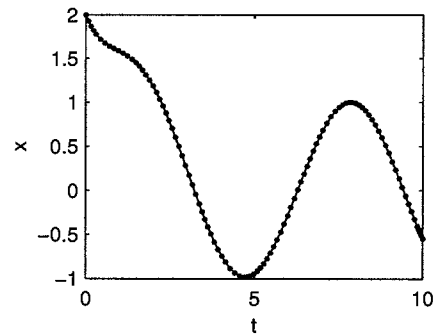
Substituting into the right-hand side of the second equation,

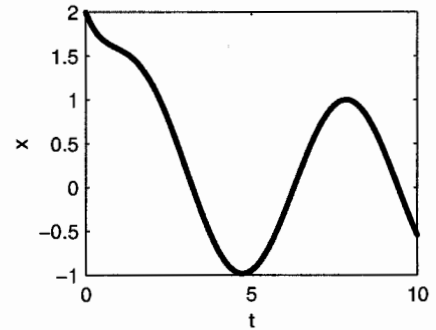
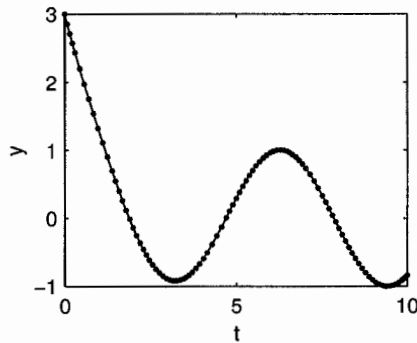
$$\begin{aligned} x - 2y + 2(\cos t - \sin t) &= 2e^{-t} + \sin t - 2(2e^{-t} + \cos t) \\ &\quad + 2 \cos t - 2 \sin t \\ &= -2e^{-t} - \sin t. \end{aligned}$$

Thus, the solutions satisfy the second equation. Finally, $x(0) = 2e^0 + \sin 0 = 2$ and $y(0) = 2e^0 + \cos 0 = 3$.

Matlab's `ode45`, a variable-step solver, produced a numerical solution in about 0.22 seconds on a 300 MHz PC. The calculation required about 15,353 floating point operations, which was calculated with Matlab's `flops` command.

Two images follow. The first contains the exact and numerical solution of x versus t . The second contains that of y versus t .





- (b) First, we'll show that $x(t) = 2e^{-t} + \sin t$ and $y(t) = 2e^{-t} + \cos t$ are also solutions of the second system. Obviously, because the first equation of the second system is identical to the first equation of the first system, both $x(t)$ and $y(t)$ still satisfy this equation. Now for the second equation. Subbing $y(t)$ into the left-hand side of the second equation,

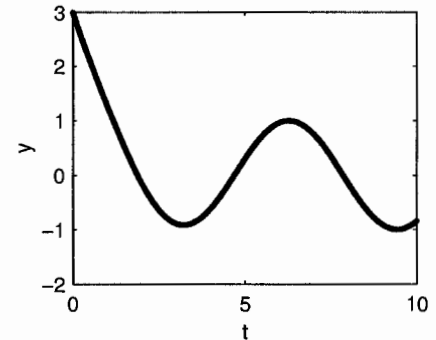
$$y' = (2e^{-t} + \cos t)' = -2e^{-t} - \sin t.$$

Substituting $x(t)$ and $y(t)$ into the right-hand side of the second equation,

$$\begin{aligned} -2x + y + 2 \sin t &= 998(2e^{-t} + \sin t) - 999(2e^{-t} + \cos t) \\ &\quad + 999 \cos t - 999 \sin t \\ &= -2e^{-t} - \sin t. \end{aligned}$$

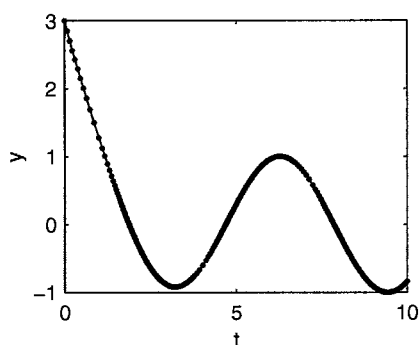
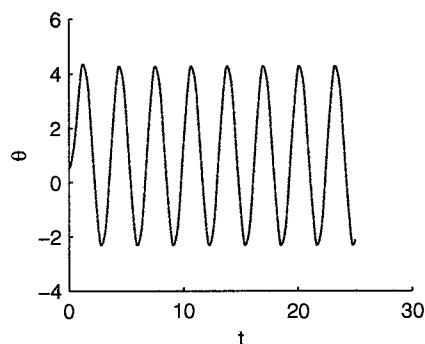
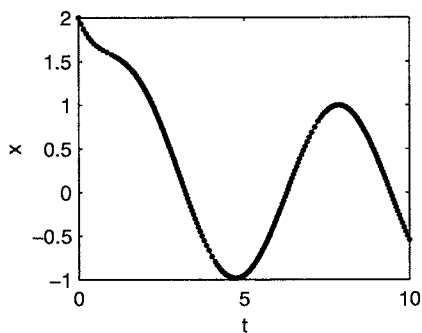
Thus, the equations satisfy the second system. Finally, both $x(t)$ and $y(t)$ still satisfy the same initial conditions.

The same solver, `ode45`, was used to produce a numerical solution of the second system, same initial conditions, same interval, and, of course, same exact solutions.

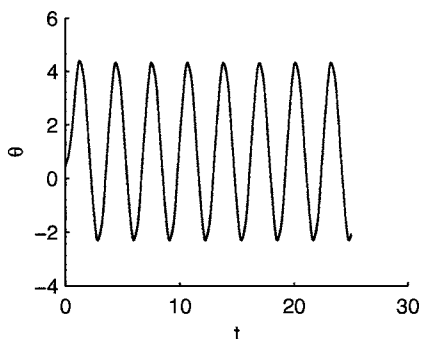


However, the time required elevated to about 12 seconds and the floating point operations required increased to 1,778,814! Two things are apparent. First, the second system is stiff. Secondly, stiffness is not a property of the solution, as both systems in this case have precisely the same solution. Rather, stiffness is some inherent part of the differential equations.

- (c) This time we solved the second system with Matlab's `ode23s`, a stiff solver. The performance improved dramatically during the run, which lasted about 2.31 seconds, using approximately 84,703 floating point operations. This considerable savings in resources is evident in the less dense plots of the second system that follow.



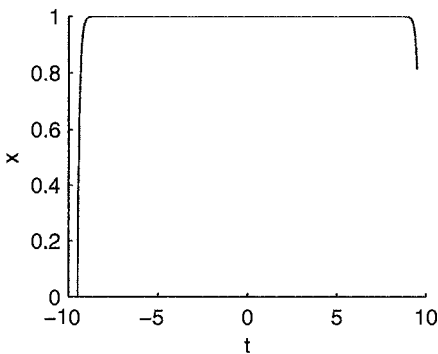
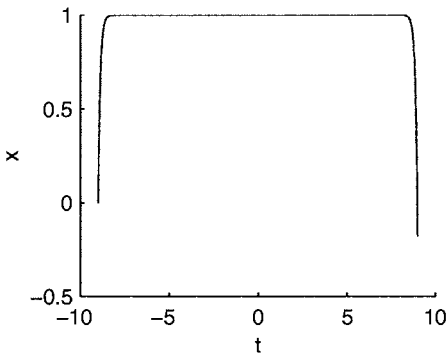
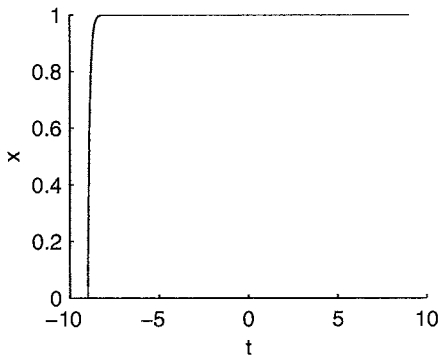
8. Using RK4, we found nice agreement on $[0, 25]$ using a step-size $h = 0.1$, but even better agreement using $h = 0.05$, as the figures that follow show.



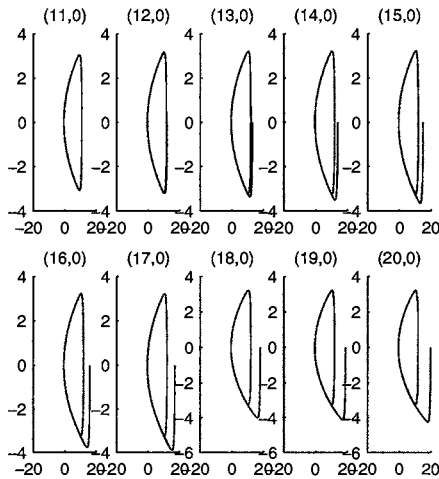
In a similar manner, on $[0, 50]$ with $h = 0.05$ some of the color of the first graph bled through the color of the second graph, but reducing the step size to $h = 0.025$ rectified the situation, producing excellent agreement between the two solutions. On $[0, 75]$, some color bled through at $h = 0.025$, but halving the step size to $h = 0.0125$ rectified the situation. Halving this again produced agreement on $[0, 100]$, but the computation was agonizingly slow on a 166 MHz machine. One would be inclined to estimate that one would have to have the step-size 4 more times to $h \approx 3.9 \times 10^{-4}$ get good agreement on $[0, 200]$.

9. The initial value problem $x' = t(x-1)$, $x(-10) = 0$, has solution $x(t) = 1 - e^{(t^2-100)/2}$. Note that $x(10) = 0$. We used Matlab and an RK4 routine to produce numerical solutions at step sizes $h = 0.1$, 0.01 , and 0.001 . Note that none of these solutions give the correct solution (namely, zero) at $t = 10$.

Step size	Run Time	Flops
0.1	0.17	6,250
0.01	2.64	62,050
0.001	105.9	620,050



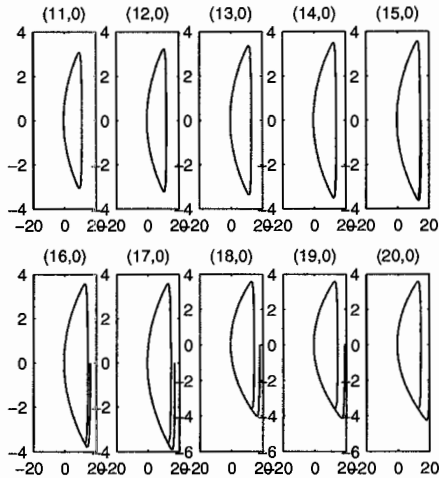
10. We decided to attack this experiment with the RK4 algorithm. Fixing a step size, $h = 0.1$, setting the time interval, $[0, 18]$, then using initial conditions $(k, 0)$, $k = 11, 11, \dots, 20$, we found an image that helped gain insight into the task at hand. Each solution in the following image is marked with its initial condition.



The run time and floating point operations at each step size are summarized in the following table.

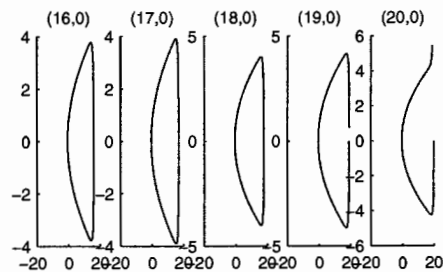
The solutions with initial conditions $(11, 0)$ and $(12, 0)$ appear to be closed curves. The others do

not. We decreased the step size to $h = 0.05$.

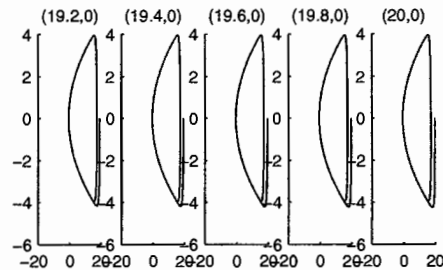


The first five appear to be closed curves, the others do not. So, now we are thinking that problems might occur on our system in the last five images; that is, starting with initial condition $(k, 0)$, where $16 \leq k \leq 20$. Again, we decrease the step size, but concentrate on the last five initial conditions. We use

step size $h = 0.01$ for our next image.



Some sort of strange behavior occurs between initial conditions $(19, 0)$ and $(20, 0)$. We increased the time interval to $[0, 20]$, then looked at initial conditions $(k, 0)$, with $k = 19.2, 19.4, \dots, 20$. We also try a step size of $h = 0.005$.

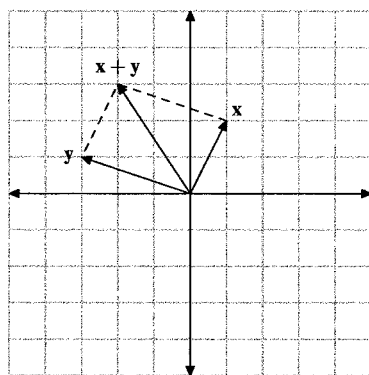


At this point, we will stop and leave the remainder of the exploration to you. Perhaps a reduction in step size will allow the solution with initial condition $(20, 0)$ to close, perhaps we can move beyond $(20, 0)$. But the machine time is quickly becoming prohibitive, at least on our system.

Chapter 7. Matrix Algebra

Section 7.1. Vectors and Matrices

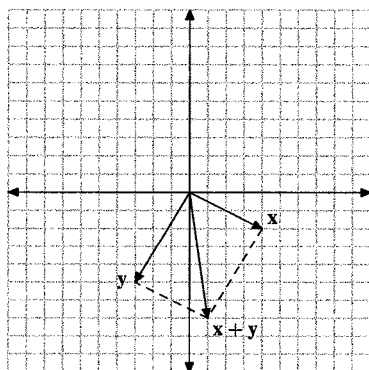
1. The geometric sum



Analytically,

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

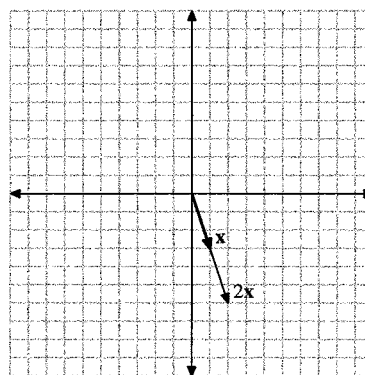
2. The geometric sum



Analytically,

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \end{pmatrix}$$

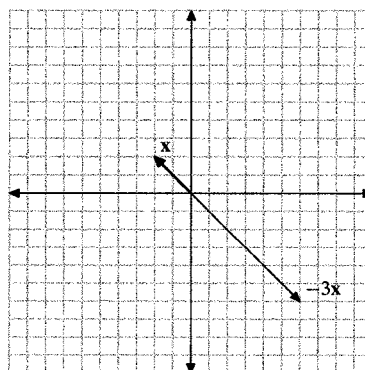
3. The geometric solution



Analytically,

$$2\mathbf{x} = 2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

4. The geometric solution

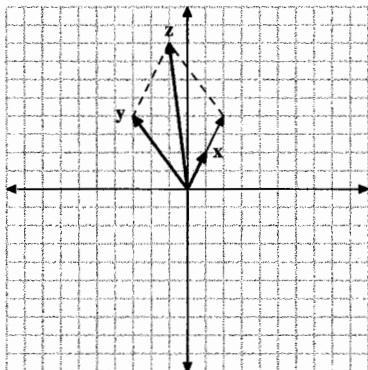


Analytically,

$$-3\mathbf{x} = -3 \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -18 \end{pmatrix}$$

5. The geometric solution

$$2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

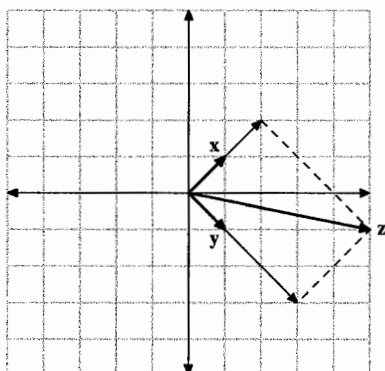


Checking,

$$2\mathbf{x} + 1\mathbf{y} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

6. The geometric solution

$$2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

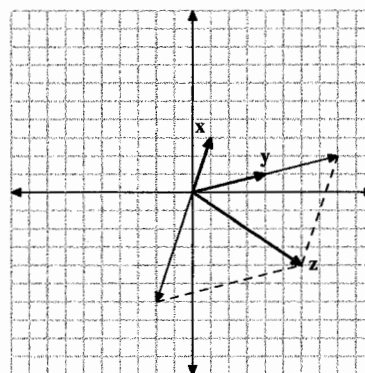


Checking,

$$2\mathbf{x} + 3\mathbf{y} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

7. The geometric solution

$$-2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

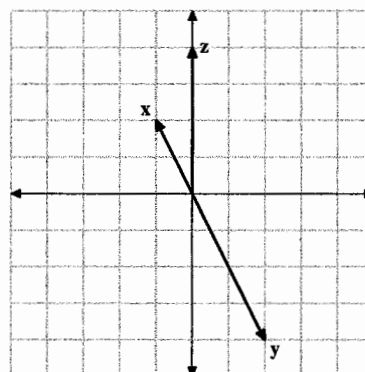


Checking,

$$-2\mathbf{x} + 2\mathbf{y} = -2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

8. You cannot write
- $\mathbf{z}$
- as a linear combination of
- $\mathbf{x}$
- and
- $\mathbf{y}$
- . It does not lie on the line spanned by
- $\mathbf{x}$
- and
- $\mathbf{y}$
- .

The geometric solution



Checking,

$$2\mathbf{x} + 1\mathbf{y} = 2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

9. If you multiply matrix
- A
- by
- $[2, 0, 0]^T$
- , then

$$A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2\mathbf{a}_1.$$

You can use similar strategies to triple and quadruple the second and third columns, respectively. Thus,

$$\begin{pmatrix} -1 & 2 & 4 \\ 0 & 5 & 2 \\ -1 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 6 & 16 \\ 0 & 15 & 8 \\ -2 & -6 & 12 \end{pmatrix}.$$

The columns of AB are

$$Ab_1 = [a_1, a_2, a_3] \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2a_1,$$

$$Ab_2 = [a_1, a_2, a_3] \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = 3a_2,$$

$$Ab_3 = [a_1, a_2, a_3] \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 4a_3.$$

Thus,

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

10. First,

$$\begin{aligned} A(\mathbf{x} + \mathbf{y}) &= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right), \\ &= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \\ &= \begin{pmatrix} 3 \\ 3 \end{pmatrix}. \end{aligned}$$

But,

$$\begin{aligned} A\mathbf{x} + A\mathbf{y} &= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 3 \end{pmatrix}. \end{aligned}$$

Secondly,

$$\begin{aligned} A(a\mathbf{x}) &= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \left(2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right), \\ &= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \\ &= \begin{pmatrix} -10 \\ 6 \end{pmatrix}. \end{aligned}$$

But

$$\begin{aligned} a(A\mathbf{x}) &= 2 \left(\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) \\ &= 2 \begin{pmatrix} -5 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -10 \\ 6 \end{pmatrix}. \end{aligned}$$

11. First,

$$\begin{aligned} A(a\mathbf{x}) &= \begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & 1 \end{pmatrix} \left(-3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 15 \\ -9 \\ 6 \end{pmatrix}. \end{aligned}$$

But,

$$\begin{aligned} a(A\mathbf{x}) &= -3 \left(\begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right) \\ &= -3 \begin{pmatrix} -5 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 15 \\ -9 \\ 6 \end{pmatrix}. \end{aligned}$$

Secondly,

$$\begin{aligned} A(\mathbf{x} + \mathbf{y}) &= \begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} \right), \\ &= \begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \\ &= \begin{pmatrix} -7 \\ -1 \\ 0 \end{pmatrix}. \end{aligned}$$

But,

$$\begin{aligned} A\mathbf{x} + A\mathbf{y} &= \begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} -1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}, \\ &= \begin{pmatrix} -5 \\ 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -7 \\ -1 \\ 0 \end{pmatrix}. \end{aligned}$$

12. (a) If $(x_1, \dots, x_n)^T, (y_1, \dots, y_n)^T \in \mathbf{R}^n$, then $(x_1 + y_1, \dots, x_n + y_n)^T \in \mathbf{R}^n$.
- (b) If $\alpha \in \mathbf{R}$ and $(x_1, \dots, x_n)^T \in \mathbf{R}^n$, then $(\alpha x_1, \dots, \alpha x_n)^T \in \mathbf{R}^n$.
- (c) If $(x_1, \dots, x_n)^T, (y_1, \dots, y_n)^T \in \mathbf{R}^n$, then $(x_1 + y_1, \dots, x_n + y_n)^T = (y_1 + x_1, \dots, y_n + x_n)^T$.
- (d) If $(x_1, \dots, x_n)^T, (y_1, \dots, y_n)^T$, and $(z_1, \dots, z_n)^T \in \mathbf{R}^n$, then $((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n)^T = (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n))^T$.
- (e) If $(x_1, \dots, x_n)^T \in \mathbf{R}^n$, then $(0, \dots, 0)^T \in \mathbf{R}^n$ and $(x_1 + 0, \dots, x_n + 0)^T = (x_1, \dots, x_n)^T$.
- (f) If $(x_1, \dots, x_n)^T \in \mathbf{R}^n$, then we know that $(-x_1, \dots, -x_n)^T \in \mathbf{R}^n$ and $(x_1 + (-x_1), \dots, x_n + (-x_n))^T = (0, \dots, 0)^T$.
- (g) If $\alpha \in \mathbf{R}, (x_1, \dots, x_n)^T, (y_1, \dots, y_n)^T \in \mathbf{R}^n$, then $(\alpha(x_1 + y_1), \dots, \alpha(x_n + y_n))^T = (\alpha x_1 + \alpha y_1, \dots, \alpha x_n + \alpha y_n)^T$.
- (h) If $\alpha, \beta \in \mathbf{R}, \mathbf{x} \in \mathbf{R}^n$, then

$$\begin{aligned} &((\alpha + \beta)x_1, \dots, (\alpha + \beta)x_n)^T \\ &= (\alpha x_1 + \beta x_1, \dots, \alpha x_n + \beta x_n)^T. \end{aligned}$$

- (i) If $\alpha, \beta \in \mathbf{R}, \mathbf{x} \in \mathbf{R}^n$, then

$$\begin{aligned} &((\alpha\beta)x_1, \dots, (\alpha\beta)x_n)^T \\ &= (\alpha(\beta x_1), \dots, \alpha(\beta x_n))^T. \end{aligned}$$

- (j) If $(x_1, \dots, x_n) \in \mathbf{R}^n$, then $(1x_1, \dots, 1x_n)^T = (x_1, \dots, x_n)^T$.

13. If A is $m \times n$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, then

$$\begin{aligned} A(\mathbf{x} + \mathbf{y}) &= [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n] \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \\ &= (x_1 + y_1)\mathbf{a}_1 + (x_2 + y_2)\mathbf{a}_2 + \dots \\ &\quad + (x_n + y_n)\mathbf{a}_n \\ &= (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n) \\ &\quad + (y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \dots + y_n\mathbf{a}_n) \\ &= [\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &\quad + [\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n] \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\ &= A\mathbf{x} + A\mathbf{y}. \end{aligned}$$

If A is $m \times n, a \in \mathbf{R}$, and $\mathbf{x} \in \mathbf{R}^n$, then

$$\begin{aligned} A(\alpha\mathbf{x}) &= [\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n] \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} \\ &= (\alpha x_1)\mathbf{a}_1 + (\alpha x_2)\mathbf{a}_2 + \dots + (\alpha x_n)\mathbf{a}_n \\ &= \alpha(x_1\mathbf{a}_1) + \alpha(x_2\mathbf{a}_2) + \dots + \alpha(x_n\mathbf{a}_n) \\ &= \alpha(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n) \\ &= \alpha \left([\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right) \\ &= \alpha(A\mathbf{x}). \end{aligned}$$

- 14.

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} \\ B + A &= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

15.

$$\begin{aligned}
 (A + B) + C &= \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \\
 A + (B + C) &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \\
 &\quad + \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.
 \end{aligned}$$

16.

$$\begin{aligned}
 (AB)C &= \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 5 & 0 \\ 1 & -4 \end{pmatrix} \\
 A(BC) &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} 5 & 0 \\ 1 & -4 \end{pmatrix}.
 \end{aligned}$$

17.

$$\begin{aligned}
 A(B + C) &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix} \\
 AB + AC &= \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix}.
 \end{aligned}$$

18.

$$\begin{aligned}
 (A + B)C &= \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \\
 AC + BC &= \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}.
 \end{aligned}$$

19.

$$\begin{aligned}
 (\alpha\beta)A &= -6 \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -6 & -12 \\ 6 & 0 \end{pmatrix} \\
 \alpha(\beta A) &= 3 \begin{pmatrix} -2 & -4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -6 & -12 \\ 6 & 0 \end{pmatrix}.
 \end{aligned}$$

20.

$$\begin{aligned}
 \alpha(AB) &= 3 \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 3 & -3 \end{pmatrix} \\
 (\alpha A)B &= \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 9 \\ 3 & -3 \end{pmatrix} \\
 A(\alpha B) &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 3 & -3 \end{pmatrix}.
 \end{aligned}$$

21.

$$\begin{aligned}
 (\alpha + \beta)A &= 1 \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \\
 \alpha A + \beta A &= \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ 2 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}.
 \end{aligned}$$

22.

$$\begin{aligned}
 \alpha(A + B) &= 3 \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 0 & 3 \end{pmatrix} \\
 \alpha A + \alpha B &= \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 3 \\ 3 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 9 \\ 0 & 3 \end{pmatrix}.
 \end{aligned}$$

23.

$$(A^T)^T = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}^T = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} = A.$$

24.

$$\begin{aligned}
 (\alpha A)^T &= \begin{pmatrix} 6 & -12 \\ -12 & 0 \end{pmatrix}^T = \begin{pmatrix} 6 & -12 \\ -12 & 0 \end{pmatrix} \\
 \alpha A^T &= -3 \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -12 \\ -12 & 0 \end{pmatrix}.
 \end{aligned}$$

25.

$$\begin{aligned}
 (A + B)^T &= \begin{pmatrix} -2 & -1 \\ 2 & 3 \end{pmatrix}^T = \begin{pmatrix} -2 & 2 \\ -1 & 3 \end{pmatrix} \\
 A^T + B^T &= \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -5 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} -2 & 2 \\ -1 & 3 \end{pmatrix}.
 \end{aligned}$$

26.

$$\begin{aligned}
 (AB)^T &= \begin{pmatrix} -8 & 22 \\ 0 & -20 \end{pmatrix}^T = \begin{pmatrix} -8 & 0 \\ 22 & -20 \end{pmatrix} \\
 B^T A^T &= \begin{pmatrix} 0 & -2 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} -8 & 0 \\ 22 & -20 \end{pmatrix}.
 \end{aligned}$$

27.

$$\begin{aligned}
 2\mathbf{x}_1 + 3\mathbf{x}_2 &= 2 \begin{pmatrix} 9 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} -6 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} 18 \\ 10 \end{pmatrix} + \begin{pmatrix} -18 \\ -3 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 7 \end{pmatrix} \\
 \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 7 \end{pmatrix}
 \end{aligned}$$

28.

$$\begin{aligned}
 -\mathbf{x}_3 + 5\mathbf{x}_4 &= -\begin{pmatrix} -1 \\ 8 \end{pmatrix} + 5 \begin{pmatrix} 7 \\ -9 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ -8 \end{pmatrix} + \begin{pmatrix} 35 \\ -45 \end{pmatrix} \\
 &= \begin{pmatrix} 36 \\ -53 \end{pmatrix} \\
 \begin{pmatrix} -1 & 7 \\ 8 & -9 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} &= \begin{pmatrix} 36 \\ -53 \end{pmatrix}
 \end{aligned}$$

29.

$$\begin{aligned}
 4\mathbf{x}_2 - 7\mathbf{x}_4 - 3\mathbf{x}_1 &= 4 \begin{pmatrix} -6 \\ -1 \end{pmatrix} - 7 \begin{pmatrix} 7 \\ -9 \end{pmatrix} - 3 \begin{pmatrix} 9 \\ 5 \end{pmatrix} \\
 &= \begin{pmatrix} -24 \\ -4 \end{pmatrix} + \begin{pmatrix} -49 \\ 63 \end{pmatrix} + \begin{pmatrix} -27 \\ -15 \end{pmatrix} \\
 &= \begin{pmatrix} -100 \\ 44 \end{pmatrix} \\
 \begin{pmatrix} -6 & 7 & 9 \\ -1 & -9 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -7 \\ -3 \end{pmatrix} &= \begin{pmatrix} -100 \\ 44 \end{pmatrix}
 \end{aligned}$$

30.

$$-2\mathbf{v}_2 + 4\mathbf{v}_3 + 5\mathbf{x}_4 = -2 \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} + 5 \begin{pmatrix} 7 \\ -9 \\ -9 \end{pmatrix}.$$

The dimensions do not allow these vectors to be added.

31.

$$\begin{aligned}
 4\mathbf{v}_1 - 3\mathbf{v}_4 + 3\mathbf{v}_2 + 4\mathbf{v}_3 &= 4 \begin{pmatrix} 10 \\ -5 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} \\
 &= \begin{pmatrix} 40 \\ -20 \\ 12 \end{pmatrix} + \begin{pmatrix} 3 \\ -9 \\ -18 \end{pmatrix} + \begin{pmatrix} 0 \\ 24 \\ 18 \end{pmatrix} + \begin{pmatrix} 0 \\ -36 \\ 28 \end{pmatrix} \\
 &= \begin{pmatrix} 43 \\ -41 \\ 40 \end{pmatrix}
 \end{aligned}$$

32.

$$\begin{aligned}
 \mathbf{v}_2 - 5\mathbf{v}_1 - 3\mathbf{v}_4 - 2\mathbf{v}_3 &= \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} - 5 \begin{pmatrix} 10 \\ -5 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} + \begin{pmatrix} -50 \\ 25 \\ -15 \end{pmatrix} + \begin{pmatrix} 3 \\ -9 \\ -18 \end{pmatrix} + \begin{pmatrix} 0 \\ 18 \\ -14 \end{pmatrix} \\
 &= \begin{pmatrix} -47 \\ 42 \\ -41 \end{pmatrix}
 \end{aligned}$$

33.

$$A\mathbf{y} = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 39 \\ 17 \end{pmatrix}.$$

34.

$$A\mathbf{x}_3 = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 8 \end{pmatrix} = \begin{pmatrix} -57 \\ -13 \end{pmatrix}.$$

35.

$$B\mathbf{z} = \begin{pmatrix} 6 & -1 & 7 \\ -1 & 8 & -9 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 20 \\ -17 \end{pmatrix}.$$

36.

$$B\mathbf{w} = \begin{pmatrix} -6 & -1 & 7 \\ -1 & 8 & -9 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -33 \\ 25 \end{pmatrix}$$

37.

$$C\mathbf{z} = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ 11 \\ 9 \end{pmatrix}.$$

38.

$$C\mathbf{w} = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 23 \\ -19 \\ -12 \end{pmatrix}.$$

39.

$$C\mathbf{v}_3 = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} = \begin{pmatrix} -7 \\ -51 \\ -12 \end{pmatrix}.$$

40.

$$\begin{aligned} D\mathbf{u} &= \begin{pmatrix} 10 & 0 & 0 & -1 \\ -5 & 8 & -9 & 3 \\ 3 & 6 & 7 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -14 \\ 59 \\ 25 \end{pmatrix}. \end{aligned}$$

41.

$$\mathbf{y}^T = \begin{pmatrix} 3 \\ -2 \end{pmatrix}^T = (3 \quad -2).$$

42.

$$\mathbf{u}^T = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}^T = (-1 \quad 3 \quad -2 \quad 4).$$

43.

$$A^T = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix}^T = \begin{pmatrix} 9 & 5 \\ -6 & -1 \end{pmatrix}.$$

44.

$$B^T = \begin{pmatrix} 6 & -1 & 7 \\ -1 & 8 & -9 \end{pmatrix}^T = \begin{pmatrix} 6 & -1 \\ -1 & 8 \\ 7 & -9 \end{pmatrix}.$$

45.

$$C^T = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix}^T = \begin{pmatrix} 10 & -5 & 3 \\ 0 & 8 & 6 \\ -1 & 3 & 6 \end{pmatrix}.$$

46.

$$\begin{aligned} D^T &= \begin{pmatrix} 10 & 0 & 0 & -1 \\ -5 & 8 & -9 & 3 \\ 3 & 6 & 7 & 6 \end{pmatrix}^T \\ &= \begin{pmatrix} 10 & -5 & 3 \\ 0 & 8 & 6 \\ 0 & -9 & 7 \\ -1 & 3 & 6 \end{pmatrix}. \end{aligned}$$

$$47. \quad 3x + 4y = 7 \text{ is equivalent to } \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (7).$$

$$48. \quad -x + 4y = 3 \text{ is equivalent to } \begin{pmatrix} -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (3).$$

49.

$$\begin{aligned} 3x + 4y &= 7 \\ -x + 3y &= 2 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 3 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}.$$

50.

$$\begin{aligned} x - 3y &= 5 \\ -2x + 3y &= -2 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 1 & -3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

51.

$$\begin{aligned} -x_1 + x_3 &= 0 \\ 2x_1 + 3x_2 &= 3 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

52.

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ -2x_1 + 3x_2 - 2x_3 &= -3 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 1 & 3 & -1 \\ -2 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

53.

$$\begin{aligned} -x_1 + x_3 &= 0 \\ 2x_1 + 3x_2 &= 3 \\ x_2 - x_3 &= 4 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}.$$

54.

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ -2x_1 + 3x_2 - 2x_3 &= -3 \\ 2x_1 - x_3 &= 0 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ -2 & 3 & -2 & -3 \\ 2 & 0 & -1 & 0 \end{pmatrix}.$$

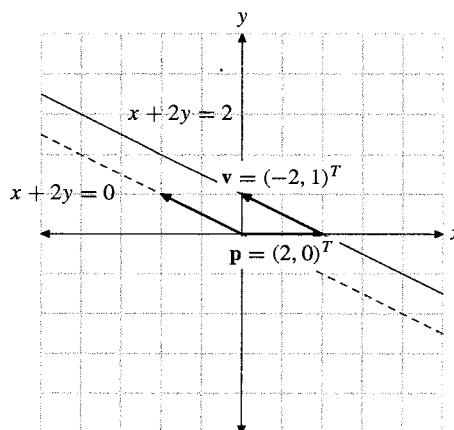
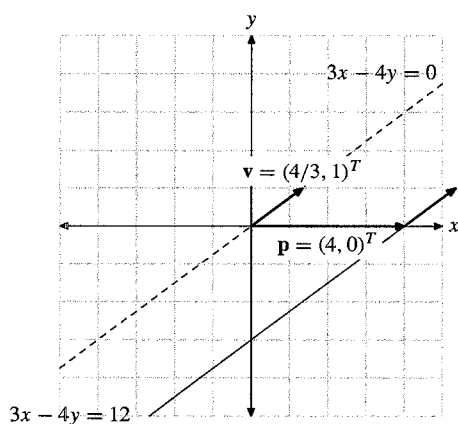
Section 7.2. Systems of Linear Equations with Two or Three Variables

1. Let $(x, y)^T$ represent a solution, the solve $3x - 4y = 12$ for x , obtaining $x = (4y + 12)/3$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (4y + 12)/3 \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + y \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}.$$

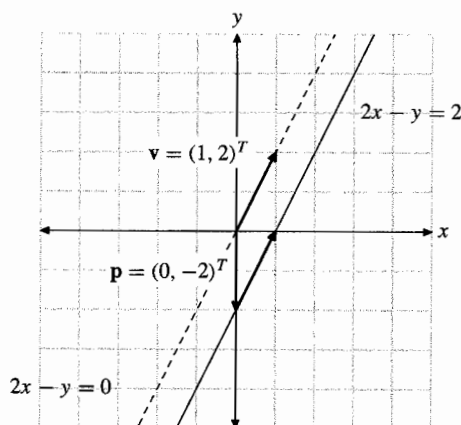
2. Let $(x, y)^T$ represent a solution, the solve $x + 2y = 2$ for x , obtaining $x = 2 - 2y$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 - 2y \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

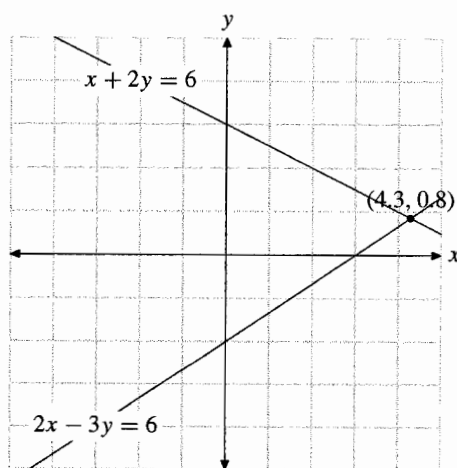


3. Let $(x, y)^T$ represent a solution, then solve $2x - y = 2$ for y , obtaining $y = 2x - 2$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + x \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$



5. The plot of the system shows a single point of intersection, approximately $(4.3, 0.8)$.



The augmented matrix for the system $x + 2y = 6$, $2x - 3y = 6$ is

$$\begin{pmatrix} 1 & 2 & 6 \\ 2 & -3 & 6 \end{pmatrix}.$$

Add -2 times the first row to the second row to obtain

$$\begin{pmatrix} 1 & 2 & 6 \\ 0 & -7 & -6 \end{pmatrix}.$$

Solve the second equation for y .

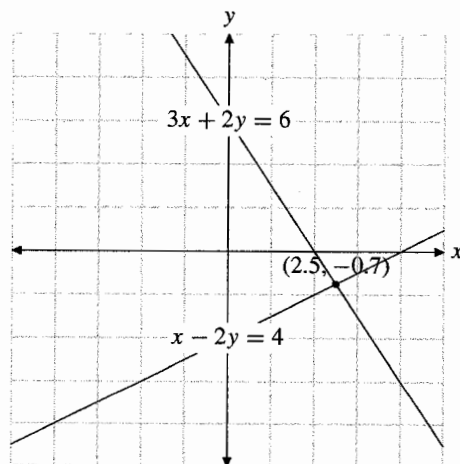
$$\begin{aligned} -7y &= -6 \\ y &= \frac{6}{7} \end{aligned}$$

Substitute $y = 6/7$ in the first equation, $x + 2y = 6$, then solve for x .

$$\begin{aligned} x + 2\left(\frac{6}{7}\right) &= 6 \\ x &= \frac{30}{7} \end{aligned}$$

Hence, the solution is $(30/7, 6/7) \approx (4.28, 0.85)$.

6. The plot of the system shows a single point of intersection, approximately $(2.5, -0.7)$.



The augmented matrix for the system $x - 2y = 4$, $3x + 2y = 6$ is

$$\begin{pmatrix} 1 & -2 & 4 \\ 3 & 2 & 6 \end{pmatrix}.$$

Add -3 times the first row to the second row to obtain

$$\begin{pmatrix} 1 & -2 & 4 \\ 0 & 8 & -6 \end{pmatrix}.$$

Solve the second equation for y .

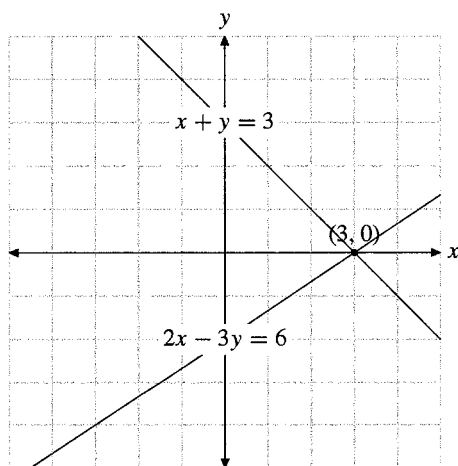
$$\begin{aligned} -8y &= -6 \\ y &= -\frac{3}{4} \end{aligned}$$

Substitute $y = -3/4$ in the first equation, $x - 2y = 4$, then solve for x .

$$\begin{aligned} x - 2\left(-\frac{3}{4}\right) &= 4 \\ x &= \frac{5}{2} \end{aligned}$$

Hence, the solution is $(5/2, -3/4) \approx (2.5, -0.75)$.

7. The plot of the system shows a single point of intersection, approximately $(3, 0)$.



The augmented matrix for the system $x + y = 3$, $2x - 3y = 6$ is

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & -3 & 6 \end{pmatrix}.$$

Add -2 times the first row to the second row to obtain

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & -5 & 0 \end{pmatrix}.$$

Solve the second equation for y .

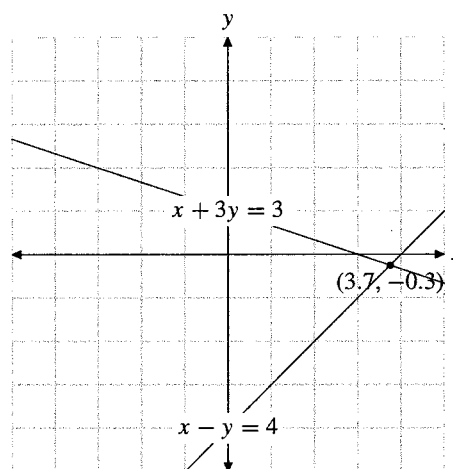
$$\begin{aligned} -5y &= 0 \\ y &= 0 \end{aligned}$$

Substitute $y = 0$ in the first equation, $x + y = 3$, then solve for x .

$$\begin{aligned} x + 0 &= 3 \\ x &= 3 \end{aligned}$$

Hence, the solution is $(3, 0)$.

8. The plot of the system shows a single point of intersection, approximately $(3.7, -0.3)$.



The augmented matrix for the system $x - y = 4$, $x + 3y = 3$ is

$$\begin{pmatrix} 1 & -1 & 4 \\ 1 & 3 & 3 \end{pmatrix}.$$

Add -1 times the first row to the second row to obtain

$$\begin{pmatrix} 1 & -1 & 4 \\ 0 & 4 & -1 \end{pmatrix}.$$

Solve the second equation for y .

$$4y = -1$$

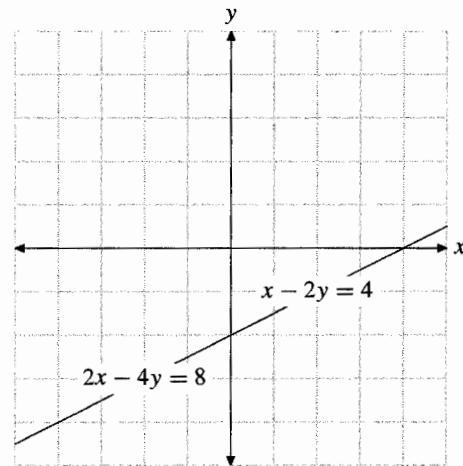
$$y = -\frac{1}{4}$$

Substitute $y = -1/4$ in the first equation, $x - y = 4$, then solve for x .

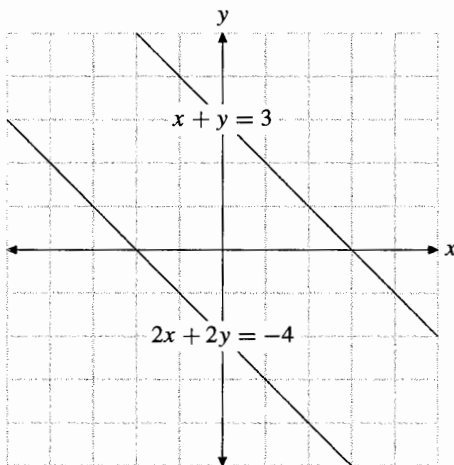
$$x - \left(\frac{1}{4}\right) = 4$$

$$x = \frac{15}{4}$$

Hence, the solution is $(15/4, -1/4) \approx (3.75, -0.25)$.



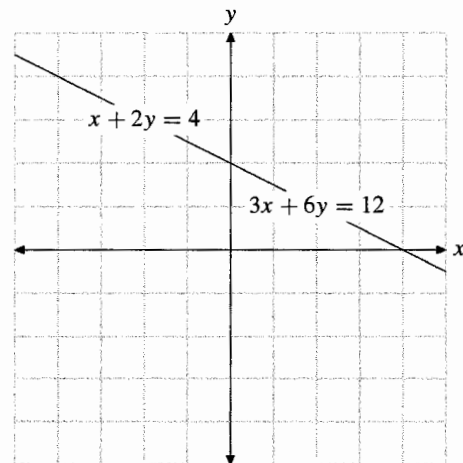
9. The lines having equations $x + y = 3$ and $2x + 2y = -4$ are parallel, as seen in the following figure.



The system has no solutions. The system is inconsistent.

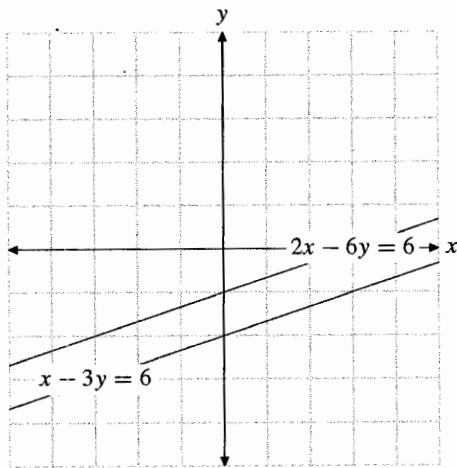
Any point on the line is a solution. The system is consistent. §

11. It is easy to see that the second equation is a multiple of the first. Thus, the lines having equations $x + 2y = 4$ and $3x + 6y = 12$ are the same line, as seen in the following figure.



Any point on the line is a solution. The system is consistent.

10. It is easy to see that the second equation is a multiple of the first. Thus, the lines having equations $x - 2y = 4$ and $2x - 4y = 8$ are the same line, as seen in the following figure.
12. The lines having equations $x - 3y = 6$ and $2x - 6y = 12$ are parallel, as seen in the following figure.



The system has no solutions. The system is inconsistent.

13. Solve the equation $x + 2y - 3z = 12$ for x .

$$x = 12 - 2y + 3z$$

Thus,

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 12 - 2y + 3z \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

14. In this case, it is easier to solve the equation $2x + y - 3z = 6$ for y .

$$y = 6 - 2x + 3z$$

Thus,

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x \\ 6 - 2x + 3z \\ z \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} + x \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}. \end{aligned}$$

15. In this case, it is easier to solve the equation $3x - 4y + z = 12$ for z .

$$z = 12 - 3x + 4y$$

Thus,

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x \\ y \\ 12 - 3x + 4y \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 12 \end{pmatrix} + x \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}. \end{aligned}$$

16. Solve the equation $2x + 3y + 4z = 6$ for x .

$$x = 3 - \frac{3}{2}y + 2z$$

Thus,

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 - (3/2)y + 2z \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

17. The augmented matrix for the system is

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & 5 & 8 & 40 \end{array} \right)$$

Subtract 2 times the first row from the second row.

$$\begin{array}{rcl} x + 2y - 3z & = & 6 \\ y + 14z & = & 28 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & 14 & 28 \end{array} \right)$$

Note that z is free. Let $z = t$ in the second equation and solve for y .

$$y + 14t = 28$$

$$y = 28 - 14t$$

Sub $z = t$ and $y = 28 - 14t$ in the first equation and solve for x .

$$\begin{aligned} x + 2y - 3z &= 6 \\ x + 2(28 - 14t) - 3t &= 6 \\ x &= 31t - 50 \end{aligned}$$

Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31t - 50 \\ 28 - 14t \\ t \end{pmatrix} = \begin{pmatrix} -50 \\ 28 \\ 0 \end{pmatrix} + t \begin{pmatrix} 31 \\ -14 \\ 1 \end{pmatrix}.$$

18. The augmented matrix for the system is

$$\left(\begin{array}{cccc|c} 1 & -3 & 4 & 12 & 12 \\ -3 & 10 & 8 & 40 & 76 \end{array} \right)$$

Add 3 times the first row to the second row.

$$\begin{array}{rcl} x - 3y + 4z = 12 & \left(\begin{array}{cccc|c} 1 & -3 & 4 & 12 & 12 \\ 0 & 1 & 20 & 76 & 76 \end{array} \right) \\ y + 20z = 76 & \end{array}$$

Note that z is free. Let $z = t$ in the second equation and solve for y .

$$\begin{aligned} y + 20t &= 76 \\ y &= 76 - 20t \end{aligned}$$

Sub $z = t$ and $y = 76 - 20t$ in the first equation and solve for x .

$$\begin{aligned} x - 3y + 4z &= 12 \\ x - 3(76 - 20t) + 4t &= 12 \\ x &= 240 - 64t \end{aligned}$$

Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 240 - 64t \\ 76 - 20t \\ t \end{pmatrix} = \begin{pmatrix} 240 \\ 76 \\ 0 \end{pmatrix} + t \begin{pmatrix} -64 \\ -20 \\ 1 \end{pmatrix}.$$

19. The augmented matrix for the system is

$$\left(\begin{array}{cccc|c} 2 & -4 & 5 & 40 & 40 \\ -4 & 10 & -4 & 20 & 100 \end{array} \right)$$

Add 2 times the first row to the second row.

$$\begin{array}{rcl} 2x - 4y + 5z = 40 & \left(\begin{array}{cccc|c} 2 & -4 & 5 & 40 & 40 \\ 0 & 2 & 6 & 100 & 100 \end{array} \right) \\ 2y + 6z = 100 & \end{array}$$

Note that z is free. Let $z = t$ in the second equation and solve for y .

$$\begin{aligned} 2y + 6t &= 100 \\ y &= 50 - 3t \end{aligned}$$

Sub $z = t$ and $y = 50 - 3t$ in the first equation and solve for x .

$$\begin{aligned} 2x - 4y + 5z &= 40 \\ 2x - 4(50 - 3t) + 5t &= 40 \\ x &= 120 - \frac{17}{2}t \end{aligned}$$

Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 120 - (17/2)t \\ 50 - 3t \\ t \end{pmatrix} = \begin{pmatrix} 120 \\ 50 \\ 0 \end{pmatrix} + t \begin{pmatrix} -17/2 \\ -3 \\ 1 \end{pmatrix}.$$

20. The augmented matrix for the system is

$$\left(\begin{array}{cccc|c} 3 & -4 & 5 & 60 & 60 \\ -6 & 9 & -4 & -150 & -30 \end{array} \right)$$

Add 2 times the first row to the second row.

$$\begin{array}{rcl} 3x - 4y + 5z = 60 & \left(\begin{array}{cccc|c} 3 & -4 & 5 & 60 & 60 \\ 0 & 1 & 6 & -30 & -30 \end{array} \right) \\ y + 6z = -30 & \end{array}$$

Note that z is free. Let $z = t$ in the second equation and solve for y .

$$\begin{aligned} y + 6t &= -30 \\ y &= -30 - 6t \end{aligned}$$

Sub $z = t$ and $y = -30 - 6t$ in the first equation and solve for x .

$$\begin{aligned} 3x - 4y + 5z &= 60 \\ 3x - 4(-30 - 6t) + 5t &= 60 \\ x &= -20 - \frac{29}{3}t \end{aligned}$$

Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -20 - (29/3)t \\ -30 - 6t \\ t \end{pmatrix} \\ = \begin{pmatrix} -20 \\ -30 \\ 0 \end{pmatrix} + t \begin{pmatrix} -29/3 \\ -6 \\ 1 \end{pmatrix}.$$

21. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & -1 & 1 & 4 \\ -1 & 2 & 2 & 6 \end{array} \right).$$

Subtract 2 times row 1 from row 2. Add row 1 to row 3.

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & -3 & -1 & -2 \\ 0 & 3 & 3 & 9 \end{array} \right)$$

Add row 2 to row 3.

$$\begin{array}{rcl} x + y + z & = & 3 \\ -3y - z & = & -2 \\ 2z & = & 7 \end{array} \quad \left(\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 2 & 7 \end{array} \right)$$

Solve the last equation for z .

$$\begin{aligned} 2z &= 7 \\ z &= \frac{7}{2} \end{aligned}$$

Substitute $z = 7/2$ in the second equation and solve for y .

$$\begin{aligned} -3y - \frac{7}{2} &= -2 \\ y &= -\frac{1}{2} \end{aligned}$$

Substitute $y = -1/2$ and $z = 7/2$ in the first equation and solve for x .

$$\begin{aligned} x - \frac{1}{2} + \frac{7}{2} &= 3 \\ x &= 0 \end{aligned}$$

Hence, the solution is $(x, y, z) = (0, -1/2, 7/2)$.

22. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 1 & -1 & 2 & 4 \\ -2 & 1 & -1 & 6 \\ 3 & 1 & -3 & 7 \end{array} \right)$$

Add 2 times row 1 to row 2. Subtract 3 times row 1 from row 3.

$$\left(\begin{array}{cccc} 1 & -1 & 2 & 4 \\ 0 & -1 & 3 & 14 \\ 0 & 4 & -9 & -5 \end{array} \right)$$

Add 4 times row 2 to row 3.

$$\begin{array}{rcl} x - y + 2z & = & 4 \\ -y + 3z & = & 14 \\ 3z & = & 51 \end{array} \quad \left(\begin{array}{cccc} 1 & -1 & 2 & 4 \\ 0 & -1 & 3 & 14 \\ 0 & 0 & 3 & 51 \end{array} \right)$$

Solve the last equation for z .

$$\begin{aligned} 3z &= 51 \\ z &= 17 \end{aligned}$$

Substitute $z = 17$ in the second equation and solve for y .

$$\begin{aligned} -y + 3(17) &= 14 \\ y &= 37 \end{aligned}$$

Substitute $y = 37$ and $z = 17$ in the first equation and solve for x .

$$\begin{aligned} x - 37 + 2(17) &= 4 \\ x &= 7 \end{aligned}$$

Hence, the solution is $(x, y, z) = (7, 37, 17)$.

23. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 1 & -2 & 5 & 10 \\ -2 & 1 & 1 & 12 \\ 2 & -1 & -2 & 6 \end{array} \right)$$

Add 2 times row 1 to row 2. Subtract 2 times row 1 from row 3.

$$\left(\begin{array}{cccc} 1 & -2 & 5 & 10 \\ 0 & -3 & 11 & 32 \\ 0 & 3 & -12 & -14 \end{array} \right)$$

Add row 2 to row 3.

$$\begin{array}{rcl} x - 2y + 5z = 10 & & \\ -3y + 11z = 32 & & \\ -z = 18 & & \end{array} \quad \left(\begin{array}{cccc} 1 & -2 & 5 & 10 \\ 0 & -3 & 11 & 32 \\ 0 & 0 & -1 & 18 \end{array} \right)$$

Solve the last equation for z .

$$\begin{aligned} -z &= 18 \\ z &= -18 \end{aligned}$$

Substitute $z = -18$ in the second equation and solve for y .

$$\begin{aligned} -3y + 11(-18) &= 32 \\ y &= -\frac{230}{3} \end{aligned}$$

Substitute $y = -230/3$ and $z = -18$ in the first equation and solve for x .

$$\begin{aligned} x - 2\left(-\frac{230}{3}\right) + 5(-18) &= 10 \\ x &= -\frac{160}{3} \end{aligned}$$

Hence, the solution is $(x, y, z) = (-160/3, -230/3, -18)$.

24. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 1 & 1 & -2 & 4 \\ -3 & 1 & -1 & 6 \\ 2 & 6 & 2 & 0 \end{array} \right)$$

Add 3 times row 1 to row 2. Subtract 2 times row 1 from row 3.

$$\left(\begin{array}{cccc} 1 & 1 & -2 & 4 \\ 0 & 4 & -7 & 18 \\ 0 & 4 & 6 & -8 \end{array} \right)$$

Subtract row 2 from row 3.

$$\begin{array}{rcl} x + y - 2z = 4 & & \\ 4y - 7z = 18 & & \\ 13z = -26 & & \end{array} \quad \left(\begin{array}{cccc} 1 & 1 & -2 & 4 \\ 0 & 4 & -7 & 18 \\ 0 & 0 & 13 & -26 \end{array} \right)$$

Solve the last equation for z .

$$\begin{aligned} 13z &= -26 \\ z &= -2 \end{aligned}$$

Substitute $z = -2$ in the second equation and solve for y .

$$\begin{aligned} 4y - 7(-2) &= 18 \\ y &= 1 \end{aligned}$$

Substitute $y = 1$ and $z = -2$ in the first equation and solve for x .

$$\begin{aligned} x + 1 - 2(-2) &= 4 \\ x &= -1 \end{aligned}$$

Hence, the solution is $(x, y, z) = (-1, 1, -2)$.

25. No. There are only three possible solution sets: a point, a line, or no solutions. A circle is none of these.
26. The set S consists of a circle of radius 2, centered at $\mathbf{x}_0$. The set also includes the center at $\mathbf{x}_0$. Again, this set is not a point or a line, so it cannot be the solution set of a system of linear equations in two unknowns.
27. The set $S = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} : t > 0 \right\}$ in the right half plane in $\mathbf{R}^2$, not including the vertical axis. This set is not a point or a line, so it cannot be the solution set of a system of linear equations in two unknowns.
28. Assuming that s is some fixed number, then $S = \left\{ \begin{pmatrix} t \\ s \end{pmatrix} : t > 0 \right\}$ is a half line in the plane. This set is not a line or point. It is not infinitely long as we cannot move infinitely in two directions as we can on a line.
29. As $y = t \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ in a line in $\mathbf{R}^2$, one would expect to find a system whose solution set is this line. Indeed, if

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 2 \\ -3 \end{pmatrix},$$

then $x = 2t$ and $y = -3t$. Eliminating the parameter t by solving one equation for t and substituting in the other equation. Then, $t = x/2$ and

$$\begin{aligned} y &= -3 \begin{pmatrix} x \\ 2 \end{pmatrix} \\ 2y &= -3x \\ 3x + 2y &= 0. \end{aligned}$$

Of course, there are other systems having this line as their solution set. For example, consider the system

$$\begin{aligned} 3x + 2y &= 0 \\ 6x + 4y &= 0. \end{aligned}$$

30. Again,

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} -1 \\ 4 \end{pmatrix},$$

so $x = -t$ and $y = 4t$. Eliminating the parameter t yields $y = -4x$ or $4x + y = 0$. Other systems will also have this line as a solution. For example, consider

$$\begin{aligned} 4x + y &= 0, \\ 8x + 2y &= 0. \end{aligned}$$

31. The line passes through $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the direction $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Write

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \end{pmatrix},$$

so that

$$\begin{aligned} x &= 2t, \\ y &= 1 - 3t. \end{aligned}$$

Multiply the first equation by 3 and the second equation by 2.

$$\begin{aligned} 3x &= 6t \\ 2y &= 2 - 6t. \end{aligned}$$

Add these to get $3x + 2y = 2$. Of course, other systems are possible. For example, consider

$$\begin{aligned} 3x + 2y &= 2 \\ 6x + 4y &= 4. \end{aligned}$$

32. Again,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

goes

$$\begin{aligned} x &= -2, \\ y &= 3 - 3t. \end{aligned}$$

This is somewhat tricky as x is fixed while y varies from $-\infty$ to $+\infty$. Thus, we are talking about the vertical line with equation $x = -2$.

33. A point has zero dimension. A line is one-dimensional.

34. A point is zero dimensional, a line is one-dimensional, and a plane is two dimensional.

35. In $\mathbf{R}^4$ we can expect dimensions 0, 1, 2, and 3.

36. Solve for x_1 ,

$$x_1 = 2 - 2x_2 + 2x_3 - x_4.$$

Thus, solutions have the form

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 2 - 2x_2 + 2x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ &\quad + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

The solution set has dimension 3 ($(2, 0, 0, 0)^T$ is just a translation).

37. Solve the second equation for x_2

$$x_2 = 3 + 3x_3 + x_4.$$

Substitute in the first equation

$$\begin{aligned} x_1 + 2(3 + 3x_3 + x_4) - 2x_3 + x_4 &= 2 \\ x_1 + 6 + 4x_3 + 3x_4 &= 2, \end{aligned}$$

and solve for

$$x_1 = -4 - 4x_3 - 3x_4.$$

Thus, solutions have the form

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} -4 - 4x_3 - 3x_4 \\ 3 + 3x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

This set has dimension 2. Think of the set as a translated plane in $\mathbf{R}^4$.

38. Solve the third equation for x_3

$$x_3 = x_4.$$

Substitute this in the second equation and solve for x_2

$$x_2 - 3x_4 - x_4 = 3$$

$$x_2 = 3 + 4x_4.$$

Substitute both of these in the first equation and solve for x_1

$$x_1 + 2(3 + 4x_4) - 2x_4 + x_4 = 2$$

$$x_1 + 6 + 7x_4 = 2$$

$$x_1 = -4 - 7x_4.$$

————— × —————

Thus, solutions have the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 - 7x_4 \\ 3 + 4x_4 \\ x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -7 \\ 4 \\ 1 \\ 1 \end{pmatrix}.$$

This solution has dimension 1. Think of the set as a translated line in $\mathbf{R}^4$.

Section 7.3. Solving Systems of Equations

1. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 4 \\ 2 & -1 & -1 & 6 \\ 4 & 1 & 1 & 14 \end{array} \right)$$

Use the pivot in the first row to zero the entries below it in the first column. Add -2 times row 1 to row 2. Add -4 times row 1 to row 3.

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 4 \\ 0 & -3 & -3 & -2 \\ 0 & -3 & -3 & -3 \end{array} \right)$$

Use the pivot in the second row to zero the entries below it in the second column. Add -1 times row 2 to row 3.

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 4 \\ 0 & -3 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 + x_3 = 4 \\ -3x_2 - 3x_3 = -2 \end{array}$$

Note that x_1 and x_2 are pivot variables and x_3 is a free variable. Set $x_3 = t$ in the second equation and solve for x_2 .

$$-3x_2 - 3t = -2$$

$$x_2 = -t + 2/3$$

Set $x_3 = t$ and $x_2 = -t + 2/3$ in the first equation and solve for x_1 .

$$\begin{aligned} x_1 + (-t + 2/3) + t &= 4 \\ x_1 &= 10/3 \end{aligned}$$

Thus,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10/3 \\ -t + 2/3 \\ t \end{pmatrix} = \begin{pmatrix} 10/3 \\ 2/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

2. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 1 & 2 & -1 & 4 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 1 & 5 \end{array} \right)$$

Use the pivot in the first row to zero the entries below it in the first column. Add -2 times row 1 to row 2. Add -3 times row 1 to row 3.

$$\left(\begin{array}{cccc} 1 & 2 & -1 & 4 \\ 0 & -5 & 4 & -6 \\ 0 & -5 & 4 & -7 \end{array} \right)$$

Use the pivot in the second row to zero the entries below it in the second column. Add -1 times row 2

to row 3.

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & -5 & 4 & -6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that there is a pivot in the last column of the third row. Therefore, the system is inconsistent.

3. The augmented matrix for the system is

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ -1 & -1 & 2 & 4 \\ 1 & 3 & 6 & 7 \end{pmatrix}$$

Use the pivot in the first row to zero the entries below it in the first column. Add 1 times row 1 to row 2. Add -1 times row 1 to row 3.

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 4 & 6 \\ 0 & 1 & 4 & 5 \end{pmatrix}$$

Use the pivot in the second row to zero the entries below it in the second column. Add -1 times row 2 to row 3.

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that there is a pivot in the last column of the third row. Therefore, the system is inconsistent.

4. The augmented matrix for the system is

$$\begin{pmatrix} 1 & 1 & -2 & 2 \\ 3 & -1 & -1 & 3 \\ 5 & 1 & -5 & 7 \end{pmatrix}$$

Use the pivot in the first row to zero the entries below it in the first column. Add -3 times row 1 to row 2. Add -5 times row 1 to row 3.

$$\begin{pmatrix} 1 & 1 & -2 & 2 \\ 0 & -4 & 5 & -3 \\ 0 & -4 & 5 & -3 \end{pmatrix}$$

Use the pivot in the second row to zero the entries below it in the second column. Add -1 times row 2 to row 3.

$$\begin{pmatrix} 1 & 1 & -2 & 2 \\ 0 & -4 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} x_1 + x_2 - 2x_3 = 2 \\ -4x_2 + 5x_3 = -3 \end{array}$$

Note that x_1 and x_2 are pivot variables and x_3 is a free variable. Set $x_3 = t$ in the second equation and solve for x_2 .

$$\begin{aligned} -4x_2 + 5t &= -3 \\ x_2 &= 3/4 + 5t/4 \end{aligned}$$

Set $x_3 = t$ and $x_2 = 3/4 + 5t/4$ in the first equation and solve for x_1 .

$$\begin{aligned} x_1 + (3/4 + 5t/4) - 2t &= 2 \\ x_1 &= 5/4 + 3t/4 \end{aligned}$$

Thus,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5/4 + 3t/4 \\ 3/4 + 5t/4 \\ t \end{pmatrix} = \begin{pmatrix} 5/4 \\ 3/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3/4 \\ 5/4 \\ 1 \end{pmatrix}.$$

5. The augmented matrix for the system is

$$\begin{pmatrix} 1 & 2 & -2 & 6 \\ 2 & 4 & -4 & 12 \\ 3 & 6 & -6 & 18 \end{pmatrix}$$

Use the pivot in the first row to zero the entries below it in the first column. Add -2 times row 1 to row 2. Add -3 times row 1 to row 3.

$$\begin{pmatrix} 1 & 2 & -2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad x_1 + 2x_2 - 2x_3 = 6$$

Note that x_1 is a pivot variable and x_2 and x_3 are free variables. Set $x_2 = s$ and $x_3 = t$ in the first equation and solve for x_1 .

$$\begin{aligned} x_1 + 2s - 2t &= 6 \\ x_1 &= 6 - 2s + 2t \end{aligned}$$

Thus,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 - 2s + 2t \\ s \\ t \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

6. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 1 & -2 & -3 & 6 \\ 4 & -8 & -12 & 24 \\ 2 & -4 & -6 & 12 \end{array}\right)$$

Use the pivot in the first row to zero the entries below it in the first column. Add -4 times row 1 to row 2. Add -2 times row 1 to row 3.

$$\left(\begin{array}{cccc} 1 & -2 & -3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad x_1 - 2x_2 - 3x_3 = 6$$

Note that x_1 is a pivot variable and x_2 and x_3 are free variables. Set $x_2 = s$ and $x_3 = t$ in the first equation and solve for x_1 .

$$\begin{aligned} x_1 - 2s - 3t &= 6 \\ x_1 &= 6 + 2s + 3t \end{aligned}$$

Thus,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 + 2s + 3t \\ s \\ t \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

7. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 0 & 1 & -2 & 4 \\ 1 & 2 & -2 & 6 \\ 1 & 4 & -6 & 14 \end{array}\right)$$

Exchange rows 1 and 2.

$$\left(\begin{array}{cccc} 1 & 2 & -2 & 6 \\ 0 & 1 & -2 & 4 \\ 1 & 4 & -6 & 14 \end{array}\right)$$

Use the pivot in the first row to zero the entries below it in the first column. Add -1 times row 1 to row 3.

$$\left(\begin{array}{cccc} 1 & 2 & -2 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 2 & -4 & 8 \end{array}\right)$$

Use the pivot in the second row to zero the entries below it in the second column. Add -2 times row 2 to row 3.

$$\left(\begin{array}{cccc} 1 & 2 & -2 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{aligned} x_1 + 2x_2 - 2x_3 &= 6 \\ x_2 - 2x_3 &= 4 \end{aligned}$$

Note that x_1 and x_2 are pivot variables and x_3 is a free variable. Set $x_3 = t$ in the second equation and solve for x_2 .

$$\begin{aligned} x_2 - 2t &= 4 \\ x_2 &= 4 + 2t \end{aligned}$$

Set $x_2 = 4 + 2t$ and $x_3 = t$ in the first equation and solve for x_1 .

$$\begin{aligned} x_1 + 2(4 + 2t) - 2t &= 6 \\ x_1 &= -2 - 2t \end{aligned}$$

Thus,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 - 2t \\ 4 + 2t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

8. The augmented matrix for the system is

$$\left(\begin{array}{cccc} 0 & 1 & -4 & 6 \\ 1 & 2 & -1 & 2 \\ 2 & 1 & -2 & 4 \end{array}\right)$$

Exchange rows 1 and 2.

$$\left(\begin{array}{cccc} 1 & 2 & -1 & 2 \\ 0 & 1 & -4 & 6 \\ 2 & 1 & -2 & 4 \end{array}\right)$$

Use the pivot in the first row to zero the entries below it in the first column. Add -2 times row 1 to row 3.

$$\left(\begin{array}{cccc} 1 & 2 & -1 & 2 \\ 0 & 1 & -4 & 6 \\ 0 & -3 & 0 & 0 \end{array}\right)$$

Use the pivot in the second row to zero the entries below it in the second column. Add 3 times row 2 to row 3.

$$\left(\begin{array}{cccc} 1 & 2 & -1 & 2 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & -12 & 18 \end{array}\right) \quad \begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_2 - 4x_3 &= 6 \\ -12x_3 &= 18 \end{aligned}$$

There are no free variables. Solve the third equation for x_3 .

$$\begin{aligned} -12x_3 &= 18 \\ x_3 &= -3/2 \end{aligned}$$

Set $x_3 = -3/2$ in the second equation and solve for x_2 .

$$\begin{aligned} x_2 - 4(-3/2) &= 6 \\ x_2 &= 0 \end{aligned}$$

Set $x_2 = 0$ and $x_3 = -3/2$ in the first equation and solve for x_1 .

$$\begin{aligned} x_1 + 2(0) - (-3/2) &= 2 \\ x_1 &= 1/2 \end{aligned}$$

Thus, the solution is $(x_1, x_2, x_3) = (1/2, 0, -3/2)$.

9. The augmented matrix for the system is

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 6 \\ 2 & 1 & -2 & -1 & 4 \\ 0 & 6 & 4 & -1 & 4 \end{array} \right)$$

Use the pivot in the first row to zero the entries below it in the first column. Add -2 times row 1 to row 2.

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 6 \\ 0 & -3 & 4 & -3 & -8 \\ 0 & 6 & 4 & -1 & 4 \end{array} \right)$$

Use the pivot in the second row to zero the entries below it in the second column. Add 2 times row 2 to row 3.

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 6 \\ 0 & -3 & 4 & -3 & -8 \\ 0 & 0 & 12 & -7 & -12 \end{array} \right)$$

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 6 \\ -3x_2 + 4x_3 - 3x_4 &= -8 \\ +12x_3 - 7x_4 &= -12 \end{aligned}$$

Note that x_1 , x_2 , and x_3 are pivot variables and x_4 is a free variable. Set $x_4 = t$ in the third equation and solve for x_3 .

$$\begin{aligned} 12x_3 - 7t &= -12 \\ x_3 &= -1 + 7t/12 \end{aligned}$$

Set $x_3 = -1 + 7t/12$ and $x_4 = t$ in the second equation and solve for x_2 .

$$\begin{aligned} -3x_2 + 4(-1 + 7t/12) - 3t &= -8 \\ x_2 &= 4/3 - 2t/9 \end{aligned}$$

Finally, set $x_2 = 4/3 - 2t/9$, $x_3 = -1 + 7t/12$, and $x_4 = t$ in the first equation and solve

$$x_1 + 2(4/3 - 2t/9) - 3(-1 + 7t/12) + t = 6$$

for x_1 , getting

$$x_1 = 1/3 + 43t/36.$$

Thus,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1/3 + 43t/36 \\ 4/3 - 2t/9 \\ -1 + 7t/12 \end{pmatrix} \\ &= \begin{pmatrix} 1/3 \\ 4/3 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 43/36 \\ -2/9 \\ 7/12 \\ 1 \end{pmatrix}. \end{aligned}$$

10. The augmented matrix for the system is

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 6 \\ 1 & -3 & 1 & -2 & 6 \\ 0 & -5 & 4 & -3 & 0 \end{array} \right)$$

Use the pivot in the first row to zero the entries below it in the first column. Add -1 times row 1 to row 2.

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 6 \\ 0 & -5 & 4 & -3 & 0 \\ 0 & -5 & 4 & -3 & 0 \end{array} \right)$$

Use the pivot in the second row to zero the entries below it in the second column. Add -1 times row 2 to row 3.

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 6 \\ 0 & -5 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 6 \\ -5x_2 + 4x_3 - 3x_4 &= 0 \end{aligned}$$

Note that x_1 and x_2 are pivot variables and x_3 and x_4 are free variables. Set $x_3 = s$ and $x_4 = t$ in the second equation and solve for x_2 .

$$\begin{aligned} -5x_2 + 4s - 3t &= 0 \\ x_2 &= 4s/5 - 3t/5 \end{aligned}$$

Set $x_2 = 4s/5 - 3t/5$, $x_3 = s$ and $x_4 = t$ in the first equation and solve for x_1 .

$$\begin{aligned} x_1 + 2(4s/5 - 3t/5) - 3s + t &= 6 \\ x_1 &= 6 + 7s/5 + t/5 \end{aligned}$$

Thus,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 + 7s/5 + t/5 \\ 4s/5 - 3t/5 \\ s \\ t \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 7/5 \\ 4/5 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1/5 \\ -3/5 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

11. We prefer to make a “forward pass” through the matrix to row echelon form, then make a “backward pass” through the matrix to reduced row echelon form. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 5 & -1 & 0 \end{array} \right).$$

Use the pivot in the first row to zero each entry in column 1 that falls below the pivot. Add -2 times row 1 to row 2.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \end{array} \right).$$

This matrix is in row echelon form. We will now use the pivot in the second row to zero every entry in column 2 that lies above the pivot. Add -2 times row 2 to row 1.

$$\left(\begin{array}{ccc|c} 1 & 0 & -13 & 5 \\ 0 & 1 & 5 & -2 \end{array} \right)$$

This last matrix is in reduced row echelon form.

12. We prefer to make a “forward pass” through the matrix to row echelon form, then make a “backward pass” through the matrix to reduced row echelon form. The augmented matrix is

$$\left(\begin{array}{cccc} 1 & -2 & 2 & 4 \\ -2 & 5 & 2 & 0 \end{array} \right)$$

Use the pivot in the first row to zero each entry in column 1 that falls below the pivot. Add 2 times row 1 to row 2.

$$\left(\begin{array}{cccc} 1 & -2 & 2 & 4 \\ 0 & 1 & 6 & 8 \end{array} \right)$$

This matrix is in row echelon form. We will now use the pivot in the second row to zero every entry in column 2 that lies above the pivot. Add 2 times row 2 to row 1.

$$\left(\begin{array}{cccc} 1 & 0 & 14 & 20 \\ 0 & 1 & 6 & 8 \end{array} \right)$$

This last matrix is in reduced row echelon form.

13. We prefer to make a “forward pass” through the matrix to row echelon form, then make a “backward pass” through the matrix to reduced row echelon form. The augmented matrix is

$$\left(\begin{array}{cccc} 1 & -2 & 4 & 0 \\ -2 & 1 & 1 & 4 \end{array} \right)$$

Use the pivot in the first row to zero each entry in column 1 that falls below the pivot. Add 2 times row 1 to row 2.

$$\left(\begin{array}{cccc} 1 & -2 & 4 & 0 \\ 0 & -3 & 9 & 4 \end{array} \right)$$

This matrix is in row echelon form. We will now use the pivot in the second row to zero every entry in column 2 that lies above the pivot. Add $-2/3$ times row 2 to row 1.

$$\left(\begin{array}{cccc} 1 & 0 & -2 & -8/3 \\ 0 & -3 & 9 & 4 \end{array} \right)$$

Finally, we need to scale each pivot to 1. Multiply the second row by $-1/3$.

$$\left(\begin{array}{cccc} 1 & 0 & -2 & -8/3 \\ 0 & 1 & -3 & -4/3 \end{array} \right)$$

This last matrix is in reduced row echelon form.

14. We prefer to make a “forward pass” through the matrix to row echelon form, then make a “backward pass” through the matrix to reduced row echelon form. The augmented matrix is

$$\left(\begin{array}{cccc} 1 & 0 & -2 & -2 \\ 3 & -1 & 3 & -6 \end{array}\right)$$

Use the pivot in the first row to zero each entry in column 1 that falls below the pivot. Add -3 times row 1 to row 2.

$$\left(\begin{array}{cccc} 1 & 0 & -2 & -2 \\ 0 & -1 & 9 & 0 \end{array}\right)$$

This matrix is in row echelon form. We need to scale each pivot to 1. Multiply the second row by -1 .

$$\left(\begin{array}{cccc} 1 & 0 & -2 & -2 \\ 0 & 1 & -9 & 0 \end{array}\right)$$

This last matrix is in reduced row echelon form.

15. We prefer to make a “forward pass” through the matrix to row echelon form, then make a “backward pass” through the matrix to reduced row echelon form. The augmented matrix is

$$\left(\begin{array}{ccccc} 1 & 1 & -1 & 2 & 3 \\ 2 & 2 & -3 & 4 & 5 \end{array}\right)$$

Use the pivot in the first row to zero each entry in column 1 that falls below the pivot. Add -2 times row 1 to row 2.

$$\left(\begin{array}{ccccc} 1 & 1 & -1 & 2 & 3 \\ 0 & 0 & -1 & 0 & -1 \end{array}\right)$$

This matrix is in row echelon form. We will now use the pivot in the second row to zero every entry that lies above the pivot in the third column. Add -1 times row 2 to row 1.

$$\left(\begin{array}{ccccc} 1 & 1 & 0 & 2 & 4 \\ 0 & 0 & -1 & 0 & -1 \end{array}\right)$$

We need to scale each pivot to 1. Multiply the second row by -1 .

$$\left(\begin{array}{ccccc} 1 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

This last matrix is in reduced row echelon form.

16. We prefer to make a “forward pass” through the matrix to row echelon form, then make a “backward pass” through the matrix to reduced row echelon form. The augmented matrix is

$$\left(\begin{array}{ccccc} 1 & -2 & 2 & -2 & -4 \\ -5 & 10 & -10 & 12 & 8 \end{array}\right)$$

Use the pivot in the first row to zero each entry in column 1 that falls below the pivot. Add 5 times row 1 to row 2.

$$\left(\begin{array}{ccccc} 1 & -2 & 2 & -2 & -4 \\ 0 & 0 & 0 & 2 & -12 \end{array}\right)$$

This matrix is in row echelon form. We will now use the pivot in the second row to zero every entry that lies above the pivot in the fourth column. Add 1 times row 2 to row 1.

$$\left(\begin{array}{ccccc} 1 & -2 & 2 & 0 & -16 \\ 0 & 0 & 0 & 2 & -12 \end{array}\right)$$

We need to scale each pivot to 1. Multiply the second row by $1/2$.

$$\left(\begin{array}{ccccc} 1 & -2 & 2 & 0 & -16 \\ 0 & 0 & 0 & 1 & -6 \end{array}\right)$$

This last matrix is in reduced row echelon form.

17. We prefer to make a “forward pass” through the matrix to row echelon form, then make a “backward pass” through the matrix to reduced row echelon form. The augmented matrix is

$$\left(\begin{array}{ccccc} 1 & 2 & 2 & -2 & 3 \\ 1 & 2 & -2 & 4 & 2 \\ -1 & -3 & 0 & 0 & 2 \end{array}\right)$$

Use the pivot in the first row to zero each entry in column 1 that falls below the pivot. Add -1 times row 1 to row 2. Add 1 times row 1 to row 3.

$$\left(\begin{array}{ccccc} 1 & 2 & 2 & -2 & 3 \\ 0 & 0 & -4 & 6 & -1 \\ 0 & -1 & 2 & -2 & 5 \end{array}\right)$$

Exchange rows 2 and 3.

$$\begin{pmatrix} 1 & 2 & 2 & -2 & 3 \\ 0 & -1 & 2 & -2 & 5 \\ 0 & 0 & -4 & 6 & -1 \end{pmatrix}$$

This matrix is in row echelon form. We will now use the pivot in the third row to zero every entry that lies above the pivot in the third column. Add $1/2$ times row 3 to row 2. Add $1/2$ times row 3 to row 1.

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 5/2 \\ 0 & -1 & 0 & 1 & 9/2 \\ 0 & 0 & -4 & 6 & -1 \end{pmatrix}$$

We will now use the pivot in the second row to zero each entry in column 2 that lies above the pivot. Add 2 times row 2 to row 1.

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 23/2 \\ 0 & -1 & 0 & 1 & 9/2 \\ 0 & 0 & -4 & 6 & -1 \end{pmatrix}$$

We need to scale each pivot to 1. Multiply the second row by -1 , the third row by $-1/4$.

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 23/2 \\ 0 & 1 & 0 & -1 & -9/2 \\ 0 & 0 & 1 & -3/2 & 1/4 \end{pmatrix}$$

This last matrix is in reduced row echelon form.

18. We prefer to make a “forward pass” through the matrix to row echelon form, then make a “backward pass” through the matrix to reduced row echelon form. The augmented matrix is

$$\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 2 & -2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 2 & 2 \end{pmatrix}$$

Use the pivot in the first row to zero each entry in column 1 that falls below the pivot. Add -2 times row 1 to row 2.

$$\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 0 & 0 & -2 & -6 & 2 \\ 0 & 0 & -1 & 2 & 2 \end{pmatrix}$$

Use the pivot in row two to zero each entry in column 3 that lies below the pivot. Add $-1/2$ times row 2 to row 3.

$$\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 0 & 0 & -2 & -6 & 2 \\ 0 & 0 & 0 & 5 & 1 \end{pmatrix}$$

This matrix is in row echelon form. We will now use the pivot in the third row to zero every entry that lies above the pivot in the fourth column. Add $6/5$ times row 3 to row 2. Add $-3/5$ times row 3 to row 1.

$$\begin{pmatrix} 1 & -1 & 2 & 0 & -8/5 \\ 0 & 0 & -2 & 0 & 16/5 \\ 0 & 0 & 0 & 5 & 1 \end{pmatrix}$$

We will now use the pivot in the second row to zero each entry in column 3 that lies above the pivot. Add 1 times row 2 to row 1.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 8/5 \\ 0 & 0 & -2 & 0 & 16/5 \\ 0 & 0 & 0 & 5 & 1 \end{pmatrix}$$

We need to scale each pivot to 1. Multiply the second row by $-1/2$, the third row by $1/5$.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 8/5 \\ 0 & 0 & 1 & 0 & -8/5 \\ 0 & 0 & 0 & 1 & 1/5 \end{pmatrix}$$

This last matrix is in reduced row echelon form.

19. The augmented matrix and corresponding system.

$$\begin{pmatrix} 1 & 0 & -2 & 2 & -2 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 - 2x_3 + 2x_4 &= -2 \\ x_2 + x_3 - x_4 &= 1 \end{aligned}$$

Thus, x_1 and x_2 are pivot variables, while x_3 and x_4 are free. Set $x_3 = s$ and $x_4 = t$ and solve each equation for its pivot variable.

$$\begin{aligned} x_1 &= -2 + 2s - 2t \\ x_2 &= 1 - s + t \end{aligned}$$

Thus, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 + 2s - 2t \\ 1 - s + t \\ s \\ t \end{pmatrix} \\ = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

20. The augmented matrix and corresponding system.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ x_1 - x_2 = 1 \\ x_3 - x_4 = 0$$

Thus, x_1 and x_3 are pivot variables, while x_2 and x_4 are free. Set $x_2 = s$ and $x_4 = t$ and solve each equation for its pivot variable.

$$x_1 = 1 + s \\ x_3 = t$$

Thus, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + s \\ s \\ t \\ t \end{pmatrix} \\ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

21. The augmented matrix and corresponding system.

$$\begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ x_1 - x_2 + x_3 - x_5 = 1 \\ x_4 + x_5 = 0$$

Thus, x_1 and x_4 are pivot variables, while x_2 , x_3 , and x_5 are free. Set $x_2 = u$, $x_3 = v$, and $x_5 = w$ and solve each equation for its pivot variable.

$$x_1 = 1 + u - v + w \\ x_4 = -w$$

Thus, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 + u - v + w \\ u \\ v \\ -w \\ w \end{pmatrix} \\ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

22. The augmented matrix and corresponding system.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ x_1 - x_2 + x_5 = 1 \\ x_3 - x_4 - x_5 = 1$$

Thus, x_1 and x_3 are pivot variables, while x_2 , x_4 , and x_5 are free. Set $x_2 = u$, $x_4 = v$, and $x_5 = w$ and solve each equation for its pivot variable.

$$x_1 = 1 + u - w \\ x_3 = 1 + v + w$$

Thus, the solution is

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} &= \begin{pmatrix} 1+u-w \\ u \\ 1+v+w \\ v \\ w \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &\quad + w \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

23. Set up the augmented matrix and reduce.

$$\left(\begin{array}{cccc|cccc} -6 & 8 & 0 & 2 & 1 & 0 & 0 & 1 \\ 4 & 8 & 8 & 20 & 0 & 1 & 0 & 1 \\ -2 & 2 & 7 & 7 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

This gives $\mathbf{x} = (1, 1, 1)^T$.

24. Set up the augmented matrix and reduce.

$$\left(\begin{array}{cccc|cccc} 3 & -3 & 1 & 0 & 1 & 0 & 0 & 1 \\ 7 & -4 & 5 & 3 & 0 & 1 & 0 & 1 \\ 4 & -3 & -3 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Thus, $\mathbf{x} = (1, 1, 0)^T$.

25. The augmented matrix is

$$M = \left(\begin{array}{cccc|cccc} 4 & 2 & -5 & -5 & 1 & 0 & 0 & 1 \\ -14 & -8 & 18 & 16 & 0 & 1 & 0 & 1 \\ -3 & -2 & 4 & 3 & 0 & 0 & 1 & 0 \end{array}\right)$$

Using a computer we reduce this to row echelon form

$$\left(\begin{array}{cccc|cccc} 4 & 2 & -5 & -5 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1/2 & -3/2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

The simplified system is

$$\begin{aligned}4y_1 + 2y_2 - 5y_3 &= -5 \\ -y_2 + y_3/2 &= -3/2.\end{aligned}$$

To backsolve, we first set the free variable $y_3 = t$. Next we solve for $y_2 = 3/2 + y_3/2 = (3+t)/2$. Finally, we solve for $y_1 = (-5 - 2y_2 + 5y_3)/4 = [-5 - (3+t) + 5t]/4 = -2+t$. Hence our solutions are all vectors of the form

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2+t \\ (3+t)/2 \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 3/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}.$$

26. Set up the augmented matrix and reduce.

$$\left(\begin{array}{cccc|cccc} -4 & 10 & -6 & -14 & 1 & 0 & -1 & 1 \\ 0 & -4 & 4 & 4 & 0 & 1 & -1 & -1 \\ 2 & -10 & 8 & 12 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\begin{aligned}x - z &= 1, \\ y - z &= -1.\end{aligned}$$

Thus, x and y are pivot variables and z is free. Solve each equation for its pivot variable.

$$\begin{aligned}x &= 1 + z \\ y &= -1 + z\end{aligned}$$

Thus,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+z \\ -1+z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where z is free.

27. The augmented matrix is

$$M = \left(\begin{array}{cccc|cccc} -3 & -3 & 1 & 4 & 1 & 0 & 0 & 1 \\ 8 & 7 & -2 & -8 & 0 & 1 & 0 & 1 \\ 8 & 6 & -1 & -5 & 0 & 0 & 1 & 0 \end{array}\right)$$

Using a computer we reduce this to the row echelon form

$$\left(\begin{array}{cccc|cccc} -3 & -3 & 1 & 4 & 1 & 0 & 0 & 1 \\ 0 & -1 & 2/3 & 8/3 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 1 & 0 \end{array}\right)$$

the simplified system is

$$\begin{aligned}-3y_1 - 3y_2 + y_3 &= 4 \\ -y_2 + 2y_3/3 &= 8/3 \\ y_3/3 &= 1/3.\end{aligned}$$

Backsolving we get $y_3 = 1$. Next $y_2 = -8/3 + 2/3 = -2$. Finally, $y_1 = (-4 - 3y_2 + y_3)/3 = 1$. Hence the only solution is $\mathbf{y} = (1, -2, 1)^T$.

28. Set up the augmented matrix and reduce.

$$\left(\begin{array}{cccc|c} 5 & 9 & 2 & 8 & -2 \\ -2 & -3 & -1 & -2 & 2 \\ 0 & -2 & 1 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

Thus, $\mathbf{x} = (-2, 2, 0)^T$.

29. Set up the augmented matrix and reduce.

$$\left(\begin{array}{cccc|c} -12 & 12 & -8 & -8 & 0 \\ -16 & 16 & -10 & -10 & 0 \\ -3 & 3 & -1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\begin{aligned} x - y &= 0, \\ z &= 1. \end{aligned}$$

Thus, x and z are pivot variables and y is free. Solve each equation for its pivot variable.

$$\begin{aligned} x &= y \\ z &= 1 \end{aligned}$$

Thus

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

where y is free.

30. Set up the augmented matrix and reduce.

$$\left(\begin{array}{cccc|c} 0 & 4 & 6 & -7 & 4 \\ -4 & 10 & 4 & -8 & 6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 11/4 & -19/8 & 1 \\ 0 & 1 & 3/2 & -7/4 & 1 \end{array}\right)$$

Thus,

$$\begin{aligned} x_1 + \frac{11}{4}x_3 - \frac{19}{8}x_4 &= 1, \\ x_2 + \frac{3}{2}x_3 - \frac{7}{4}x_4 &= 1. \end{aligned}$$

Thus, x_1 and x_2 are pivot variables, while x_3 and x_4 are free. Solve each equation for its pivot variable.

$$\begin{aligned} x_1 &= 1 - \frac{11}{4}x_3 + \frac{19}{8}x_4 \\ x_2 &= 1 - \frac{3}{2}x_3 + \frac{7}{4}x_4 \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 1 - (11/4)x_3 + (19/8)x_4 \\ 1 - (3/2)x_3 + (7/4)x_4 \\ x_3 \\ x_4 \end{pmatrix}, \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -11/4 \\ -3/2 \\ 1 \\ 0 \end{pmatrix} \\ &\quad + x_4 \begin{pmatrix} 19/8 \\ 7/4 \\ 0 \\ 1 \end{pmatrix}, \end{aligned}$$

where x_3 and x_4 are free.

31. Set up the augmented matrix and reduce.

$$\left(\begin{array}{ccccc|c} -5 & -4 & 4 & 3 & 17 \\ 7 & 6 & -5 & 3 & 12 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -2 & -15 & -75 \\ 0 & 1 & 3/2 & 18 & 179/2 \end{array}\right)$$

This gives

$$\begin{aligned} x_1 - 2x_3 - 15x_4 &= -75, \\ x_2 + \frac{3}{2}x_3 + 18x_4 &= \frac{179}{2}. \end{aligned}$$

Thus, x_1 and x_2 are pivot variables, while x_3 and x_4 are free. Solve each equation for its pivot variable.

$$\begin{aligned} x_1 &= -75 + 2x_3 + 15x_4 \\ x_2 &= \frac{179}{2} - \frac{3}{2}x_3 - 18x_4 \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} -75 + 2x_3 + 15x_4 \\ 179/2 - (3/2)x_3 - 18x_4 \\ x_3 \\ x_4 \end{pmatrix}, \\ &= \begin{pmatrix} -75 \\ 179/2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -3/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 15 \\ -18 \\ 0 \\ 1 \end{pmatrix}, \end{aligned}$$

where x_3 and x_4 are free.

32. Set up the augmented matrix and reduce.

$$\left(\begin{array}{ccccc|c} 2 & -3 & -2 & 2 & -4 & \\ -4 & 4 & 0 & 3 & -7 & \\ 8 & -8 & -1 & -7 & 13 & \end{array}\right) \\ \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -25/4 & 29/4 & \\ 0 & 1 & 0 & -11/2 & 11/2 & \\ 0 & 0 & 1 & 1 & 1 & \end{array}\right)$$

This gives

$$\begin{aligned} x_1 - \frac{25}{4}x_4 &= \frac{29}{4}, \\ x_2 - \frac{11}{2}x_4 &= \frac{11}{2}, \\ x_3 + x_4 &= 1. \end{aligned}$$

Thus, x_1 , x_2 , and x_3 are pivot variables and x_4 is free. Solve each equation for its pivot variable.

$$\begin{aligned} x_1 &= \frac{29}{4} + \frac{25}{4}x_4 \\ x_2 &= \frac{11}{2} + \frac{11}{2}x_4 \\ x_3 &= 1 - x_4 \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 29/4 + (25/4)x_4 \\ 11/2 + (11/2)x_4 \\ 1 - x_4 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} 29/4 \\ 11/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 25/4 \\ 11/2 \\ -1 \\ 1 \end{pmatrix}, \end{aligned}$$

where x_4 is free.

33. The augmented matrix is

$$M = \left(\begin{array}{ccccc|c} -7 & 7 & -8 & -3 & 37 & \\ 9 & -5 & 8 & -2 & -35 & \\ 5 & 0 & 2 & 8 & -9 & \end{array}\right).$$

Using a computer to perform row operations this is reduced to the row echelon form

$$\left(\begin{array}{ccccc|c} -7 & 7 & -8 & -3 & 37 & \\ 0 & 4 & -16/7 & -41/7 & 88/7 & \\ 0 & 0 & -6/7 & 369/28 & 12/7 & \end{array}\right)$$

The simplified system of equations is

$$\begin{aligned} -7y_1 + 7y_2 - 8y_3 - 3y_4 &= 37 \\ 4y_2 - 16y_3/7 - 41y_4/7 &= 88/7 \\ -6y_3/7 + 369y_4/28 &= 12/7 \end{aligned}$$

To backsolve, we first set the free variable $y_4 = t$. Then we solve for $y_3 = (-12 + 369t/4)/6 = -2 + 123t/8$. Next,

$$\begin{aligned} y_2 &= (88/7 + 16y_3/7 + 41y_4/7)/4 \\ &= [88 + 16(-2 + 123t/8) + 41t]/28 \\ &= [56 + 287t]/28 \\ &= 2 + 41t/4. \end{aligned}$$

Finally

$$\begin{aligned} y_1 &= [-37 + 7y_2 - 8y_3 - 3y_4]/7 \\ &= [-37 + 7(2 + 41t/4) \\ &\quad - 8(-2 + 123t/8) - 3t]/7 \\ &= -1 - 31t/4. \end{aligned}$$

Hence the solutions are the vectors

$$\begin{aligned} \mathbf{y} &= \begin{pmatrix} -1 - 31t/4 \\ 2 + 41t/4 \\ -2 + 123t/8 \\ t \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -31/4 \\ 41/4 \\ 123/8 \\ 1 \end{pmatrix}. \end{aligned}$$

34. Set up the augmented matrix and reduce.

$$\left(\begin{array}{ccccc|c} -7 & -4 & -5 & -9 & 31 & \\ 1 & 10 & 7 & 10 & -18 & \\ -2 & 0 & -3 & 0 & -2 & \end{array}\right) \\ \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 75/55 & -379/77 & \\ 0 & 1 & 0 & 19/14 & -57/14 & \\ 0 & 0 & 1 & -50/77 & 304/77 & \end{array}\right)$$

This gives

$$\begin{aligned} x_1 + \frac{75}{77}x_4 &= -\frac{379}{77}, \\ x_2 + \frac{19}{14}x_4 &= -\frac{57}{14}, \\ x_3 - \frac{50}{77}x_4 &= \frac{304}{77}. \end{aligned}$$

Therefore, x_1 , x_2 , and x_3 are pivot variables and x_4 is free. Solve each equation for its pivot variable.

$$\begin{aligned}x_1 &= -\frac{379}{77} - \frac{75}{77}x_4 \\x_2 &= -\frac{57}{14} - \frac{19}{14}x_4 \\x_3 &= \frac{304}{77} + \frac{50}{77}x_4\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -379/77 - (75/77)x_4 \\ -57/14 - (19/14)x_4 \\ 304/77 + (50/77)x_4 \\ x_4 \end{pmatrix} \\&= \begin{pmatrix} -379/77 \\ -57/14 \\ 304/77 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -75/77 \\ -19/14 \\ 50/77 \\ 1 \end{pmatrix},\end{aligned}$$

where x_4 is free.

35. The augmented matrix is

$$M = \begin{pmatrix} 8 & -6 & 9 & 8 & -1 & 15 \\ -9 & 5 & -7 & 9 & 0 & -30 \\ 1 & -4 & 1 & -3 & -7 & 9 \end{pmatrix}$$

With a computer we reduce this to row echelon form

$$\begin{pmatrix} 8 & -6 & 9 & 8 & -1 & 15 \\ 0 & -14 & 25 & 144 & -9 & -105 \\ 0 & 0 & -83 & -524 & -67 & 441 \end{pmatrix}$$

The simplified system of equation is

$$\begin{aligned}8y_1 - 6y_2 + 9y_3 + 8y_4 - y_5 &= 15 \\ -14y_2 + 25y_3 + 144y_4 - 9y_5 &= -105 \\ -83y_3 - 524y_4 - 67y_5 &= 441\end{aligned}$$

To backsolve, we first set the free variables $y_4 = s$ and $y_5 = t$. Then we solve for

$$y_3 = [-441 - 524s - 67t]/83.$$

Next,

$$\begin{aligned}y_2 &= [105 + 25y_3 + 144y_4 - 9y_5]/14 \\&= [105 + 25(-441 - 524s - 67t)/83 \\&\quad + 144s - 9t]/14 \\&= [-165 - 82s - 173t]/83.\end{aligned}$$

and finally,

$$\begin{aligned}y_1 &= [15 + 6y_2 - 9y_3 - 8y_4 + y_5]/8 \\&= [15 + 6(-165 - 82s + 173t)/83 \\&\quad - 9(-441 - 524s - 67t)/83 - 8s + t]/8 \\&= [528 + 445s - 44t]/83.\end{aligned}$$

Hence the solutions are the vectors

$$\begin{aligned}\mathbf{y} &= \begin{pmatrix} (528 + 445s - 44t)/83 \\ (-165 - 82s - 173t)/83 \\ (-441 - 524s - 67t)/83 \\ s \\ t \end{pmatrix} \\&= \begin{pmatrix} 528/83 \\ -165/83 \\ -441/83 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 445/83 \\ -82/83 \\ -524/83 \\ 1 \\ 0 \end{pmatrix} \\&\quad + t \begin{pmatrix} -44/83 \\ -173/83 \\ -67/83 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

36. The augmented matrix is

$$M = \begin{pmatrix} -2 & 3 & 6 & -7 & -1 & 3 \\ -1 & -6 & 1 & -6 & -8 & 1 \\ 0 & -9 & -5 & -1 & 9 & 16 \end{pmatrix}.$$

With a computer we reduce this to row echelon form

$$\begin{pmatrix} -2 & 3 & 6 & -7 & -1 & 3 \\ 0 & 15 & 4 & 5 & 15 & 1 \\ 0 & 0 & 13 & -10 & -90 & -83 \end{pmatrix}.$$

The simplified system of equations is

$$\begin{aligned}-2y_1 + 3y_2 + 6y_3 - 7y_4 - y_5 &= 3 \\ 15y_2 + 4y_3 + 5y_4 + 15y_5 &= 1 \\ 13y_3 - 10y_4 - 90y_5 &= -83\end{aligned}$$

To backsolve, we first set the free variables $y_4 = s$ and $y_5 = t$. Then we solve for

$$y_3 = [-83 + 10s + 90t]/13.$$

Next,

$$\begin{aligned} y_2 &= [1 - 4y_3 - 5y_4 - 15y_5]/15 \\ &= [1 - 4(-83 + 10s + 90t)/13 \\ &\quad - 5s - 15t]/15 \\ &= [23 - 7s - 37t]/13. \end{aligned}$$

and finally,

$$\begin{aligned} y_1 &= [3 - 3y_2 - 6y_3 + 7y_4 + y_5]/2 \\ &= [3 - 3(23 - 7s - 37t)/13 \\ &\quad - 6(-83 + 10s + 90t)/13 + 7s + t]/2 \\ &= -18 - 2s + 16t. \end{aligned}$$

Hence the solutions are the vectors

$$\begin{aligned} \mathbf{y} &= \begin{pmatrix} -18 - 2s + 16t \\ [23 - 7s - 37t]/13 \\ [-83 + 10s + 90t]/13 \\ s \\ t \end{pmatrix} \\ &= \begin{pmatrix} -18 \\ 23/13 \\ -83/13 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ -7/13 \\ +10/13 \\ 1 \\ 0 \end{pmatrix} \\ &\quad + t \begin{pmatrix} 16 \\ -37/13 \\ 90/13 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

————— × —————

Section 7.4. Homogeneous and Inhomogeneous Systems

1. We know that $A(c\mathbf{b}) = c(A\mathbf{b})$, where A is a matrix, c is a scalar, and $\mathbf{b}$ is a vector. Thus,

$$\begin{aligned} &\begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} t \\ 1/2 \\ 1 \end{pmatrix} \\ &= t \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix} \\ &= t \left(1 \begin{pmatrix} -3 \\ 14 \\ 4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2 \\ 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -18 \\ -4 \end{pmatrix} \right) \\ &= t \begin{pmatrix} -3 - 1 + 4 \\ 14 + 4 - 18 \\ 4 + 1 - 5 \end{pmatrix} \\ &= t \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

Hence the solution in Example 4.1 is in the null space

of matrix A .

2. We know that $A(s\mathbf{x} + t\mathbf{y}) = s(A\mathbf{x}) + t(A\mathbf{y})$, where A is a matrix, s and t are scalars, and $\mathbf{x}$ and $\mathbf{y}$ are vectors. Thus,

$$\begin{aligned} &(1 \ 3 \ 2) \left[s \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= s(1 \ 3 \ 2) \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t(1 \ 3 \ 2) \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \\ &= s(-3 + 3 + 0) + t(2 + 0 - 2) \\ &= \mathbf{0}. \end{aligned}$$

Hence the solution in Example 4.25 is in the null space of matrix A .

3. The matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_1 - 2x_3 = 0$$

$$x_2 + 3x_3 = 0$$

is in reduced row echelon form. The variables x_1 and x_2 are pivot variables, while x_3 is free. Set $x_3 = t$ and solve each equation for its pivot variable.

$$x_1 = 2t$$

$$x_2 = -3t$$

Thus, all solutions of $A\mathbf{x} = \mathbf{0}$ are given parametrically by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t \\ -3t \\ t \end{pmatrix} = t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

4. The matrix

$$A = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} x_1 - 5x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}$$

is in reduced row echelon form. The variables x_1 and x_2 are pivot variables, while x_3 is free. Set $x_3 = t$ and solve each equation for its pivot variable.

$$x_1 = 5t$$

$$x_2 = -2t$$

Thus, all solutions of $A\mathbf{x} = \mathbf{0}$ are given parametrically by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5t \\ -2t \\ t \end{pmatrix} = t \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

5. The matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$

$$x_1 + 2x_3 - 2x_4 = 0$$

$$x_2 + 3x_3 - x_4 = 0$$

is in reduced row echelon form. The variables x_1 and x_2 are pivot variables, while x_3 and x_4 are free. Set $x_3 = s$ and $x_4 = t$ and solve each equation for its pivot variable.

$$x_1 = -2s + 2t$$

$$x_2 = -3s + t$$

Thus, all solutions of $A\mathbf{x} = \mathbf{0}$ are given parametrically by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2s + 2t \\ -3s + t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

6. The matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} x_1 + x_2 + 4x_4 = 0 \\ x_3 + 2x_4 = 0 \end{array}$$

is in reduced row echelon form. The variables x_1 and x_3 are pivot variables, while x_2 and x_4 are free. Set $x_2 = s$ and $x_4 = t$ and solve each equation for its pivot variable.

$$x_1 = -s - 4t$$

$$x_3 = -2t$$

Thus, all solutions of $A\mathbf{x} = \mathbf{0}$ are given parametrically by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s - 4t \\ s \\ -2t \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

7. The matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$x_1 - x_3 + 3x_5 = 0$$

$$x_2 + 2x_3 - 5x_5 = 0$$

$$x_4 + 2x_5 = 0$$

is in reduced row echelon form. The variables x_1 , x_2 , and x_4 are pivot variables, while x_3 and x_5 are

free. Set $x_3 = s$ and $x_5 = t$ and solve each equation for its pivot variable.

$$\begin{aligned}x_1 &= s - 3t \\x_2 &= -2s + 5t \\x_4 &= -2t\end{aligned}$$

Thus, all solutions of $A\mathbf{x} = \mathbf{0}$ are given parametrically by

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} &= \begin{pmatrix} s - 3t \\ -2s + 5t \\ s \\ -2t \\ t \end{pmatrix} \\ &= s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 5 \\ 0 \\ -2 \\ 1 \end{pmatrix}.\end{aligned}$$

8. The matrix

$$\begin{aligned}A &= \begin{pmatrix} 1 & 2 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} \\ x_1 + 2x_2 &\quad -5x_5 = 0 \\ x_3 &\quad -4x_5 = 0 \\ x_4 + 3x_5 &= 0\end{aligned}$$

is in reduced row echelon form. The variables x_1 , x_3 , and x_4 are pivot variables, while x_2 and x_5 are free. Set $x_2 = s$ and $x_5 = t$ and solve each equation for its pivot variable.

$$\begin{aligned}x_1 &= -2s + 5t \\x_3 &= 4t \\x_4 &= -3t\end{aligned}$$

Thus, all solutions of $A\mathbf{x} = \mathbf{0}$ are given parametrically by

cally by

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} &= \begin{pmatrix} -2s + 5t \\ s \\ 4t \\ -3t \\ t \end{pmatrix} \\ &= s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ 0 \\ 4 \\ -3 \\ 1 \end{pmatrix}.\end{aligned}$$

9. The matrix

$$A = \begin{pmatrix} 1 & 0 & -2 & 4 \end{pmatrix} \quad x_1 - 2x_3 + 4x_4 = 0$$

is in reduced row echelon form. The variable x_1 is a pivot variable, while x_2 , x_3 , and x_4 are free. Set $x_2 = u$, $x_3 = v$, and $x_4 = w$ and solve the equation for its pivot variable.

$$x_1 = 2v - 4w$$

Thus, all solutions of $A\mathbf{x} = \mathbf{0}$ are given parametrically by

$$\begin{aligned}\text{vec } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 2v - 4w \\ u \\ v \\ w \end{pmatrix} \\ &= u \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

10. The matrix

$$\begin{aligned}A &= \begin{pmatrix} 1 & 2 & -3 & 4 \end{pmatrix} \\ x_1 + 2x_2 - 3x_3 + 4x_4 &= 0\end{aligned}$$

is in reduced row echelon form. The variable x_1 is a pivot variable, while x_2 , x_3 , and x_4 are free. Set $x_2 = u$, $x_3 = v$, and $x_4 = w$ and solve the equation for its pivot variable.

$$x_1 = -2u + 3v - 4w$$

Thus, all solutions of $A\mathbf{x} = \mathbf{0}$ are given parametrically by

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} -2u + 3v - 4w \\ u \\ v \\ w \end{pmatrix} \\ &= u \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

11. The reduced row echelon form of A is

$$\begin{aligned}A &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 2 & 0 & 0 & -2 \\ 3 & -4 & 4 & 1 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

Note that x_1 and x_2 are pivot variables, while x_3 and x_4 are free. Solve the first two equations (represented by the first two rows of the reduced row echelon form) for the pivot variables in terms of the free variables.

$$\begin{aligned}x_1 &= x_4 \\ x_2 &= x_3 + x_4\end{aligned}$$

Thus,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, the vectors $(0, 1, 1, 0)^T$ and $(1, 1, 0, 1)^T$ are “special.” Note that there are two of these special vectors. It is also important to note that there are two free variables in the reduced row echelon form of the matrix A . This is not a coincidence.

12. The reduced row echelon form of A is

$$\begin{aligned}A &= \begin{pmatrix} -3 & -5 & -1 & -5 \\ -2 & -3 & -1 & -3 \\ 0 & 3 & -3 & 3 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

Note that x_1 and x_2 are pivot variables, while x_3 and x_4 are free. Solve the first two equations (represented by the first two rows of the reduced row echelon form) for the pivot variables in terms of the free variables.

$$\begin{aligned}x_1 &= -2x_3 \\ x_2 &= x_3 - x_4\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} -2x_3 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} \\ &= x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

Thus, the vectors $(-2, 1, 1, 0)^T$ and $(0, -1, 0, 1)^T$ are “special.” Note that there are two of these special vectors. It is also important to note that there are two free variables in the reduced row echelon form of the matrix A . This is not a coincidence.

13. The reduced row echelon form of A is

$$\begin{aligned}A &= \begin{pmatrix} 0 & -2 & -2 & 0 & 2 \\ -3 & 1 & 4 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

Note that x_1 and x_2 are pivot variables, while x_3 , x_4 , and x_5 are free. Solve the first two equations (represented by the first two rows of the reduced row echelon form) for the pivot variables in terms of the free variables.

$$\begin{aligned}x_1 &= x_3 \\ x_2 &= -x_3 + x_5\end{aligned}$$

Thus,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \\ = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, the vectors $(1, -1, 1, 0, 0)^T$, $(0, 0, 0, 1, 0)^T$, and $(0, 1, 0, 0, 1)^T$ are “special.” Note that there are three of these special vectors. It is also important to note that there are three free variables in the reduced row echelon form of the matrix A . This is not a coincidence.

14. The reduced row echelon form of A is

$$A = \begin{pmatrix} -1 & -1 & -2 & 2 & -3 \\ 3 & 3 & 1 & -1 & 4 \\ 0 & 0 & 2 & -2 & 2 \end{pmatrix} \\ \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that x_1 and x_3 are pivot variables, while x_2 , x_4 , and x_5 are free. Solve the first two equations (represented by the first two rows of the reduced row echelon form) for the pivot variables in terms of the free variables.

$$x_1 = -x_2 - x_5 \\ x_3 = x_4 - x_5$$

Thus,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_2 - x_5 \\ x_2 \\ x_4 - x_5 \\ x_4 \\ x_5 \end{pmatrix} \\ = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, the vectors $(-1, 1, 0, 0, 0)^T$, $(0, 0, 1, 1, 0)^T$, and $(-1, 0, -1, 0, 1)^T$ are “special.” Note that there are three of these special vectors. It is also important to note that there are three free variables in the reduced row echelon form of the matrix A . This is not a coincidence.

15. (a) There are six variables, but only three equations. There must be at least three free variables. The reduced form of the augmented matrix will contain only three rows. Therefore, there can be at most three pivot variables, leaving room for three or more free variables.
(b) The augmented matrix reduces.

$$\begin{pmatrix} -1 & 1 & 1 & -1 & 0 & -1 \\ 1 & -4 & 2 & 4 & -3 & -2 \\ 3 & -3 & -3 & 3 & 0 & 3 \end{pmatrix} \\ \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -2 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As you can see, x_1 and x_2 are pivot variables. So there are actually four free variables, x_3 , x_4 , x_5 , and x_6 .

16. (a) There are seven variables, but only four equations. There must be at least three free variables. The reduced form of the augmented matrix will contain only four rows. Therefore, there can be at most four pivot variables, leaving room for three or more free variables.
(b) The augmented matrix reduces.

$$\begin{pmatrix} 3 & 0 & 3 & -3 & 0 & -3 & 3 \\ -1 & 3 & 2 & 1 & 3 & 4 & 2 \\ 1 & -2 & -1 & -1 & -2 & -3 & -1 \\ -1 & 1 & 0 & 1 & 1 & 2 & 0 \end{pmatrix} \\ \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As you can see, x_1 and x_2 are pivot variables. So there are actually five free variables, x_3 , x_4 , x_5 , x_6 , and x_7 .

17. Let $\mathbf{x}$ be in $\text{null}(A)$ and α be a scalar (number). Then, since $\mathbf{x} \in \text{null}(A)$, we know that $A\mathbf{x} = \mathbf{0}$. Consequently,

$$\begin{aligned} A(\alpha\mathbf{x}) &= \alpha(A\mathbf{x}) \\ &= \alpha\mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

Thus, $\alpha\mathbf{x}$ is in $\text{null}(A)$.

18. Reduce the augmented matrix.

$$\left(\begin{array}{cccc} 2 & -2 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{array}\right) \xrightarrow{\text{ref}} \left(\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus x_1 and x_3 are pivot variables and x_2 is free. Set $x_2 = t$, then solve the first two equations (rows) for their pivot variables.

$$\begin{aligned} x_1 &= t \\ x_3 &= -1 \end{aligned}$$

Thus, the solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Note that $\mathbf{p} = (0, 0, -1)^T$ and $\mathbf{v} = t(1, 1, 0)^T$ and

$$\begin{aligned} &\left(\begin{array}{ccc} 2 & -2 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 3 \end{array}\right) \left[t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] \\ &= t \left(\begin{array}{ccc} 2 & -2 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 3 \end{array}\right) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= t \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

19. Reduce the augmented matrix.

$$\left(\begin{array}{cccc} 5 & -2 & -5 & 2 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{array}\right) \xrightarrow{\text{ref}} \left(\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus x_1 and x_2 are pivot variables and x_3 is free. Set $x_3 = t$, then solve the first two equations (rows) for their pivot variables.

$$\begin{aligned} x_1 &= t \\ x_2 &= -1 \end{aligned}$$

Thus, the solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -1 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Note that $\mathbf{p} = (0, -1, 0)^T$ and $\mathbf{v} = t(1, 0, 1)^T$ and

$$\begin{aligned} &\left(\begin{array}{ccc} 5 & -2 & -5 \\ -3 & 0 & 3 \\ 0 & -3 & 0 \end{array}\right) \left[t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= t \left(\begin{array}{ccc} 5 & -2 & -5 \\ -3 & 0 & 3 \\ 0 & -3 & 0 \end{array}\right) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= t \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

20. Reduce the augmented matrix.

$$\left(\begin{array}{ccccc} -3 & 3 & 3 & 0 & 3 \\ -4 & 2 & 4 & -2 & 4 \\ 1 & 0 & -1 & 1 & -1 \end{array}\right) \xrightarrow{\text{ref}} \left(\begin{array}{ccccc} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Thus x_1 and x_2 are pivot variables and x_3 and x_4 are free. Set $x_3 = s$ and $x_4 = t$, then solve the first two equations (rows) for their pivot variables.

$$\begin{aligned} x_1 &= -1 + s - t \\ x_2 &= -t \end{aligned}$$

Thus, the solution is

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} -1 + s - t \\ -t \\ s \\ t \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

Note that $\mathbf{p} = (-1, 0, 0, 0)^T$ and $\mathbf{v} = s(1, 0, 1, 0)^T + t(-1, -1, 0, 1)^T$ and

$$\begin{aligned}&\begin{pmatrix} -3 & 3 & 3 & 0 \\ -4 & 2 & 4 & -2 \\ 1 & 0 & -1 & 1 \end{pmatrix} \left[s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= s \begin{pmatrix} -3 & 3 & 3 & 0 \\ -4 & 2 & 4 & -2 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ &\quad + t \begin{pmatrix} -3 & 3 & 3 & 0 \\ -4 & 2 & 4 & -2 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \\ &= s \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \mathbf{0}.\end{aligned}$$

21. Reduce the augmented matrix.

$$\begin{aligned}&\begin{pmatrix} 2 & 0 & 0 & -4 & 0 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \end{pmatrix} \\ &\xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

Thus x_1 and x_2 are pivot variables and x_3 and x_4 are free. Set $x_3 = s$ and $x_4 = t$, then solve the first two equations (rows) for their pivot variables.

$$\begin{aligned}x_1 &= 2t \\ x_2 &= 1 + s + t\end{aligned}$$

Thus, the solution is

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 2t \\ 1 + s + t \\ s \\ t \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

Note that $\mathbf{p} = (0, 1, 0, 0)^T$ and $\mathbf{v} = s(0, 1, 1, 0)^T + t(2, 1, 0, 1)^T$ and

$$\begin{aligned}&\begin{pmatrix} 2 & 0 & 0 & -4 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \left[s \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= s \begin{pmatrix} 2 & 0 & 0 & -4 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &\quad + t \begin{pmatrix} 2 & 0 & 0 & -4 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ &= s \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \mathbf{0}.\end{aligned}$$

22. Reduce the given matrix.

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$$

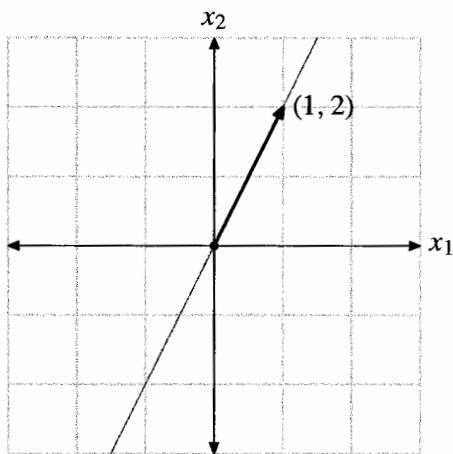
Thus, x_1 is a pivot variable and x_2 is free. Setting $x_2 = t$ and solving the first equation for x_1 ,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t/2 \\ t \end{pmatrix} = t \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}.$$

However, any multiple of $(1/2, 1)^T$ will do, so let's use $(1, 2)^T$ and write

$$\mathbf{x} = t \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This is a line through the origin in $\mathbf{R}^2$, as shown in the following figure.



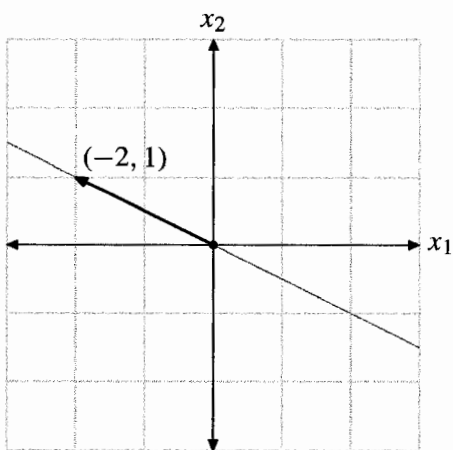
23. Reduce the given matrix.

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Thus, x_1 is a pivot variable and x_2 is free. Setting $x_2 = t$ and solving the first equation for x_1 ,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

This is a line through the origin in $\mathbf{R}^2$, as shown in the following figure.



24. Reduce the given matrix.

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 3/2 \\ 0 & 0 \end{pmatrix}$$

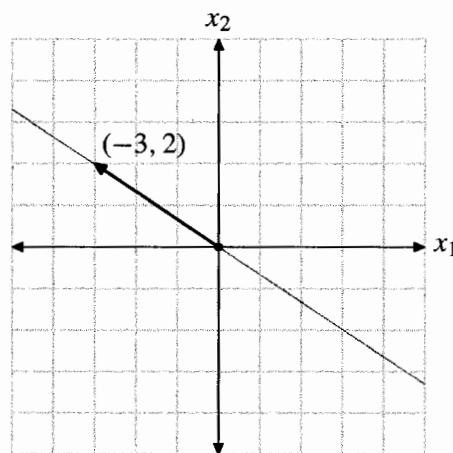
Thus, x_1 is a pivot variable and x_2 is free. Setting $x_2 = t$ and solving the first equation for x_1 ,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3t/2 \\ t \end{pmatrix} = t \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}.$$

However, any multiple of $(-3/2, 1)^T$ will do, so let's use $(-3, 2)^T$ and write

$$\mathbf{x} = t \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

This is a line through the origin in $\mathbf{R}^2$, as shown in the following figure.



25. Reduce the given matrix.

$$A = \begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix}$$

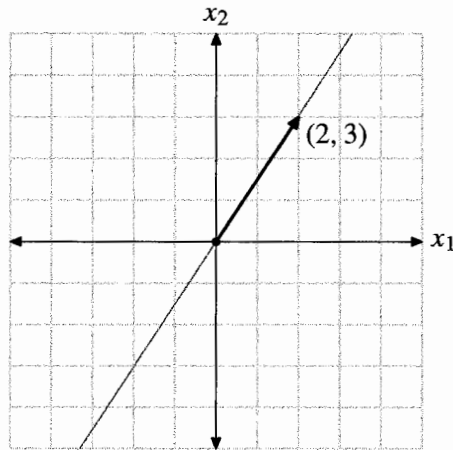
Thus, x_1 is a pivot variable and x_2 is free. Setting $x_2 = t$ and solving the first equation for x_1 ,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2t/3 \\ t \end{pmatrix} = t \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}.$$

However, any multiple of $(2/3, 1)^T$ will do, so let's use $(2, 3)^T$ and write

$$\mathbf{x} = t \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

This is a line through the origin in $\mathbf{R}^2$, as shown in the following figure.



26. Reduce the given matrix.

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 0 & 2 \\ 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, the only element in the nullspace of A is $(0, 0, 0)^T$. This is a point in $\mathbf{R}^3$.

27. Reduce the given matrix.

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & -2 \\ 2 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, x_1 and x_2 are pivot variables and x_3 is free. Set $x_3 = t$ and solve the first two equations (rows) for their pivot variables.

$$\begin{aligned} x_1 &= t \\ x_2 &= -t \end{aligned}$$

Thus, the null space solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

This is a line through the origin in $\mathbf{R}^3$.

28. Reduce the given matrix.

$$\begin{pmatrix} 4 & -4 & -4 \\ -5 & 5 & 5 \\ 3 & -3 & -3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, x_1 is a pivot variable, but x_2 and x_3 are free. Set $x_2 = s$ and $x_3 = t$ and solve the first equation (row) for its pivot variable.

$$x_1 = s + t$$

Thus, the null space solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s + t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

This is a plane passing through the origin in $\mathbf{R}^3$.

29. Reduce the given matrix.

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -4 & 4 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, x_1 is a pivot variable, but x_2 and x_3 are free. Set $x_2 = s$ and $x_3 = t$ and solve the first equation (row) for its pivot variable.

$$x_1 = s + t$$

Thus, the null space solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s + t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

This is a plane passing through the origin in $\mathbf{R}^3$.

Section 7.5. Bases of a Subspace

1. Arrange the appropriate augmented matrix and reduce.

$$[\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}] = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4/3 \end{pmatrix}$$

This system has a unique solution $(1, 4/3)^T$, so

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \mathbf{w} = 1\mathbf{u}_1 + \frac{4}{3}\mathbf{u}_2 = 1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{4}{3} \begin{pmatrix} 3 \\ 0 \end{pmatrix},$$

which is easily checked.

2. Arrange the appropriate augmented matrix and reduce.

$$[\mathbf{u}_1, \mathbf{u}_3, \mathbf{w}] = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

This system has solutions $(3 - 2t, t)^T$. For example, pick $t = 1$ to get $(1, 1)^T$ as a possible solution. Hence,

$$\begin{pmatrix} 3 \\ -6 \end{pmatrix} = \mathbf{w} = 1\mathbf{u}_1 + 1\mathbf{u}_3 = 1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ -4 \end{pmatrix},$$

which is easily checked. There are other solutions. For example, choose $t = 2$ and $(3 - 2t, t)^T = (-1, 2)^T$. Again, it is easy to check that

$$\begin{pmatrix} 3 \\ -6 \end{pmatrix} = \mathbf{w} = -1\mathbf{u}_1 + 2\mathbf{u}_3 = -1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -4 \end{pmatrix},$$

3. Arrange the appropriate augmented matrix and reduce.

$$[\mathbf{u}_1, \mathbf{u}_3, \mathbf{w}] = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This system is inconsistent. Hence, $\mathbf{w}$ is not in $\text{span}\{\mathbf{u}_1, \mathbf{u}_3\}$.

4. Arrange the appropriate augmented matrix and reduce.

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}] = \begin{pmatrix} 1 & 0 & -3 \\ -4 & -2 & 2 \\ 4 & 1 & -7 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

This system has the unique solution $(-3, 5)^T$. Hence,

$$\begin{pmatrix} -3 \\ 2 \\ -7 \end{pmatrix} = \mathbf{w} = -3\mathbf{v}_1 + 5\mathbf{v}_2 = -3 \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix},$$

as is easily checked.

5. Arrange the appropriate augmented matrix and reduce.

$$[\mathbf{v}_1, \mathbf{v}_3, \mathbf{w}] = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -2 & 4 \\ 4 & 3 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This system is inconsistent. Hence, $\mathbf{w}$ is not in $\text{span}\{\mathbf{v}_1, \mathbf{v}_3\}$.

6. Arrange the appropriate augmented matrix and reduce.

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}] = \begin{pmatrix} 1 & 0 & 1 & 1 \\ -4 & -2 & -2 & 1 \\ 4 & 1 & 3 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This system is inconsistent. Hence, $\mathbf{w}$ is not in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. Arrange the appropriate augmented matrix and reduce.

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}] = \begin{pmatrix} 1 & 0 & 1 & 1 \\ -4 & -2 & -2 & 0 \\ 4 & 1 & 3 & 2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This system has solution $(1 - t, -2 + t, t)^T$. For example, choose $t = 3$ and $(1 - t, -2 + t, t)^T = (-2, 1, 3)^T$. Hence,

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \mathbf{w} = -2\mathbf{v}_1 + 1\mathbf{v}_2 + 3\mathbf{v}_3 \\ = -2 \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix},$$

as is easily checked. There are other solutions. For example, choose $t = 0$ and $(1 - t, -2 + t, t)^T = (1, -2, 0)^T$. Hence,

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \mathbf{w} = 1\mathbf{v}_1 - 2\mathbf{v}_2 + 0\mathbf{v}_3 \\ = 1 \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix},$$

as is easily checked.

8. Arrange the appropriate augmented matrix and reduce.

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}] = \begin{pmatrix} 1 & 0 & 1 & -7 \\ -4 & -2 & -2 & 22 \\ 4 & 1 & 3 & -25 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 1 & -7 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This system has solution $(-7 - t, 3 + t, t)^T$. For example, choose $t = 1$ and $(-7 - t, 3 + t, t)^T =$

$(-8, 4, 1)^T$. Hence,

$$\begin{pmatrix} -7 \\ 22 \\ -25 \end{pmatrix} = \mathbf{w} = -8\mathbf{v}_1 + 4\mathbf{v}_2 + 1\mathbf{v}_3 \\ = -8 \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix},$$

as is easily checked. There are other solutions. For example, choose $t = 0$ and $(-7 - t, 3 + t, t)^T = (-7, 3, 0)^T$. Hence,

$$\begin{pmatrix} -7 \\ 22 \\ -25 \end{pmatrix} = \mathbf{w} = -7\mathbf{v}_1 + 3\mathbf{v}_2 + 0\mathbf{v}_3 \\ = -7 \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix},$$

as is easily checked.

9. Set up the augmented matrix

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}] = \begin{pmatrix} 1 & 2 & w_1 \\ -2 & 3 & w_2 \end{pmatrix}.$$

Add 2 times row 1 to row 2.

$$\begin{pmatrix} 1 & 2 & w_1 \\ 0 & 7 & 2w_1 + w_2 \end{pmatrix}$$

Backsolving, the second equation (row) gives

$$7x_2 = 2w_1 + w_2 \\ x_2 = \frac{2}{7}w_1 + \frac{1}{7}w_2.$$

Substitute this result in the first equation (row).

$$x_1 + 2 \left(\frac{2}{7}w_1 + \frac{1}{7}w_2 \right) = w_1 \\ x_1 = \frac{3}{7}w_1 - \frac{2}{7}w_2.$$

Hence,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathbf{w} \\ = \left(\frac{3}{7}w_1 - \frac{2}{7}w_2 \right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ + \left(\frac{2}{7}w_1 + \frac{1}{7}w_2 \right) \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

10. Set up the augmented matrix

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}] = \begin{pmatrix} 0 & -2 & -2 & w_1 \\ -1 & 1 & -2 & w_2 \\ -2 & -4 & 0 & w_3 \end{pmatrix}$$

Exchange rows 1 and 2.

$$\begin{pmatrix} -1 & 1 & -2 & w_2 \\ 0 & -2 & -2 & w_1 \\ -2 & -4 & 0 & w_3 \end{pmatrix}$$

Subtract 2 times row 1 from row 3.

$$\begin{pmatrix} -1 & 1 & -2 & w_2 \\ 0 & -2 & -2 & w_1 \\ 0 & -6 & 4 & -2w_2 + w_3 \end{pmatrix}$$

Subtract 3 times row 2 from row 3.

$$\begin{pmatrix} -1 & 1 & -2 & w_2 \\ 0 & -2 & -2 & w_1 \\ 0 & 0 & 10 & -3w_1 - 2w_2 + w_3 \end{pmatrix}$$

Backsolving, solve the first equation (row) for

$$\begin{aligned} 10x_3 &= -3w_1 - 2w_2 + w_3 \\ x_3 &= -\frac{3}{10}w_1 - \frac{1}{5}w_2 + \frac{1}{10}w_3. \end{aligned}$$

Substitute this value in the second equation (row)

$$-2x_2 - 2\left(-\frac{3}{10}w_1 - \frac{1}{5}w_2 + \frac{1}{10}w_3\right) = w_1$$

and solve for

$$x_2 = -\frac{1}{5}w_1 + \frac{1}{5}w_2 - \frac{1}{10}w_3.$$

Substitute these last two results in the first equation (row) to find

$$\begin{aligned} -x_1 + \left(-\frac{1}{5}w_1 + \frac{1}{5}w_2 - \frac{1}{10}w_3\right) \\ - 2\left(-\frac{3}{10}w_1 - \frac{1}{5}w_2 + \frac{1}{10}w_3\right) \\ = w_2 \quad \text{or} \\ x_1 = \frac{2}{5}w_1 - \frac{2}{5}w_2 - \frac{3}{10}w_3. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{w} &= \left(\frac{2}{5}w_1 - \frac{2}{5}w_2 - \frac{3}{10}w_3\right)\mathbf{v}_1 \\ &\quad + \left(-\frac{1}{5}w_1 + \frac{1}{5}w_2 - \frac{1}{10}w_3\right)\mathbf{v}_2 \\ &\quad + \left(-\frac{3}{10}w_1 - \frac{1}{5}w_2 + \frac{1}{10}w_3\right)\mathbf{v}_3, \end{aligned}$$

as can be checked.

11. Set up the augmented matrix and reduce.

$$\begin{aligned} [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] &= \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ -2 & 2 & -2 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This system has solutions $(-2t, -t, t)^T$, t a number. Thus, the nullspace is spanned by the single vector $(-2, -1, 1)^T$. Thus,

$$\begin{aligned} -2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 &= -2\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

12. Set up the augmented matrix and reduce.

$$\begin{aligned} [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] &= \begin{pmatrix} 2 & 5 & -3 \\ -3 & -2 & -1 \\ 3 & 5 & -2 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This system has solutions $(-t, t, t)^T$, t a number. Thus, the nullspace is spanned by the single

vector $(-1, 1, 1)^T$. Thus,

$$\begin{aligned} -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 &= -\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

13. Set up the augmented matrix and reduce.

$$\begin{aligned} [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] &= \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -1 \\ -2 & 5 & 11 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This system has solutions $(-2t, -3t, t)^T$, t a number. Thus, the nullspace is spanned by the single vector $(-2, -3, 1)^T$. Thus,

$$\begin{aligned} -2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 &= -2\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 11 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

14. Set up the augmented matrix and reduce.

$$\begin{aligned} [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] &= \begin{pmatrix} -3 & 2 & 7 \\ -2 & 0 & 2 \\ -1 & 2 & 5 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This system has solutions $(t, -2t, t)^T$, t a number. Thus, the nullspace is spanned by the single vector $(1, -2, 1)^T$. Thus,

$$\begin{aligned} \mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 &= \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

15. Set up the augmented matrix and reduce.

$$\begin{aligned} [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4] &= \begin{pmatrix} 1 & 2 & -2 & 6 \\ 2 & 0 & 4 & -4 \\ -2 & 2 & -8 & 12 \\ 0 & 3 & -6 & 12 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This system has solutions $(-2s + 2t, 2s - 4t, s, t)^T$, s, t numbers. This can be written $s(-2, 2, 1, 0)^T + t(2, -4, 0, 1)^T$, so the null space is spanned by the vectors $(-2, 2, 1, 0)^T$ and $(2, -4, 0, 1)^T$. If we choose $s = 2$ and $t = 1$, then $s(-2, 2, 1, 0)^T + t(2, -4, 0, 1)^T = (-2, 0, 2, 1)^T$. Thus,

$$\begin{aligned} -2\mathbf{v}_1 + 0\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4 &= -2\begin{pmatrix} 1 \\ 2 \\ -2 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 0 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 4 \\ -8 \\ -6 \end{pmatrix} + \begin{pmatrix} 6 \\ -4 \\ 12 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

16. Set up the augmented matrix and reduce.

$$\begin{aligned} [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4] &= \begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & 1 & 0 & 3 \\ 2 & 0 & -2 & 8 \\ -1 & -1 & 0 & -3 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This system has solutions $(s - 4t, -s + t, s, t)^T$, s, t numbers. This can be written $s(1, -1, 1, 0)^T + t(-4, 1, 0, 1)^T$, so the null space is spanned by the vectors $(1, -1, 1, 0)^T$ and $(-4, 1, 0, 1)^T$. If we

choose $s = 1$ and $t = 1$, then $s(1, -1, 1, 0)^T + t(-4, 1, 0, 1)^T = (-3, 0, 1, 1)^T$. Thus,

$$\begin{aligned} & -3\mathbf{v}_1 + 0\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 \\ &= -3 \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 8 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

17. It is easy to see that $(1, 2)^T$ is not a scalar multiple of $(-1, 3)^T$. Thus, the vectors are independent. More formally, if

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then the augmented matrix reduces as follows.

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus, $c_1 = c_2 = 0$ and the vectors are independent.

18. It is easy to see that $(-2, 3)^T$ is not a scalar multiple of $(2, -6)^T$. Thus, the vectors are independent. More formally, if

$$c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then the augmented matrix reduces as follows.

$$\begin{pmatrix} -2 & 2 & 0 \\ 3 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus, $c_1 = c_2 = 0$ and the vectors are independent.

19. The vector $(-1, 7, 7)^T$ is not a scalar multiple of the vector $(-3, 7, -4)^T$. They are independent.
20. The vector $\mathbf{v}_1 = (-8, 9, -6)^T$ is not a scalar multiple of the vector $\mathbf{v}_2 = (-2, 0, 7)^T$. They are independent.

21. Set up the equation

$$c_1 \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 7 \\ -4 \end{pmatrix} + c_3 \begin{pmatrix} -4 \\ -14 \\ 23 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces as follows.

$$\begin{pmatrix} -1 & -3 & -4 & 0 \\ 7 & 7 & -14 & 0 \\ 7 & -4 & 23 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Therefore, $c_1 = c_2 = c_3 = 0$ and the vectors are independent.

22. Set up the equation

$$c_1 \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} + c_3 \begin{pmatrix} 8 \\ -18 \\ 40 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces as follows.

$$\begin{pmatrix} -8 & -2 & 8 & 0 \\ 9 & 0 & -18 & 0 \\ -6 & 7 & 40 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$c_1 = 2c_3$$

$$c_2 = -4c_3$$

$$c_3 = \text{free}$$

For example, select $c_3 = 1$. Then, $(c_1, c_2, c_3) = (2, -4, 1)$ and the calculation

$$2 \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix} - 4 \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} + 1 \begin{pmatrix} 8 \\ -18 \\ 40 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

shows that the vectors are dependent.

23. Set up the equation

$$c_1 \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 8 \\ -4 \end{pmatrix} + c_3 \begin{pmatrix} -4 \\ -14 \\ 23 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Reduce the coefficient matrix.

$$\begin{pmatrix} -1 & -3 & -4 \\ 7 & 8 & -14 \\ 7 & -4 & 23 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $c_1 = c_2 = c_3 = 0$ and the vectors are independent.

24. Set up the equation

$$c_1 \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ -1 \\ 7 \end{pmatrix} + c_3 \begin{pmatrix} 8 \\ -18 \\ 40 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Reduce the coefficient matrix.

$$\begin{pmatrix} -8 & -2 & 8 \\ 9 & -1 & -18 \\ -6 & 7 & 40 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $c_1 = c_2 = c_3 = 0$ and the vectors are independent.

25. Reduce.

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1/2 \end{pmatrix}$$

This system has solutions $(t/2, t)^T$, t a number. This can be written $t(1/2, 1)^T$, so the nullspace is spanned by the single vector $(1/2, 1)^T$. Since a single vector is linearly independent, a basis for the nullspace is

$$B = \left\{ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\}.$$

26. Reduce.

$$\begin{pmatrix} -3 & 5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -5/3 \end{pmatrix}$$

This system has solutions $(5t/3, t)^T$, t a number. This can be written $t(5/3, 1)^T$, so the nullspace is spanned by the single vector $(5/3, 1)^T$. Since a single vector is linearly independent, a basis for the nullspace is

$$B = \left\{ \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} \right\}.$$

27. Reduce.

$$\begin{pmatrix} 4 & 4 \\ -2 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

This system has solutions $(-t, t)^T$, t a number. This can be written $t(-1, 1)^T$, so the nullspace is spanned by the single vector $(-1, 1)^T$. Since a single vector is linearly independent, a basis for the nullspace is

$$B = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

28. Reduce.

$$\begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, the nullspace contains only the zero vector and a basis is

$$B = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

29. Reduce.

$$\begin{pmatrix} 1 & 1 & 1 \\ -5 & -2 & -5 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This system has solutions $(-t, 0, t)^T$, t a number. This can be written $t(-1, 0, 1)^T$, so the nullspace is spanned by the single vector $(-1, 0, 1)^T$. Since a single vector is linearly independent, a basis for the nullspace is

$$B = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

30. Reduce.

$$\begin{pmatrix} -3 & 8 & -11 \\ -4 & 10 & -14 \\ -2 & 5 & -7 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This system has solutions $(-t, t, t)^T$, t a number. This can be written $t(-1, 1, 1)^T$, so the nullspace is spanned by the single vector $(-1, 1, 1)^T$. Since a single vector is linearly independent, a basis for the nullspace is

$$B = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

31. Reduce.

$$\begin{pmatrix} 2 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 \\ -3 & 3 & 3 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This system has solutions $(-s-t, -2s-t, s, t)^T$, s, t numbers. This can be written $s(-1, -2, 1, 0)^T + t(-1, -1, 0, 1)^T$, so the nullspace is spanned by the

vectors $(-1, -2, 1, 0)^T$ and $(-1, -1, 0, 1)^T$. Because these vectors are not multiples of one another, they are independent. Thus, a basis for this nullspace is

$$B = \left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

32. Reduce.

$$\begin{pmatrix} -8 & 14 & -24 & 14 \\ 4 & -10 & 18 & -10 \\ 4 & -8 & 14 & -8 \\ -2 & 5 & -9 & 5 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This system has solutions $(s/2, 2s - t, s, t)^T$, s, t numbers. This can be written $s(1/2, 2, 1, 0)^T + t(0, -1, 0, 1)^T$, so the nullspace is spanned by the vectors $(1/2, 2, 1, 0)^T$ and $(0, -1, 0, 1)^T$. Because these vectors are not multiples of one another, they are independent. Thus, a basis for this nullspace is

$$B = \left\{ \begin{pmatrix} 1/2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

33. Since the vectors $\mathbf{v}_1 = (1, 2)^T$ and $\mathbf{v}_2 = (-1, 3)^T$ are not multiples of one another, they are linearly independent and they form a basis for their span. Since there are two vectors in the basis, the dimension is 2.

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$$

34. Since the vectors $\mathbf{v}_1 = (-2, 3)^T$ and $\mathbf{v}_2 = (2, 6)^T$ are not multiples of one another, they are linearly independent and they form a basis for their span. Since there are two vectors in the basis, the dimension is 2.

$$B = \left\{ \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$$

35. Since the vectors $\mathbf{v}_1 = (-1, 7, 7)^T$ and $\mathbf{v}_2 = (-3, 7, -4)^T$ are not multiples of one another, they are linearly independent and they form a basis for their span. Since there are two vectors in the basis, the dimension is 2.

$$B = \left\{ \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 7 \\ -4 \end{pmatrix} \right\}$$

36. Since the vectors $\mathbf{v}_1 = (-8, 9, -6)^T$ and $\mathbf{v}_2 = (-2, 0, 7)^T$ are not multiples of one another, they are linearly independent and they form a basis for their span. Since there are two vectors in the basis, the dimension is 2.

$$B = \left\{ \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} \right\}$$

37. Place the vectors in a matrix and reduce.

$$\begin{pmatrix} -1 & -3 & -4 \\ 7 & 7 & -14 \\ 7 & -4 & 23 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, the nullspace contains only the zero vector and the vectors are independent. Thus, a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is

$$B = \left\{ \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 7 \\ -4 \end{pmatrix}, \begin{pmatrix} -4 \\ -14 \\ 23 \end{pmatrix} \right\}$$

The dimension is 3.

38. Place the vectors in a matrix and reduce.

$$\begin{pmatrix} -8 & -2 & 8 \\ 9 & 0 & -18 \\ -6 & 7 & 40 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

It is easy to see that $(2, -4, 1)^T$ is in the nullspace. Thus, $2\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, or equivalently, $\mathbf{v}_3 = -2\mathbf{v}_1 + 4\mathbf{v}_2$. Thus, $\mathbf{v}_3$ is redundant information and can be thrown away. The remaining two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of one another, so they are independent and form a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$B = \left\{ \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} \right\}$$

The dimension is 2.

39. Place the vectors in a matrix and reduce.

$$\begin{pmatrix} -1 & -3 & -4 \\ 7 & 8 & -14 \\ 7 & -4 & 23 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, the nullspace contains only the zero vector and the vectors are independent and form a basis for the $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$B = \left\{ \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 8 \\ -4 \end{pmatrix}, \begin{pmatrix} -4 \\ -14 \\ 23 \end{pmatrix} \right\}.$$

The dimension is 3.

40. Place the vectors in a matrix and reduce.

$$\begin{pmatrix} -8 & -2 & 8 \\ 9 & -1 & -18 \\ -6 & 7 & 40 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the nullspace contains only the zero vector and the vectors are independent and form a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$B = \left\{ \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 8 \\ -18 \\ 40 \end{pmatrix} \right\}.$$

The dimension is 3.

41. Note that $\mathbf{v}_1 = (1, -2)^T$ is a multiple of $\mathbf{v}_2 = (2, -4)^T$. Therefore, we can throw away the vector $\mathbf{v}_2$ without losing any information and $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1\}$. Geometrically, all multiples of $\mathbf{v}_1 = (1, 2)^T$ form a line through the origin in $\mathbf{R}^2$.

42. Note that $\mathbf{v}_1 = (2, -2)^T$ and $\mathbf{v}_2 = (2, -4)^T$ are not multiples of one another. Therefore, these vectors are independent and form a basis for $\mathbf{R}^2$. That is, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbf{R}^2$.

43. Place the vectors in a matrix and reduce.

$$\begin{pmatrix} -1 & -3 & 4 \\ 3 & -2 & -1 \\ 3 & 2 & -5 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This system has solutions $(t, t, t)^T$, t a number. This can be written $t(1, 1, 1)^T$, so the nullspace is spanned by the single vector $(1, 1, 1)^T$. Hence, $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, or equivalently, $\mathbf{v}_3 = -\mathbf{v}_1 - \mathbf{v}_2$. Hence, $\mathbf{v}_3$ is redundant information and $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Because the remaining vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of one another, they are independent and form a basis of dimension two for the span. Hence, this describes a plane in $\mathbf{R}^3$ that passes through the origin.

44. Note that $\mathbf{v}_2 = (2, 4, 6)^T$ and $\mathbf{v}_3 = (-5, -10, -15)^T$ are both multiples of $\mathbf{v}_1 = (1, 2, 3)^T$. Hence, $\mathbf{v}_2$ and $\mathbf{v}_3$ contain redundant information and can be thrown away, so $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1\}$. Hence, $\mathbf{v}_1 = (1, 2, 3)^T$ forms a basis of dimension one for the span. Thus the span is a line passing through the origin in $\mathbf{R}^3$.

—————x—————

Section 7.6. Square Matrices

1. Reduce the coefficient matrix.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

It is easy to see that $(2, 1)^T$ is in the nullspace. Hence, $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$, or equivalently, $\mathbf{a}_2 = -2\mathbf{a}_1$, where

$\mathbf{a}_1$ and $\mathbf{a}_2$ are the columns of the coefficient matrix A . Hence, $\text{span}\{\mathbf{a}_1, \mathbf{a}_2\} = \text{span}\{\mathbf{a}_1\}$, so the span is a line through the origin in the direction of $(1, 2)^T$. Thus, the system $A\mathbf{x} = \mathbf{b}$ has solutions only if $\mathbf{b}$ lies on this line. Thus, $(b_1, b_2)^T$ must be a multiple of $(1, 2)^T$. However, $(b_1, b_2)^T = \lambda(1, 2)^T$ implies that

$b_1 = \lambda$ and $b_2 = 2\lambda$. Dividing, $b_1/b_2 = 1/2$, which leads to $-2b_1 + b_2 = 0$. Because the system only has solutions for $\mathbf{b} = (b_1, b_2)^T$ on this line, and not for all values of $\mathbf{b}$, the coefficient matrix A is singular.

Alternatively, start with the augmented matrix

$$\begin{pmatrix} 1 & -2 & b_1 \\ 2 & -4 & b_2 \end{pmatrix}.$$

Add -2 times row 1 to row 2 to obtain

$$\begin{pmatrix} 1 & -2 & b_1 \\ 0 & 0 & -2b_1 + b_2 \end{pmatrix}.$$

This system is inconsistent unless $-2b_1 + b_2 = 0$.

2. Reduce the coefficient matrix.

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, the only element in the null space is the zero vector and the columns of A are independent, forming a basis of dimension two. Thus, the columns of A span all of $\mathbf{R}^2$. Because of this, the vector $\mathbf{b} = (b_1, b_2)^T$ can be written as a linear combination of the columns of A , regardless of the values of b_1 and b_2 . Therefore, the system has solutions for all values of $\mathbf{b} = (b_1, b_2)^T$. Thus, the coefficient matrix A is nonsingular.

Alternatively, start with the augmented matrix

$$\begin{pmatrix} 1 & 2 & b_1 \\ -2 & 3 & b_2 \end{pmatrix}.$$

Add 2 times row 1 to row 2 to obtain

$$\begin{pmatrix} 1 & -2 & b_1 \\ 0 & 7 & 2b_1 + b_2 \end{pmatrix}.$$

The presence of the pivot in the second row, second column, insures that this system has solutions for all values of b_1 and b_2 .

3. Reduce the coefficient matrix.

$$\begin{pmatrix} 3 & 3 & -3 \\ -1 & -1 & 1 \\ 3 & 5 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

It is easy to see that $(2, -1, 1)$ is in the nullspace of the coefficient matrix A . Thus, $2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$, or $\mathbf{a}_3 = -2\mathbf{a}_1 + \mathbf{a}_2$, where $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are the columns of the matrix A . Thus, $\mathbf{a}_3$ is redundant, and $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = \text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$. Thus, the span is the set of all linear combinations of $\mathbf{a}_1$ and $\mathbf{a}_2$, which is a plane in $\mathbf{R}^3$ passing through the origin. Thus, unless $\mathbf{b} = (b_1, b_2, b_3)^T$ lies on this plane, the system is inconsistent.

Now, set up the augmented matrix

$$\begin{pmatrix} 3 & 3 & -3 & b_1 \\ -1 & -1 & 1 & b_2 \\ 3 & 5 & -1 & b_3 \end{pmatrix}$$

Add $1/3$ times row 1 to row 2.

$$\begin{pmatrix} 3 & 3 & -3 & b_1 \\ 0 & 0 & 0 & (1/3)b_1 + b_2 \\ 3 & 5 & -1 & b_3 \end{pmatrix}$$

Note the second row. The system will be inconsistent unless $(1/3)b_1 + b_2 = 0$, which is a plane in $\mathbf{R}^3$. Therefore, the coefficient matrix is singular.

Can you show that the plane $(1/3)b_1 + b_2 = 0$ is spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$? *Hint: Consider the nullspace of the matrix $\begin{pmatrix} 1/3 & 1 & 0 \end{pmatrix}$.*

4. Subtract 2 times row 1 from row 2.

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$$

Note the presence of a zero on the diagonal. Thus, the matrix is singular.

5. Subtract 3 times row 1 from row 2.

$$\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -10 \end{pmatrix}$$

Each diagonal element is nonzero. Thus, the matrix is nonsingular.

6. Add 2 times row 1 to row 3.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ -2 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix}$$

Subtract row 2 from row 3.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Note the presence of a zero on the diagonal. Thus, this matrix is singular.

7. Add 2 times row 1 to row 2. Add 2 times row 1 to row 3.

$$\begin{pmatrix} 1 & 0 & -1 \\ -2 & 3 & 3 \\ -2 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 3 & -1 \end{pmatrix}$$

Subtract row 2 from row 3.

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

Each diagonal element is nonzero. Thus, the matrix is nonsingular.

8. Add $1/2$ times row 1 to row 2. Add $-1/2$ times row 1 to row 3.

$$\begin{pmatrix} 2 & 0 & 2 \\ -1 & 1 & 3 \\ 1 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & -4 & 2 \end{pmatrix}$$

Add 4 times row 2 to row 3.

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & -4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 18 \end{pmatrix}$$

Each diagonal element is nonzero. Thus, the matrix is nonsingular.

9. Add $1/2$ times row 1 to row 2. Add $3/2$ times row 1 to row 3.

$$\begin{pmatrix} 2 & 1 & -1 \\ -1 & -3 & -2 \\ -3 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & -1 \\ 0 & -5/2 & -5/2 \\ 0 & -1/2 & -1/2 \end{pmatrix}$$

Add $-1/5$ times row 2 to row 3.

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & -5/2 & -5/2 \\ 0 & -1/2 & -1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & -1 \\ 0 & -5/2 & -5/2 \\ 0 & 0 & 0 \end{pmatrix}$$

Note the presence of a zero element on the diagonal. Thus, the matrix is singular.

10. Add -1 times row 1 to row 2. Add row 1 to row 3.

$$\begin{pmatrix} -2 & 1 & 0 \\ -2 & 4 & 2 \\ 2 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & 1 \end{pmatrix}$$

Add $1/3$ times row 2 to row 3.

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 5/3 \end{pmatrix}$$

Each diagonal element is nonzero. Thus, the matrix is nonsingular.

11. Add 3 times row 1 to row 3.

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 4 \\ 3 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 4 \\ 0 & -3 & 6 \end{pmatrix}$$

Add $-3/2$ times row 2 to row 3.

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 4 \\ 0 & -3 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Note the presence of a zero element on the diagonal. Thus, the matrix is singular.

12. Reduce A .

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We now see that the only solution of the homogeneous system is the zero vector $(0, 0)^T$. Hence the matrix is nonsingular.

13. Reduce A .

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}.$$

The solutions are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

where x_2 is free. The coefficient matrix is singular.

14. Reduce A .

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now see that the only solution of the homogeneous system is the zero vector $(0, 0)^T$. Hence the matrix is nonsingular.

15. Reduce A .

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solutions are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

where x_3 is free. The coefficient matrix is singular.

16. Reduce A .

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The solutions are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

where x_3 is free. The coefficient matrix is singular.

17. Reduce A .

$$A = \begin{pmatrix} 0 & -1 & -2 \\ -5 & 2 & -1 \\ -4 & 2 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The solutions are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix},$$

where x_3 is free. The coefficient matrix is singular.

18. Reduce A .

$$A = \begin{pmatrix} -2 & -2 & -2 & 0 \\ -1 & -4 & 2 & -3 \\ 3 & 1 & 5 & -2 \\ 3 & 3 & 3 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solutions are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_3 + x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

where x_3 and x_4 are free. The coefficient matrix is singular.

19. Reduce A .

$$A = \begin{pmatrix} 0 & 1 & 3 & 0 \\ -2 & 1 & -3 & 1 \\ -1 & -2 & 3 & 0 \\ 2 & -1 & -1 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The only solution is $\mathbf{x} = (0, 0, 0, 0)^T$. The matrix is nonsingular.

20. The augmented matrix $[A \ I]$ reduces to

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Matrix A is singular and has no inverse.

21. The augmented matrix $[A \ I]$ reduces to

$$\begin{pmatrix} 0 & -4 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1/2 & -1 \\ 0 & 1 & -1/4 & 0 \end{pmatrix}$$

Hence A is nonsingular and

$$A^{-1} = \begin{pmatrix} -1/2 & -1 \\ -1/4 & 0 \end{pmatrix}.$$

22. The augmented matrix $[A \ I]$ reduces to

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Matrix A is singular and has no inverse.

23. The augmented matrix $[A \ I]$ reduces to

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, matrix A is nonsingular and

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

24. We augment A with the identity and perform row operations:

$$[A, I] \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Hence A is nonsingular and

$$A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1/2 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

25. The augmented matrix $[A \ I]$ reduces to

$$\begin{pmatrix} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -5 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Matrix A is singular and has no inverse.

26. The augmented matrix $[A \ I]$ reduces to

$$\begin{pmatrix} 0 & -3 & -1 & 2 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & -2 & 0 & 0 & 1 & 0 \\ -3 & -1 & 3 & 4 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/5 & 1/15 & 1 & 3/5 \\ 0 & 0 & 1 & 0 & -1/5 & -4/15 & -1 & -2/5 \\ 0 & 0 & 0 & 1 & 1/10 & -1/30 & 1 & 7/10 \end{pmatrix}$$

Hence A is nonsingular and

$$A^{-1} = \begin{pmatrix} 0 & -1/3 & 0 & 0 \\ -1/5 & 1/15 & 1 & 3/5 \\ -1/5 & -4/15 & -1 & -2/5 \\ 1/10 & -1/30 & 1 & 7/10 \end{pmatrix}$$

27. The augmented matrix $[A \ I]$ reduces to

$$\begin{pmatrix} 3 & -1 & -3 & -1 & 1 & 0 & 0 & 0 \\ 0 & -3 & -4 & 1 & 0 & 1 & 0 & 0 \\ -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & -3 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1/12 & -5/12 & 1/6 \\ 0 & 1 & 0 & 1 & 0 & -2/3 & -1/3 & -2/3 \\ 0 & 0 & 1 & -1 & 0 & 1/4 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/3 & 5/3 & 1/3 \end{pmatrix}$$

Hence A is singular.

28. The coefficient matrix, in row echelon form, is

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}.$$

The row echelon form has nonzero entries on its diagonal, so the coefficient matrix is nonsingular and the system has a unique solution.

29. The coefficient matrix, in row echelon form, is

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

The presence of a zero diagonal element in the row echelon form indicates that the coefficient matrix is singular. The system does not have a unique solution.

30. In matrix form,

$$\begin{pmatrix} -1 & -3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix},$$

and the coefficient matrix has row echelon form

$$\begin{pmatrix} -1 & -3 \\ 2 & 6 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}.$$

The presence of a zero on the diagonal of the row echelon form indicates that the coefficient matrix is singular. The system does not have a unique solution.

31. In matrix form,

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the coefficient matrix has row echelon form

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}.$$

All diagonal elements in the row echelon form are nonzero, so the coefficient matrix is nonsingular and the system has a unique solution.

32. The coefficient matrix has row echelon form

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

All diagonal elements in the row echelon form are nonzero, so the coefficient matrix is nonsingular and the system has a unique solution.

33. The coefficient matrix has row echelon form

$$\begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The presence of a zero on the diagonal of the row echelon form indicates that the coefficient matrix is singular. The system does not have a unique solution.

34. The row echelon form is

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 8 & x \\ 0 & 8 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & 8 & 5 \\ 0 & 0 & x + 11/4 \end{pmatrix}.$$

The matrix is invertible only if it is nonsingular. This is the case only if $x + 11/4 \neq 0$, or equivalently, $x \neq -11/4$.

35. Suppose that
- A
- is invertible. Then:

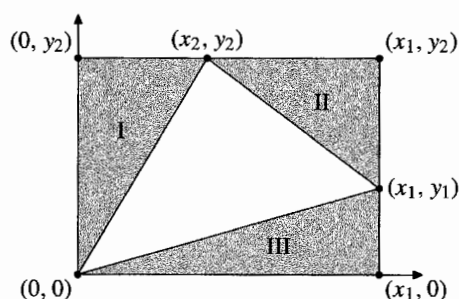
- A is nonsingular.
- The only solution of the homogeneous system $A\mathbf{y} = \mathbf{0}$ is the zero vector $\mathbf{0}$.
- The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side $\mathbf{b}$.
- If A is put into row echelon form then the diagonal entries of the result are nonzero.
- If A is put into reduced row echelon form then the result is the identity matrix.

36. Suppose that
- A
- is nonsingular. Then:

- A is invertible.
- The only solution of the homogeneous system $A\mathbf{y} = \mathbf{0}$ is the zero vector $\mathbf{0}$.
- The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side $\mathbf{b}$.
- If A is put into row echelon form then the diagonal entries of the result are nonzero.
- If A is put into reduced row echelon form then the result is the identity matrix.

Section 7.7. Determinants

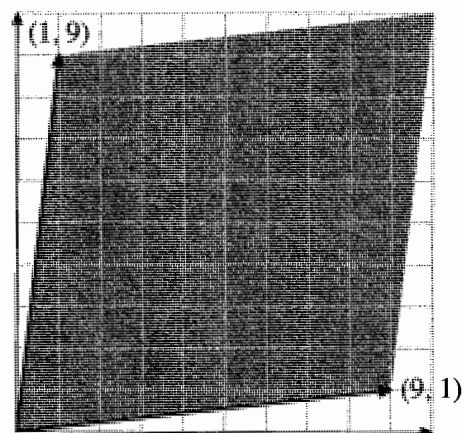
1. The area of the triangle spanned by the vectors $\mathbf{x}_1 = (x_1, y_1)^T$ and $\mathbf{x}_2 = (x_2, y_2)^T$ is calculated by subtracting the areas of the triangular regions I, II, and III from the area of the bounding rectangle.



Thus, the area of the triangle is

$$\begin{aligned}
 A &= x_1 y_2 - \frac{1}{2} x_2 y_2 - \frac{1}{2} (x_1 - x_2)(y_2 - y_1) \\
 &\quad - \frac{1}{2} x_1 y_1, \\
 &= \frac{1}{2} x_1 y_2 - \frac{1}{2} x_2 y_1, \\
 &= \frac{1}{2} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}.
 \end{aligned}$$

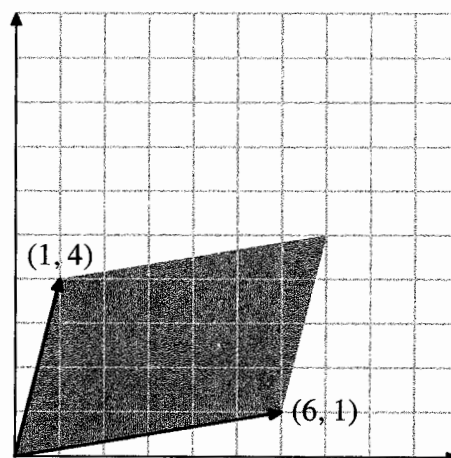
2. Estimate the area by counting square units inside the parallelogram in



The determinant is

$$\begin{aligned}
 |\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} 9 & 1 \\ 1 & 9 \end{vmatrix} = (9)(9) - (1)(1) \\
 &= 81 - 1 = 80.
 \end{aligned}$$

3. Estimate the area by counting square units inside the parallelogram in

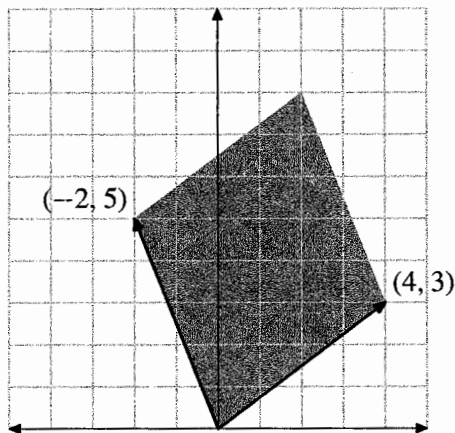


The determinant is

$$\begin{aligned} |\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} 1 & 6 \\ 4 & 1 \end{vmatrix} = (1)(1) - (4)(6) \\ &= 1 - 24 = -23. \end{aligned}$$

Note that the determinant is the negative of the area.

4. Estimate the area by counting square units inside the parallelogram in

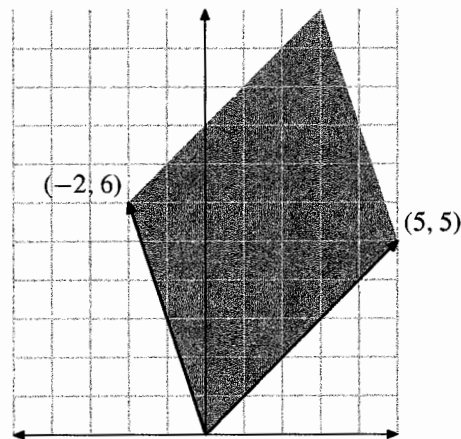


The determinant is

$$\begin{aligned} |\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = (-2)(3) - (5)(4) \\ &= -6 - 20 = -26. \end{aligned}$$

Note that the determinant is the negative of the area.

5. Estimate the area by counting square units inside the parallelogram in



The determinant is

$$\begin{aligned} |\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} 5 & -2 \\ 5 & 6 \end{vmatrix} = (5)(6) - (5)(-2) \\ &= 30 + 10 = 40. \end{aligned}$$

6. First note the determinant of A .

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

To prove part (1), construct B by adding r times row 1 to row 2.

$$B = \begin{pmatrix} a & b \\ c + ra & d + rb \end{pmatrix}.$$

Then,

$$\begin{aligned} |B| &= \begin{vmatrix} a & b \\ c + ra & d + rb \end{vmatrix}, \\ &= a(d + rb) - b(c + ra), \\ &= ad - bc, \\ &= |A|. \end{aligned}$$

To prove part (2), craft B by swapping rows 1 and 2 of matrix A .

$$B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Then,

$$\begin{aligned}|B| &= \begin{vmatrix} c & d \\ a & b \end{vmatrix}, \\ &= bc - ad, \\ &= -(ad - bc), \\ &= -|A|.\end{aligned}$$

Finally, to prove part (3), build matrix B by multiplying row 1 of matrix A by r .

$$B = \begin{pmatrix} ra & rb \\ c & d \end{pmatrix}$$

Then,

$$\begin{aligned}|B| &= \begin{vmatrix} ra & rb \\ c & d \end{vmatrix}, \\ &= rad - rbc, \\ &= r(ad - bc), \\ &= r|A|.\end{aligned}$$

7. Using row operations we get

$$\begin{aligned}\begin{pmatrix} 3 & 0 & 0 \\ 3 & 6 & 3 \\ -18 & -18 & -9 \end{pmatrix} &\rightarrow \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ -18 & -18 & -9 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & -18 & -9 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

Since the final matrix has a row of zeros, the determinant is 0.

8. Add $4/5$ times row 1 to row 2; add $-4/5$ times row 1 to row 3.

$$\begin{vmatrix} 5 & 6 & 4 \\ -4 & -9 & -8 \\ 4 & 6 & 5 \end{vmatrix} = \begin{vmatrix} 5 & 6 & 4 \\ 0 & -21/5 & -24/5 \\ 0 & 6/5 & 9/5 \end{vmatrix}$$

Add $2/7$ times row 2 to row 3.

$$\begin{aligned}&= \begin{vmatrix} 5 & 6 & 4 \\ 0 & -21/5 & -24/5 \\ 0 & 0 & 3/7 \end{vmatrix} \\ &= 5 \left(-\frac{21}{5} \right) \left(\frac{3}{7} \right) \\ &= -9\end{aligned}$$

9. Add 3 times row 1 to row 2; add -4 times row 1 to row 3.

$$\begin{vmatrix} 1 & 0 & 4 \\ -3 & 3 & -2 \\ 4 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 4 \\ 0 & 3 & 10 \\ 0 & -1 & -18 \end{vmatrix}$$

Add $1/3$ times row 2 to row 3.

$$\begin{aligned}&= \begin{vmatrix} 1 & 0 & 4 \\ 0 & 3 & 10 \\ 0 & 0 & -44/3 \end{vmatrix} \\ &= (1)(3) \left(-\frac{44}{3} \right) \\ &= -44\end{aligned}$$

10. Using row operations we get

$$\begin{aligned}\begin{pmatrix} 1 & 2 & -3 \\ 0 & 6 & -2 \\ -2 & 3 & 2 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 6 & -2 \\ 0 & 7 & -4 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 6 & -2 \\ 0 & 0 & -5/3 \end{pmatrix}\end{aligned}$$

The determinant of the final, upper triangular matrix is the product of the diagonal elements, or -10 .

11. Swap rows 1 and 3.

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & -1 & 3 & 4 \\ -1 & 3 & 1 & 2 \end{vmatrix}$$

Add 2 times row 1 to row 3; add -1 times row 1 to row 4.

$$= - \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 3 & 3 & 4 \\ 0 & 1 & 1 & 2 \end{vmatrix}$$

Factor a 2 out of row 2.

$$= -2 \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 3 & 4 \\ 0 & 1 & 1 & 2 \end{vmatrix}$$

Add -3 times row 2 to row 3; add -1 times row 2 to row 4.

$$= -2 \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 2 & 2 \end{vmatrix}$$

Swap rows 3 and 4; then factor out a 2 from the new row 3.

$$= (2)(2) \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 6 & 4 \end{vmatrix}$$

Add -6 times row 3 to row 4.

$$\begin{aligned} &= (2)(2) \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{vmatrix} \\ &= (2)(2)(-1)(1)(1)(-2) \\ &= 8 \end{aligned}$$

12. Swap rows 1 and 3.

$$\begin{vmatrix} 3 & -3 & -2 & -1 \\ 2 & 0 & -2 & -1 \\ 1 & -2 & 0 & 0 \\ 4 & -1 & -4 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 0 & 0 \\ 2 & 0 & -2 & -1 \\ 3 & -3 & -2 & -1 \\ 4 & -1 & -4 & -2 \end{vmatrix}$$

Add -2 times row 1 to row 2; add -3 times row 1 to row 3; add -4 times row 1 to row 4.

$$= - \begin{vmatrix} 1 & -2 & 0 & 0 \\ 0 & 4 & -2 & -1 \\ 0 & 3 & -2 & -1 \\ 0 & 7 & -4 & -2 \end{vmatrix}$$

Factor out a 4 from row 2.

$$= -4 \begin{vmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/4 \\ 0 & 3 & -2 & -1 \\ 0 & 7 & -4 & -2 \end{vmatrix}$$

Add -3 times row 2 to row 3; add -7 times row 2 to row 4.

$$= -4 \begin{vmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/4 \\ 0 & 0 & -1/2 & -1/4 \\ 0 & 0 & -1/2 & -1/4 \end{vmatrix}$$

Add -1 times row 3 to row 4.

$$\begin{aligned} &= -4 \begin{vmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/4 \\ 0 & 0 & -1/2 & -1/4 \\ 0 & 0 & 0 & 0 \end{vmatrix} \\ &= -4(1)(1) \left(-\frac{1}{2} \right) (0) \\ &= 0 \end{aligned}$$

13. (a) Suppose that row i is a scalar multiple of row j , say

$$|A| = \begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ ra_{j1} & ra_{j2} & \dots & ra_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

Then, adding $-r$ times for j to row i does not change the value of the determinant. Thus,

$$\begin{aligned} &= \begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} \\ &= 0, \end{aligned}$$

because it has a zero row. The same result applies to the columns. Suppose column i is a scalar multiple of column j , say

$$|A| = \begin{vmatrix} \dots & ra_{1j} & \dots & a_{1j} & \dots \\ \dots & ra_{2j} & \dots & a_{2j} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & ra_{nj} & \dots & a_{nj} & \dots \end{vmatrix}$$

But adding $-r$ times column j to column i does not change the value of the determinant. Thus,

$$\begin{aligned} &= \begin{vmatrix} \dots & 0 & \dots & a_{1j} & \dots \\ \dots & 0 & \dots & a_{2j} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & \dots & a_{nj} & \dots \end{vmatrix} \\ &= 0, \end{aligned}$$

because A has a zero column.

- (b) The first matrix has zero determinant because the third row has all zeros. Similarly, the second matrix has zero determinant because the third column has all zeros. The third matrix has zero determinant because its second row is a scalar multiple of its first row. The fourth matrix has zero determinant because the second column is a scalar multiple of its first column.

14. (a) Suppose matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$ is $n \times n$, where $\mathbf{a}_j$ denotes the j th column. Assume that column j is a linear combination of the columns that precede it,

$$\mathbf{a}_j = c_1 \mathbf{a}_1 + \dots + c_{j-1} \mathbf{a}_{j-1}.$$

Thus,

$$\mathbf{a}_j - c_1 \mathbf{a}_1 - \dots - c_{j-1} \mathbf{a}_{j-1} = \vec{0}.$$

Thus, if we perform the following

- Add $-c_1$ times column 1 and add to column j .
- Add $-c_2$ times column 2 and add to column j .

$\vdots$

- Add $-c_{j-1}$ times column $j-1$ to column j .

Then,

$$\begin{aligned} & |\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_j, \dots, \mathbf{a}_n|, \\ &= |\mathbf{a}_1, \mathbf{a}_2, \dots, \\ & \quad (\mathbf{a}_j - c_1 \mathbf{a}_1 - \dots - c_{j-1} \mathbf{a}_{j-1}), \dots, \mathbf{a}_n|, \\ &= |\mathbf{a}_1, \mathbf{a}_2, \dots, \vec{0}, \dots, \mathbf{a}_n|, \\ &= 0, \end{aligned}$$

because there is a zero column.

If the j th row of A is a linear combination of its preceding rows, then the j th column of A^T is a linear combination of its preceding columns and has zero determinant. Thus,

$$|A| = |A^T| = 0.$$

- (b) The first matrix has zero determinant because its third row is the sum of its first two rows. The

second matrix has zero determinant because its third column is the sum of its first two columns. Matrix three has zero determinant because its third row equals twice its first row added to its second row. The fourth matrix has zero determinant because its third column is the sum of twice its first column and three times its second column.

15. The transpose of a lower triangular matrix is upper triangular. The determinant of the transpose equals the determinant of the original. Note that transposing a lower triangular matrix does not dislodge the diagonal elements. Thus, the determinant of a lower triangular matrix is the product of its diagonal elements.

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 4 \end{vmatrix} = (1)(2)(3)(4) = 24$$

16. Since there are two zeros in the first row we expand by that row.

$$\det A = 3 \det \begin{pmatrix} 6 & 3 \\ -18 & -9 \end{pmatrix} = 3(-54 + 54) = 0.$$

17. Expand across the first row.

$$\begin{aligned} & \begin{vmatrix} 5 & 6 & 4 \\ -4 & -9 & -8 \\ 4 & 6 & 5 \end{vmatrix} \\ &= 5 \begin{vmatrix} -9 & -8 \\ 6 & 5 \end{vmatrix} - 6 \begin{vmatrix} -4 & -8 \\ 4 & 5 \end{vmatrix} \\ & \quad + 4 \begin{vmatrix} -4 & -9 \\ 4 & 6 \end{vmatrix} \\ &= 5(-45 + 48) - 6(-20 + 32) \\ & \quad + 4(-24 + 36) \\ &= 15 - 72 + 48 \\ &= -9 \end{aligned}$$

18. Expand across the first row.

$$\begin{vmatrix} 1 & 0 & 4 \\ -3 & 3 & -2 \\ 4 & -1 & -2 \end{vmatrix} = 1 \begin{vmatrix} 3 & -2 \\ -1 & -2 \end{vmatrix} + 4 \begin{vmatrix} -3 & 3 \\ 4 & -1 \end{vmatrix} \\
 = 1(-6 - 2) + 4(3 - 12) \\
 = -8 - 36 \\
 = -44$$

19. There is a zero in the first column, so we will expand by that column.

$$\det A = 1 \cdot \det \begin{pmatrix} 6 & -2 \\ 3 & 2 \end{pmatrix} + (-2) \cdot \det \begin{pmatrix} 2 & -3 \\ 6 & -2 \end{pmatrix} \\
 = 18 - 28 = -10$$

20. Expand across the third row.

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 2 \end{vmatrix} \\
 = -1 \begin{vmatrix} -1 & 3 & 4 \\ 2 & -2 & 0 \\ 3 & 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 4 \\ 0 & -2 & 0 \\ -1 & 1 & 2 \end{vmatrix}$$

But, expanding across the second row of the first and second matrices,

$$= -1 \left\{ -2 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} -1 & 4 \\ 3 & 2 \end{vmatrix} \right\} \\
 - 2 \left\{ -2 \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} \right\} \\
 = 1\{-2(2) - 2(-14)\} - 2\{-2(8)\} \\
 = 4 - 28 + 32 \\
 = 8$$

21. Expand across the third row.

$$\begin{vmatrix} 3 & -3 & -2 & 1 \\ 2 & 0 & -2 & -1 \\ 1 & -2 & 0 & 0 \\ 4 & -1 & -4 & -2 \end{vmatrix} \\
 = 1 \begin{vmatrix} -3 & -2 & -1 \\ 0 & -2 & -1 \\ -1 & -4 & -2 \end{vmatrix} + 2 \begin{vmatrix} 3 & -2 & -1 \\ 2 & -2 & -1 \\ 4 & -4 & -2 \end{vmatrix}$$

Expand down the first column of the first matrix and the first row of the second matrix.

$$= 1 \left\{ -3 \begin{vmatrix} -2 & -1 \\ -4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & -1 \\ -2 & -1 \end{vmatrix} \right\} \\
 + 2 \left\{ 3 \begin{vmatrix} -2 & -1 \\ -4 & -2 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} 2 & -2 \\ 4 & -4 \end{vmatrix} \right\}$$

Note that in each 2×2 determinant, the second row is a multiple of the first. Therefore, they all equal zero and the determinant of the original 4×4 is also zero.

22. The determinant of
- A
- is

$$\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 0 & 7 \end{vmatrix} = 7.$$

To find the nullspace, reduce the augmented matrix $[A \ \vec{0}]$.

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The nullspace contains only the zero vector. Thus, the only solution of

$$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is $x_1 = x_2 = 0$ and the columns are independent.

23. The determinant of
- A
- is

$$|A| = \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} = 0.$$

To find the nullspace, reduce the augmented matrix $[A \ \vec{0}]$.

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The solutions are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

where x_2 is free. A basis for the nullspace is

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

Because the nullspace is nontrivial,

$$x_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has nontrivial solutions, so the columns of A are dependent.

24. The determinant of A is

$$|A| = \begin{vmatrix} 1 & 3 \\ -1 & -3 \end{vmatrix} = 0.$$

To find the nullspace, reduce the augmented matrix $[A \ \vec{0}]$.

$$\begin{pmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The solutions are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

where x_2 is free. Thus,

$$B = \left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$$

is a basis for the nullspace. Because the nullspace is nontrivial,

$$x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has nontrivial solutions, so the columns of A are dependent.

25. The determinant of A is

$$|A| = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

To find the nullspace, reduce the augmented matrix $[A \ \vec{0}]$.

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The nullspace contains only the zero vector. Thus, the only solution of

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is $x_1 = x_2 = 0$ and the columns are independent.

26. Taking advantage of the zeros we expand by the third row.

$$\det A = \det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = -1.$$

Since $\det A \neq 0$, $\text{null}(A)$ is trivial and the column vectors of A are linearly independent.

27. The determinant is 0. The reduced row echelon form is

$$\begin{pmatrix} 1 & -2 & -4 \\ 2 & 1 & 2 \\ 3 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the nullspace is

$$B = \left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Thus, the equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has nontrivial solutions and the columns are dependent.

28. To take advantage of the zero we could expand by the first row or the second column. Let's choose the second column.

$$\begin{aligned} \det A &= -1 \cdot \det \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix} \\ &\quad + 1 \cdot \det \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \\ &= 1 - 1 = 0. \end{aligned}$$

Since $\det A = 0$, $\text{null}(A)$ is nontrivial, and the column vectors of A are linearly dependent. $\text{null}(A)$ is the subspace with basis $(1, 1, -1)^T$.

29. The determinant is 1. The reduced row echelon form is

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 5 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The nullspace contains only the zero vector. Thus, the equation

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has only the trivial solution and the columns are independent.

30. To have a nontrivial nullspace, the determinant must equal zero.

$$\begin{aligned} 0 &= \begin{vmatrix} 2 & x \\ 3 & -2 \end{vmatrix} \\ 0 &= -4 - 3x \\ x &= -\frac{4}{3} \end{aligned}$$

31. To have a nontrivial nullspace, the determinant must equal zero.

$$\begin{aligned} 0 &= \begin{vmatrix} 2 & x \\ x & 3 \end{vmatrix} \\ 0 &= 6 - x^2 \\ x &= \pm\sqrt{6} \end{aligned}$$

32. To have a nontrivial nullspace the determinant must equal zero.

$$\begin{aligned} 0 &= \begin{vmatrix} x & 4 \\ 3 & -2 \end{vmatrix} \\ 0 &= -2x - 12 \\ 2x &= -12 \\ x &= -6 \end{aligned}$$

33. To have a nontrivial nullspace, the determinant must equal zero.

$$\begin{aligned} 0 &= \begin{vmatrix} -1 & x \\ -x & 4 \end{vmatrix} \\ 0 &= -4 + x^2 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

34. To have a nontrivial nullspace, the determinant must

equal zero.

$$\begin{aligned} 0 &= \begin{vmatrix} 2-x & 1 \\ 0 & -1-x \end{vmatrix} \\ 0 &= (2-x)(-1-x) \\ x &= 2, 1 \end{aligned}$$

35. To have a nontrivial nullspace, the determinant must equal zero.

$$\begin{aligned} 0 &= \begin{vmatrix} -1-x & 0 \\ 3 & 2-x \end{vmatrix} \\ 0 &= (-1-x)(2-x) \\ x &= -1, 2 \end{aligned}$$

36. To have a nontrivial nullspace, the determinant must equal zero.

$$0 = \begin{vmatrix} -1-x & 5 & 2 \\ 0 & -x & -1 \\ 0 & 6 & -5-x \end{vmatrix}$$

Expand down the first column.

$$\begin{aligned} 0 &= (-1-x) \begin{vmatrix} -x & -1 \\ 6 & -5-x \end{vmatrix} \\ 0 &= (-1-x)\{-x(-5-x) + 6\} \\ 0 &= -(x+1)(x^2 + 5x + 6) \\ 0 &= -(x+1)(x+3)(x+2) \end{aligned}$$

Thus, $x = -1, -3, -2$.

37. To have a nontrivial nullspace, the determinant must equal zero.

$$0 = \begin{vmatrix} 2-x & 0 & 0 \\ -1 & -x & 2 \\ 0 & -2 & 5-x \end{vmatrix}$$

Expand across the first row.

$$\begin{aligned} 0 &= (2-x) \begin{vmatrix} -x & 2 \\ -2 & 5-x \end{vmatrix} \\ 0 &= (2-x)\{-x(5-x) + 4\} \\ 0 &= -(x-2)(x^2 - 5x + 4) \\ 0 &= -(x-2)(x-1)(x-4) \end{aligned}$$

Thus, $x = 2, 1, 4$.

38. To have a nontrivial nullspace, the determinant must equal zero.

$$0 = \begin{vmatrix} 2-x & 0 & 1 \\ -3 & -1-x & -1 \\ -2 & 0 & -1-x \end{vmatrix}$$

Expand across the first row.

$$\begin{aligned} 0 &= (2-x) \begin{vmatrix} -1-x & -1 \\ 0 & -1-x \end{vmatrix} \\ &\quad + 1 \begin{vmatrix} -3 & -1-x \\ -2 & 0 \end{vmatrix} \\ 0 &= (2-x)(-1-x)(-1-x) + 2(-1-x) \\ 0 &= (-1-x)[(2-x)(-1-x) + 2] \\ 0 &= (-1-x)(x^2 - x) \\ 0 &= -(x+1)x(x-1) \end{aligned}$$

Thus, $x = -1, 0, 1$.

39. To have a nontrivial nullspace, the determinant must equal zero.

$$0 = \begin{vmatrix} -1-x & 2 & 2 \\ 0 & -2-x & 0 \\ -1 & 4 & 2-x \end{vmatrix}$$

Expand across the second row.

$$\begin{aligned} 0 &= (-2-x) \begin{vmatrix} -1-x & 2 \\ -1 & 2-x \end{vmatrix} \\ 0 &= -(x+2)((-1-x)(2-x) + 2) \\ 0 &= -(x+2)(-x+x^2) \\ 0 &= -(x+2)x(x-1) \end{aligned}$$

Thus, $x = -2, 0, 1$.

40. The determinant is

$$\begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 3.$$

Therefore, the nullspace is trivial, containing only the zero vector.

41. The determinant is

$$\begin{vmatrix} -2 & 3 \\ -2 & 4 \end{vmatrix} = -2.$$

Therefore, the nullspace is trivial, containing only the zero vector.

42. The determinant is

$$\begin{vmatrix} -1 & -9 & 10 \\ -7 & -19 & 26 \\ -2 & -10 & 12 \end{vmatrix} = 0.$$

This is easily seen by noticing that the third column is the negative of the sum of the first two columns. The nullspace is not trivial. It contains nonzero vectors.

43. Add 3 times the first row to the second; add -4 times the first row to the third.

$$\begin{vmatrix} 1 & 0 & 4 \\ -3 & 3 & -2 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 4 \\ 0 & 3 & 10 \\ 0 & 0 & -18 \end{vmatrix} = -54$$

The nullspace is trivial, containing only the zero vector.

44. Add 2 times row 2 to row 2; add -3 times row 1 to row 3.

$$\begin{vmatrix} 1 & 2 & -3 \\ -2 & 0 & -1 \\ 3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & -7 \\ 0 & -6 & 11 \end{vmatrix}$$

Add $3/2$ times row 2 to row 3.

$$\begin{aligned} &= \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & -7 \\ 0 & 0 & 1/2 \end{vmatrix} \\ &= (1)(4)\left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

The nullspace is trivial, containing only the zero vector.

45. Swap rows 1 and 3.

$$\begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & -2 \\ -1 & 0 & 3 \end{vmatrix} = - \begin{vmatrix} -1 & 0 & 3 \\ 2 & 0 & -2 \\ 0 & 1 & 2 \end{vmatrix}$$

Add 2 times row 1 to row 2.

$$= - \begin{vmatrix} -1 & 0 & 3 \\ 0 & 0 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

Swap rows 2 and 3.

$$= \begin{vmatrix} -1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix} \\ = (-1)(1)(4) \\ = -4$$

The nullspace is trivial, containing only the zero vector.

46. Swap columns 1 and 2.

$$\begin{vmatrix} 10 & -1 & -3 & 9 \\ 3 & 2 & -3 & 3 \\ 3 & 1 & -2 & 3 \\ -10 & 2 & 2 & -9 \end{vmatrix} \\ = - \begin{vmatrix} -1 & 10 & -3 & 9 \\ 2 & 3 & -3 & 3 \\ 1 & 3 & -2 & 3 \\ 2 & -10 & 2 & -9 \end{vmatrix}$$

Add 2 times row 1 to row 2; add row 1 to row 3; add 2 times row 1 to row 4.

$$= - \begin{vmatrix} -1 & 10 & -3 & 9 \\ 0 & 23 & -9 & 21 \\ 0 & 13 & -5 & 11 \\ 0 & 10 & -4 & 9 \end{vmatrix}$$

Too hard to continue with row reductions. Expand down first column.

$$= -(-1) \begin{vmatrix} 23 & -9 & 21 \\ 13 & -5 & 11 \\ 10 & -4 & 9 \end{vmatrix}$$

Expand across first row.

$$= 1 \left\{ 23 \begin{vmatrix} -5 & 11 \\ -4 & 9 \end{vmatrix} + 9 \begin{vmatrix} 13 & 11 \\ 10 & 9 \end{vmatrix} + 21 \begin{vmatrix} 13 & -5 \\ 10 & -4 \end{vmatrix} \right\} \\ = 23(-1) + 9(7) - 21(-2) \\ = -23 + 63 - 42 \\ = -2$$

The nullspace is trivial and contains only the zero vector.

47. Swap rows 1 and 2.

$$\begin{vmatrix} 3 & 0 & 20 & -8 \\ 2 & 3 & -2 & 0 \\ 6 & 4 & 17 & -8 \\ 16 & 10 & 50 & -23 \end{vmatrix} = - \begin{vmatrix} 2 & 3 & -2 & 0 \\ 3 & 0 & 20 & -8 \\ 6 & 4 & 17 & -8 \\ 16 & 10 & 50 & -23 \end{vmatrix}$$

Add $-3/2$ times row 1 to row 2; add -3 times row 1 to row 3; add -8 times row 1 to row 4.

$$= - \begin{vmatrix} 2 & 3 & -2 & 0 \\ 0 & -9/2 & 23 & -8 \\ 0 & -5 & 23 & -8 \\ 0 & -14 & 66 & -23 \end{vmatrix}$$

Too hard to continue with reduction. Expand down first column.

$$= -2 \begin{vmatrix} -9/2 & 23 & -8 \\ -5 & 23 & -8 \\ -14 & 66 & -23 \end{vmatrix} \\ = \begin{vmatrix} 9 & -46 & 16 \\ -5 & 23 & -8 \\ -14 & 66 & -23 \end{vmatrix}$$

Expand across first row.

$$= 9 \begin{vmatrix} 23 & -8 \\ 66 & -23 \end{vmatrix} + 46 \begin{vmatrix} -5 & -8 \\ -14 & -23 \end{vmatrix} \\ + 16 \begin{vmatrix} -5 & 23 \\ -14 & 66 \end{vmatrix} \\ = 9(-1) + 46(3) + 16(-8) \\ = 1$$

The nullspace is trivial, containing only the zero vector.

48. The second and fourth row are identical. The determinant is zero and the nullspace contains nonzero vectors.

49. Rows 2 and 5 are identical, so the determinant is 0. Therefore there is a nonzero vector in the nullspace.

50. (a) $-2U$ multiplies each of 4 rows by -2 . Thus,

$$\begin{aligned}\det(-2U) &= (-2)^4 \det(U) \\ &= (-2)^4 (-3) \\ &= -48\end{aligned}$$

- (b) The determinant of a product is the product of the determinants. Thus,

$$\begin{aligned}\det(U^3) &= \det(UUU) \\ &= \det(U) \det(U) \det(U) \\ &= (-3)(-3)(-3) \\ &= -27\end{aligned}$$

- (c) Because

$$\begin{aligned}1 &= \det(I) = \det(UU^{-1}) \\ &= \det(U) \det(U^{-1}),\end{aligned}$$

we know

$$\det(U^{-1}) = \frac{1}{\det U} = \frac{1}{-3} = -\frac{1}{3}.$$

51. False. For a counterexample, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$\det(A + B) = \det \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = -2.$$

But,

$$\begin{aligned}\det(A) + \det(B) &= \det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= -1 + 0 = -1.\end{aligned}$$

52. (a) If A and B are similar, then there is a nonsingular S such that $A = S^{-1}BS$. Thus,

$$\begin{aligned}\det A &= \det(S^{-1}BS) \\ &= \det(S^{-1}) \det(B) \det(S) \\ &= \frac{1}{\det(S)} \det(B) \det(S) \\ &= \det(B).\end{aligned}$$

- (b) Then

$$\det(A - \lambda I) = \det(S^{-1}BS - \lambda I).$$

But, I commutes with S , so

$$\begin{aligned}&= \det(S^{-1}BS - S^{-1}(\lambda I)S), \\ &= \det(S^{-1}(B - \lambda I)S), \\ &= \det(S^{-1}) \det(B - \lambda I) \det(S), \\ &= \frac{1}{\det(S)} \det(B - \lambda I) \det(S), \\ &= \det(B - \lambda I).\end{aligned}$$

53. $\det A \neq 0$ is equivalent to each of the following:

- A is nonsingular.
- A is invertible.
- $\text{null}(A)$ is trivial.
- The system $A\mathbf{x} = \vec{b}$ has a unique solution for every right hand side $\vec{b}$.
- If A is $n \times n$, the column vectors in A are a basis for $\mathbf{R}^n$.
- When A is reduced to row echelon form the diagonal entries are all nonzero.
- When A is reduced to reduced row echelon form the result is the identity matrix.

54. $\det A = 0$ is equivalent to each of the following:

- A is singular.
- A is noninvertible.
- $\text{null}(A)$ contains a nonzero vector.
- There is a vector $\vec{b}$ for which the system $A\mathbf{x} = \vec{b}$ is inconsistent.
- If A is $n \times n$, the column vectors in A are linearly dependent.
- When A is reduced to row echelon form there is a row of zeros.

Chapter 8. An Introduction to Systems

Section 8.1. Definitions and Examples

1. The system

$$\begin{aligned}x' &= v \\v' &= -x - 0.02v + 2 \cos t,\end{aligned}$$

has dependent variables (unknowns) x and v . Therefore, the dimension is 2. The right side depends explicitly on the independent variable t , so the system is nonautonomous.

2. The dimension is 2 since there are two equations and two unknown functions. The system is autonomous since there is no explicit dependence of the right hand sides on the independent variable.

3. The system

$$\begin{aligned}x' &= -ax + ay \\y' &= rx - y - xz \\z' &= -bz + xy,\end{aligned}$$

has dependent variables (unknowns) x , y , and z . Therefore, the dimension is 3. The right side does not depend explicitly on t , so the system is autonomous.

4. The dimension is 3 since there are three equations and three unknown functions. The system is autonomous since there is no explicit dependence of the right hand sides on the independent variable.

5. The system

$$\begin{aligned}u_1' &= u_2 \\u_2' &= -\frac{1}{2}u_1 + \frac{1}{2}u_3 \\u_3' &= u_4 \\u_4' &= \frac{3}{2}u_1 + \frac{1}{2}u_3,\end{aligned}$$

has dependent variables (unknowns) u_1 , u_2 , u_3 and u_4 . Therefore, the dimension is 4. The right side does not depend explicitly on t , so the system is autonomous.

6. The system

$$\begin{aligned}u_1' &= u_2 \\u_2' &= -\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_1}u_3 \\u_3' &= u_4 \\u_4' &= \frac{k_2}{m_2}u_1 - \frac{k_2 + k_3}{m_2}u_3 + \cos t\end{aligned}$$

has dependent variables u_1 , u_2 , u_3 , and u_4 . Therefore, the dimension is 4. The right sides explicitly reference the independent variable t , so the system is not autonomous.

7. If $x(t) = 2e^{2t} - 2e^{-t}$ and $y(t) = -e^{-t} + 2e^{2t}$, then

$$x' = (2e^{2t} - 2e^{-t})' = 4e^{2t} + 2e^{-t},$$

and

$$\begin{aligned}-4x + 6y &= -4(2e^{2t} - 2e^{-t}) + 6(-e^{-t} + 2e^{2t}) \\&= 4e^{2t} + 2e^{-t},\end{aligned}$$

so the first equation is satisfied. Further,

$$y' = (-e^{-t} + 2e^{2t})' = e^{-t} + 4e^{2t},$$

and

$$\begin{aligned}-3x + 5y &= -3(2e^{2t} - 2e^{-t}) + 5(-e^{-t} + 2e^{2t}) \\&= 4e^{2t} + e^{-t},\end{aligned}$$

so the second equation is satisfied. Finally,

$$\begin{aligned}x(0) &= 2e^{2(0)} - 2e^{-0} = 0 \\y(0) &= -e^{-0} + 2e^{2(0)} = 1,\end{aligned}$$

so the initial conditions are also satisfied.

8. If $x(t) = (1+t)e^{-t}$ and $y(t) = -te^{-t}$, then

$$\begin{aligned} x' &= ((1+t)e^{-t})' \\ &= -(1+t)e^{-t} + e^{-t} \\ &= -te^{-t} \\ &= y, \end{aligned}$$

so the first equation is satisfied. Further,

$$y' = (-te^{-t})' = te^{-t} - e^{-t},$$

and

$$\begin{aligned} -x - 2y &= -(1+t)e^{-t} - 2(-te^{-t}) \\ &= te^{-t} - e^{-t}, \end{aligned}$$

and the second equation is satisfied. Finally,

$$\begin{aligned} x(0) &= (1+0)e^{-0} = 1 \\ y(0) &= -0e^{-0} = 0, \end{aligned}$$

and the initial conditions are satisfied.

9. If $x(t) = e^{-t}(-\cos t - \sin t)$ and $v(t) = 2e^{-t} \sin t$, then

$$\begin{aligned} x' &= (e^{-t}(-\cos t - \sin t))' \\ &= e^{-t}(\sin t - \cos t) - e^{-t}(-\cos t - \sin t) \\ &= 2e^{-t} \sin t \\ &= v, \end{aligned}$$

and the first equation is satisfied. Further,

$$v' = (2e^{-t} \sin t)' = 2e^{-t} \cos t - 2e^{-t} \sin t,$$

and

$$\begin{aligned} -2x - 2v &= -2(e^{-t}(-\cos t - \sin t)) \\ &\quad - 2(2e^{-t} \sin t) \\ &= 2e^{-t} \cos t - 2e^{-t} \sin t, \end{aligned}$$

so the second equation is satisfied. Finally,

$$\begin{aligned} x(0) &= e^{-0}(-\cos 0 - \sin 0) = -1 \\ v(0) &= 2e^{-0} \sin 0 = 0, \end{aligned}$$

so the initial conditions are satisfied.

10. If $x(t) = e^t$ and $y(t) = e^{-t}$, then

$$x' = (e^t)' = e^t,$$

and

$$x^2 y = (e^t)^2 (e^{-t}) = e^t,$$

and the first equation is satisfied. Further,

$$y' = (e^{-t})' = -e^{-t},$$

and

$$-xy^2 = -e^t (e^{-t})^2 = -e^{-t},$$

so the second equation is satisfied. Finally, $x(0) = e^0 = 1$ and $y(0) = e^{-0} = 1$, so the initial conditions are satisfied.

11. We know that $y'' = -2y' - 4y + 3 \cos 2t$, $y(0) = 1$, $y'(0) = 0$. If we let $u_1 = y$ and $u_2 = y'$, then

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -2u_2 - 4u_1 + 3 \cos 2t. \end{aligned}$$

Hence with $\mathbf{u} = (u_1, u_2)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ -2u_2 - 4u_1 + 3 \cos 2t \end{pmatrix}.$$

Furthermore $\mathbf{u}(0) = (u_1(0), u_2(0))^T = (y(0), y'(0))^T = (1, 0)^T$.

12. We know that $x'' = -(\mu/m)x' - (k/m)x + F_0 \cos \omega t$, $x(0) = x_0$, $x'(0) = v_0$. Hence, with $u_1 = x$, and $u_2 = x'$,

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{\mu}{m}u_2 - \frac{k}{m}u_1 + F_0 \cos \omega t. \end{aligned}$$

Therefore, with $\mathbf{u} = (u_1, u_2)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ -(\mu/m)u_2 - (k/m)u_1 + F_0 \cos \omega t \end{pmatrix}.$$

Furthermore, $\vec{u}(0) = (u_1(0), u_2(0))^T = (x(0), x'(0))^T = (x_0, v_0)^T$.

13. We know that $x'' = -\delta x' + x - x^3 + \gamma \cos \omega t$, $x(0) = x_0$, $x'(0) = v_0$. If we let $u_1 = x$ and $u_2 = x'$, then

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\delta u_2 + u_1 - u_1^3 + \gamma \cos \omega t. \end{aligned}$$

Furthermore, $u_1(0) = x(0) = x_0$ and $u_2(0) = x'(0) = v_0$.

14. Set
- $u_1 = x$
- and
- $u_2 = x'$
- . Then

$$\begin{aligned}u_1' &= x' = u_2 \\u_2' &= x'' = -\mu(x^2 - 1)x' - x \\&= -\mu(u_1^2 - 1)u_2 - u_1.\end{aligned}$$

Hence with $\mathbf{u} = (u_1, u_2)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ -\mu(u_1^2 - 1)u_2 - u_1 \end{pmatrix}.$$

Furthermore $\mathbf{u}(0) = (u_1(0), u_2(0))^T = (x(0), x'(0))^T = (x_0, v_0)^T$.

15. We know that
- $\omega''' = \omega$
- ,
- $\omega(0) = \omega_0$
- ,
- $\omega'(0) = \alpha_0$
- , and
- $\omega''(0) = \gamma_0$
- . If we let
- $u_1 = \omega$
- ,
- $u_2 = \omega'$
- , and
- $u_3 = \omega''$
- , then

$$\begin{aligned}u_1' &= u_2 \\u_2' &= u_3 \\u_3' &= u_1.\end{aligned}$$

Further, $u_1(0) = \omega(0) = \omega_0$, $u_2(0) = \omega'(0) = \alpha_0$, and $u_3(0) = \omega''(0) = \gamma_0$.

16. We know that
- $y''' = -y'y'' + \sin \omega t$
- ,
- $y(0) = \alpha$
- ,
- $y'(0) = \beta$
- , and
- $y''(0) = \gamma$
- . If we let
- $u_1 = y$
- ,
- $u_2 = y'$
- , and
- $u_3 = y''$
- , then

$$\begin{aligned}u_1' &= u_2 \\u_2' &= u_3 \\u_3' &= -u_2 u_3 + \sin \omega t.\end{aligned}$$

With $\mathbf{u} = (u_1, u_2, u_3)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ u_3 \\ -u_2 u_3 + \sin \omega t \end{pmatrix}.$$

Furthermore, $\vec{u}(0) = (u_1(0), u_2(0), u_3(0))^T = (y(0), y'(0), y''(0))^T = (\alpha, \beta, \gamma)^T$.

17. The right side of the system

$$\begin{aligned}u' &= v \\v' &= -3u - 2v + 5 \cos t\end{aligned}$$

depends explicitly on the independent variable t . Therefore, the system is nonautonomous.

18. The right side of the system

$$\begin{aligned}u' &= v(u^2 + v^2) \\v' &= -v(u^2 + v^2)\end{aligned}$$

makes no explicit mention of the independent variable. Therefore, the system is autonomous.

19. The right side of the system

$$\begin{aligned}u' &= v \cos u \\v' &= tv\end{aligned}$$

depends explicitly on the independent variable t . Therefore, the system is nonautonomous.

20. The right-hand side in the system

$$\begin{aligned}u' &= v \cos u \\v' &= u^2 e^u\end{aligned}$$

makes no explicit mention of the independent variable. Therefore, the system is autonomous.

21. In the system

$$\begin{aligned}u' &= v + \cos u \\v' &= v - t\omega \\ \omega' &= 5u - 9v + 8\omega,\end{aligned}$$

the right side depends explicitly on the independent variable t . Therefore, the system is nonautonomous.

22. The right-hand side of the system

$$\begin{aligned}u' &= w + \cos u - 2v \\v' &= u^2 e^v \\w' &= u + v + w\end{aligned}$$

makes no explicit mention of the independent variable. Therefore, the system is autonomous.

23. Let
- $x_1 = u$
- ,
- $x_2 = v$
- . If we let
- $\mathbf{x} = (x_1, x_2)'$
- , then the system

$$\begin{aligned}x_1' &= x_2 \\x_2' &= -3x_1 - 2x_2 + 5 \cos t\end{aligned}$$

can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 \\ -3x_1 - 2x_2 + 5 \cos t \end{pmatrix}.$$

24. Let $x_1 = u$, $x_2 = v$. If we let $\mathbf{x} = (x_1, x_2)'$, then the system

$$\begin{aligned} x_1' &= x_1(x_1^2 + x_2^2) \\ x_2' &= -x_2(x_1^2 + x_2^2) \end{aligned}$$

can be written

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1(x_1^2 + x_2^2) \\ -x_2(x_1^2 + x_2^2) \end{pmatrix}.$$

25. Let $x_1 = u$, $x_2 = v$. If we let $\mathbf{x} = (x_1, x_2)'$, then the system

$$\begin{aligned} x_1' &= x_2 \cos x_1 \\ x_2' &= tx_2 \end{aligned}$$

can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 \cos x_1 \\ tx_2 \end{pmatrix}.$$

26. Let $x_1 = u$ and $x_2 = v$. If we let $\mathbf{x} = (x_1, x_2)^T$, then the system

$$\begin{aligned} x_1' &= x_2 \cos x_1 \\ x_2' &= x_1^2 e^{x_1} \end{aligned}$$

can be written

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_2 \cos x_1 \\ x_1^2 e^{x_1} \end{pmatrix}.$$

27. Let $x_1 = u$, $x_2 = v$, and $x_3 = w$. If we let $\mathbf{x} = (x_1, x_2, x_3)'$, then the system

$$\begin{aligned} x_1' &= x_2 + \cos x_1 \\ x_2' &= x_2 - tx_3 \\ x_3' &= 5x_1 - 9x_2 + 8x_3 \end{aligned}$$

can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 + \cos x_1 \\ x_2 - tx_3 \\ 5x_1 - 9x_2 + 8x_3 \end{pmatrix}.$$

28. Let $x_1 = u$, $x_2 = v$, and $x_3 = w$. If we let $\vec{x} = (x_1, x_2, x_3)^T$, then the system

$$\begin{aligned} x_1' &= x_3 + \cos x_1 - 2x_2 \\ x_2' &= x_1^2 e^{x_2} \\ x_3' &= x_1 + x_2 + x_3 \end{aligned}$$

can be written

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_3 + \cos x_1 - 2x_2 \\ x_1^2 e^{x_2} \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

29. Let $x_1 = S$, $x_2 = I$, and $x_3 = R$. If we let $\mathbf{x} = (x_1, x_2, x_3)^T$, then the system

$$\begin{aligned} x_1' &= -x_1 x_2 \\ x_2' &= x_1 x_2 - x_2 \\ x_3' &= x_1 \end{aligned}$$

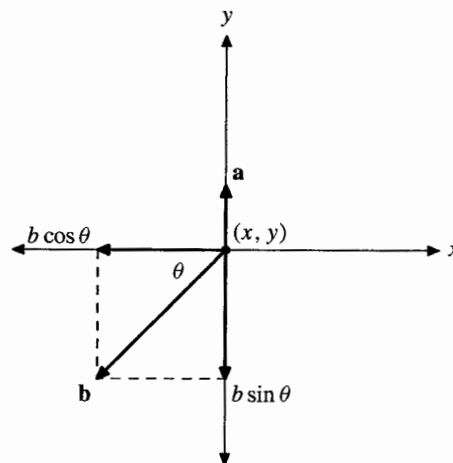
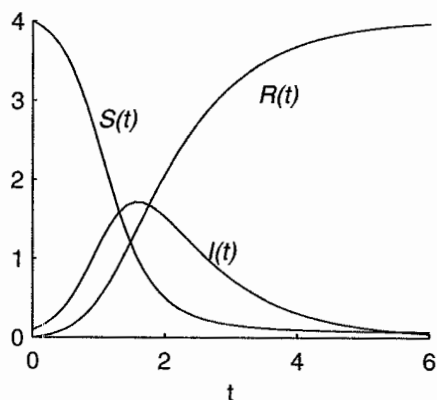
can be written

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_1 x_2 \\ x_1 x_2 - x_2 \\ x_1 \end{pmatrix}.$$

Furthermore, $(x_1(0), x_2(0), x_3(0))^T = (S(0), I(0), R(0))^T = (4, 0, 1, 0)^T$. These equations can be used to produce the following component solutions, $x_1(t) = S(t)$, $x_2(t) = I(t)$, and $x_3(t) = R(t)$.



30. To normalize quantities in the SIR model, let $s = S/N$, $i = I/N$, and $r = R/N$. Starting with

$$\begin{aligned} S' &= -aSI \\ I' &= aSI - bI \\ R' &= bI, \end{aligned}$$

and substituting,

$$\begin{aligned} (sN)' &= -a(sN)(iN) \\ (iN)' &= a(sN)(iN) - b(iN) \\ (rN)' &= b(iN), \end{aligned}$$

or,

$$\begin{aligned} s' &= -asiN \\ i' &= asiN - bi \\ r' &= bi. \end{aligned}$$

31. First, break each velocity vector into vertical and horizontal components, letting b represent the magnitude of the vector $\mathbf{b}$.

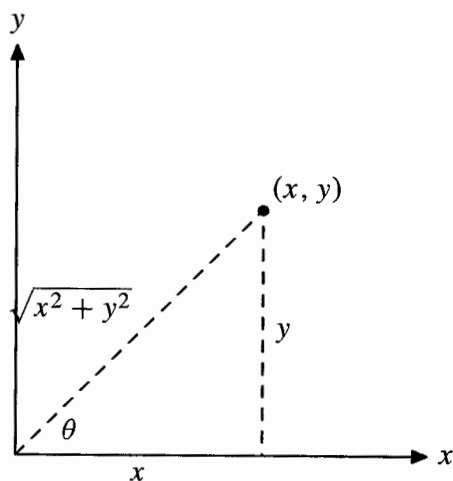
As you can see in the figure, the velocity in the x -direction is $-b \cos \theta$. Thus,

$$\frac{dx}{dt} = -b \cos \theta.$$

In the y -direction, we resolve two components.

$$\frac{dy}{dt} = a - b \sin \theta$$

A second image,



gives us

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}$$

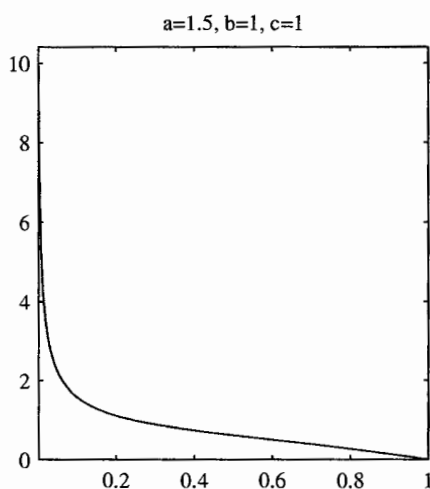
$$\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

Substituting,

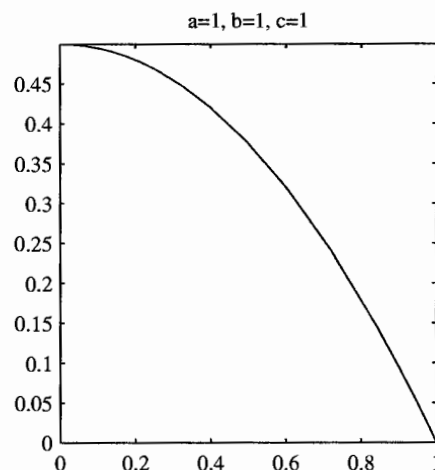
$$\frac{dx}{dt} = \frac{-bx}{\sqrt{x^2 + y^2}}$$

$$\frac{dy}{dt} = a - \frac{by}{\sqrt{x^2 + y^2}}.$$

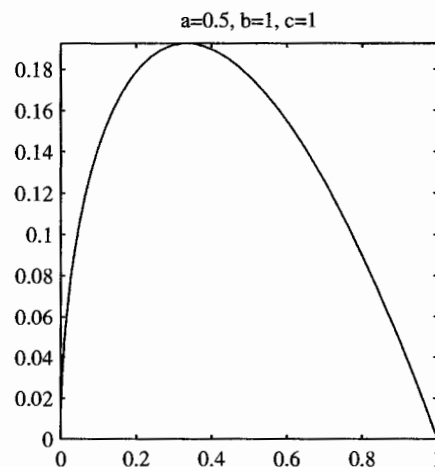
32. Here are the interesting cases. In the first figure, the speed of the current is greater than that of the boat. As a result, the boat drifts downstream.



In the second case, the boat speed and current speed are equal. The boat eventually makes it to shore, but a significant distance downstream from its destination.

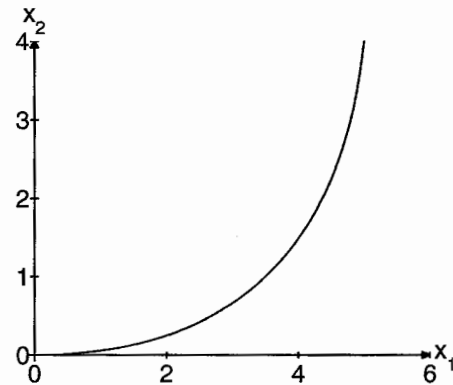
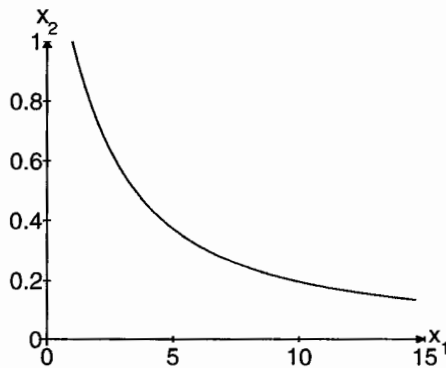
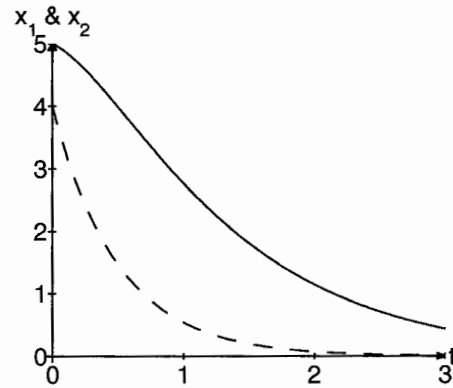
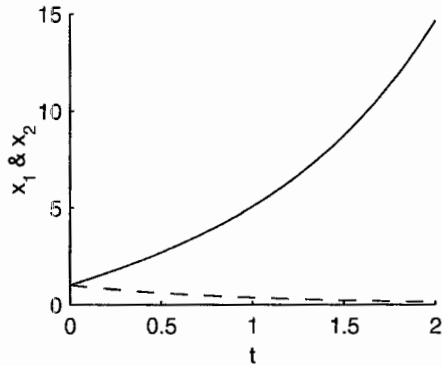


The last case, with the boat speed greater than the current speed, was tricky. We managed to get our variable step solver to stop by solving on the time interval $[0, 1.3]$. Answers will vary on different systems.

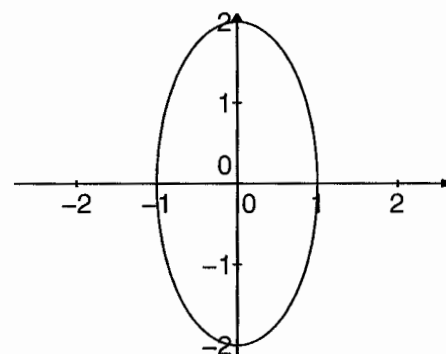
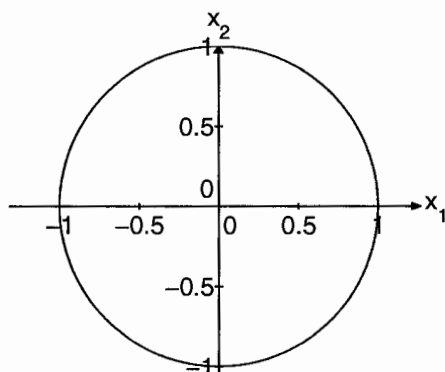
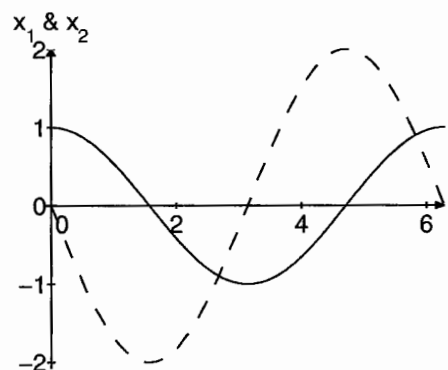
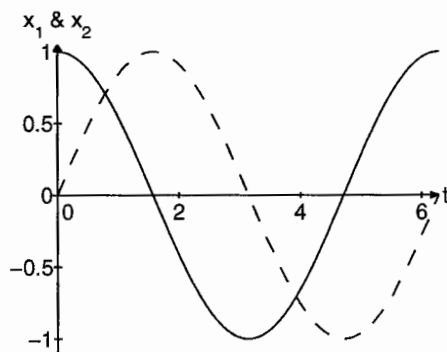


Section 8.2. Geometric Interpretation of Solutions

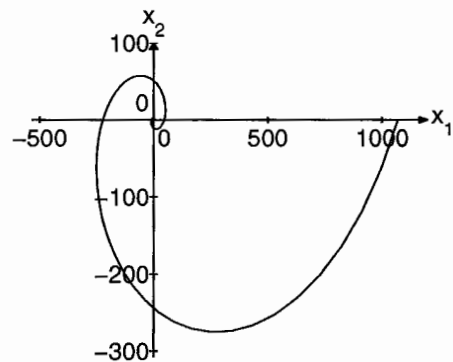
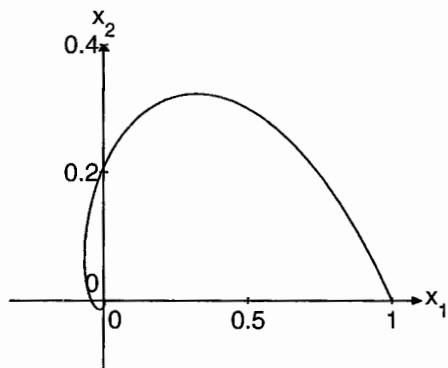
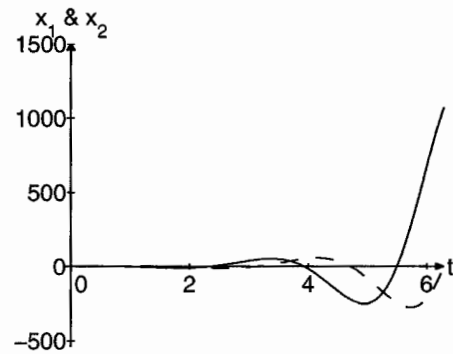
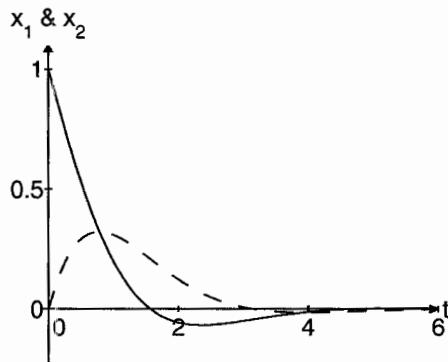
1. The first figure is a plot of $x_1(t) = 2e^t - e^{-t}$ (the solid curve) and $x_2(t) = e^{-t}$ (the dashed curve) versus t on the time interval $[0, 2]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.



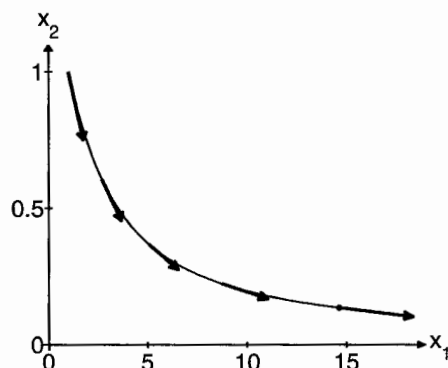
2. The first figure is a plot of $x_1(t) = 9e^{-t} - 4e^{-2t}$ (the solid curve) and $x_2(t) = 4e^{-2t}$ (the dashed curve) versus t on the time interval $[0, 3]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.
3. The first figure is a plot of $x_1(t) = \cos t$ (the solid curve) and $x_2(t) = \sin t$ (the dashed curve) versus t on the time interval $[0, 2\pi]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.



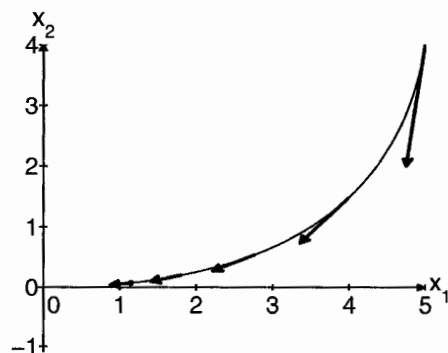
4. The first figure is a plot of $x_1(t) = \cos t$ (the solid curve) and $x_2(t) = -2 \sin t$ (the dashed curve) versus t on the time interval $[0, 2\pi]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.
5. The first figure is a plot of $x_1(t) = e^{-t} \cos t$ (the solid curve) and $x_2(t) = e^{-t} \sin t$ (the dashed curve) versus t on the time interval $[0, 2\pi]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.



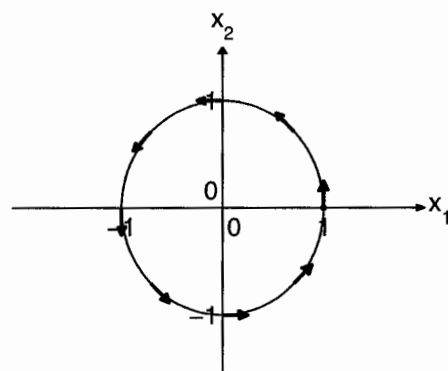
6. The first figure is a plot of $x_1(t) = 2e^t \cos 2t$ (the solid curve) and $x_2(t) = e^t \sin 2t$ (the dashed curve) versus t on the time interval $[0, 2\pi]$. The second figure right uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.
7. If $\mathbf{x}(t) = (2e^t - e^{-t}, e^{-t})$, then $\mathbf{x}'(t) = (2e^t + e^{-t}, -e^{-t})$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



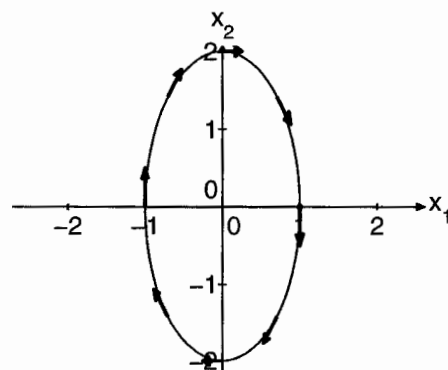
8. If $\mathbf{x}(t) = (9e^{-t} - 4e^{-2t}, 4e^{-2t})$, then $\mathbf{x}'(t) = (-9e^{-t} + 8e^{-2t}, -8e^{-2t})$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



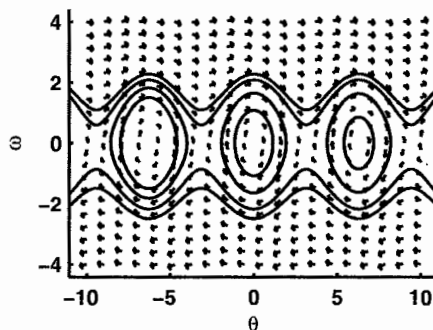
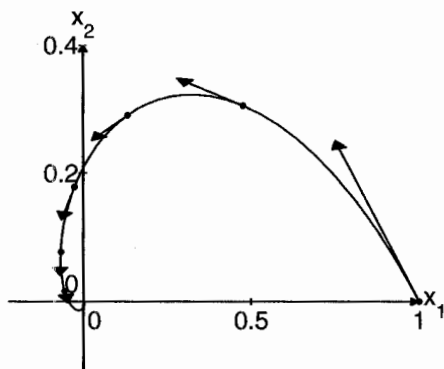
9. If $\mathbf{x}(t) = (\cos t, \sin t)$, then $\mathbf{x}'(t) = (-\sin t, \cos t)$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



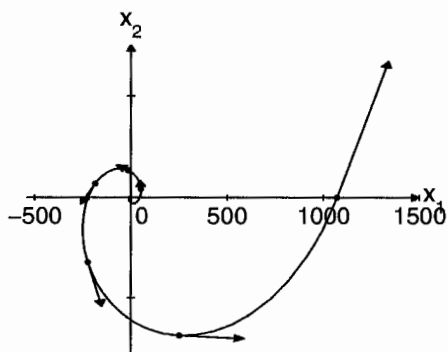
10. If $\mathbf{x}(t) = (\cos t, -2 \sin t)$, then $\mathbf{x}'(t) = (-\sin t, -2 \cos t)$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



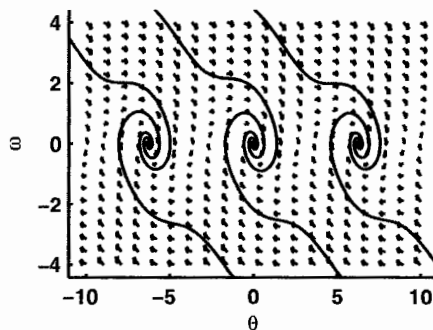
11. If $\mathbf{x}(t) = (e^{-t} \cos t, e^{-t} \sin t)$, then $\mathbf{x}'(t) = (e^{-t}(-\sin t - \cos t), e^{-t}(\cos t - \sin t))$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



12. If $\mathbf{x}(t) = (2e^t \cos 2t, e^t \sin 2t)$, then $\mathbf{x}'(t) = (e^t(-4 \sin 2t + 2 \cos 2t), e^t(2 \cos 2t + \sin 2t))$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.

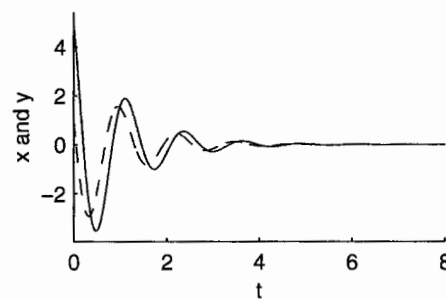
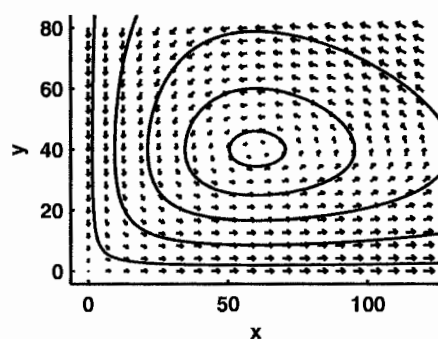


13. The directions and several solution trajectories for $\theta' = \omega$ and $\omega' = -\sin \theta$ follow.

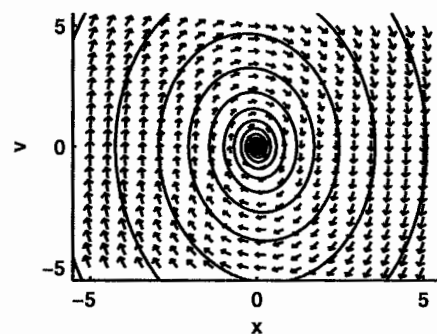
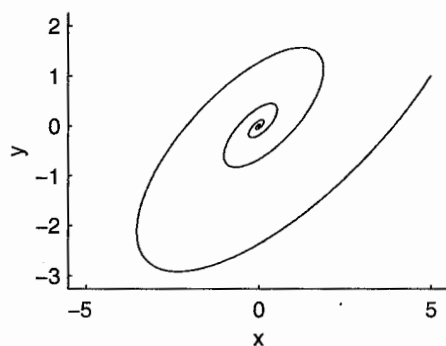


14. The directions and several solution trajectories for $\theta' = \omega$ and $\omega' = -\sin \theta - 0.5\omega$ follow.

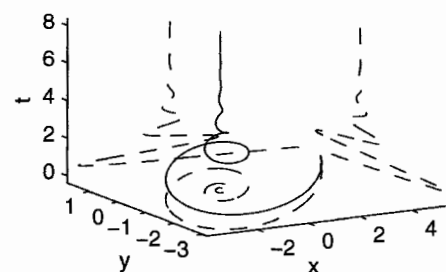
15. The directions and several solution trajectories for $x' = (0.4 - 0.01y)x$ and $y' = (0.005x - 0.3)y$ follow.



16. The directions and several solution trajectories $x' = v$ and $v' = -3x - 0.2v$ follow.

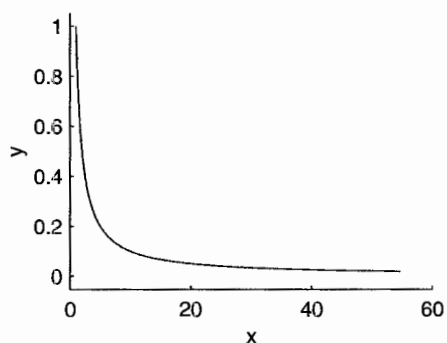
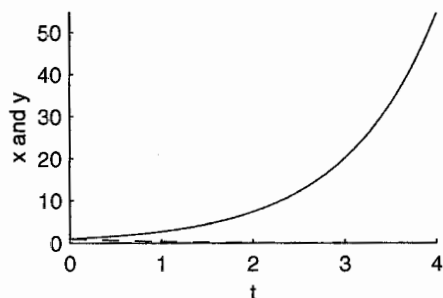


Finally, a composite plot, with the 3D plot solid and the others dashed.

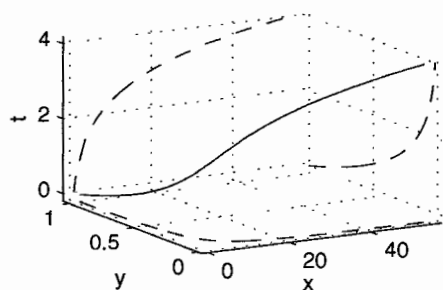


17. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.

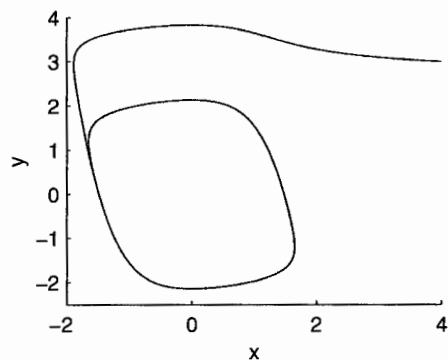
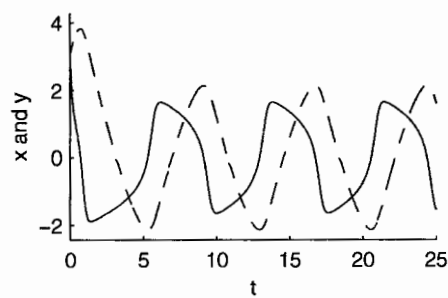
18. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.



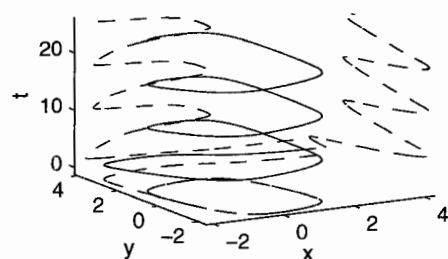
Finally, a composite plot, with the 3D plot solid and the others dashed.



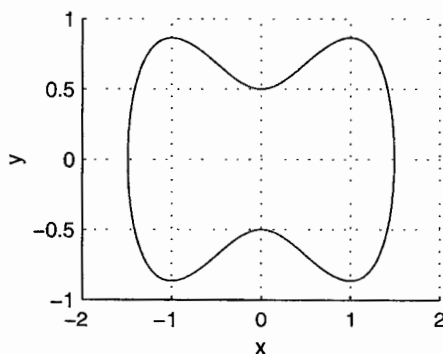
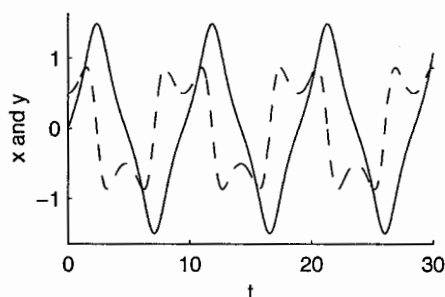
19. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.



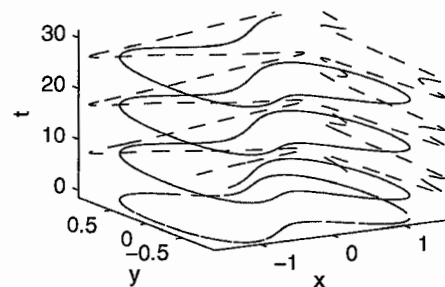
Finally, a composite plot, with the 3D plot solid and the others dashed.



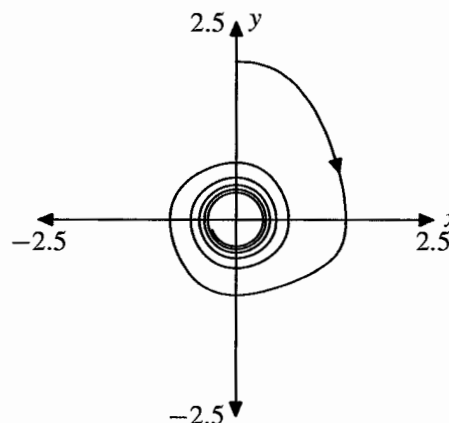
20. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.



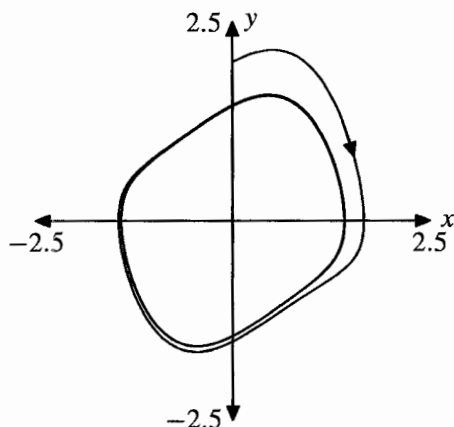
Finally, a composite plot, with the 3D plot solid and the others dashed.



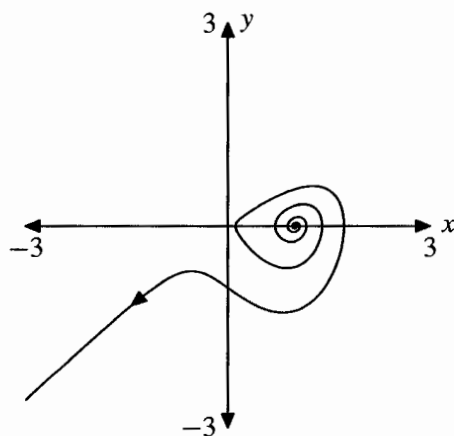
21. In I and IV, x and y exhibit oscillatory behavior. In I, the oscillatory behavior decays, while in IV, the oscillatory behavior eventually becomes periodic. Thus, $I \mapsto D$ and $IV \mapsto C$. In II and III, one component decays to zero while the other component levels asymptotically to some value. In II, y decays to zero, so $II \mapsto A$. But, in III, x decays to zero, so $III \mapsto B$.
22. Initially, $x(0) = 0$ and $y(0) = 2$, then y decays as x increases, thereafter both x and y oscillate as they decay toward zero.



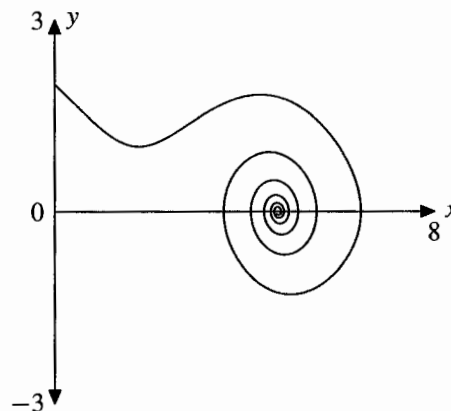
23. Initially, $x(0) = 0$ and $y(0) = 2$. Shortly thereafter, y decays as x increases. Soon, both x and y begin a seemingly periodic motion.



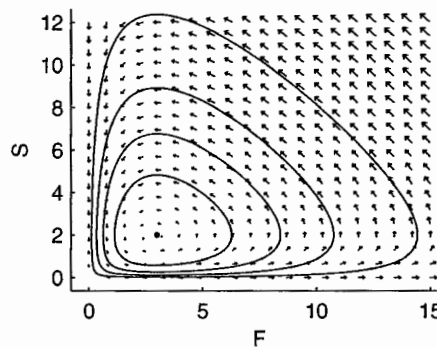
24. At first, x oscillates mildly about 1, while y oscillates mildly about zero. This would indicate a turning about $(1, 0)$ in the phase plane. The oscillations grow larger until both x and y shoot off to $-\infty$. One possible solution follows.



25. Initially, $x(0) = 0$ and $y(0) = 2$. Thereafter, x increases rapidly, then decays asymptotically in an oscillatory manner to about 5 or 6. Meanwhile, y decays, eventually oscillating about zero. One possible solution follows.



26. (a) The solutions appear to be closed, indicating a periodic oscillation of the two species, which eventually return to starting levels.



- (b) The solution starting at $(3, 2)$ remains at $(3, 2)$ for all time. This solution, as we shall soon see, is called an equilibrium solution.
27. If the prey population grows according to the logistic model, then

$$F' = r \left(1 - \frac{F}{K} \right) F.$$

Expanding, $F' = rF - rF^2/K$, and replacing r with a and r/K with e ,

$$F' = aF - eF^2.$$

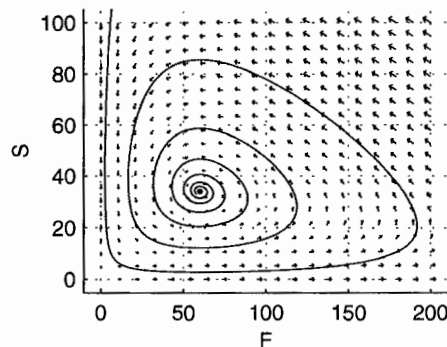
Now, because interaction between species is harmful to the prey,

$$F' = aF - eF^2 - bFS.$$

If predators still follow the Malthusian model, and interactions between species is still considered beneficial to the predators, then

$$S' = -cS + dFS.$$

28. (a) The tricky part on this problem is finding a good viewing window. After some experimentation, we found that $\{(F, S) : 0 \leq F \leq 200, 0 \leq S \leq 100\}$ provided a nice window into the phase plane. It appears that solutions, rather than being periodic as in the standard predator-prey model, now spiral into a stable point.



- (b) The solution starting at $(60, 34)$ remains at $(60, 34)$ for all time. As we shall soon see, this type of solution is called an equilibrium solution.
29. (a) Given that $dF/dt = aF - bFS$ and $dS/dt =$

$$-cS + dFS,$$

$$\begin{aligned} \frac{dS}{dF} &= \frac{dS/dt}{dF/dt} \\ &= \frac{-cS + dFS}{aF - bFS} \\ &= \frac{(-c + dF)S}{(a - bS)F}. \end{aligned}$$

- (b) Using the result in part (a), we separate the variables.

$$\begin{aligned} \frac{a - bS}{S} dS &= \frac{-c + dF}{F} dF \\ \left(\frac{a}{S} - b\right) dS &= \left(-\frac{c}{F} + d\right) dF \end{aligned}$$

Integrate.

$$a \ln S - bS = -c \ln F + dF + C,$$

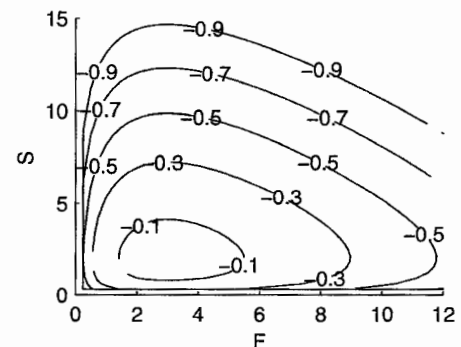
or, equivalently,

$$a \ln S - bS + c \ln F - dF = C.$$

- (c) An implicit function plotter was used to sketch

$$0.2 \ln S - 0.1S + 0.3 \ln F - 0.1F = C,$$

for $C = -0.2, -0.4, -0.6, -0.8$, and -1 .



Section 8.3. Qualitative Analysis

1. Set the right hand side of $x' = 0.2x - 0.04xy$ equal to zero.

$$0.2x - 0.04xy = 0$$

$$20x - 4xy = 0$$

$$4x(5 - y) = 0$$

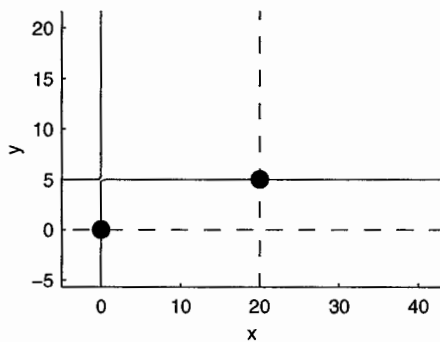
Thus, $x = 0$ and $y = 5$ are the x -nullclines. They appear in a solid line style in the figure. Set the right hand side of $y' = -0.1y + 0.005xy$ equal to zero.

$$-0.1y + 0.005xy = 0$$

$$-100y + 5xy = 0$$

$$5y(-20 + x) = 0$$

Thus, $y = 0$ and $x = 20$ are the y -nullclines. They appear in a dashed line style in the figure.



The equilibrium points appear where the x -nullclines intersect the y -nullclines. These are easily seen to be $(0, 0)$ and $(20, 5)$.

2. Set the right-hand side of $x' = 4x - 2x^2 - xy$ equal to zero.

$$4x - 2x^2 - xy = 0$$

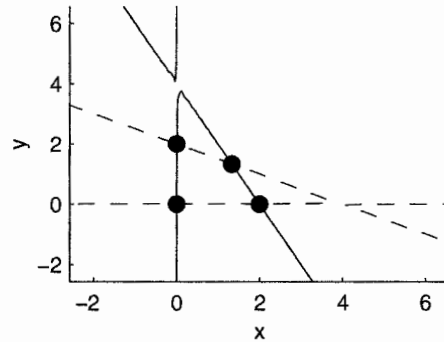
$$x(4 - 2x - y) = 0$$

Thus, $x = 0$ and $4 - 2x - y = 0$ are the x -nullclines. They appear in a solid line style in the figure. Set the right-hand side of $y' = 4y - xy - 2y^2$ equal to zero.

$$4y - xy - 2y^2 = 0$$

$$y(4 - x - 2y) = 0$$

Thus, $y = 0$ and $4 - x - 2y = 0$ are the y -nullclines. They appear in a dashed line style in the figure.



The equilibrium points appear where the x -nullclines intersect the y -nullclines. These are $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(4/3, 4/3)$.

3. Set the right hand side of $x' = x - y - x^3$ equal to zero.

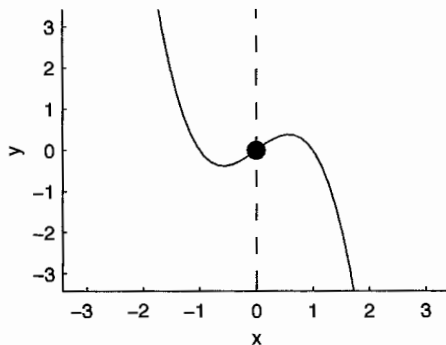
$$x - y - x^3 = 0$$

$$y = x - x^3$$

Thus, $y = x - x^3$ is the x -nullcline. It appears in a solid line style in the figure. Set the right hand side of $y' = x$ equal to zero.

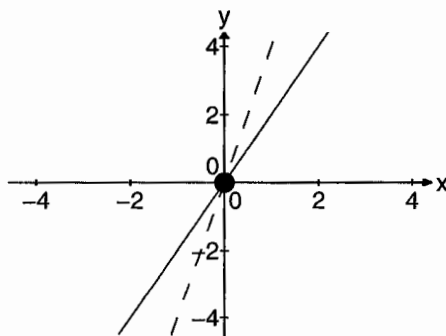
$$x = 0$$

Thus, $x = 0$ is the y -nullcline. It appears in a dashed line style in the figure.



The equilibrium point appears where the x -nullcline intersects the y -nullcline. This is easily seen to be $(0, 0)$.

4. The x -nullcline is the line defined by $2x - y = 0$. It is shown below as the solid line. The y -nullcline is the line defined by $-4x + 2y = 0$. It is the dashed line in the following figure.



5. Set the right hand side of $x' = y$ equal to zero.

$$y = 0$$

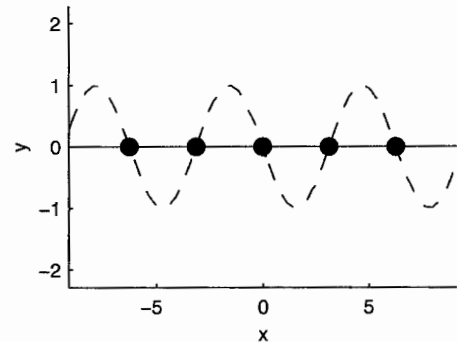
Thus, $y = 0$ is the x -nullcline. It appears in a solid line style in the figure. Set the right hand side of

$$y' = -\sin x - y \text{ equal to zero.}$$

$$-\sin x - y = 0$$

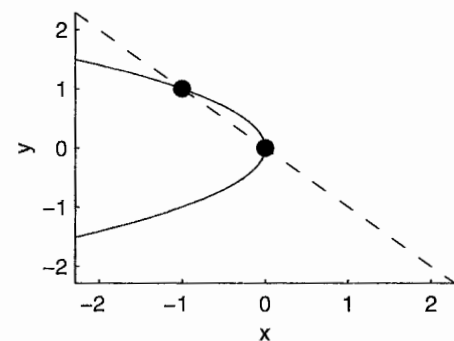
$$y = -\sin x$$

Thus, $y = -\sin x$ is the y -nullcline. It appears in a dashed line style in the figure.



The equilibrium points appear where the x -nullcline intersects the y -nullcline. These occur at the points $(k\pi, 0)$, where k is an integer. A few are shown in the figure.

6. Set the right-hand side of $x' = x + y^2$ equal to zero. Thus, $x + y^2 = 0$ is the x -nullcline, set in a solid line style in the figure. Next, set the right-hand side of $y' = x + y$ equal to zero. Thus, $x + y = 0$ is the y -nullcline, set in a dashed line style in the figure.



The equilibrium points appear where the x -nullclines intersect the y -nullclines. These are at $(0, 0)$ and $(-1, 1)$.

7. (a) If $x(t) = t$ and $y(t) = \sin t$, then

$$x' = (t)' = 1,$$

and

$$1 - (y - \sin x) \cos x = 1 - (\sin t - \sin t) \cos t = 1,$$

so the first equation is satisfied. Further,

$$y' = (\sin t)' = \cos t,$$

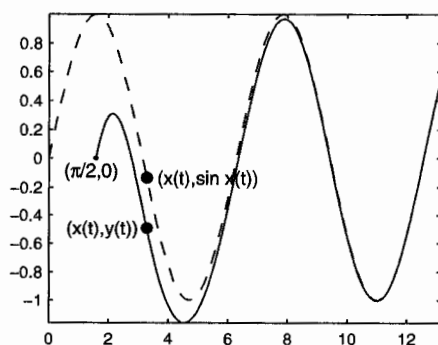
and

$$\cos x - y + \sin x = \cos t - \sin t + \sin t = \cos t,$$

so the second equation is satisfied.

- (b) See the figure in part (c).

- (c) Because of uniqueness, the solution with initial condition $x(0) = \pi/2$, $y(0) = 0$, cannot cross the solution $x = t$, $y = \sin t$ found in part (a). Thus, it must remain below the solution in part (a) for all time. Therefore, if $(x(t), y(t))$ denotes the second solution, we must have $y(t) < \sin x(t)$ for all time, as shown in the figure.



8. (a) If $x = t$ and $y = e^t$, then

$$x' = (t)' = 1$$

and

$$1 - e^x(y - e^x) = 1 - e^t(e^t - e^t) = 1,$$

so the first equation is satisfied. Further,

$$y' = (e^t)' = e^t$$

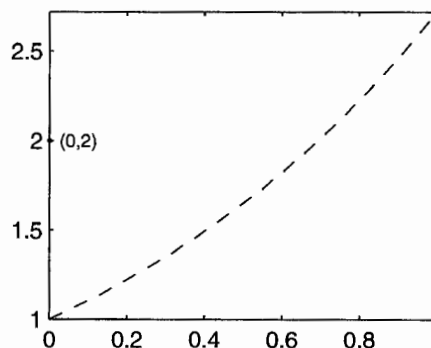
and

$$2e^x - y = 2e^t - e^t = e^t,$$

so the second equation is satisfied.

- (b) See the figure in part (c).

- (c) Note that the solution in part (a) has $x(0) = 0$, $y(0) = e^0 = 1$. Because of uniqueness, the solution with initial condition $x(0) = 0$, $y(0) = 2$ must not intersect the solution in part (a). Thus, this solution must remain above the solution in part (a) for all time. Actually, things are a little tricky here, as $(0, 2)$ is an equilibrium solution of the system. Thus, the solution with initial condition $(0, 2)$ remains at $(0, 2)$ for all time.



9. (a) If $x = e^t$ and $y = -e^t$, then

$$x' = (e^t)' = e^t,$$

and

$$\begin{aligned} -y - x(x^2 - y^2) &= -(-e^t) - e^t((e^t)^2 - (-e^t)^2) \\ &= e^t - e^t(e^{2t} - e^{2t}) \\ &= e^t, \end{aligned}$$

so the first equation is satisfied. Further,

$$y' = (-e^t)' = -e^t,$$

and

$$\begin{aligned} -x - y(x^2 - y^2) &= -e^t - (-e^t)((e^t)^2 - (-e^t)^2) \\ &= -e^t + e^t(e^{2t} - e^{2t}) \\ &= -e^t, \end{aligned}$$

so the second equation is satisfied.

(b) If $x(t) = e^{-t}$ and $y(t) = e^{-t}$, then

$$x' = (e^{-t})' = -e^{-t},$$

and

$$\begin{aligned} -y - x(x^2 - y^2) &= -e^{-t} - e^{-t}((e^{-t})^2 - (e^{-t})^2) \\ &= -e^{-t} - e^{-t}(e^{-2t} - e^{-2t}) \\ &= -e^{-t}, \end{aligned}$$

so the first equation is satisfied. Further,

$$y' = (e^{-t})' = -e^{-t},$$

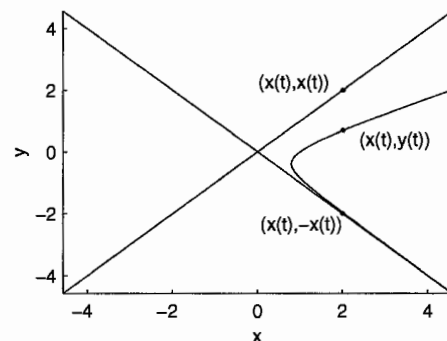
and

$$\begin{aligned} -x - y(x^2 - y^2) &= -e^{-t} - e^{-t}((e^{-t})^2 - (e^{-t})^2) \\ &= -e^{-t} - e^{-t}(e^{-2t} - e^{-2t}) \\ &= -e^{-t}, \end{aligned}$$

and the second equation is satisfied.

(c) See the figure in part (d).

(d) It is important to recognize that solution (a), $x(t) = e^t$, $y(t) = -e^t$, is a parametric description of the line $y = -x$. Also, solution (b) lies on the line $y = -x$. Uniqueness tells us that this third solution must remain trapped between the solutions of parts (a) and (b) (solutions may not cross). Thus, if $(x(t), y(t))$ is a point on this third solution, the figure shows why $-x(t) < y(t) < x(t)$ for all t .



10. (a) Notice that if $x(t) = \sin t$ and $y(t) = \cos t$, then $x^2 + y^2 = 1$. Therefore we have

$$x' = \cos t \quad \text{and}$$

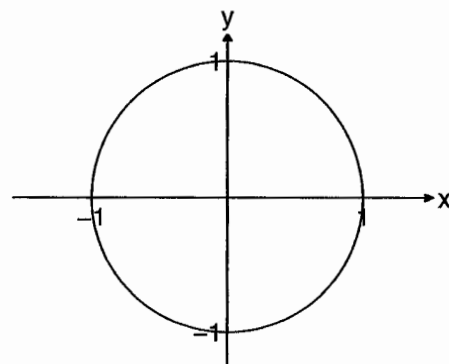
$$y - x(x^2 + y^2 - 1) = y = \cos t.$$

Next

$$y' = -\sin t \quad \text{and}$$

$$-x - y(x^2 + y^2 - 1) = -x = -\sin t.$$

- (b) The solution curve in part (a) is the unit circle and is plotted below.



- (c) The point $(x(0), y(0)) = (0.5, 0)$ is inside the unit circle, which is the solution curve from part (a). By the uniqueness theorem the solution curve starting at $(0.5, 0)$ cannot cross the

unit circle. It must therefore stay inside the unit circle for all time. Hence $x^2(t) + y^2(t) < 1$ for all t .

11. (a) Set the right side of $x' = ax - bxy - \epsilon x$ equal to zero.

$$ax - bxy - \epsilon x = 0$$

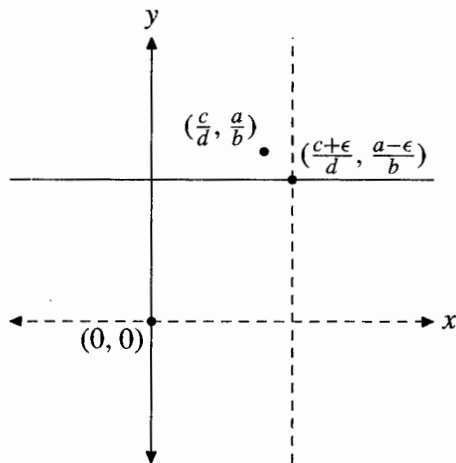
$$x((a - \epsilon) - by) = 0$$

Thus, $x = 0$ and $y = (a - \epsilon)/b$ are the x -nullclines. Set the right side of $y' = -cy + dxy - \epsilon y$ equal to zero.

$$-cy + dxy - \epsilon y = 0$$

$$y((-c - \epsilon) + dx) = 0$$

Thus, $y = 0$ and $x = (c + \epsilon)/d$ are the y -nullclines. The equilibrium points are intersections of the x - and y -nullclines, easily seen to be $(0, 0)$ and $((c + \epsilon)/d, (a - \epsilon)/b)$.

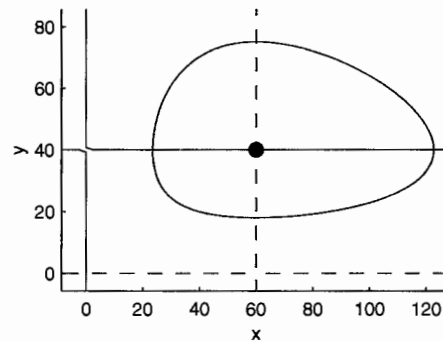


- (b) As you can see in the figure in part (a), the harvesting strategy has shifted the equilibrium point. It has moved from

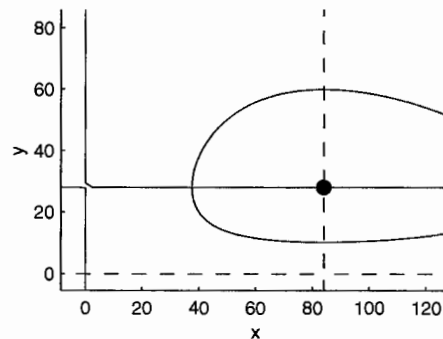
$$\left(\frac{c}{d}, \frac{a}{b}\right) \text{ to } \left(\frac{c + \epsilon}{d}, \frac{a - \epsilon}{b}\right).$$

Therefore, the predator population decreases somewhat, but the prey population actually increases. So, this harvesting strategy is a poor one if the intent was to decrease the level of prey (such as spraying with insecticides).

- (c) First, the original system, $x' = (0.4 - 0.01y)x$, $y' = (-0.3 + 0.005x)y$, without harvesting, provides this image.



Note that the equilibrium point is $(60, 40)$ for this system. Now, if we add harvesting, as in system 11c, then we get this result. Note that both pictures were crafted with the same initial condition; namely, $x(0) = 40$ and $y(0) = 20$.



Clearly, the predator population has decreased on average, while the prey population has increased on average. The equilibrium point is now at $(84, 28)$.

12. (a) Set the right-hand side of $x' = ax(1 - x/K) -$

bxy equal to zero.

$$ax \left(1 - \frac{x}{K}\right) - bxy = 0$$

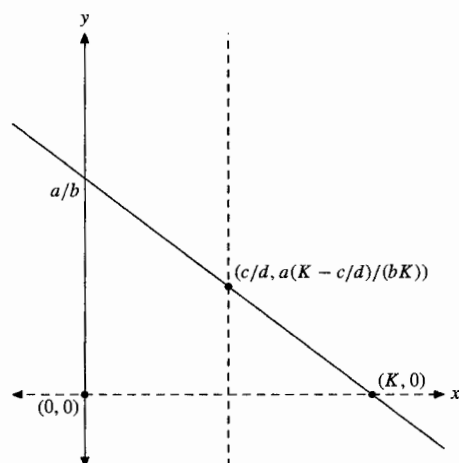
$$x \left[a \left(1 - \frac{x}{K}\right) - by \right] = 0$$

Thus, $x = 0$ and $a(1 - x/K) - by = 0$ are the x -nullclines (solid). Set the right-hand side of $y' = -cy + dxy$ equal to zero.

$$-cy + dxy = 0$$

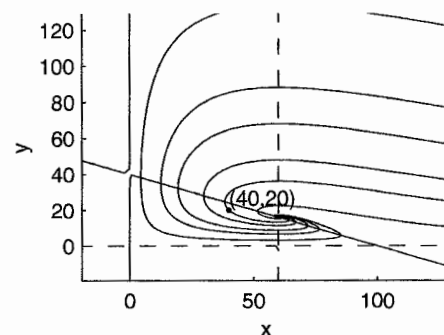
$$y(-c + dx) = 0$$

Thus, $y = 0$ and $-c + dx = 0$ are the y -nullclines (dashed). If $K > c/d$, then the equilibrium point $(c/d, a(K - c/d)/(bK))$ falls in the first quadrant.



There are also equilibrium points at $(0, 0)$ and $(K, 0)$.

- (b) This particular trajectory is labelled with initial condition $(40, 20)$ in the phase portrait in part (c).
- (c) The phase portrait follows.



In this adjustment of the predator-prey model, note that solutions are no longer periodic. Rather, solutions “spiral” and “sink” into the equilibrium point at $(c/d, a(K - c/d)/(bK))$, in this case, the point $(60, 16)$.

13. If x grows according to the logistic model, then

$$x' = r \left(1 - \frac{x}{K}\right) x = rx - \frac{rx^2}{K}.$$

Of course, this becomes

$$x' = ax - bx^2,$$

if we replace r with a and r/K with b . Similarly, if y grows according to the logistic model, we can eventually write

$$y' = Ay - By^2.$$

Now, in light of the competition for resources, interaction between the two species is harmful to each. Thus, a certain percentage of contacts between species will decrease the rates dx/dt and dy/dt (perhaps differently for each rate), so we write

$$x' = ax - bx^2 - cxy$$

$$y' = Ay - By^2 - Cxy.$$

14. To name a few:

- We’ve seen systems where the solutions are closed periodic solutions that spiral about a “central” equilibrium point.
- We’ve seen systems where solutions seemingly “sink” into an equilibrium point.
- There are others.

15. Setting the right side of
- $x' = \sigma(y - x)$
- equal to zero,

$$\begin{aligned}\sigma(y - x) &= 0 \\ y &= x.\end{aligned}$$

Setting the right side of $z' = -\beta z + xy$ equal to zero,

$$\begin{aligned}-\beta z + xy &= 0 \\ xy &= \beta z.\end{aligned}$$

However, $y = x$, so we can write

$$x^2 = \beta z.$$

Setting the right side of $y' = \rho x - y - xz$ equal to zero,

$$\rho x - y - xz = 0.$$

However, $y = x$, so

$$\begin{aligned}\rho x - x - xz &= 0 \\ x(\rho - 1 - z) &= 0.\end{aligned}$$

Thus, either $x = 0$ or $z = \rho - 1$. If $x = 0$, then y and z are easily shown to also equal zero, giving $(0, 0, 0)$ as one equilibrium point. If $z = \rho - 1$, then

$$\begin{aligned}x^2 &= \beta z \\ x^2 &= \beta(\rho - 1) \\ x &= \pm\sqrt{\beta(\rho - 1)}.\end{aligned}$$

Of course, this last expression is real only if $\rho > 1$, which it is. Thus, the second equilibrium point is $(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1)$ and the third equilibrium point is $(-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1)$.

Section 8.4. Linear Systems

1. Note that the system

$$\begin{aligned}x_1' &= -x_2 \\ x_2' &= -x_1 - 2x_2 + 5 \sin t\end{aligned}$$

has the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + f_2(t).\end{aligned}$$

Hence, the system is linear. Although $f_1(t) = 0$, because $f_2(t) = 5 \sin t$ is not zero, the system is inhomogeneous.

2. Note that the system

$$\begin{aligned}x_1' &= -2x_1 + x_1x_2 \\ x_2' &= -3x_1 - x_2,\end{aligned}$$

because of the nonlinear term x_1x_2 , does not have the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + f_2(t).\end{aligned}$$

Hence, the system is nonlinear.

3. Note that the system

$$\begin{aligned}x_1' &= -x_2 \\ x_2' &= \sin x_1,\end{aligned}$$

because of the nonlinear term $\sin x_1$, does not have the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + f_2(t).\end{aligned}$$

Hence, the system is nonlinear.

4. Note that the system

$$\begin{aligned}x_1' &= x_1 + (\sin t)x_2 \\ x_2' &= 2tx_1 - x_2\end{aligned}$$

has the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + f_2(t).\end{aligned}$$

Hence, the system is linear. Because $f_1(t) = 0$ and $f_2(t) = 0$, the system is homogeneous.

5. Note that the system

$$\begin{aligned}x_1' &= x_2 \\x_2' &= x_3 \\x_3' &= -x_1 - x_2 + \sin t\end{aligned}$$

has the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 + f_1(t) \\x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3 + f_2(t) \\x_3' &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3 + f_3(t).\end{aligned}$$

Hence, the system is linear. Although $f_1(t) = 0$ and $f_2(t) = 0$, because $f_3(t) = \sin t$ is nonzero, the system is inhomogeneous.

6. Note that the system

$$\begin{aligned}x_1' &= -x_1 + tx_2 - t^2x_3 \\x_2' &= -2x_1 + 2tx_2 \\x_3' &= -t^2x_1 + (\sin t)x_2 - \sin x_3,\end{aligned}$$

because of the nonlinear term $\sin x_3$, does not have the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 + f_1(t) \\x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3 + f_2(t) \\x_3' &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3 + f_3(t).\end{aligned}$$

Hence, the system is nonlinear. Note that all the other terms are linear.

7. If $x_1(t) = e^t$ and $x_2(t) = -e^t$, then the left side of $x_1' = 3x_1 + 2x_2$ is

$$x_1' = e^t.$$

Substituting in the right hand side,

$$3x_1 + 2x_2 = 3(e^t) + 2(-e^t) = e^t.$$

Hence, the first equation is satisfied. In a similar fashion, the left side of $x_2' = -4x_1 - 3x_2$ is $x_2' = -e^t$ and the right side is $-4x_1 - 3x_2 = -4(e^t) - 3(-e^t) = -e^t$. Hence, the second equation is satisfied.

8. If $x_1(t) = e^{-2t}$ and $x_2(t) = -e^{-2t}$, then the left side of $x_1' = -5x_1 - 3x_2$ is

$$x_1' = -2e^{-2t}.$$

Substituting in the right hand side,

$$-5x_1 - 3x_2 = -5(e^{-2t}) - 3(-e^{-2t}) = -2e^{-2t}.$$

Hence, the first equation is satisfied. In a similar fashion, the left side of $x_2' = 6x_1 + 4x_2$ is $x_2' = 2e^{-2t}$ and the right side is $6x_1 + 4x_2 = 6(e^{-2t}) + 4(-e^{-2t}) = 2e^{-2t}$. Hence, the second equation is satisfied.

9. If $x_1(t) = 2e^{-t} - e^{-2t}$ and $x_2(t) = -2e^{-t}$, then the left side of $x_1' = -2x_1 - x_2$ is

$$x_1' = -2e^{-t} + 2e^{-2t}.$$

Substituting in the right hand side,

$$\begin{aligned}-2x_1 - x_2 &= -2(2e^{-t} - e^{-2t}) - (-2e^{-t}) \\&= -2e^{-t} + 2e^{-2t}.\end{aligned}$$

Hence, the first equation is satisfied. In a similar fashion, the left side of $x_2' = -x_2$ is $x_2' = 2e^{-t}$ and the right side is $-x_2 = 2e^{-t}$. Hence, the second equation is satisfied.

10. If $x_1(t) = -2e^{-2t} + 3e^{3t}$ and $x_2(t) = -2e^{-2t}$, then the left side of $x_1' = 3x_1 - 5x_2$ is

$$x_1' = 4e^{-2t} + 9e^{3t}.$$

Substituting in the right hand side,

$$\begin{aligned}3x_1 - 5x_2 &= 3(-2e^{-2t} + 3e^{3t}) - 5(-2e^{-2t}) \\&= 4e^{-2t} + 9e^{3t}.\end{aligned}$$

Hence, the first equation is satisfied. In a similar fashion, the left side of $x_2' = -2x_2$ is $x_2' = 4e^{-2t}$ and the right side is $-2x_2 = -2(-2e^{-2t}) = 4e^{-2t}$. Hence, the second equation is satisfied.

11. We can write

$$\begin{aligned}x_1' &= -2x_1 + 3x_2 \\x_2' &= x_1 - 4x_2\end{aligned}$$

as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Therefore, the system is linear.

12. We can write

$$\begin{aligned}x_1' &= tx_1 + (t \cos t)x_2 \\x_2' &= (\sin t)x_1 + 3x_2\end{aligned}$$

as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} t & t \cos t \\ \sin t & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Therefore, the system is linear.

13. The system

$$\begin{aligned}x_1' &= -2x_1 + x_2^2 \\x_2' &= 3x_1 - x_2\end{aligned}$$

is nonlinear. Note the term x_2^2 . It cannot be written in the form $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$.

14. We can write

$$\begin{aligned}x_1' &= -2x_1 + 3tx_2 + \cos t \\x_2' &= \frac{1}{t}x_1 - 4x_2 + \frac{\sin t}{t}.\end{aligned}$$

as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -2 & 3t \\ 1/t & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \cos t \\ (\sin t)/t \end{pmatrix}$$

Therefore, the system is linear.

15. The system

$$\begin{aligned}t^2 x_1' &= -2tx_1x_2 + 3x_2 \\(1/t)x_2' &= tx_1 - (4/t)x_2\end{aligned}$$

is nonlinear. Note the nonlinear term $-2tx_1x_2$. Therefore, the system cannot be written in the form $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$.

16. We can write

$$\begin{aligned}x_1' &= -2tx_1 + 3x_2 - (\cos t)x_3 \\x_2' &= x_1 - 4t^2x_3 \\x_3' &= -\frac{3}{t}x_2 - \cos t\end{aligned}$$

as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -2t & 3 & -\cos t \\ 1 & 0 & -4t^2 \\ 0 & -3/t & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\cos t \end{pmatrix}.$$

Therefore, the system is linear.

17. Exercises 11, 12, 14, and 16 are linear; 13 and 15 are nonlinear. Of the linear systems, only 11 and 12 are homogeneous.

18. Write the system
- $x_1' = -3x_1$
- ,
- $x_2' = -2x_1 - x_2$
- in matrix-vector form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is in the form $\mathbf{x}' = A\mathbf{x}$. Next, if $\mathbf{x} = (e^{-3t}, e^{-3t})^T$, then $\mathbf{x}' = (-3e^{-3t}, -3e^{-3t})^T$ and

$$A\mathbf{x} = \begin{pmatrix} -3 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix}.$$

Thus, $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x} = (e^{-3t}, e^{-3t})^T$ is a solution.

19. Write the system
- $x_1' = 8x_1 - 10x_2$
- ,
- $x_2' = 5x_1 - 7x_2$
- in matrix-vector form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 8 & -10 \\ 5 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is in the form $\mathbf{x}' = A\mathbf{x}$. Next, if $\mathbf{x} = (2e^{3t}, e^{3t})^T$, then $\mathbf{x}' = (6e^{3t}, 3e^{3t})^T$ and

$$A\mathbf{x} = \begin{pmatrix} 8 & -10 \\ 5 & -7 \end{pmatrix} \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 6e^{3t} \\ 3e^{3t} \end{pmatrix}.$$

Thus, $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x} = (2e^{3t}, e^{3t})^T$ is a solution.

20. Write the system
- $x_1' = -3x_1 + x_2$
- ,
- $x_2' = -2x_1$
- in matrix-vector form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is in the form $\mathbf{x}' = A\mathbf{x}$. Next, if $\mathbf{x} = (-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t})^T$, then $\mathbf{x}' = (2e^{-2t} - e^{-t}, 2e^{-2t} - 2e^{-t})^T$ and

$$\begin{aligned}A\mathbf{x} &= \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} -e^{-2t} + e^{-t} \\ -e^{-2t} + 2e^{-t} \end{pmatrix} \\&= \begin{pmatrix} 2e^{-2t} - e^{-t} \\ 2e^{-2t} - 2e^{-t} \end{pmatrix}.\end{aligned}$$

Thus, $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x} = (-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t})^T$ is a solution.

21. Write the system $x_1' = -x_1 + 4x_2$, $x_2' = 3x_2$ in matrix-vector form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Next, if $\mathbf{x} = (e^{3t} - e^{-t}, e^{3t})^T$, then $\mathbf{x}' = (3e^{3t} + e^{-t}, 3e^{3t})^T$ and

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} e^{3t} - e^{-t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} + e^{-t} \\ 3e^{3t} \end{pmatrix}.$$

Thus, $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{x} = (e^{3t} - e^{-t}, e^{3t})^T$ is a solution.

22. Let's begin by setting a node at a (see Figure ??) and labeling current coming out of node a with I_1 , I_2 , and I_3 . Further, let the voltage drops across the capacitor, resistor, and inductor be represented by V_1 , V_2 , and V_3 , respectively. Kirchhoff's second law requires that the sum of the currents at node a equal zero. Thus,

$$I_1 + I_2 + I_3 = 0. \quad (4.1)$$

Next, the sum of the voltage drops around the leftmost loop must equal zero. Thus,

$$V_1 - V_2 = 0. \quad (4.2)$$

Also, the sum of the voltage drops around the rightmost loop must equal zero. Thus,

$$V_2 - V_3 = 0. \quad (4.3)$$

Furthermore, the voltage drops across each circuit element are given by the equations

$$CV_1' = I_1, \quad V_2 = RI_2, \quad \text{and} \quad LI_3' = V_3. \quad (4.4)$$

Note that the independent variables in equations (4.18) and (4.19) are V and I , represented by V_1 and I_3 , respectively. Therefore, the goal is to eliminate V_2 , V_3 , I_1 and I_2 from equations (4.1), (4.2), (4.3), and (4.4), keeping only V_1 and I_3 .

First, note that equations (4.2) and (4.3) imply that $V_1 = V_2 = V_3$, an anticipated consequence of the fact that the circuit elements are in parallel. Taking $LI_3' = V_3$ from equation (4.4), then replacing V_3 with V_1 yields the equation

$$LI_3' = V_1,$$

or, alternatively,

$$I_3' = \frac{V_1}{L}. \quad (4.5)$$

Secondly, start with $CV_1' = I_1$ from equation (4.4), then use equation (4.1) to write

$$CV_1' = -I_2 - I_3.$$

Now, from equation (4.4), substitute $I_2 = V_2/R$ to get

$$CV_1' = -\frac{V_2}{R} - I_3.$$

But, $V_1 = V_2$, so

$$CV_1' = -\frac{V_1}{R} - I_3,$$

or, alternatively,

$$V_1' = -\frac{V_1}{RC} - \frac{I_3}{C}. \quad (4.6)$$

Because, $V_1 = V$ and $I_3 = I$, equations (4.5) and (4.6) provide us with

$$I' = \frac{V}{L} \quad (4.7)$$

$$V' = -\frac{V}{RC} - \frac{I}{C}. \quad (4.8)$$

23. Label the currents coming out of node "a", as shown in Figure 8. The sum of the current flowing out of node "a" must equal zero.

$$I + I_2 + I_3 = 0 \quad (4.9)$$

Proceeding in a clockwise direction around the top loop, the sum of the voltage drops across each element must equal zero. Thus,

$$R_1 I + LI' - R_2 I_2 = 0. \quad (4.10)$$

Because we are moving in the direction of the current I , the voltage drops across the resistor R_1 and the inductor L are positive. However, as we continue in a clockwise direction, we move **against** the current I_2 and thus experience a voltage **gain** across the resistor R_2 . Hence, the negative sign in equation (4.10).

Secondly, we proceed in a clockwise direction around the bottom loop. Again, the sum of the voltage drops must equal zero.

$$R_2 I_2 - V = 0 \quad (4.11)$$

Again, we move opposite the current when moving clockwise through the capacitor. This, coupled with the fact that V represents the voltage drop across the capacitor, explains $-V$ in equation (4.11). Solve equation (4.10) for LI' .

$$LI' = -R_1 I + R_2 I_2 \quad (4.12)$$

>From equation (4.11), $R_2 I_2 = V$, which, when substituted in (4.12), produces the following result.

$$LI' = -R_1 I + V \quad (4.13)$$

Note that this is our first equation (4.20).

Next, solve equation (4.11) for I_2 .

$$I_2 = \frac{V}{R_2} \quad (4.14)$$

>From equation (4.9), $I_2 = -I - I_3$. With this substitution, equation (4.14) becomes

$$\frac{V}{R_2} = -I - I_3. \quad (4.15)$$

However, the voltage drop across the capacitor is given by

$$V = \frac{q}{C},$$

where q is the charge on the capacitor. Differentiating,

$$CV' = q',$$

and using the fact that $I_3 = q'$,

$$CV' = I_3.$$

Substituting this result in equation (4.15), we obtain

$$\frac{V}{R_2} = -I - CV',$$

or equivalently,

$$CV' = -I - \frac{V}{R_2}, \quad (4.16)$$

which is identical to equation (4.21).

24. **Solution.** We'll represent the current across E and R_1 with the variable I . This leads to the adjustment in the schematic shown in Figure ?? . Kirchoff's second law says that the current into the node at D equals the current out at the same node. Thus,

$$I = I_1 + I_2. \quad (4.17)$$

Traversing the loop $ABCD A$ in a clockwise direction, the sum of the voltage drops at each element must equal zero. Thus,

$$-E + R_1 I + L_1 I_1' = 0,$$

or equivalently,

$$L_1 I_1' = -R_1 I + E. \quad (4.18)$$

Use equation (4.17) to replace I with $I_1 + I_2$ in equation (4.18). Thus,

$$L_1 I_1' = -R_1 (I_1 + I_2) + E,$$

or equivalently,

$$L_1 I_1' = -R_1 I_1 - R_2 I_2 + E.$$

This last equation is the requested equation (4.22).

Next, proceed in a clockwise direction about the loop $ABCJKDA$.¹ Thus,

$$-E + R_1 I + R_2 I_2 + L_2 I_2' = 0,$$

or equivalently,

$$L_2 I_2' = -R_1 I - R_2 I_2 + E. \quad (4.19)$$

We again use equation (4.17) to replace I with $I_1 + I_2$ in equation (4.19).

$$L_2 I_2' = -R_1 (I_1 + I_2) - R_2 I_2 + E$$

A little algebra provides

$$L_2 I_2' = -R_1 I_1 - (R_1 + R_2) I_2 + E,$$

which is the needed equation (4.23).

¹We could traverse the loop $CJKDC$, but this would involve both I_1' and I_2' . Although this loop would eventually provide us with the needed result, the loop $ABCJKDA$ only involves I_2' , a better strategy for this problem.

25. Let $x_1(t)$ and $x_2(t)$ represent the salt content (in pounds) of the upper and lower tanks, respectively. Because pure water flows into the upper tank, no salt is coming into the tank. However, the rate at which salt is leaving the tank is

$$\begin{aligned}\text{Rate Out} &= 4 \text{ gal/min} \times \frac{x_1(t)}{100} \text{ lb/gal} \\ &= \frac{x_1(t)}{25} \text{ lb/min.}\end{aligned}$$

Because $dx_1/dt = \text{Rate In} - \text{Rate Out}$, we write

$$\frac{dx_1}{dt} = -\frac{x_1}{25}. \quad (4.20)$$

Salt enters the lower tank at the same rate as it is leaving the upper tank. Further, the rate at which salt leaves the lower tank is

$$\begin{aligned}\text{Rate Out} &= 4 \text{ gal/min} \times \frac{x_2(t)}{100} \text{ lb/gal} \\ &= \frac{x_2(t)}{25} \text{ lb/min.}\end{aligned}$$

Again, because $dx_2/dt = \text{Rate In} - \text{Rate Out}$, we write

$$\frac{dx_2}{dt} = \frac{x_1}{25} - \frac{x_2}{25}. \quad (4.21)$$

In matrix form, the system defined by equations (4.20) and (4.21) is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1/25 & 0 \\ 1/25 & -1/25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (4.22)$$

Recall that the initial salt content in the upper and lower tanks is 10 and 20 pounds, respectively. Thus, the initial condition is $\mathbf{x}(0) = (x_1(0), x_2(0)) = (10, 20)$.

Finally, when the system (4.22) is compared with $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, we see that $\mathbf{f} = \mathbf{0}$ and the system is homogeneous.

26. Let $x_1(t)$ and $x_2(t)$ represents the amount of salt (in pounds) in the upper and lower tanks, respectively. The rate at which salt flows into the upper tank is found by multiplying the solution rate by the concentration of the salt solution entering the upper tank.

$$\text{Rate In} = 5 \text{ gal/min} \times 0.2 \text{ lb/gal} = 1 \text{ lb/min}$$

Similarly, salt leaves the upper tank at the following rate.

$$\text{Rate Out} = 5 \text{ gal/min} \times \frac{x_1}{200} \text{ lb/gal} = \frac{x_1}{40} \text{ lb/min}$$

Thus, the rate at which the salt content is changing in the upper tank is $dx_1/dt = \text{Rate In} - \text{Rate Out}$, or

$$\frac{dx_1}{dt} = 1 - \frac{x_1}{40}. \quad (4.23)$$

A similar argument shows that the salt content in the lower tank is changing according to the equation

$$\frac{dx_2}{dt} = \frac{x_1}{40} - \frac{x_2}{40}. \quad (4.24)$$

In matrix-vector form, the system becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1/40 & 0 \\ 1/40 & -1/40 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Because each tank contains pure water at time zero, the initial condition is $\mathbf{x}(0) = (0, 0)^T$. When we compare the system 4.24 with $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, we see that $\mathbf{f} = (1, 0)^T$, so the system is inhomogeneous.

27. Let $x_1(t)$, $x_2(t)$, and $x_3(t)$ represent the salt content (in pounds) of the first, second, and third tanks, respectively. Salt enters the first tank at a rate

$$5 \text{ gal/min} \times 2 \text{ lb/gal} = 10 \text{ lb/min.}$$

Salt leaves the first tank at a rate

$$5 \text{ gal/min} \times \frac{x_1}{100} \text{ lb/gal} = \frac{x_1}{20} \text{ lb/gal.}$$

Because $dx_1/dt = \text{Rate In} - \text{Rate Out}$, the differential equation governing the salt content in the first tank is

$$\frac{dx_1}{dt} = 10 - \frac{x_1}{20}. \quad (4.25)$$

Similarly, the equations governing the salt content in the second and third tanks are

$$\frac{dx_2}{dt} = \frac{x_1}{20} - \frac{x_2}{16}, \quad (4.26)$$

and

$$\frac{dx_3}{dt} = \frac{x_2}{16} - \frac{x_3}{12}, \quad (4.27)$$

respectively. In matrix-vector form, the system created by equations (4.25), (4.26), and (4.27), becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{bmatrix} -1/20 & 0 & 0 \\ 1/20 & -1/16 & 0 \\ 0 & 1/16 & -1/12 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}.$$

Comparing this system with $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, we see that $\mathbf{f}(t) = (10, 0, 0)^T$, so the system is inhomogeneous. Because each tank contains pure water at time $t = 0$, the initial condition is $\mathbf{x}(0) = (0, 0, 0)^T$.

28. As in Example 4.8, the forces on mass m_1 are $-k_1x$ and $k_2(y - x)$. Thus, by Newton's law,

$$m_1 x'' = -k_1 x + k_2(y - x),$$

or equivalently,

$$x'' = -\frac{k_1}{m_1}x + \frac{k_2}{m_1}(y - x). \quad (4.28)$$

On the second mass m_2 , the spring must exert an equal but opposite force to that which it exerts on the first mass m_1 ; i.e., $-k_2(y - x)$. There is also the driving force $F = A \cos \omega t$ exerted on this second mass. Thus, by Newton's law,

$$m_2 y'' = -k_2(y - x) + A \cos \omega t,$$

or equivalently,

$$y'' = -\frac{k_2}{m_2}(y - x) + \frac{A}{m_2} \cos \omega t. \quad (4.29)$$

To change to a system of first order equations, let $x_1 = x$, $x_2 = x'$, $x_3 = y$, and $x_4 = y'$. Then,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_3 - x_1) \\ x_3' &= x_4 \\ x_4' &= -\frac{k_2}{m_2}(x_3 - x_1) + \frac{A}{m_2} \cos \omega t. \end{aligned} \quad (4.30)$$

In matrix form, this can be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(k_1 + k_2)/m_1 & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -k_2/m_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (A/m_2) \cos \omega t \end{pmatrix}.$$

Note that this has the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, with $\mathbf{f}(t) = (0, 0, 0, (A/m_2) \cos \omega t)^T$. Hence, the system is inhomogeneous. Because the masses are released from rest from their equilibrium positions,

$$\begin{aligned} \mathbf{x}(0) &= (x_1(0), x_2(0), x_3(0), x_4(0))^T \\ &= (x(0), x'(0), y(0), y'(0))^T \\ &= (0, 0, 0, 0)^T. \end{aligned}$$

29. As in Example 4.8, the forces on mass the leftmost mass are $-kx$ and $k(y - x)$. Thus, by Newton's law,

$$mx'' = -kx + k(y - x) = -2kx + ky,$$

or equivalently,

$$x'' = -\frac{2k}{m}x + \frac{k}{m}y. \quad (4.31)$$

On the middle mass m , the spring connecting it to the left most mass must exert an equal but opposite force to that which it exerts on the leftmost mass; i.e., $-k(y - x)$. On the other side, the spring connecting the middle mass to the rightmost mass exerts a force $k(z - y)$. Thus, by Newton's law,

$$my'' = -k(y - x) + k(z - y) = kx - 2ky + kz,$$

or equivalently,

$$y'' = \frac{k}{m}x - \frac{2k}{m}y + \frac{k}{m}z. \quad (4.32)$$

Finally, the spring connecting the rightmost mass to the middle mass must exert an equal but opposite force to that which it exerts on the middle mass; i.e.,

$-k(z - y)$. On the other side, the spring connecting the right most mass to the vertical support on the right exerts a force of $-kz$. Hence, by Newton's law,

$$mz'' = -k(z - y) - kz = ky - 2kz,$$

or equivalently,

$$z'' = \frac{k}{m}y - \frac{2k}{m}z. \quad (4.33)$$

To change to a system of first order equations, let $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$, $x_5 = z$, and $x_6 = z'$. Then,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{2k}{m}x_1 + \frac{k}{m}x_3 \\ x_3' &= x_4 \\ x_4' &= \frac{k}{m}x_1 - \frac{2k}{m}x_3 + \frac{k}{m}x_5 \\ x_5' &= x_6 \\ x_6' &= \frac{k}{m}x_3 - \frac{2k}{m}x_5. \end{aligned} \quad (4.34)$$

In matrix form, this can be written $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2k/m & 0 & k/m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k/m & 0 & -2k/m & 0 & k/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & k/m & 0 & -2k/m & 0 \end{pmatrix}.$$

Note that this has the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, with $\mathbf{f}(t) = \mathbf{0}$. Hence, the system is homogeneous. Because the masses are released from rest, each displaced 10 cm to the right of their equilibrium positions,

$$\begin{aligned} \mathbf{x}(0) &= (x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0))^T \\ &= (x(0), x'(0), y(0), y'(0), z(0), z'(0))^T \\ &= (10, 0, 10, 0, 10, 0)^T. \end{aligned}$$

Section 8.5. Properties of Linear Systems

1. The system

$$\begin{aligned} x_1' &= -x_1 + 3x_2 \\ x_2' &= 2x_2 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

2. The system

$$\begin{aligned} x_1' &= 6x_1 + 4x_2 \\ x_2' &= -8x_1 - 6x_2 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

3. The system

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -x_1 + x_2 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

4. The system

$$\begin{aligned} x_1' &= -x_2 \\ x_2' &= x_1 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

5. The system

$$\begin{aligned}x_1' &= x_1 + x_2 \\x_2' &= -x_1 + x_2 + e^t\end{aligned}$$

can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

6. The system

$$\begin{aligned}x_1' &= -x_2 + \sin t \\x_2' &= x_1\end{aligned}$$

can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}.$$

7. We saw that the system in Exercise 1 can be written as $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}.$$

If $\mathbf{x}(t) = (e^{-t}, 0)^T$, then

$$\mathbf{x}'(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix},$$

and

$$A\mathbf{x}(t) = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Thus, $\mathbf{x}$ satisfies the system. If $\vec{y}(t) = (e^{2t}, e^{2t})^T$, then

$$\mathbf{y}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix},$$

and

$$A\mathbf{y}(t) = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Thus, $\mathbf{y}$ satisfies the system. Now,

$$C_1\mathbf{x}(t) + C_2\mathbf{y}(t) = \begin{pmatrix} C_1e^{-t} + C_2e^{2t} \\ C_2e^{2t} \end{pmatrix}.$$

Thus,

$$(C_1\mathbf{x}(t) + C_2\mathbf{y}(t))' = \begin{pmatrix} -C_1e^{-t} + 2C_2e^{2t} \\ 2C_2e^{2t} \end{pmatrix}.$$

But,

$$\begin{aligned}A(C_1\mathbf{x}(t) + C_2\mathbf{y}(t)) &= \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C_1e^{-t} + C_2e^{2t} \\ C_2e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -C_1e^{-t} + 2C_2e^{2t} \\ 2C_2e^{2t} \end{pmatrix},\end{aligned}$$

So $C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$ is a solution of the system.

8. We saw that the system in Exercise 2 can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix}.$$

If $\mathbf{x}(t) = (-e^{2t}, e^{2t})^T$, then

$$\mathbf{x}'(t) = \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} -2e^{2t} \\ 2e^{2t} \end{pmatrix},$$

and

$$A\mathbf{x}(t) = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} -2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Thus, $\mathbf{x}$ satisfies the system. If $\mathbf{y}(t) = (-e^{-2t}, 2e^{-2t})^T$, then

$$\mathbf{y}'(t) = \begin{pmatrix} -e^{-2t} \\ 2e^{-2t} \end{pmatrix}' = \begin{pmatrix} 2e^{-2t} \\ -4e^{-2t} \end{pmatrix},$$

and

$$A\mathbf{y}(t) = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix} \begin{pmatrix} -e^{-2t} \\ 2e^{-2t} \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ -4e^{-2t} \end{pmatrix}.$$

Thus, $\mathbf{y}$ satisfies the system. Now,

$$C_1\mathbf{x}(t) + C_2\mathbf{y}(t) = \begin{pmatrix} -C_1e^{2t} - C_2e^{-2t} \\ C_1e^{2t} + 2C_2e^{-2t} \end{pmatrix}.$$

Thus,

$$(C_1\mathbf{x}(t) + C_2\mathbf{y}(t))' = \begin{pmatrix} -2C_1e^{2t} + 2C_2e^{-2t} \\ 2C_1e^{2t} - 4C_2e^{-2t} \end{pmatrix}.$$

But,

$$\begin{aligned} A(C_1\mathbf{x}(t) + C_2\mathbf{y}(t)) &= \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix} \begin{pmatrix} -C_1e^{2t} - C_2e^{-2t} \\ C_1e^{2t} + 2C_2e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} -2C_1e^{2t} + 2C_2e^{-2t} \\ 2C_1e^{2t} - 4C_2e^{-2t} \end{pmatrix}, \end{aligned}$$

so $C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$ is a solution of the system.

9. We saw that the system in Exercise 3 can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

If $\mathbf{x}(t) = (e^t \cos t, -e^t \sin t)^T$, then

$$\mathbf{x}(t) = \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix}' = \begin{pmatrix} -e^t \sin t + e^t \cos t \\ -e^t \cos t - e^t \sin t \end{pmatrix}.$$

But,

$$\begin{aligned} A\mathbf{x}(t) &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix} \\ &= \begin{pmatrix} e^t \cos t - e^t \sin t \\ -e^t \cos t - e^t \sin t \end{pmatrix}, \end{aligned}$$

so $\mathbf{x}(t)$ is a solution of the system. Next, if $\mathbf{y}(t) = (e^t \sin t, e^t \cos t)^T$, then

$$\mathbf{y}'(t) = \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}' = \begin{pmatrix} e^t \cos t + e^t \sin t \\ -e^t \sin t + e^t \cos t \end{pmatrix}.$$

But,

$$\begin{aligned} A\mathbf{y}(t) &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix} \\ &= \begin{pmatrix} e^t \sin t + e^t \cos t \\ -e^t \sin t + e^t \cos t \end{pmatrix}, \end{aligned}$$

so $\mathbf{y}(t)$ is a solution of the system. Now,

$$C_1\mathbf{x}(t) + C_2\mathbf{y}(t) = \begin{pmatrix} C_1e^t \cos t + C_2e^t \sin t \\ -C_1e^t \sin t + C_2e^t \cos t \end{pmatrix}.$$

Thus,

$$\begin{aligned} (C_1\mathbf{x}(t) + C_2\mathbf{y}(t))' &= \begin{pmatrix} -C_1e^t \sin t + C_1e^t \cos t \\ +C_2e^t \cos t + C_2e^t \sin t \\ -C_1e^t \cos t - C_1e^t \sin t \\ -C_2e^t \sin t + C_2e^t \cos t \end{pmatrix} \\ &= \begin{pmatrix} (C_1 + C_2)e^t \cos t \\ +(-C_1 + C_2)e^t \sin t \\ (-C_1 + C_2)e^t \cos t \\ +(-C_1 - C_2)e^t \sin t \end{pmatrix}. \end{aligned}$$

But,

$$\begin{aligned} A(C_1\mathbf{x}(t) + C_2\mathbf{y}(t)) &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} C_1e^t \cos t + C_2e^t \sin t \\ -C_1e^t \sin t + C_2e^t \cos t \end{pmatrix} \\ &= \begin{pmatrix} (C_1 + C_2)e^t \cos t \\ +(-C_1 + C_2)e^t \sin t \\ (-C_1 + C_2)e^t \cos t \\ +(-C_1 - C_2)e^t \sin t \end{pmatrix}, \end{aligned}$$

so $C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$ is a solution of the system.

10. We saw that the system in Exercise 4 can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If $\mathbf{x}(t) = (\cos t, \sin t)^T$, then

$$\mathbf{x}'(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}' = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},$$

and

$$A\mathbf{x}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

Thus, $\mathbf{x}$ is a solution of the system in Exercise 4. Next, if $\mathbf{y}(t) = (\sin t, -\cos t)^T$, then

$$\mathbf{y}'(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}' = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix},$$

and

$$A\mathbf{y}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Thus, $\mathbf{y}$ is a solution of the system in Exercise 4. Now,

$$C_1\mathbf{x}(t) + C_2\mathbf{y}(t) = \begin{pmatrix} C_1 \cos t + C_2 \sin t \\ C_1 \sin t - C_2 \cos t \end{pmatrix},$$

and

$$(C_1\mathbf{x}(t) + C_2\mathbf{y}(t))' = \begin{pmatrix} -C_1 \sin t + C_2 \cos t \\ C_1 \cos t + C_2 \sin t \end{pmatrix}.$$

But,

$$\begin{aligned} A(C_1\mathbf{x}(t) + C_2\mathbf{y}(t)) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \cos t + C_2 \sin t \\ C_1 \sin t - C_2 \cos t \end{pmatrix} \\ &= \begin{pmatrix} -C_1 \sin t + C_2 \cos t \\ C_1 \cos t + C_2 \sin t \end{pmatrix} \end{aligned}$$

so $C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$ is a solution of the system in Exercise 4.

11. We saw in Exercise 5 that the system can be written as $\mathbf{x}' = A\mathbf{x} + \mathbf{w}$, where

$$A' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

If $\mathbf{x}_p(t) = (e^t, 0)^T$, then

$$\mathbf{x}'_p(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}' = \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

But,

$$\begin{aligned} A\mathbf{x}_p + \mathbf{w} &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} e^t \\ -e^t \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} e^t \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, $\mathbf{x}_p$ is a solution of the system. Next, if $\mathbf{z}(t) = \mathbf{x}_p + C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$, then

$$\begin{aligned} \mathbf{z}' &= \mathbf{x}'_p + C_1\mathbf{x}'(t) + C_2\mathbf{y}'(t) \\ &= A\mathbf{x}_p + \mathbf{w} + C_1A\mathbf{x} + C_2A\mathbf{y} \\ &= A[\mathbf{x}_p + C_1\mathbf{x} + C_2\mathbf{y}] + \mathbf{w} \\ &= A\mathbf{z} + \mathbf{w}. \end{aligned}$$

Thus, $\mathbf{z}$ is a solution of the system.

12. We saw in Exercise 6 that the system can be written $\mathbf{x}' = A\mathbf{x} + \mathbf{w}$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} \sin t \\ 0 \end{pmatrix}.$$

If $\mathbf{x}_p(t) = (1/2)(t \sin t - \cos t, -t \cos t)^T$, then

$$\begin{aligned} \mathbf{x}'_p(t) &= \frac{1}{2} \begin{pmatrix} t \sin t - \cos t \\ -t \cos t \end{pmatrix}' \\ &= \frac{1}{2} \begin{pmatrix} t \cos t + 2 \sin t \\ t \sin t - \cos t \end{pmatrix}. \end{aligned}$$

But,

$$\begin{aligned} A\mathbf{x}_p + \mathbf{w} &= \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \sin t - \cos t \\ -t \cos t \end{pmatrix} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} t \cos t + 2 \sin t \\ t \sin t - \cos t \end{pmatrix}. \end{aligned}$$

Therefore, $\mathbf{x}_p$ is a solution of the system in Exercise 10. Next, if $\mathbf{z}(t) = \mathbf{x}_p(t) + C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$, then

$$\begin{aligned} \mathbf{z}' &= \mathbf{x}'_p + C_1\mathbf{x}'(t) + C_2\mathbf{y}'(t) \\ &= A\mathbf{x}_p + \mathbf{w} + C_1A\mathbf{x} + C_2A\mathbf{y} \\ &= A[\mathbf{x}_p + C_1\mathbf{x} + C_2\mathbf{y}] + \mathbf{w} \\ &= A\mathbf{z} + \mathbf{w}. \end{aligned}$$

Thus, $\mathbf{z}$ is a solution of the system in Exercise 10.

13. In Exercise 7 we saw that $\mathbf{x}(t) = (e^{-t}, 0)^T$ and $\mathbf{y}(t) = (e^{2t}, e^{2t})^T$ were solutions of

$$\mathbf{x}' = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \mathbf{x}.$$

If we evaluate these solutions at $t = 0$, then

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Because,

$$\det([\mathbf{x}(0), \mathbf{y}(0)]) = \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \neq 0,$$

these vectors are independent and form a fundamental solution set for the system. Hence, the general solution of the system is

$$\mathbf{z}(t) = C_1 \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

To find the solution with initial condition $\mathbf{z}(0) = (0, 1)^T$, substitute the initial condition in the general solution to get

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

This system has solutions $C_1 = -1$ and $C_2 = 1$. Thus, the solution is

$$\mathbf{z}(t) = -1 \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} + 1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} -e^{-t} + e^{2t} \\ e^{2t} \end{pmatrix}.$$

14. In Exercise 8, we saw that $\mathbf{x}(t) = (-e^{2t}, e^{2t})^T$ and $\mathbf{y}(t) = (-e^{-2t}, 2e^{-2t})^T$ were solutions of

$$\mathbf{x}' = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix} \mathbf{x}.$$

If we evaluate these solutions at $t = 0$, then

$$\mathbf{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Because

$$\det([\mathbf{x}(0), \mathbf{y}(0)]) = \det \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = -1 \neq 0,$$

these vectors are independent and form a fundamental solution set for the system. Hence, the general solution of the system is

$$\mathbf{z}(t) = C_1 \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} -e^{-2t} \\ 2e^{-2t} \end{pmatrix}.$$

To find the solution with initial condition $\mathbf{z}(0) = (1, -4)^T$, substitute $t = 0$ in the general solution to get

$$\begin{pmatrix} 1 \\ -4 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

This system has solution $C_1 = 2$ and $C_2 = -3$. Thus, the solution is

$$\mathbf{z}(t) = 2 \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} - 3 \begin{pmatrix} -e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{2t} + 3e^{-2t} \\ 2e^{2t} - 6e^{-2t} \end{pmatrix}.$$

15. In Exercise 9, we saw that $\mathbf{x}(t) = (e^t \cos t, -e^t \sin t)^T$ and $\mathbf{y}(t) = (e^t \sin t, e^t \cos t)^T$ were solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}.$$

If we evaluate the solutions at $t = 0$, then

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Because $\det([\mathbf{x}(0), \mathbf{y}(0)]) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$, the vectors are independent and form a fundamental set of solutions. Hence, then general solution of the system is

$$\mathbf{z}(t) = C_1 \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix} + C_2 \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}.$$

To find the particular solution, substitute the initial condition $\mathbf{z}(0) = (-2, 3)^T$.

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, $C_1 = -2$ and $C_2 = 3$ and the solution is

$$\mathbf{z}(t) = -2 \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix} + 3 \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}$$

$$= \begin{pmatrix} -2e^t \cos t + 3e^t \sin t \\ 2e^t \sin t + 3e^t \cos t \end{pmatrix}.$$

16. In Exercise 10, we saw that $\mathbf{x}(t) = (\cos t, \sin t)^T$ and $\mathbf{y}(t) = (\sin t, -\cos t)^T$ were solutions of

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

If we evaluate the solutions at $t = 0$, then

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Because

$$\det([\mathbf{x}(0), \mathbf{y}(0)]) = \det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \neq 0,$$

the vectors are independent and form a fundamental solution set for the system. Hence, the general solution of the system is

$$\mathbf{z}(t) = C_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

To find the solution with initial condition $\mathbf{z}(0) = (3, 2)^T$, substitute $t = 0$ in the general solution to get

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

This system has solutions $C_1 = 3$ and $C_2 = -2$. Thus, the solution is

$$\mathbf{z}(t) = 3 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} - 2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

$$\mathbf{z}(t) = \begin{pmatrix} 3 \cos t - 2 \sin t \\ 3 \sin t + 2 \cos t \end{pmatrix}.$$

17. (a) If $\mathbf{x} = (x_1, x_2)^T = (0, e^t)^T$, then

$$x'_1 = 0 = 0 \cdot e^t = x_1 x_2, \text{ and}$$

$$x'_2 = e^t = x_2.$$

Hence, $\mathbf{x}$ is a solution of the system

$$x'_1 = x_1 x_2$$

$$x'_2 = x_2.$$

If $\mathbf{y} = (y_1, y_2)^T = (1, 0)^T$, then

$$y'_1 = 0 = 1 \cdot 0 = y_1 y_2, \text{ and}$$

$$y'_2 = 0 = y_2,$$

so $\mathbf{y}$ is also a solution of the system.

- (b) If $\mathbf{z}(t) = \mathbf{x}(t) + \mathbf{y}(t)$, then $\tilde{\mathbf{z}} = (z_1, z_2)^T = (1, e^t)^T$. But,

$$z'_1 = 0 \neq 1 \cdot e^t = z_1 z_2,$$

so $\mathbf{z}$ is *not* a solution of the system. There is no contradiction of Theorem 5.1 because the system is *not* linear.

18. Let $\mathbf{x}(t)$ be a solution of

$$\mathbf{y}' = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \mathbf{y}$$

with initial condition $\mathbf{x}(0) = (1, -1)^T$. We know that

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

is a solution of the system. Set $\mathbf{y}(0) = (1, -1)^T$.

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

This system has solution $C_1 = 2$ and $C_2 = -3$. Thus,

$$\mathbf{y}(t) = 2 \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} - 3 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

is a solution of the system with $\mathbf{y}(0) = (1, -1)^T$. By the uniqueness theorem,

$$\mathbf{x}(t) = \mathbf{y}(t) = 2 \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} - 3 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

for all t .

19. Evaluate each solution $\mathbf{y}_1(t) = (-e^{-t}, -e^{-t}, e^{-t})^T$, $\mathbf{y}_2(t) = (0, -e^t, 2e^t)^T$, and $\mathbf{y}_3(t) = (e^{2t}, 0, 2e^{2t})^T$ at $t = 0$.

$$\mathbf{y}_1(0) = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2(0) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad \text{and}$$

$$\mathbf{y}_3(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Because

$$\det \begin{pmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix} = 1 \neq 0,$$

these vectors are independent. Therefore, by Proposition 5.12, the solutions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are independent for all t .

20. Evaluate each solution $\mathbf{y}_1(t) = (e^{-t}, e^{-t}, 2e^{-t})^T$, $\mathbf{y}_2(t) = (-e^t, 3e^t, 0)^T$, and $\mathbf{y}_3(t) = (2e^{-t} - 3e^t, 2e^{-t} + 9e^t, 4e^{-t})^T$ at $t = 0$.

$$\mathbf{y}_1(0) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{y}_2(0) = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$\mathbf{y}_3(0) = \begin{pmatrix} -1 \\ 11 \\ 4 \end{pmatrix}$$

Because

$$\det \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 11 \\ 2 & 0 & 4 \end{pmatrix} = 0,$$

these vectors are dependent. Therefore, by Proposition 5.12, the solutions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are dependent for all t .

21. Evaluate each solution $\mathbf{y}_1(t) = (-e^{-t} + e^{2t}, -e^{-t} + e^{2t}, e^{2t})^T$, $\mathbf{y}_2(t) = (-e^t + e^{-t}, e^{-t}, 0)^T$, and $\mathbf{y}_3(t) = (e^t, 0, 0)^T$ at $t = 0$.

$$\mathbf{y}_1(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$\mathbf{y}_3(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Because

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -1 \neq 0,$$

then these vectors are independent. Therefore, by Proposition 5.12, the solutions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are independent for all t .

22. Evaluate each solution $\mathbf{y}_1(t) = (e^{-t}, 2e^{-t}, 2e^{-t})^T$, $\mathbf{y}_2(t) = (e^t, e^t, 0)^T$, $\mathbf{y}_3(t) = (0, 0, e^t)^T$ at $t = 0$.

$$\mathbf{y}_1(0) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{y}_2(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$\mathbf{y}_3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Because

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = -1,$$

these vectors are independent. Therefore, by Proposition 5.12, the solutions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are independent for all t .

23. If $\mathbf{y}_1(t) = (-e^{2t}, 2e^{2t})^T$, then

$$\mathbf{y}_1' = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}' = \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix},$$

and

$$\begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} \mathbf{y}_1 = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix}.$$

Therefore, $\mathbf{y}_1$ is a solution. In a similar manner, you can also show that $\mathbf{y}_2(t) = (-e^{-2t}, e^{-2t})^T$ is a solution. Evaluating each solution at $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

These vectors are independent ($\mathbf{y}_1$ is not a multiple of $\mathbf{y}_2$), so the solutions are independent for all t and form a fundamental set of solutions. Thus, the general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

Substitute the initial condition $\mathbf{y}(0) = (-5, 8)^T$.

$$\begin{pmatrix} -5 \\ 8 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -5 \\ 8 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

This system has solution $C_1 = 3$ and $C_2 = 2$, so the final solution is

$$\begin{aligned}\mathbf{y}(t) &= 3 \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + 2 \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} -3e^{2t} - 2e^{-2t} \\ 6e^{2t} + 2e^{-2t} \end{pmatrix}.\end{aligned}$$

24. If $\mathbf{y}_1(t) = (e^{-2t}, 0)^T$, then

$$\mathbf{y}'_1 = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}' = \begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix},$$

and

$$\begin{aligned}\begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix} \mathbf{y}_1 &= \begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix}.\end{aligned}$$

Therefore, $\mathbf{y}_1$ is a solution. If $\tilde{\mathbf{y}}_2(t) = (e^{-4t}, e^{-4t})^T$, then

$$\mathbf{y}'_2 = \begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix}' = \begin{pmatrix} -4e^{-4t} \\ -4e^{-4t} \end{pmatrix},$$

and

$$\begin{aligned}\begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix} \mathbf{y}_2 &= \begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} -4e^{-4t} \\ -4e^{-4t} \end{pmatrix}\end{aligned}$$

Therefore, $\mathbf{y}_2$ is a solution. Further, evaluating each solution at $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These vectors are clearly independent ($\mathbf{y}_1$ is not a multiple of $\mathbf{y}_2$), so the solutions $\mathbf{y}_1$ and $\tilde{\mathbf{y}}_2$ are independent for all t and form a fundamental set of solutions. Thus, the general solution is

$$\mathbf{y} = C_1 \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix}.$$

Substitute the initial condition $\mathbf{y}(0) = (3, 1)^T$.

$$\begin{aligned}\begin{pmatrix} 3 \\ 1 \end{pmatrix} &= C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\end{aligned}$$

This system has solution $C_1 = 2$ and $C_2 = 1$, so the final solution is

$$\mathbf{y}(t) = 2 \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} + \begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix} = \begin{pmatrix} 2e^{-2t} + e^{-4t} \\ e^{-4t} \end{pmatrix}.$$

25. If $\mathbf{y}_1(t) = (e^{2t}, e^{2t})^T$, then

$$\mathbf{y}'_1 = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix},$$

and

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{y}_1 = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

In a similar manner, you can also show that $\tilde{\mathbf{y}}_2(t) = (e^{2t}(t+2), e^{2t}(t+1))^T$ is a solution of the system. Further, evaluating each solution at $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The vectors are independent ($\mathbf{y}_1$ is not a multiple of $\mathbf{y}_2$), so the solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent for all t and form a fundamental set of solutions. Thus, the general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}.$$

Substitute the initial condition $\mathbf{y}(0) = (0, 1)^T$.

$$\begin{aligned}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\end{aligned}$$

This system has solution $C_1 = 2$ and $C_2 = -1$. Thus, the final solution is

$$\mathbf{y}(t) = 2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} - \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix} = \begin{pmatrix} -te^{2t} \\ (1-t)e^{2t} \end{pmatrix}.$$

26. If $\mathbf{y}_1(t) = ((1/2) \cos t - (1/2) \sin t, \cos t)^T$, then

$$\begin{aligned}\mathbf{y}_1' &= \begin{pmatrix} (1/2) \cos t - (1/2) \sin t \\ \cos t \end{pmatrix}' \\ &= \begin{pmatrix} -(1/2) \sin t - (1/2) \cos t \\ -\sin t \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{y}_1 &= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} (1/2) \cos t - (1/2) \sin t \\ \cos t \end{pmatrix} \\ &= \begin{pmatrix} -(1/2) \sin t - (1/2) \cos t \\ -\sin t \end{pmatrix}.\end{aligned}$$

Therefore, $\mathbf{y}_1$ is a solution. If $\mathbf{y}_2((1/2) \sin t + (1/2) \cos t, \sin t)^T$, then

$$\begin{aligned}\mathbf{y}_2' &= \begin{pmatrix} (1/2) \sin t + (1/2) \cos t \\ \sin t \end{pmatrix}' \\ &= \begin{pmatrix} (1/2) \cos t - (1/2) \sin t \\ \cos t \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{y}_2 &= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} (1/2) \sin t + (1/2) \cos t \\ \sin t \end{pmatrix} \\ &= \begin{pmatrix} (1/2) \cos t - (1/2) \sin t \\ \cos t \end{pmatrix}.\end{aligned}$$

Therefore, $\mathbf{y}_2$ is a solution. Further, evaluating each solution at $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

These vectors are clearly independent ($\mathbf{y}_1$ is not a multiple of $\mathbf{y}_2$), so the solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent for all t and form a fundamental set of solutions. Thus, the general solution is

$$\begin{aligned}\mathbf{y}(t) &= C_1 \begin{pmatrix} (1/2) \cos t - (1/2) \sin t \\ \cos t \end{pmatrix} \\ &\quad + C_2 \begin{pmatrix} (1/2) \sin t + (1/2) \cos t \\ \sin t \end{pmatrix}.\end{aligned}$$

Substitute the initial condition $\mathbf{y}(0) = (1, 0)^T$.

$$\begin{aligned}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= C_1 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\end{aligned}$$

This system has solution $C_1 = 0$ and $C_2 = 2$, so the final solution is

$$\begin{aligned}\mathbf{y}(t) &= 2 \begin{pmatrix} ((1/2) \sin t + (1/2) \cos t) \\ \sin t \end{pmatrix} \\ &= \begin{pmatrix} \sin t + \cos t \\ 2 \sin t \end{pmatrix}.\end{aligned}$$

27. A careful examination of the flow rates in and out of each tank will indicate that the volume of solution in each tank remains unchanged (3L) over time. Let $x_A(t)$ and $x_B(t)$ represent the salt content (kg) of tanks A and B, respectively. Let's begin with an analysis of tank A. We use the so-called "balance law,"

$$\frac{dx_A}{dt} = \text{Rate in} - \text{Rate out}.$$

Pure water enters tank A at 5 L/min, so there is no salt entering there. However, solution from tank B enters tank A at a rate of 4 L/min. The salt concentration in tank B is $x_B(t)/3$ kg/L, so the rate at which salt is entering tank A from tank B is

$$4 \text{ L/min} \times x_B(t)/3 \text{ kg/L} = 4x_B(t)/3 \text{ kg/min}.$$

Solution is leaving tank A for tank B at a rate of 9 L/min. The salt concentration of tank A is $x_A(t)/3$ kg/L. Thus, salt is leaving tank A at a rate of

$$9 \text{ L/min} \times x_A(t)/3 \text{ kg/L} = 3x_A(t) \text{ kg/min}.$$

Thus, using the balance law above,

$$\frac{dx_A}{dt} = \frac{4x_B}{3} - 3x_A.$$

On tank B,

$$\frac{dx_B}{dt} = \text{Rate in} - \text{Rate out}.$$

We already know that salt enters tank B from tank A at a rate of $3x_A(t)$ kg/min. We also know that salt

is leaving tank B for tank A at a rate of $4x_B(t)/3$ kg/min. Let's examine the drain, where solution leaves tank B at a rate of 5 L/min. Because the concentration of salt in tank B is $x_B(t)/3$ kg/L, salt leaves tank B through the drain at

$$5 \text{ L/min} \times x_B/3 = 5x_B/3 \text{ kg/min.}$$

Using the balance law,

$$\frac{dx_B}{dt} = 3x_A - \frac{4x_B}{3} - \frac{5x_B}{3} = 3x_A - 3x_B.$$

28. A careful examination of the flow rates in and out of each tank will indicate that the volume of solution in each tank remains unchanged (200 gal) over time. We let $x_A(t)$, $x_B(t)$, and $x_C(t)$ represent the salt content (in lb) of Tank A, Tank B, and Tank C, respectively. Let's begin with an analysis of Tank A. We use the so-called "balance law."

$$\frac{dx_A}{dt} = \text{Rate in} - \text{Rate out.}$$

Pure water is poured into Tank A at 5 gal/min, so there is no salt entering there. However, solution from Tank B enters Tank A at a rate of 4 gal/min. The salt concentration in Tank B is $x_B(t)/200$ lb/gal, so the rate at which salt is entering Tank A from Tank B is

$$4 \text{ gal/min} \times x_B(t)/200 \text{ lb/gal} = x_B(t)/50 \text{ lb/min.}$$

Solution is leaving Tank A for Tank B at a rate of 9 gal/min. The salt concentration of Tank A is $x_A(t)/200$ lb/gal. Thus, salt is leaving Tank A at a rate of

$$9 \text{ gal/min} \times x_A(t)/200 \text{ lb/gal} = 9x_A(t)/200 \text{ lb/min.}$$

Thus, using the balance law above,

$$\frac{dx_A}{dt} = \frac{x_B}{50} - \frac{9x_A}{200}.$$

On to Tank B.

$$\frac{dx_B}{dt} = \text{Rate in} - \text{Rate out}$$

We already have salt exchanges from Tank A to Tank B and vice versa, so let's analyze the salt exchange

between Tank B and Tank C. Solution enters Tank B from Tank C at a rate of 4 gal/min. But the concentration of salt in Tank C is $x_C(t)/200$ lb/gal. Thus, salt enters Tank B from Tank C at a rate

$$4 \text{ gal/min} \times x_C(t)/200 \text{ lb/gal} = x_C(t)/50 \text{ lb/min.}$$

Solution leaves Tank B and enters Tank C at a rate of 9 gal/min. However, the concentration of salt in Tank B is $x_B(t)/200$ lb/gal. Thus, salt leaves Tank B and enters Tank C at a rate

$$9 \text{ gal/min} \times x_B(t)/200 \text{ lb/gal} = 9x_B(t)/200 \text{ lb/min.}$$

Substituting salt exchanges in the balance law above,

$$\begin{aligned} \frac{dx_B}{dt} &= \left(\frac{9x_A}{200} + \frac{x_C}{50} \right) - \left(\frac{x_B}{50} + \frac{9x_B}{200} \right) \\ &= \frac{9x_A}{200} - \frac{13x_B}{200} + \frac{x_C}{50}. \end{aligned}$$

One tank to go.

$$\frac{dx_C}{dt} = \text{Rate in} - \text{Rate out}$$

We already know the salt exchanges between Tank B and Tank C, so we only need to compute the rate at which salt is leaving from Tank C through the remaining spigot. Solution leaves Tank C through this spigot at a rate of 5 gal/min. As the concentration of salt in Tank C is $x_C(t)/200$ lb/gal, the rate at which salt leaves Tank C through this spigot is

$$5 \text{ gal/min} \times x_C(t)/200 \text{ lb/gal} = x_C(t)/40 \text{ lb/min.}$$

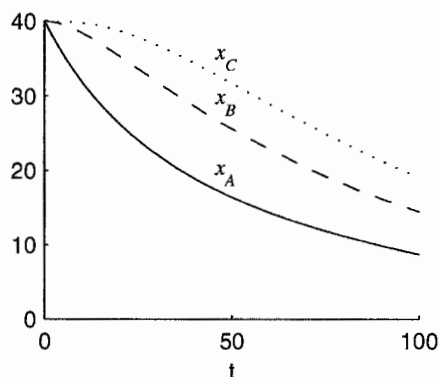
Thus, using the balance law above,

$$\begin{aligned} \frac{dx_C}{dt} &= \frac{9x_B}{200} - \left(\frac{x_C}{50} + \frac{x_C}{40} \right) \\ &= \frac{9x_B}{200} - \frac{9x_C}{200}. \end{aligned}$$

29. In Exercise 28, the salt content in tanks A, B, and C was modeled by the system

$$\begin{aligned} \frac{dx_A}{dt} &= \frac{x_B}{50} - \frac{9x_A}{200} \\ \frac{dx_B}{dt} &= \frac{9x_A}{200} - \frac{13x_B}{200} + \frac{x_C}{50} \\ \frac{dx_C}{dt} &= \frac{9x_B}{200} - \frac{9x_C}{200}. \end{aligned}$$

Setting $x_A(0) = x_B(0) = x_C(0) = 40$, the salt content in each tank is pictured below.



30. Because solution is entering and leaving each tank at the same rate (10 gal/min), each tank maintains a constant volume of solution (200 gal). Let $x(t)$, $y(t)$, and $z(t)$ represent the amount of salt in the first, second, and third tanks in the cascade, respectively. We invoke the balance law for the first tank.

$$\frac{dx}{dt} = \text{Rate in} - \text{Rate out}$$

No salt enters the first tank, but salt solution leaves the first tank at a rate of 10 gal/min. The concentration of salt in the first tank is $x(t)/200$, so the rate at which salt leaves the first tank is

$$10 \text{ gal/min} \times x(t)/200 \text{ lb/gal} = x(t)/20 \text{ lb/min.}$$

Using these salt rates in the above balance law,

$$\frac{dx}{dt} = -\frac{x}{20}.$$

The balance law for the second tank is

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out.}$$

The rate at which salt enters the second tank is the same as the rate at which salt leaves the first tank, $x/20$ lb/min. Salt solution leaves the second tank at a rate of 10 gal/min. The concentration of salt in the second tank is $y(t)/200$ lb/gal, so the rate at which salt leaves the second tank is

$$10 \text{ gal/min} \times y(t)/200 \text{ lb/gal} = y(t)/20 \text{ lb/min.}$$

Using these salt rates in the above balance law,

$$\frac{dy}{dt} = \frac{x}{20} - \frac{y}{20}.$$

The balance law for the third tank is

$$\frac{dz}{dt} = \text{Rate in} - \text{Rate out.}$$

The rate at which salt enters the third tank is the same as the rate at which salt leaves the second tank, $y(t)/20$ lb/min. Salt solution leaves the third tank at a rate of 10 gal/min. The concentration of salt in the third tank is $z(t)/200$ lb/gal, so the rate at which salt leaves the third tank is

$$10 \text{ gal/min} \times z(t)/200 \text{ lb/gal} = z(t)/20 \text{ lb/min.}$$

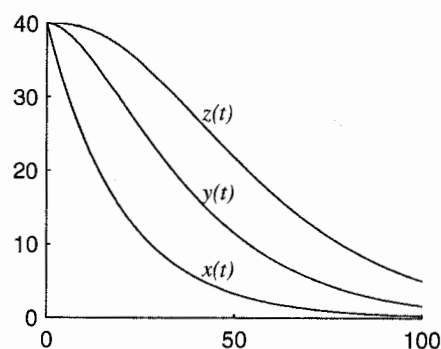
Using these salt rates in the above balance law,

$$\frac{dz}{dt} = \frac{y}{20} - \frac{z}{20}.$$

31. In Exercise 30, we saw that

$$\begin{aligned} \frac{dx}{dt} &= -\frac{x}{20} \\ \frac{dy}{dt} &= \frac{x}{20} - \frac{y}{20} \\ \frac{dz}{dt} &= \frac{y}{20} - \frac{z}{20} \end{aligned}$$

modeled the salt content in three cascading tanks. If $x(0) = y(0) = z(0) = 40$, then the following figure shows the salt content in each tank over time.



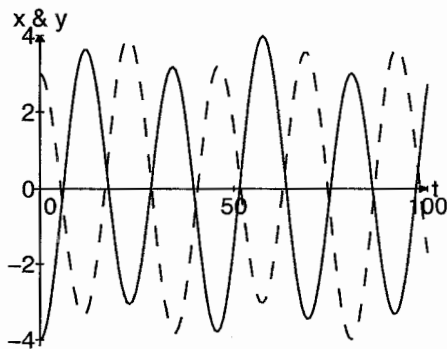
32. Given $k_1 = k_2 = k_3 = 1$ dyne/cm, and $m_1 = m_2 = 40$ g, the equations of motion for the coupled oscillator become

$$\begin{aligned} u'_1 &= u_2 \\ u'_2 &= -\frac{k_1 + k_2}{m_1}u_1 + \frac{k_2}{m_1}u_3 = -\frac{1}{20}u_1 + \frac{1}{40}u_3 \\ u'_3 &= u_4 \\ u'_4 &= \frac{k_2}{m_2}u_1 - \frac{k_2 + k_3}{m_2}u_3 = \frac{1}{40}u_1 - \frac{1}{20}u_3. \end{aligned}$$

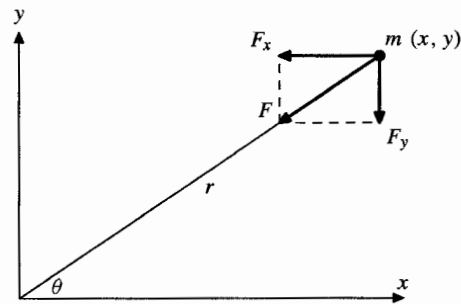
Initial displacements are $x(0) = -4$ cm and $y(0) = 3$ cm. Both masses are released from rest, so $x'(0) = y'(0) = 0$. These lead to initial conditions

$$\begin{aligned} u_1(0) &= x(0) = -4 \\ u_2(0) &= x'(0) = 0 \\ u_3(0) &= y(0) = 3 \\ u_4(0) &= y'(0) = 0 \end{aligned}$$

This system and initial conditions lead to the solution that follows. The solid curve is x and the dashed curve is y .



33. In the figure,



F represents the magnitude of the radially inward force. It is easy to see that

$$\begin{aligned} F_x &= F \cos \theta \\ F_y &= F \sin \theta. \end{aligned}$$

Furthermore,

$$\begin{aligned} \cos \theta &= \frac{x}{r} \\ \sin \theta &= \frac{y}{r}, \end{aligned}$$

and with these substitutions and the fact that $F = km/r^2$, the former equations become

$$\begin{aligned} F_x &= -\frac{km}{r^2} \cdot \frac{x}{r} = -\frac{kmx}{r^3}, \\ F_y &= -\frac{km}{r^2} \cdot \frac{y}{r} = -\frac{kmy}{r^3}, \end{aligned}$$

the minus signs being present because the force is directed opposite the displacement. Next, Newton gives us $F_x = ma_x = mx''$ and $F_y = ma_y = my''$, so

$$\begin{aligned} mx'' &= -\frac{kmx}{r^3} \\ my'' &= -\frac{kmy}{r^3}, \end{aligned}$$

or,

$$\begin{aligned} x'' &= -\frac{kx}{r^3} \\ y'' &= -\frac{ky}{r^3}. \end{aligned}$$

If we now let $u_1 = x$, $u_2 = x'$, $u_3 = y$, and $u_4 = y'$,

then

$$\begin{aligned}u_1' &= u_2 \\u_2' &= -\frac{ku_1}{r^3} \\u_3' &= u_4 \\u_4' &= -\frac{ku_3}{r^3}.\end{aligned}$$

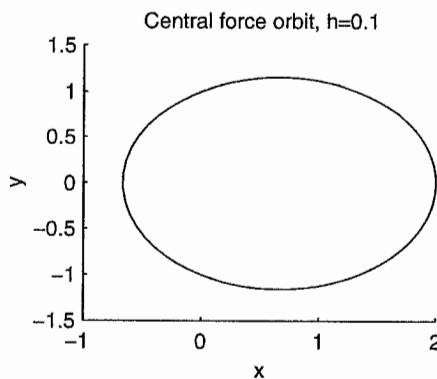
Note that $r = (x^2 + y^2)^{1/2} = (u_1^2 + u_3^2)^{1/2}$, so we actually have

$$\begin{aligned}u_1' &= u_2 \\u_2' &= -\frac{ku_1}{(u_1^2 + u_3^2)^{3/2}} \\u_3' &= u_4 \\u_4' &= -\frac{ku_3}{(u_1^2 + u_3^2)^{3/2}}.\end{aligned}$$

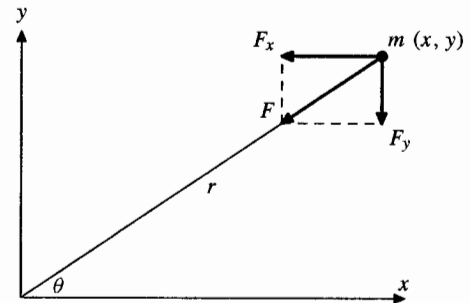
34. In Exercise 33, we saw that (with $k = 1$)

$$\begin{aligned}u_1' &= u_2 \\u_2' &= -\frac{u_1}{(u_1^2 + u_3^2)^{3/2}} \\u_3' &= u_4 \\u_4' &= -\frac{u_3}{(u_1^2 + u_3^2)^{3/2}}\end{aligned}$$

modeled the motion of the particle in the central force field. With $u_1(0) = x(0) = 2$, $u_2(0) = x'(0) = 0$, $u_3(0) = y(0) = 0$, and $u_4(0) = y'(0) = -0.5$, the motion of the particle in the xy plane follows. We used RK4 with step size $h = 0.1$ to produce the orbit.



35. In the figure,



F represents the magnitude of the radially inward force. It is easy to see that

$$\begin{aligned}F_x &= F \cos \theta \\F_y &= F \sin \theta.\end{aligned}$$

Furthermore,

$$\begin{aligned}\cos \theta &= \frac{x}{r} \\ \sin \theta &= \frac{y}{r},\end{aligned}$$

and with these substitutions and the fact that $F = km/r^3$, the former equations become

$$\begin{aligned}F_x &= -\frac{km}{r^3} \cdot \frac{x}{r} = -\frac{kmx}{r^4}, \\F_y &= -\frac{km}{r^3} \cdot \frac{y}{r} = -\frac{kmy}{r^4},\end{aligned}$$

the minus signs being present because the force is directed opposite the displacement. Next, Newton gives us $F_x = ma_x = mx''$ and $F_y = ma_y = my''$, so

$$\begin{aligned}mx'' &= -\frac{kmx}{r^4} \\my'' &= -\frac{kmy}{r^4},\end{aligned}$$

or,

$$\begin{aligned}x'' &= -\frac{kx}{r^4} \\y'' &= -\frac{ky}{r^4}.\end{aligned}$$

If we now let $u_1 = x$, $u_2 = x'$, $u_3 = y$, and $u_4 = y'$, then

$$\begin{aligned}u'_1 &= u_2 \\u'_2 &= -\frac{ku_1}{r^4} \\u'_3 &= u_4 \\u'_4 &= -\frac{ku_3}{r^4}.\end{aligned}$$

Note that $r = (x^2 + y^2)^{1/2} = (u_1^2 + u_3^2)^{1/2}$, so we

actually have

$$\begin{aligned}u'_1 &= u_2 \\u'_2 &= -\frac{ku_1}{(u_1^2 + u_3^2)^2} \\u'_3 &= u_4 \\u'_4 &= -\frac{ku_3}{(u_1^2 + u_3^2)^2}.\end{aligned}$$

Chapter 9. Linear Systems with Constant Coefficients

Section 9.1. Overview of the Technique

1. If

$$A = \begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 12 - \lambda & 14 \\ -7 & -9 - \lambda \end{pmatrix} \\ &= (12 - \lambda)(-9 - \lambda) + 98 \\ &= \lambda^2 - 3\lambda - 10 \\ &= (\lambda - 5)(\lambda + 2). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -2$.

2. If

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 2 - \lambda & 0 \\ 0 & 2 - \lambda \end{pmatrix} \\ &= (2 - \lambda)(2 - \lambda). \end{aligned}$$

Thus, $\lambda = 2$ is a repeated eigenvalue of algebraic multiplicity 2.

3. If

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} -2 - \lambda & 3 \\ 0 & -5 - \lambda \end{pmatrix} \\ &= (-2 - \lambda)(-5 - \lambda) \\ &= (\lambda + 2)(\lambda + 5). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -5$.

4. If

$$A = \begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} -4 - \lambda & 1 \\ -2 & 1 - \lambda \end{pmatrix} \\ &= (-4 - \lambda)(1 - \lambda) + 2 \\ &= \lambda^2 + 3\lambda - 2. \end{aligned}$$

The quadratic formula provides $\lambda_1 = (-3 - \sqrt{17})/2$ and $\lambda_2 = (-3 + \sqrt{17})/2$.

5. If

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 5 - \lambda & 3 \\ -6 & -4 - \lambda \end{pmatrix} \\ &= (5 - \lambda)(-4 - \lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$.

6. If

$$A = \begin{pmatrix} -2 & 5 \\ 0 & 2 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} -2 - \lambda & 5 \\ 0 & 2 - \lambda \end{pmatrix} \\ &= (\lambda - 2)(\lambda + 2). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$.

7. If

$$A = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} -3 - \lambda & 0 \\ 0 & -3 - \lambda \end{pmatrix} \\ &= (-3 - \lambda)(-3 - \lambda) \\ &= (\lambda + 3)^2. \end{aligned}$$

Thus, $\lambda = -3$ is a repeated eigenvalue of algebraic multiplicity 2.

8. If

$$A = \begin{pmatrix} 6 & 10 \\ -5 & -9 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 6 - \lambda & 10 \\ -5 & -9 - \lambda \end{pmatrix} \\ &= (6 - \lambda)(-9 - \lambda) + 50 \\ &= \lambda^2 + 3\lambda - 4 \\ &= (\lambda + 4)(\lambda - 1). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 1$.

9. If

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix},$$

then

$$p(\lambda) = - \begin{vmatrix} 1 - \lambda & 2 & 3 \\ 0 & 0 - \lambda & 2 \\ 0 & 3 & 1 - \lambda \end{vmatrix}.$$

Expanding down the first column,

$$\begin{aligned} p(\lambda) &= -(1 - \lambda) \begin{vmatrix} 0 - \lambda & 2 \\ 3 & 1 - \lambda \end{vmatrix} \\ &= (\lambda - 1)(-\lambda(1 - \lambda) - 6) \\ &= (\lambda - 1)(\lambda^2 - \lambda - 6) \\ &= (\lambda - 1)(\lambda - 3)(\lambda + 2). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = -2$.

10. If

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix},$$

then

$$p(\lambda) = - \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 4 & 3 - \lambda & 2 \\ -8 & -4 & -3 - \lambda \end{vmatrix}.$$

Expanding across the first row,

$$\begin{aligned} p(\lambda) &= -(1 - \lambda) \begin{vmatrix} 3 - \lambda & 2 \\ -4 & -3 - \lambda \end{vmatrix} \\ &= (\lambda - 1)((3 - \lambda)(-3 - \lambda) + 8) \\ &= (\lambda - 1)(\lambda^2 - 1) \\ &= (\lambda - 1)^2(\lambda + 1). \end{aligned}$$

Thus, the eigenvalues are -1 and 1 , the latter a repeated eigenvalue of algebraic multiplicity 2.

11. If

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix},$$

then

$$p(\lambda) = - \begin{vmatrix} -1 - \lambda & -4 & -2 \\ 0 & 1 - \lambda & 1 \\ -6 & -12 & 2 - \lambda \end{vmatrix}.$$

Expanding down the first column,

$$\begin{aligned} p(\lambda) &= - \left\{ (-1 - \lambda) \begin{vmatrix} 1 - \lambda & 1 \\ -12 & 2 - \lambda \end{vmatrix} \right. \\ &\quad \left. + 6 \begin{vmatrix} -4 & -2 \\ 1 - \lambda & 1 \end{vmatrix} \right\} \\ &= (\lambda + 1)((1 - \lambda)(2 - \lambda) + 12) \\ &\quad + 6(-4 + 2(1 - \lambda)) \\ &= (\lambda + 1)((1 - \lambda)(2 - \lambda) + 12) + 6(-2 - 2\lambda) \\ &= (\lambda + 1)(\lambda^2 - 3\lambda + 14) - 12(\lambda + 1) \\ &= (\lambda + 1)(\lambda^2 - 3\lambda + 14 - 12) \\ &= (\lambda + 1)(\lambda^2 - 3\lambda + 2) \\ &= (\lambda + 1)(\lambda - 1)(\lambda - 2). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$, and $\lambda_3 = 2$.

12. If

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & -1 & 3 \\ -4 & 0 & 4 \end{pmatrix},$$

Then

$$p(\lambda) = - \begin{vmatrix} 1-\lambda & 0 & -1 \\ -2 & -1-\lambda & 3 \\ -4 & 0 & 4-\lambda \end{vmatrix}.$$

Expanding down the second column,

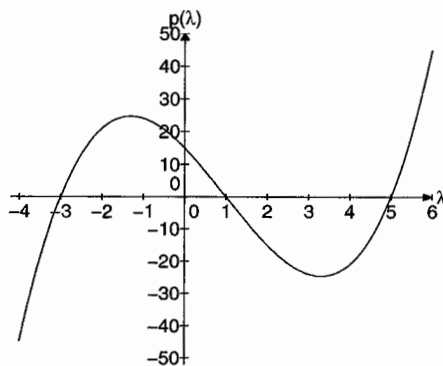
$$\begin{aligned} p(\lambda) &= -(-1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ -4 & 4-\lambda \end{vmatrix} \\ &= (\lambda+1)((1-\lambda)(4-\lambda)-4) \\ &= (\lambda+1)(\lambda^2-5\lambda) \\ &= \lambda(\lambda+1)(\lambda-5). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = 5$.

13. We used a computer to calculate the characteristic polynomial of matrix A .

$$p_A(\lambda) = \lambda^3 - 3\lambda^2 - 13\lambda + 15$$

A computer was used to calculate the eigenvalues: $\lambda_1 = -3$, $\lambda_2 = 1$, and $\lambda_3 = 5$. Next, a computer was used to draw the plot of p_A .

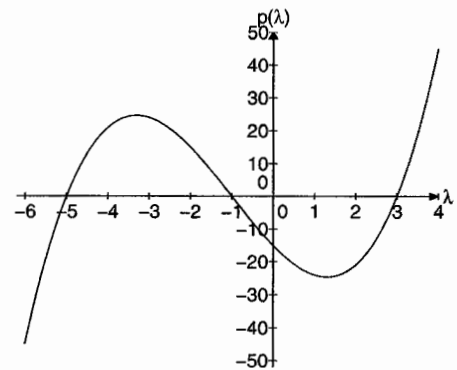


The graph of the characteristic polynomial appears to cross the horizontal axis at -3 , 1 , and 5 . Thus,

the zeros of the characteristic polynomial p_A are the eigenvalues of the matrix A . In a similar manner, the characteristic polynomial of matrix B is

$$p_B(\lambda) = \lambda^3 + 3\lambda^2 - 13\lambda - 15.$$

A computer was used to calculate the eigenvalues: $\lambda_1 = -5$, $\lambda_2 = -1$, and $\lambda_3 = 3$. A computer drawn graph of p_B follows.



The graph of the characteristic polynomial p_B crosses the horizontal axis at -5 , -1 , and 3 . Again, the zeros of the polynomial are the eigenvalues.

14. The matrix

$$A = \begin{pmatrix} 5 & 4 \\ -8 & -7 \end{pmatrix},$$

has characteristic polynomial $p(\lambda) = \lambda^2 - 2\lambda - 3$. Note that

$$\begin{aligned} p(A) &= A^2 + 2A - 3I \\ &= \begin{pmatrix} -7 & -8 \\ 16 & 17 \end{pmatrix} + \begin{pmatrix} 10 & 8 \\ -16 & -14 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

15. Using Matlab, for example, you would execute the commands

```
>> A=[12,14;-7,-9]; p=poly(A);  
polyvalm(p,A)
```

for the matrix in Exercise 1. This will result in the zero matrix. A similar command works for the matrices in the other problems.

16. If

$$A = \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} 2-\lambda & 0 \\ -4 & -2-\lambda \end{pmatrix} \\ &= (2-\lambda)(-2-\lambda). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$. For $\lambda_1 = 2$,

$$A - 2I = \begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix}$$

and $\mathbf{v}_1 = (1, -1)^T$ is an eigenvector. Thus,

$$\mathbf{y}_1(t) = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution. For $\lambda_2 = -2$,

$$A + 2I = \begin{pmatrix} 4 & 0 \\ -4 & 0 \end{pmatrix},$$

and $\mathbf{v}_2 = (0, 1)^T$ is an eigenvector. Thus,

$$\mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is a solution. Because $\mathbf{y}_1(0) = (1, -1)^T$ and $\mathbf{y}_2(0) = (0, 1)^T$ are independent, the solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are independent for all t and form a fundamental set of solutions.

17. If

$$A = \begin{pmatrix} 6 & -8 \\ 0 & -2 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} 6-\lambda & -8 \\ 0 & -2-\lambda \end{pmatrix} \\ &= -12 - 6\lambda + 2\lambda + \lambda^2 \\ &= \lambda^2 - 4\lambda - 12 \\ &= (\lambda - 6)(\lambda + 2). \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -2$. For $\lambda_1 = 6$,

$$A - 6I = \begin{pmatrix} 0 & -8 \\ 0 & -8 \end{pmatrix}.$$

It is easily seen that the nullspace of $A - 6I$ is generated by the vector $(1, 0)^T$. Thus,

$$\mathbf{y}_1(t) = e^{6t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a solution. For $\lambda_2 = -2$,

$$A + 2I = \begin{pmatrix} 8 & -8 \\ 0 & 0 \end{pmatrix}.$$

It is easily seen that the nullspace of $A + 2I$ is generated by the vector $(1, 1)^T$. Thus,

$$\mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a solution. Because $\mathbf{y}_1(0) = (1, 0)^T$ and $\mathbf{y}_2(0) = (1, 1)^T$ are independent, the solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are independent for all t and form a fundamental set of solutions.

18. If

$$A = \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} -3-\lambda & -4 \\ 2 & 3-\lambda \end{pmatrix} \\ &= (-3-\lambda)(3-\lambda) + 8 \\ &= \lambda^2 - 1 \\ &= (\lambda + 1)(\lambda - 1). \end{aligned}$$

Thus, $\lambda_1 = -1$ and $\lambda_2 = 1$ are eigenvalues. For $\lambda_1 = -1$,

$$A + I = \begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix}$$

and $\mathbf{v}_1 = (-2, 1)^T$ is an eigenvector. Thus,

$$\mathbf{y}_1(t) = e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

is a solution. For $\lambda_2 = 1$,

$$A - I = \begin{pmatrix} -4 & -4 \\ 2 & 2 \end{pmatrix}$$

and $\mathbf{v}_2 = (1, -1)^T$ is an eigenvector. Thus,

$$\mathbf{y}_2(t) = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution. Because $\mathbf{y}_1(0) = (-2, 1)^T$ and $\mathbf{y}_2(0) = (1, -1)^T$ are independent, the solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are independent for all t and form a fundamental set of solutions.

19. If

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} \\ &= (-1-\lambda)^2 \\ &= (1+\lambda)^2. \end{aligned}$$

Thus, $\lambda = -1$ is an eigenvalue. For $\lambda = -1$,

$$A + I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easily seen that both $(1, 0)^T$ and $(0, 1)^T$ are elements of the nullspace of $A + I$. Thus,

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are solutions. Because $\mathbf{y}_1(0) = (1, 0)^T$ and $\mathbf{y}_2(0) = (0, 1)^T$ are independent, $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are independent for all t and form a fundamental set of solutions.

20. If

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{pmatrix} \\ &= (3-\lambda)(-3-\lambda) + 8 \\ &= \lambda^2 - 1 \\ &= (\lambda+1)(\lambda-1). \end{aligned}$$

Thus, $\lambda_1 = -1$ and $\lambda_2 = 1$ are eigenvalues. For $\lambda_1 = -1$,

$$A + I = \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$$

and $\mathbf{v}_1 = (1, 2)^T$ is an eigenvector. Thus,

$$\mathbf{y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a solution. For $\lambda_2 = 1$,

$$A - I = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}$$

and $\mathbf{v}_2 = (1, 1)^T$ is an eigenvector. Thus,

$$\mathbf{y}_2(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a solution. Because $\mathbf{y}_1(0) = (1, 2)^T$ and $\mathbf{y}_2(0) = (1, 1)^T$ are independent, the solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are independent for all t and form a fundamental set of solutions.

21. If

$$A = \begin{pmatrix} 7 & 10 \\ -5 & -8 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} 7-\lambda & 10 \\ -5 & -8-\lambda \end{pmatrix} \\ &= -56 - 7\lambda + 8\lambda + \lambda^2 + 50 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda+3)(\lambda-2). \end{aligned}$$

Thus, $\lambda_1 = -3$ and $\lambda_2 = 2$ are eigenvalues. For $\lambda_1 = -3$,

$$A + 3I = \begin{pmatrix} 10 & 10 \\ -5 & -5 \end{pmatrix}.$$

It is easily seen that the nullspace of $A + 3I$ is generated $(1, -1)^T$. Thus,

$$\mathbf{y}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution. For $\lambda = 2$,

$$A - 2I = \begin{pmatrix} 5 & 10 \\ -5 & -10 \end{pmatrix}.$$

It is easily seen that the nullspace of $A - 2I$ is generated by $(2, -1)^T$. Thus,

$$\mathbf{y}_2(t) = e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

is a solution. Because $\mathbf{y}_1(0) = (1, -1)^T$ and $\mathbf{y}_2(0) = (2, -1)^T$ are independent, the solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are independent for all t and form a fundamental set of solutions.

22. If

$$A = \begin{pmatrix} -3 & 14 \\ 0 & 4 \end{pmatrix},$$

then

$$p(\lambda) = \det \begin{pmatrix} -3 - \lambda & 14 \\ 0 & 4 - \lambda \end{pmatrix} \\ = (-3 - \lambda)(4 - \lambda).$$

Thus, $\lambda_1 = -3$ and $\lambda_2 = 4$ are eigenvalues. For $\lambda_1 = -3$,

$$A + 3I = \begin{pmatrix} 0 & 14 \\ 0 & 7 \end{pmatrix}$$

and $\mathbf{v}_1 = (1, 0)^T$ is an eigenvector. Thus,

$$\mathbf{y}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a solution. For $\lambda_2 = 4$,

$$A - 4I = \begin{pmatrix} -7 & 14 \\ 0 & 0 \end{pmatrix}$$

and $\mathbf{v}_2 = (2, 1)^T$ is an eigenvector. Thus,

$$\mathbf{y}_2(t) = e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is a solution. Because $\mathbf{y}_1(0) = (1, 0)^T$ and $\mathbf{y}_2(0) = (2, 1)^T$ are independent, the solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are independent for all t and form a fundamental set of solutions.

23. If

$$A = \begin{pmatrix} 5 & -4 \\ 8 & -7 \end{pmatrix},$$

then

$$p(\lambda) = \det \begin{pmatrix} 5 - \lambda & -4 \\ 8 & -7 - \lambda \end{pmatrix} \\ = -35 - 5\lambda + 7\lambda + \lambda^2 + 32 \\ = \lambda^2 + 2\lambda - 3 \\ = (\lambda + 3)(\lambda - 1).$$

Thus, $\lambda_1 = -3$ and $\lambda_2 = 1$ are eigenvalues. For $\lambda_1 = -3$,

$$A + 3I = \begin{pmatrix} 8 & -4 \\ 8 & -4 \end{pmatrix}.$$

It is easily seen that the nullspace of $A + 3I$ is generated by $(1, 2)^T$. Thus,

$$\mathbf{y}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a solution. For $\lambda = 1$,

$$A - I = \begin{pmatrix} 4 & -4 \\ 8 & -8 \end{pmatrix}.$$

It is easily seen that the nullspace of $A - I$ is generated by $(1, 1)^T$. Thus,

$$\mathbf{y}_2(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a solution. Because $\mathbf{y}_1(0) = (1, 2)^T$ and $\mathbf{y}_2(0) = (1, 1)^T$ are independent, the solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are independent for all t and form a fundamental set of solutions.

24. If

$$A = \begin{pmatrix} -5 & 0 & -6 \\ 26 & -3 & 38 \\ 4 & 0 & 5 \end{pmatrix},$$

then

$$p(\lambda) = -\det \begin{pmatrix} -5 - \lambda & 0 & -6 \\ 26 & -3 - \lambda & 38 \\ 4 & 0 & 5 - \lambda \end{pmatrix} \\ = -(-3 - \lambda) \begin{vmatrix} -5 - \lambda & -6 \\ 4 & 5 - \lambda \end{vmatrix} \\ = (\lambda + 3)((-5 - \lambda)(5 - \lambda) + 24) \\ = (\lambda + 3)(\lambda^2 - 1) \\ = (\lambda + 3)(\lambda + 1)(\lambda - 1).$$

Thus, $\lambda_1 = -3$, $\lambda_2 = -1$, and $\lambda_3 = 1$ are eigenvalues. For $\lambda_1 = -3$,

$$A + 3I = \begin{pmatrix} -2 & 0 & -6 \\ 26 & 0 & 38 \\ 4 & 0 & 8 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{v}_1 = (0, 1, 0)^T$ is an eigenvector and

$$\mathbf{y}_1(t) = e^{-3t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is a solution. For $\lambda_1 = -1$,

$$A + I = \begin{pmatrix} -4 & 0 & -6 \\ 26 & -2 & 38 \\ 4 & 0 & 6 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{v}_2 = (-3, -1, 2)^T$ is an eigenvector and

$$\mathbf{y}_2(t) = e^{-t} \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$$

is a solution. For $\lambda_3 = 1$,

$$A - I = \begin{pmatrix} -6 & 0 & -6 \\ 26 & -4 & 38 \\ 4 & 0 & 4 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{v}_3 = (-1, 3, 1)^T$ is an eigenvector and

$$\mathbf{y}_3(t) = e^t \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

is a solution. Because

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \det \begin{pmatrix} 0 & -3 & -1 \\ 1 & -1 & 3 \\ 0 & 2 & 1 \end{pmatrix} = 1,$$

the solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, and $\mathbf{y}_3(t)$ are independent for all t and form a fundamental set of solutions.

25. If

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det \begin{pmatrix} -1-\lambda & 0 & 0 \\ 2 & -5-\lambda & -6 \\ -2 & 3 & 4-\lambda \end{pmatrix} \\ &= -(-1-\lambda) \begin{vmatrix} -5-\lambda & -6 \\ 3 & 4-\lambda \end{vmatrix} \\ &= (\lambda+1)(-20+5\lambda-4\lambda+\lambda^2+18) \\ &= (\lambda+1)(\lambda^2+\lambda-2) \\ &= (\lambda+1)(\lambda+2)(\lambda-1) \end{aligned}$$

Thus, $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = 1$ are eigenvalues. For $\lambda_1 = -1$,

$$A + I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -6 \\ -2 & 3 & 5 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easily seen that the nullspace of $A + I$ is generated $(1, -1, 1)^T$. Thus,

$$\mathbf{y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

is a solution. For $\lambda = -2$,

$$A + 2I = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easily seen that the nullspace of $A + 2I$ is generated by $(0, -2, 1)^T$. Thus,

$$\mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

is a solution. For $\lambda_3 = 1$,

$$A - I = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easily seen that the nullspace of $A - I$ is generated by $(0, -1, 1)$. Thus,

$$\mathbf{y}_3(t) = e^t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

is a solution. Because

$$\begin{aligned} \det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] &= \det \begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= -1, \end{aligned}$$

the solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, and $\mathbf{y}_3(t)$ are independent for all t and form a fundamental set of solutions.

26. If

$$A = \begin{pmatrix} -1 & 2 & 0 \\ -19 & 14 & 18 \\ 17 & -11 & -17 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det \begin{pmatrix} -1-\lambda & 2 & 0 \\ -19 & 14-\lambda & 18 \\ 17 & -11 & -17-\lambda \end{pmatrix} \\ &= -\left\{ (-1-\lambda) \begin{vmatrix} 14-\lambda & 18 \\ -11 & -17-\lambda \end{vmatrix} \right. \\ &\quad \left. - 2 \begin{vmatrix} -19 & 18 \\ 17 & -17-\lambda \end{vmatrix} \right\} \\ &= (\lambda+1)(\lambda^2+3\lambda-40) + 2(19\lambda+17) \\ &= \lambda^3+4\lambda^2+\lambda-6 \\ &= (\lambda-1)(\lambda+3)(\lambda+2) \end{aligned}$$

Thus, $\lambda_1 = 1$, $\lambda_2 = -3$, and $\lambda_3 = -2$ are eigenvalues. For $\lambda_1 = 1$,

$$A - I = \begin{pmatrix} -2 & 2 & 0 \\ -19 & 13 & 18 \\ 17 & -11 & -18 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{v}_1 = (3, 3, 1)^T$ is an eigenvector and

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

is a solution. For $\lambda_2 = -3$,

$$A + 3I = \begin{pmatrix} 2 & 2 & 0 \\ -19 & 17 & 18 \\ 17 & -11 & -14 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{v}_2 = (1, -1, 2)^T$ is an eigenvector and

$$\mathbf{y}_2(t) = e^{-3t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

is a solution. For $\lambda_3 = -2$,

$$A + 2I = \begin{pmatrix} 1 & 2 & 0 \\ -19 & 16 & 18 \\ 17 & -11 & -15 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{v}_3 = (2, -1, 3)^T$ is an eigenvector and

$$\mathbf{y}_3(t) = e^{-2t} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

is a solution. Because

$$\begin{aligned} \det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] &= \det \begin{pmatrix} 3 & 1 & 2 \\ 3 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix} \\ &= 1, \end{aligned}$$

the solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, and $\mathbf{y}_3(t)$ are independent for all t and form a fundamental set of solutions.

27. If

$$A = \begin{pmatrix} -3 & 0 & 2 \\ 6 & 3 & -12 \\ 2 & 2 & -6 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det \begin{pmatrix} -3-\lambda & 0 & 2 \\ 6 & 3-\lambda & -12 \\ 2 & 2 & -6-\lambda \end{pmatrix} \\ &= -\left\{ (-3-\lambda) \begin{vmatrix} 3-\lambda & -12 \\ 2 & -6-\lambda \end{vmatrix} \right. \\ &\quad \left. + 2 \begin{vmatrix} 6 & 3-\lambda \\ 2 & 2 \end{vmatrix} \right\} \\ &= (\lambda+3)(\lambda^2+3\lambda+6) - 2(6+2\lambda) \\ &= (\lambda+3)(\lambda^2+3\lambda+6) - 4(\lambda+3) \\ &= (\lambda+3)(\lambda^2+3\lambda+6-4) \\ &= (\lambda+3)(\lambda^2+3\lambda+2) \\ &= (\lambda+3)(\lambda+2)(\lambda+1). \end{aligned}$$

Thus, $\lambda_1 = -3$, $\lambda_2 = -2$, and $\lambda_3 = -1$ are eigenvalues. For $\lambda_1 = -3$,

$$A + 3I = \begin{pmatrix} 0 & 0 & 2 \\ 6 & 6 & -12 \\ 2 & 2 & -3 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easily seen that the nullspace of $A + 3I$ is generated by $(-1, 1, 0)^T$. Thus,

$$\mathbf{y}_1(t) = e^{-3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

is a solution. For $\lambda = -2$,

$$A + 2I = \begin{pmatrix} -1 & 0 & 2 \\ 6 & 5 & -12 \\ 2 & 2 & -4 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easily seen that the nullspace of $A + 2I$ is generated by $(2, 0, 1)^T$. Thus,

$$\mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

is a solution. For $\lambda_3 = -1$,

$$A + I = \begin{pmatrix} -2 & 0 & 2 \\ 6 & 4 & -12 \\ 2 & 2 & -5 \end{pmatrix},$$

which has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easily seen that the nullspace of $A + I$ is generated by $(1, 3/2, 1)$. Thus,

$$\mathbf{y}_3(t) = e^{-t} \begin{pmatrix} 1 \\ 3/2 \\ 1 \end{pmatrix}$$

is a solution. Because

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \det \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 3/2 \\ 0 & 1 & 1 \end{pmatrix} = \frac{1}{2},$$

the solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, and $\mathbf{y}_3(t)$ are independent for all t and form a fundamental set of solutions.

28. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$0 \rightarrow \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad -2 \rightarrow \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \quad 2 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

29. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$3 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad 1 \rightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad -2 \rightarrow \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

30. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$-3 \rightarrow \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad -2 \rightarrow \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \quad -1 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

31. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$3 \rightarrow \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \quad 2 \rightarrow \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \quad -1 \rightarrow \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

32. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$-2 \rightarrow \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad 3 \rightarrow \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad -4 \rightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

33. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$1 \rightarrow \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad -3 \rightarrow \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \quad -1 \rightarrow \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

34. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$1 \rightarrow \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix}, \quad -3 \rightarrow \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \\ -2 \rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad -1 \rightarrow \begin{pmatrix} -3 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

35. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$3 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad -1 \rightarrow \begin{pmatrix} 0 \\ 1 \\ 3/2 \\ 1 \end{pmatrix}, \\ -2 \rightarrow \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad -4 \rightarrow \begin{pmatrix} -1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

36. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$2 \rightarrow \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad -1 \rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \\ 1 \rightarrow \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \quad 0 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

37. Using a computer, we find the following eigenvalue-eigenvector pairs.

$$4 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad -2 \rightarrow \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix},$$

$$2 \rightarrow \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad -1 \rightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

38. Using a computer, a fundamental set of solutions is found.

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{2t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

39. Using a computer, a fundamental set of solutions is found.

$$\mathbf{y}_1(t) = e^{3t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{-2t} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

40. Using a computer, a fundamental set of solutions is found.

$$\mathbf{y}_1(t) = e^{-3t} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{-2t} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

41. Using a computer, a fundamental set of solutions is

found.

$$\mathbf{y}_1(t) = e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-3t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{2t} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

42. Using a computer, a fundamental set of solutions is found.

$$\mathbf{y}_1(t) = e^{4t} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-5t} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{-t} \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

43. Using a computer, a fundamental set of solutions is found.

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-3t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{4t} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

44. Using a computer, a fundamental set of solutions is found.

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 0 \\ 1 \\ -2 \\ -2 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{3t} \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^t \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_4(t) = e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

45. Using a computer, a fundamental set of solutions is

found.

$$\mathbf{y}_1(t) = e^{-4t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{2t} \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_4(t) = e^{-t} \begin{pmatrix} -3/2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

46. Using a computer, a fundamental set of solutions is found.

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-4t} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{3t} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{y}_4(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

47. Using a computer, a fundamental set of solutions is found.

$$\mathbf{y}_1(t) = e^{-5t} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_4(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

48. If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors associated to the eigenvalue λ , then

$$A\mathbf{v} = \lambda\mathbf{v} \quad \text{and} \quad A\mathbf{w} = \lambda\mathbf{w}.$$

Thus, if $\mathbf{y} = a\mathbf{v} + b\mathbf{w}$, then

$$\begin{aligned} A\mathbf{y} &= A(a\mathbf{v} + b\mathbf{w}) \\ &= A(a\mathbf{v}) + A(b\mathbf{w}) \\ &= a(A\mathbf{v}) + b(A\mathbf{w}) \\ &= a(\lambda\mathbf{v}) + b(\lambda\mathbf{w}) \\ &= \lambda(a\mathbf{v} + b\mathbf{w}) \\ &= \lambda\mathbf{y}. \end{aligned}$$

Thus, $\mathbf{y} = a\mathbf{v} + b\mathbf{w}$ is also an eigenvector associated with λ .

49. If

$$A = \begin{pmatrix} 6 & -8 \\ 4 & -6 \end{pmatrix},$$

then A has eigenvalues 2 and -2 , and determinant $D = -4$. Note that the product of the eigenvalues equals the determinant. If

$$B = \begin{pmatrix} -11 & -16 \\ 8 & 13 \end{pmatrix},$$

then B has eigenvalues -3 and 5 , and determinant $D = -15$. Note that the product of the eigenvalues equals the determinant. If

$$C = \begin{pmatrix} 7 & -21 & -11 \\ 5 & -13 & -5 \\ -5 & 9 & 1 \end{pmatrix},$$

then C has eigenvalues 2, -3 , and -4 , and determinant $D = 24$. Note that the product of the eigenvalues equals the determinant.

50. In the case

$$B = \begin{pmatrix} -11 & -16 \\ 8 & 13 \end{pmatrix},$$

the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -3$. Thus,

$$\lambda_1 + \lambda_2 = 2.$$

The trace of B is also

$$\text{tr}(B) = -11 + 13 = 2.$$

Thus, the trace of matrix B equals the sum of its eigenvalues. This statement is also true when applied to the matrices A and C .

51. If

$$A = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix},$$

then the eigenvalues of A are 2 and -4 . Note that the eigenvalues lie on the main diagonal. If

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{pmatrix},$$

then the eigenvalues of B are 1, -1 , and 5. Note that the eigenvalues lie on the main diagonal. If

$$C = \begin{pmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

then the eigenvalues of C are 2, 3, -4 and 2. Note that the eigenvalues lie on the main diagonal. Here is an example of a lower triangular matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 1 & 4 \end{pmatrix}$$

A computer shows that the eigenvalues are 1, -2 , and 4. Again, note that the main diagonal contains the eigenvalues.

52. Consider an $n \times n$ matrix that is upper triangular ($a_{ij} = 0$ for $i > j$). Then

$$\begin{aligned} p(\lambda) &= (-1)^n \det(A - \lambda I) \\ &= (-1)^n \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{pmatrix} \end{aligned}$$

Expanding down the first column,

$$\begin{aligned} p(\lambda) &= (-1)^n (a_{11} - \lambda) \\ &\quad \times \begin{vmatrix} a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} - \lambda & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix}. \end{aligned}$$

Expanding down the first column,

$$\begin{aligned} p(\lambda) &= (-1)^n (a_{11} - \lambda)(a_{22} - \lambda) \\ &\quad \times \begin{vmatrix} a_{33} - \lambda & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} - \lambda & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix}. \end{aligned}$$

Continuing in this manner,

$$\begin{aligned} p(\lambda) &= (-1)^n (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \cdots (a_{nn} - \lambda) \\ \text{and the eigenvalues are } \lambda_1 &= a_{11}, \lambda_2 = a_{22}, \lambda_3 = a_{33}, \dots, \text{ and } \lambda_n = a_{nn}. \end{aligned}$$

53. If

$$V = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix},$$

then

$$\begin{aligned} VDV^{-1} &= \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 4 & 3 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 4 & 3 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 10 \\ 0 & -2 \end{pmatrix} \\ &= A. \end{aligned}$$

54. If

$$A = \begin{pmatrix} 6 & 0 \\ 8 & -2 \end{pmatrix},$$

then a computer reveals the following eigenvalue-eigenvector pairs.

$$-2 \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad 6 \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, the matrices

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix}$$

diagonalize matrix A . That is $A = VDV^{-1}$.

55. If

$$A = \begin{pmatrix} -1 & -2 \\ 4 & -7 \end{pmatrix},$$

then a computer reveals the following eigenvalue-eigenvector pairs.

$$-5 \rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad -3 \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, the matrices

$$V = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -5 & 0 \\ 0 & -3 \end{pmatrix}$$

diagonalize matrix A . That is, $A = VDV^{-1}$.

56. The matrix

$$A = \begin{pmatrix} 5 & 1 \\ -1 & 3 \end{pmatrix}$$

has a repeated eigenvalue $\lambda = 4$ but only 1 independent eigenvector $\mathbf{v} = (1, -1)^T$.

Section 9.2. Planar Systems

1. The matrix

$$A = \begin{pmatrix} 2 & -6 \\ 0 & -1 \end{pmatrix},$$

has the following eigenvalue-eigenvector pairs.

$$\lambda_1 = 2 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -1 \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

2. The matrix

$$A = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix}$$

has the following eigenvalue-eigenvector pairs.

$$\lambda_1 = 2 \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 5 \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3. The matrix

$$A = \begin{pmatrix} -5 & 1 \\ -2 & -2 \end{pmatrix}$$

has the following eigenvalue-eigenvector pairs.

$$\lambda_1 = -4 \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -3 \rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

4. The matrix

$$A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

has the following eigenvalue-eigenvector pairs.

$$\lambda_1 = -3 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -1 \rightarrow \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

5. The matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$

has the following eigenvalue-eigenvector pairs.

$$\lambda_1 = 2 \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 3 \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

6. The matrix

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

has the following eigenvalue-eigenvector pairs.

$$\lambda_1 = 0 \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -2 \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

7. The system in Exercise 1 had general solution

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, if $\mathbf{y}(0) = (0, 1)^T$, then

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

Therefore, $C_1 = -2$ and $C_2 = 1$, giving particular solution

$$\mathbf{y}(t) = -2e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

8. The system in Exercise 2 had the general solution

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, if $\mathbf{y}(0) = (1, -2)^T$, then

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

The augmented matrix reduces

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -5 \end{pmatrix}.$$

Thus, $C_1 = 3$ and $C_2 = -5$, giving particular solution

$$\mathbf{y}(t) = 3e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 5e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

9. The system in Exercise 3 had general solution

$$\mathbf{y}(t) = C_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus, if $\mathbf{y}(0) = (0, -1)^T$, then

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Therefore, $C_1 = 1$ and $C_2 = -1$, giving particular solution

$$\mathbf{y}(t) = e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

10. The system in Exercise 4 had the general solution

$$\mathbf{y}(t) = C_1 e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Thus, if $\mathbf{y}(0) = (1, 1)^T$, then

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus, $C_1 = 4$ and $C_2 = 1$, giving particular solution

$$\mathbf{y}(t) = 4e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

11. The system in Exercise 5 had general solution

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, if $\mathbf{y}(0) = (3, 2)^T$, then

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Therefore, $C_1 = 1$ and $C_2 = 1$, giving particular solution

$$\mathbf{y}(t) = e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

12. The system in Exercise 6 had the general solution

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus, if $\mathbf{y}(0) = (1, 5)^T$, then

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

Thus, $C_1 = 3$ and $C_2 = 2$, giving particular solution

$$\mathbf{y}(t) = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

13. If $z(t) = e^{2it}(1, 1 + i)^T$, then

$$\begin{aligned} z(t) &= (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ 1 + i \end{bmatrix} \\ &= (\cos 2t + i \sin 2t) \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &= \left[\cos 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &\quad + i \left[\cos 2t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]. \end{aligned}$$

Therefore, $\operatorname{Re}(z(t)) = (\cos 2t, \cos 2t - \sin 2t)^T$ and $\operatorname{Im}(z(t)) = (\sin 2t, \cos 2t + \sin 2t)^T$.

14. If $z(t) = e^{(1+i)t}(-1 + i, 2)^T$, then

$$\begin{aligned} z(t) &= e^t e^{it} \begin{pmatrix} -1 + i \\ 2 \end{pmatrix} \\ &= e^t (\cos t + i \sin t) \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &= e^t \left[\cos t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &\quad + i \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \sin t \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= e^t \begin{pmatrix} -\cos t - \sin t \\ 2 \cos t \end{pmatrix} \\ &\quad + i e^t \begin{pmatrix} \cos t - \sin t \\ 2 \sin t \end{pmatrix}. \end{aligned}$$

Therefore, $\operatorname{Re}(z(t)) = e^t(-\cos t - \sin t, 2 \cos t)^T$ and $\operatorname{Im}(z(t)) = e^t(\cos t - \sin t, 2 \sin t)^T$.

15. If $z(t) = e^{3it}(-1 - i, 2)^T$, then

$$\begin{aligned} z(t) &= (\cos 3t + i \sin 3t) \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= \left[\cos 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &\quad + i \left[\sin 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \cos 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \end{aligned}$$

The real part of $z(t)$ is

$$y_1(t) = \begin{pmatrix} -\cos 3t + \sin 3t \\ 2 \cos 3t \end{pmatrix}$$

and

$$y_1'(t) = \begin{pmatrix} 3 \sin 3t + 3 \cos 3t \\ -6 \sin 3t \end{pmatrix}.$$

However,

$$\begin{aligned} &\begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} -\cos 3t + \sin 3t \\ 2 \cos 3t \end{pmatrix} \\ &= \begin{pmatrix} 3 \sin 3t + 3 \cos 3t \\ -6 \sin 3t \end{pmatrix} \end{aligned}$$

as well, so y_1 is a solution of $y' = Ay$. The imaginary part of $z(t)$ is

$$y_2(t) = \begin{pmatrix} -\sin 3t - \cos 3t \\ 2 \sin 3t \end{pmatrix}$$

and

$$y_2'(t) = \begin{pmatrix} -3 \cos 3t + 3 \sin 3t \\ 6 \cos 3t \end{pmatrix}.$$

However,

$$\begin{aligned} &\begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} -\sin 3t - \cos 3t \\ 2 \sin 3t \end{pmatrix} \\ &= \begin{pmatrix} -3 \cos 3t + 3 \sin 3t \\ 6 \cos 3t \end{pmatrix} \end{aligned}$$

as well, so y_2 is a solution of $y' = Ay$. Finally, because

$$y_1(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{and} \quad y_2(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

are independent, $y_1(t)$ and $y_2(t)$ are independent for all values of t and form a fundamental set of solutions.

16. If

$$A = \begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix},$$

then the characteristic polynomial of A is $p(\lambda) = \lambda^2 + 4$ and the eigenvalues are $\lambda_1 = 4i$ and $\lambda_2 = -4i$. Trusting that

$$A - (4i)I = \begin{pmatrix} -4 - 4i & -8 \\ 4 & 4 - 4i \end{pmatrix}$$

is singular, examination of the second row shows that $(-1 + i, 1)^T$ generates the nullspace of $A - (4i)I$.

Thus, we have a complex solution which we must break into real and imaginary parts.

$$\begin{aligned}
 \mathbf{z}(t) &= e^{4it} \begin{pmatrix} -1+i \\ 1 \end{pmatrix} \\
 &= (\cos 4t + i \sin 4t) \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\
 &= \cos 4t \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin 4t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &\quad + i \cos 4t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \sin 4t \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} \\
 &\quad + i \begin{pmatrix} \cos 4t - \sin 4t \\ \sin 4t \end{pmatrix}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \mathbf{y}_1(t) &= \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} \quad \text{and} \\
 \mathbf{y}_2(t) &= \begin{pmatrix} \cos 4t - \sin 4t \\ \sin 4t \end{pmatrix}
 \end{aligned}$$

form a fundamental set of real solutions.

17. If

$$A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix},$$

then the characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda + 5$ and the eigenvalues are $1 \pm 2i$. Trusting that

$$A - (1 + 2i)I = \begin{pmatrix} -2 - 2i & -2 \\ 4 & 2 - 2i \end{pmatrix}$$

is singular, examination of the first row reveals the eigenvector $\mathbf{v} = (1, -1 - i)^T$. Thus,

$$\begin{aligned}
 \mathbf{z}(t) &= e^{(1+2i)t} \begin{pmatrix} 1 \\ -1-i \end{pmatrix} \\
 &= e^t (\cos 2t + i \sin 2t) \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
 &= e^t \left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
 &\quad + i e^t \left[\cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right].
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \mathbf{y}_1(t) &= e^t \begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} \quad \text{and} \\
 \mathbf{y}_2(t) &= e^t \begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}
 \end{aligned}$$

form a fundamental set of solutions.

18. If

$$A = \begin{pmatrix} -1 & 1 \\ -5 & -5 \end{pmatrix},$$

then the characteristic polynomial of A is $p(\lambda) = \lambda^2 + 6\lambda + 10$ and the eigenvalues are $\lambda_1 = -3 + i$ and $\lambda_2 = -3 - i$. Trusting that

$$A - (-3 + i)I = \begin{pmatrix} 2 - i & 1 \\ -5 & -2 - i \end{pmatrix}$$

is singular, examination of the first row shows that $(1, -2+i)^T$ generates the nullspace of $A - (-3+i)I$. Thus, we have a complex solution which we must break into real and imaginary parts.

$$\begin{aligned}
 \mathbf{z}(t) &= e^{(-3+i)t} \begin{pmatrix} 1 \\ -2+i \end{pmatrix} \\
 &= e^{-3t} (\cos t + i \sin t) \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\
 &= e^{-3t} \left[\cos t \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right. \\
 &\quad \left. + i \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \sin t \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right] \\
 &= e^{-3t} \begin{pmatrix} \cos t \\ -2 \cos t - \sin t \end{pmatrix} \\
 &\quad + i e^{-3t} \begin{pmatrix} \sin t \\ \cos t - 2 \sin t \end{pmatrix}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \mathbf{y}_1(t) &= e^{-3t} \begin{pmatrix} \cos t \\ -2 \cos t - \sin t \end{pmatrix} \quad \text{and} \\
 \mathbf{y}_2(t) &= e^{-3t} \begin{pmatrix} \sin t \\ \cos t - 2 \sin t \end{pmatrix}
 \end{aligned}$$

form a fundamental set of real solutions.

19. The characteristic polynomial of

$$A = \begin{pmatrix} 0 & 4 \\ -2 & -4 \end{pmatrix}$$

is $p(\lambda) = \lambda^2 - 4\lambda + 8$, which has complex roots $\lambda = -2 \pm 2i$. For the eigenvalue $\lambda = 2 + 2i$, we have the eigenvector $\mathbf{w} = (-1 - i, 1)^T$. The corresponding exponential solution is

$$\begin{aligned} \mathbf{z}(t) &= e^{(-2+2i)t} \begin{pmatrix} -1-i \\ 1 \end{pmatrix} \\ &= e^{-2t} [\cos 2t + i \sin 2t] \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= e^{-2t} \left[\cos 2t \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin 2t \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &\quad + i e^{-2t} \left[\cos 2t \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin 2t \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]. \end{aligned}$$

The real and imaginary parts of $\mathbf{z}$,

$$\begin{aligned} \mathbf{y}_1(t) &= e^{-2t} \begin{pmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ \mathbf{y}_2(t) &= e^{-2t} \begin{pmatrix} -\cos 2t - \sin 2t \\ \sin 2t \end{pmatrix} \end{aligned}$$

are a fundamental set of solutions.

20. If

$$A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix},$$

then the characteristic polynomial of A is $p(\lambda) = \lambda^2 + 2\lambda + 10$ and the eigenvalues are $\lambda_1 = -1 + 3i$ and $\lambda_2 = -1 - 3i$. Trusting that

$$A - (-1 + 3i)I = \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix}$$

is singular, examination of the first row shows that $(1, i)^T$ generates the nullspace of $A - (-1 + 3i)I$. Thus, we have a complex solution which we must

break into real and imaginary parts.

$$\begin{aligned} \mathbf{z}(t) &= e^{(-1+3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= e^{-t} (\cos 3t + i \sin 3t) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-t} \left[\cos 3t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin 3t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right. \\ &\quad \left. + i \cos 3t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \sin 3t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &= e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + i e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{y}_1(t) &= e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} \quad \text{and} \\ \mathbf{y}_2(t) &= e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} \end{aligned}$$

form a fundamental set of real solutions.

21. If

$$A = \begin{pmatrix} 3 & -6 \\ 3 & 5 \end{pmatrix},$$

then the characteristic polynomial is $p(\lambda) = \lambda^2 - 8\lambda + 33$ and the eigenvalues are $4 \pm \sqrt{17}i$. Trusting that

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -6 \\ 3 & 5-\lambda \end{pmatrix}$$

is singular, examination of the first row reveals the eigenvector $\mathbf{v} = (6, 3 - \lambda)^T$. Substituting $\lambda =$

$4 + \sqrt{17}i$ give $\mathbf{v} = (6, -1 - \sqrt{17}i)^T$. Thus,

$$\begin{aligned} z(t) &= e^{(4+\sqrt{17}i)t} \begin{pmatrix} 6 \\ -1 - \sqrt{17}i \end{pmatrix} \\ &= e^{4t} \left[\cos \sqrt{17}t + i \sin \sqrt{17}t \right] \\ &\quad \times \left[\begin{pmatrix} 6 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -\sqrt{17} \end{pmatrix} \right] \\ &= e^{4t} \left[\cos \sqrt{17}t \begin{pmatrix} 6 \\ -1 \end{pmatrix} - \sin \sqrt{17}t \begin{pmatrix} 0 \\ -\sqrt{17} \end{pmatrix} \right] \\ &\quad + i e^{4t} \left[\cos \sqrt{17}t \begin{pmatrix} 0 \\ -\sqrt{17} \end{pmatrix} \right. \\ &\quad \left. + \sin \sqrt{17}t \begin{pmatrix} 6 \\ -1 \end{pmatrix} \right] \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{y}_1(t) &= e^{4t} \begin{pmatrix} 6 \cos \sqrt{17}t \\ -\cos \sqrt{17}t + \sqrt{17} \sin \sqrt{17}t \end{pmatrix} \\ \text{and} \\ \mathbf{y}_2(t) &= e^{4t} \begin{pmatrix} 6 \sin \sqrt{17}t \\ -\sqrt{17} \cos \sqrt{17}t - \sin \sqrt{17}t \end{pmatrix} \end{aligned}$$

form a fundamental set of solutions.

22. The fundamental solutions found in Exercise 16 allows the formation of the general solution

$$\mathbf{y}(t) = C_1 \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} + C_2 \begin{pmatrix} \cos 4t - \sin 4t \\ \sin 4t \end{pmatrix}.$$

If $\mathbf{y}(0) = (0, 2)^T$, then

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$$

Thus, $C_1 = 2$ and $C_2 = 2$ and

$$\begin{aligned} \mathbf{y}(t) &= 2 \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} \\ &\quad + 2 \begin{pmatrix} \cos 4t - \sin 4t \\ \sin 4t \end{pmatrix} \\ &= \begin{pmatrix} -4 \sin 4t \\ 2 \cos 4t + 2 \sin 4t \end{pmatrix}. \end{aligned}$$

23. The fundamental solutions found in Exercise 17 allows the formation of the general solution

$$\mathbf{y}(t) = C_1 e^t \begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}.$$

If $\mathbf{y}(0) = (0, 1)^T$, then

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Thus, $C_1 = 0$ and $C_2 = -1$ and

$$\mathbf{y}(t) = -e^t \begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}.$$

24. The fundamental solutions found in Exercise 18 allows the formation of the general solution

$$\mathbf{y}(t) = C_1 e^{-3t} \begin{pmatrix} \cos t \\ -2 \cos t - \sin t \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} \sin t \\ \cos t - 2 \sin t \end{pmatrix}.$$

If $\mathbf{y}(0) = (1, -5)^T$, then

$$\begin{pmatrix} 1 \\ -5 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}.$$

Thus, $C_1 = 1$ and $C_2 = -3$ and

$$\begin{aligned}\mathbf{y}(t) &= e^{-3t} \begin{pmatrix} \cos t \\ -2 \cos t - \sin t \end{pmatrix} \\ &\quad - 3e^{-3t} \begin{pmatrix} \sin t \\ \cos t - 2 \sin t \end{pmatrix} \\ &= e^{-3t} \begin{pmatrix} \cos t - 3 \sin t \\ -5 \cos t + 5 \sin t \end{pmatrix}.\end{aligned}$$

25. A fundamental set of solutions was found in Exercise 19, so the solution has the form $\mathbf{y}(t) = C_1 \mathbf{y}_1(t) + C_2 \mathbf{y}_2(t)$, where

$$\begin{aligned}\mathbf{y}_1(t) &= e^{-2t} \begin{pmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ \mathbf{y}_2(t) &= e^{-2t} \begin{pmatrix} -\cos 2t - \sin 2t \\ \sin 2t \end{pmatrix}.\end{aligned}$$

At $t = 0$ we have

$$\begin{aligned}\begin{pmatrix} -1 \\ 2 \end{pmatrix} &= \mathbf{y}(0) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.\end{aligned}$$

This system can be readily solved, getting $C_1 = 2$ and $C_2 = -1$. Hence the solution is

$$\mathbf{y}(t) = 2\mathbf{y}_1(t) - \mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} -\cos 2t + 3 \sin 2t \\ 2 \cos 2t - \sin 2t \end{pmatrix}.$$

26. The fundamental solutions found in Exercise 20 allows the formation of the general solution

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}.$$

If $\mathbf{y}(0) = (3, 2)^T$, then

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, $C_1 = 3$ and $C_2 = 2$ and

$$\begin{aligned}\mathbf{y}(t) &= 3e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + 2e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} 3 \cos 3t + 2 \sin 3t \\ -3 \sin 3t + 2 \cos 3t \end{pmatrix}.\end{aligned}$$

27. The fundamental solutions found in Exercise 21 allows the formation of the general solution.

$$\begin{aligned}\mathbf{y}(t) &= C_1 e^{4t} \begin{pmatrix} 6 \cos \sqrt{17}t \\ -\cos \sqrt{17}t + \sqrt{17} \sin \sqrt{17}t \end{pmatrix} \\ &\quad + C_2 e^{4t} \begin{pmatrix} 6 \sin \sqrt{17}t \\ -\sqrt{17} \cos \sqrt{17}t - \sin \sqrt{17}t \end{pmatrix}.\end{aligned}$$

If $\mathbf{y}(0) = (1, 3)^T$, then

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 6 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -\sqrt{17} \end{pmatrix}.$$

The augmented matrix reduces .

$$\begin{pmatrix} 6 & 0 & 1 \\ -1 & -\sqrt{17} & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/6 \\ 0 & 1 & -19\sqrt{17}/102 \end{pmatrix}$$

Thus, $C_1 = 1/6$ and $C_2 = -19\sqrt{17}/102$ and

$$\begin{aligned}\mathbf{y}(t) &= \frac{1}{6} e^{4t} \begin{pmatrix} 6 \cos \sqrt{17}t \\ -\cos \sqrt{17}t + \sqrt{17} \sin \sqrt{17}t \end{pmatrix} \\ &\quad - \frac{19\sqrt{17}}{102} e^{4t} \begin{pmatrix} 6 \sin \sqrt{17}t \\ -\sqrt{17} \cos \sqrt{17}t - \sin \sqrt{17}t \end{pmatrix}.\end{aligned}$$

28. A is a real 2×2 matrix with one eigenvalue λ_1 of multiplicity 2. If

$$\mathbf{y}(t) = e^{\lambda_1 t} [\mathbf{v} + t(A - \lambda_1 I)\mathbf{v}],$$

then

$$\mathbf{y}(0) = e^0 [\mathbf{v} + 0(A - \lambda_1 I)\mathbf{v}] = \mathbf{v}.$$

Moreover, using the product rule,

$$\begin{aligned}y'(t) &= \lambda_1 e^{\lambda_1 t} [\mathbf{v} + t(A - \lambda_1 I)\mathbf{v}] \\ &\quad + e^{\lambda_1 t} (A - \lambda_1 I)\mathbf{v} \\ &= e^{\lambda_1 t} [\lambda_1 \mathbf{v} + t\lambda_1(A - \lambda_1 I)\mathbf{v} + (A - \lambda_1 I)\mathbf{v}] \\ &= e^{\lambda_1 t} [A\mathbf{v} + t\lambda_1(A - \lambda_1 I)\mathbf{v}]\end{aligned}$$

However, because A has a repeated eigenvalue λ_1 , its characteristic polynomial has the form

$$p(\lambda) = (\lambda - \lambda_1)^2.$$

By the Cayley-Hamilton Theorem,

$$(A - \lambda_1 I)^2 = 0I$$

and

$$\begin{aligned}(A - \lambda_1 I)(A - \lambda_1 I) &= 0I \\ A(A - \lambda_1 I) - \lambda_1(A - \lambda_1 I) &= 0I \\ A(A - \lambda_1 I) &= \lambda_1(A - \lambda_1 I).\end{aligned}$$

Thus, we can write

$$\begin{aligned}A\mathbf{y}(t) &= Ae^{\lambda_1 t} [\mathbf{v} + t(A - \lambda_1 I)\mathbf{v}] \\ &= e^{\lambda_1 t} [A\mathbf{v} + tA(A - \lambda_1 I)\mathbf{v}] \\ &= e^{\lambda_1 t} [A\mathbf{v} + t\lambda_1(A - \lambda_1 I)\mathbf{v}].\end{aligned}$$

Therefore, $\mathbf{y}'(t) = A\mathbf{y}(t)$ and $\mathbf{y}(t) = e^{\lambda_1 t} [\mathbf{v} + t(A - \lambda_1 I)\mathbf{v}]$ is a solution of $\mathbf{y}' = A\mathbf{y}$ with $\mathbf{y}(0) = \mathbf{v}$.

29. The matrix

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

has a single eigenvalue $\lambda = -2$. However,

$$A - (-2)I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so all nonzero vectors are eigenvectors. Choose $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$ as eigenvectors. Then,

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is the general solution.

30. The matrix

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$$

has one eigenvalue, $\lambda = -2$. However, the nullspace of

$$A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

is generated by a single eigenvector, $\mathbf{v}_1 = (1, 1)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To find another solution, we need to find a vector $\mathbf{v}_2$ which satisfies $(A + 2I)\mathbf{v}_2 = \mathbf{v}_1$. Choose

$\mathbf{w} = (1, 0)^T$, which is independent of $\mathbf{v}_1$ and note that

$$(A + 2I)\mathbf{w} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -\mathbf{v}_1.$$

Thus, choose $\mathbf{v}_2 = -\mathbf{w} = (-1, 0)^T$. Our second solution is

$$\begin{aligned}\mathbf{y}_2(t) &= e^{-2t}(\mathbf{v}_1 + t\mathbf{v}_2) \\ &= e^{-2t} \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].\end{aligned}$$

Thus, the general solution can be written

$$\begin{aligned}\mathbf{y}(t) &= C_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \\ &= e^{-2t} \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right].\end{aligned}$$

31. The matrix

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

has one eigenvalue, $\lambda = 2$. However, the nullspace of

$$A - 2I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_1 = (1, 1)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To find another solution, we need to find a vector $\mathbf{v}_2$ which satisfies $(A - 2I)\mathbf{v}_2 = \mathbf{v}_1$. Choose $\mathbf{w} = (1, 0)^T$, which is independent of $\mathbf{v}_1$, and note that

$$(A - 2I)\mathbf{w} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{v}_1.$$

Thus, choose $\mathbf{v}_2 = \mathbf{w} = (1, 0)^T$. Our second solution is

$$\begin{aligned}\mathbf{y}_2(t) &= e^{2t}(\mathbf{v}_2 + t\mathbf{v}_1) \\ &= e^{2t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].\end{aligned}$$

Thus, the general solution can be written

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \\ &= e^{2t} \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \end{aligned}$$

32. The matrix

$$A = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$$

has one eigenvalue, $\lambda = 0$. However, the nullspace of

$$A + 0I = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$$

is generated by a single eigenvector, $\mathbf{v}_1 = (1, -2)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{0t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

To find another solution, we need to find a vector $\mathbf{v}_2$ which satisfies $(A + 0I)\mathbf{v}_2 = \mathbf{v}_1$. Choose $\mathbf{w} = (1, 0)^T$, which is independent of $\mathbf{v}_1$ and note that

$$(A + 0I)\mathbf{w} = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = -2\mathbf{v}_1.$$

Thus, chose $\mathbf{v}_2 = -(1/2)\mathbf{w} = (-1/2, 0)^T$. Our second solution is

$$\begin{aligned} \mathbf{y}_2(t) &= e^{0t}(\mathbf{v}_2 + t\mathbf{v}_1) \\ &= \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \end{aligned}$$

Thus, the general solution can be written

$$\begin{aligned} \mathbf{y}(t) &= C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \left[\begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right] \\ &= (C_1 + C_2 t) \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}. \end{aligned}$$

33. The matrix

$$A = \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix}$$

has one eigenvalue, $\lambda = 1$. However, the nullspace of

$$A - I = \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_1 = (1, 3)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

To find another solution, we need to find a vector $\mathbf{v}_2$ which satisfies $(A - I)\mathbf{v}_2 = \mathbf{v}_1$. Choose $\mathbf{w} = (1, 0)^T$, which is independent of $\mathbf{v}_1$, and note that

$$\begin{aligned} (A - I)\mathbf{w} &= \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -9 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = -3\mathbf{v}_1. \end{aligned}$$

Thus, choose $\mathbf{v}_2 = -(1/3)\mathbf{w} = (-1/3, 0)^T$. Our second solution is

$$\begin{aligned} \mathbf{y}_2(t) &= e^t(\mathbf{v}_2 + t\mathbf{v}_1) \\ &= e^t \left[\begin{pmatrix} -1/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right]. \end{aligned}$$

Thus, the general solution can be written

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^t \left[\begin{pmatrix} -1/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right] \\ &= e^t \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} \right] \end{aligned}$$

34. The matrix

$$A = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$$

has one eigenvalue, $\lambda = 3$. However, the nullspace of

$$A - 3I = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$$

is generated by a single eigenvector, $\mathbf{v}_1 = (1, -2)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

To find another solution, we need to find a vector $\mathbf{v}_2$ which satisfies $(A - 3I)\mathbf{v}_2 = \mathbf{v}_1$. Choose

$\mathbf{w} = (1, 0)^T$, which is independent of $\mathbf{v}_1$ and note that

$$(A - 3I)\mathbf{w} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2\mathbf{v}_1.$$

Thus, choose $\mathbf{v}_2 = (1/2)\mathbf{w} = (1/2, 0)^T$. Our second solution is

$$\begin{aligned} \mathbf{y}_2(t) &= e^{3t}(\mathbf{v}_2 + t\mathbf{v}_1) \\ &= e^{3t} \left[\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right]. \end{aligned}$$

Thus, the general solution can be written

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &\quad + C_2 e^{3t} \left[\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right] \\ &= e^{3t} \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right]. \end{aligned}$$

35. From Exercise 29,

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If $\mathbf{y}(0) = (3, -2)^T$, then

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and $C_1 = 3$ and $C_2 = -2$. Thus, the particular solution is

$$\begin{aligned} \mathbf{y}(t) &= 3e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \end{aligned}$$

36. From Exercise 30,

$$\mathbf{y}(t) = e^{-2t} \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right].$$

If $\mathbf{y}(0) = (0, -3)^T$, then

$$\begin{pmatrix} 0 \\ -3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces,

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix},$$

and $C_1 = -3$ and $C_2 = -3$. Thus, the particular solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{-2t} \left[(-3 - 3t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} -3t \\ -3 - 3t \end{pmatrix}. \end{aligned}$$

37. From Exercise 31,

$$\mathbf{y}(t) = e^{2t} \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$

If $\mathbf{y}(0) = (2, -1)^T$, then

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix},$$

and $C_1 = -1$ and $C_2 = 3$. Thus, the particular solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{2t} \left[(-1 + 3t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &= e^{2t} \begin{pmatrix} 2 + 3t \\ -1 + 3t \end{pmatrix}. \end{aligned}$$

38. From Exercise 32,

$$\mathbf{y}(t) = (C_1 + C_2 t) \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}.$$

If $\mathbf{y}(0) = (1, 1)^T$, then

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces,

$$\begin{pmatrix} 1 & -1/2 & 1 \\ -2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3 \end{pmatrix}$$

and $C_1 = -1/2$ and $C_2 = -3$. Thus, the particular solution is

$$\begin{aligned}\mathbf{y}(t) &= \left(-\frac{1}{2} - 3t\right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 3t \\ 1 + 6t \end{pmatrix}.\end{aligned}$$

39. From Exercise 33,

$$\mathbf{y}(t) = e^t \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} \right].$$

If $\mathbf{y}(0) = (5, 3)^T$, then

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces,

$$\begin{pmatrix} 1 & -1/3 & 5 \\ 3 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \end{pmatrix},$$

and $C_1 = 1$ and $C_2 = -12$. Thus, the particular solution is

$$\begin{aligned}\mathbf{y}(t) &= e^t \left[(1 - 12t) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 12 \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} 5 - 12t \\ 3 - 36t \end{pmatrix}.\end{aligned}$$

40. From Exercise 34,

$$\mathbf{y}(t) = e^{3t} \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right].$$

If $\mathbf{y}(0) = (0, 2)^T$, then

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces,

$$\begin{pmatrix} 1 & 1/2 & 0 \\ -2 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

and $C_1 = -1$ and $C_2 = 2$. Thus, the particular solution is

$$\begin{aligned}\mathbf{y}(t) &= e^{3t} \left[(-1 + 2t) \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 2 \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right] \\ &= e^{3t} \begin{pmatrix} 2t \\ 2 - 4t \end{pmatrix}.\end{aligned}$$

41. The matrix

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 - 8\lambda + 16$ and one eigenvalue, $\lambda = 4$. Moreover, the nullspace of

$$A - 4I = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_1 = (2, 1)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

To find another solution, we need to find a vector $\mathbf{v}_2$ which satisfies $(A - 4I)\mathbf{v}_2 = \mathbf{v}_1$. Choose $\vec{w} = (1, 0)^T$, which is independent of $\mathbf{v}_1$, and note that

$$(A - 4I)\vec{w} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -\mathbf{v}_1.$$

Thus, choose $\mathbf{v}_2 = -\vec{w} = (-1, 0)^T$. Our second solution is

$$\begin{aligned}\mathbf{y}_2(t) &= e^{4t}(\mathbf{v}_2 + t\mathbf{v}_1) \\ &= e^{4t} \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right].\end{aligned}$$

Thus, the general solution can be written

$$\begin{aligned}\mathbf{y}(t) &= C_1 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{4t} \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] \\ &= e^{4t} \left[(C_1 + C_2 t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]\end{aligned}$$

42. The matrix

$$A = \begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 + \lambda - 6$ with eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$. The nullspace of

$$A + 3I = \begin{pmatrix} -5 & -10 \\ 5 & 10 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_1 = (-2, 1)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The nullspace of

$$A - 2I = \begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_2 = (1, -1)^T$, with corresponding solution

$$\mathbf{y}_2(t) = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus, the general solution can be written

$$\mathbf{y}(t) = C_1 e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

43. The matrix

$$A = \begin{pmatrix} 5 & 12 \\ -4 & -9 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 + 4\lambda + 3$ and eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. The nullspace of

$$A - (-1)I = \begin{pmatrix} 6 & 12 \\ -4 & -8 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_1 = (-2, 1)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The nullspace of

$$A - (-3)I = \begin{pmatrix} 8 & 12 \\ -4 & -6 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_2 = (-3/2, 1)^T$, with corresponding solution

$$\mathbf{y}_2(t) = e^{-3t} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}.$$

Thus, the general solution can be written

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}.$$

$$A = \begin{pmatrix} -6 & 1 \\ 0 & -6 \end{pmatrix}$$

has repeated eigenvalue, $\lambda = -6$, but the nullspace of

$$A + 6I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_1 = (1, 0)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{-6t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We need a solution $\mathbf{v}_2$ satisfying $(A + 6I)\mathbf{v}_2 = \mathbf{v}_1$. Choose $\mathbf{w} = (0, 1)$, which is independent of $\mathbf{v}_1$, and note that

$$(A + 6I)\mathbf{w} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{v}_1.$$

Thus, choose $\mathbf{v}_2 = (0, 1)^T$, giving a second solution

$$\mathbf{y}_2(t) = e^{-6t}(\mathbf{v}_1 + t\mathbf{v}_2) = e^{-6t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$

Thus, the general solution can be written

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{-6t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-6t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &= e^{-6t} \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]. \end{aligned}$$

45. The matrix

$$A = \begin{pmatrix} -4 & -5 \\ 2 & 2 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 + 2\lambda + 2$ and eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. The nullspace of

$$A - (-1 + i)I = \begin{pmatrix} -3 - i & -5 \\ 2 & 3 - i \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_1 = (5, -3 - i)^T$, with corresponding solution

$$\mathbf{z}(t) = e^{(-1+i)t} \begin{pmatrix} 5 \\ -3 - i \end{pmatrix}.$$

Breaking this solution into real and imaginary parts,

$$\begin{aligned}
 \mathbf{z}(t) &= e^{(-1+i)t} \begin{pmatrix} 5 \\ -3-i \end{pmatrix} \\
 &= e^{-t}(\cos t + i \sin t) \left[\begin{pmatrix} 5 \\ -3 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
 &= e^{-t} \left[\cos t \begin{pmatrix} 5 \\ -3 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right. \\
 &\quad \left. + i \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + i \sin t \begin{pmatrix} 5 \\ -3 \end{pmatrix} \right] \\
 &= e^{-t} \begin{pmatrix} 5 \cos t \\ -3 \cos t + \sin t \end{pmatrix} \\
 &\quad + i e^{-t} \begin{pmatrix} 5 \sin t \\ -\cos t - 3 \sin t \end{pmatrix}.
 \end{aligned}$$

Thus, the general solution is

$$\begin{aligned}
 \mathbf{y}(t) &= C_1 e^{-t} \begin{pmatrix} 5 \cos t \\ -3 \cos t + \sin t \end{pmatrix} \\
 &\quad + C_2 e^{-t} \begin{pmatrix} 5 \sin t \\ -\cos t - 3 \sin t \end{pmatrix}.
 \end{aligned}$$

46. The matrix

$$A = \begin{pmatrix} -6 & 4 \\ -8 & 2 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 + 4\lambda + 20$ with eigenvalues $-2 \pm 4i$. Trusting that

$$A - (-2 + 4i)I = \begin{pmatrix} -4 - 4i & 4 \\ -8 & 4 - 4i \end{pmatrix}$$

is nonsingular, examination of the first row shows that $\mathbf{v} = (1, 1 + i)^T$ is an eigenvector. Thus,

$$\begin{aligned}
 \mathbf{z} &= e^{(-2+4i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \\
 &= e^{-2t}(\cos 4t + i \sin 4t) \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\
 &= e^{-2t} \left[\cos 4t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 4t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right. \\
 &\quad \left. + i \cos 4t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \sin 4t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]
 \end{aligned}$$

is a complex solution. The real and imaginary parts of $\mathbf{z}$ form a fundamental set of solutions that lead to

the general solution

$$\begin{aligned}
 \mathbf{y}(t) &= e^{-2t} \left(C_1 \begin{pmatrix} \cos 4t \\ \cos 4t - \sin 4t \end{pmatrix} \right. \\
 &\quad \left. + C_2 \begin{pmatrix} \sin 4t \\ \cos 4t + \sin 4t \end{pmatrix} \right)
 \end{aligned}$$

47. The matrix

$$A = \begin{pmatrix} -10 & 4 \\ -12 & 4 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 + 6\lambda + 8$ and eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -2$. The nullspace of

$$A - (-4)I = \begin{pmatrix} -6 & 4 \\ -12 & 8 \end{pmatrix}$$

is generated by the single eigenvector, $\mathbf{v}_1 = (2, 3)^T$, with corresponding solution

$$\mathbf{y}_1(t) = e^{-4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

The nullspace of

$$A - (-2)I = \begin{pmatrix} -8 & 4 \\ -12 & 6 \end{pmatrix}$$

is generated by a single eigenvector, $\mathbf{v}_2 = (1, 2)^T$, with corresponding solution

$$\mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{-4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

48. The matrix

$$A = \begin{pmatrix} -1 & 5 \\ -5 & -1 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = \lambda^2 + 2\lambda + 26$ with eigenvalues $-1 \pm 5i$. Trusting that

$$A - (-1 + 5i)I = \begin{pmatrix} -5i & 5 \\ -5 & -5i \end{pmatrix}$$

is nonsingular, examination of the first row shows that $\mathbf{v} = (1, i)^T$ is an eigenvector. Thus,

$$\begin{aligned}\mathbf{z} &= e^{(-1+5i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= e^{-t}(\cos 5t + i \sin 5t) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-t} \left(\cos 5t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin 5t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &\quad + i e^{-t} \left(\cos 5t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin 5t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right),\end{aligned}$$

is a complex solution. The real and imaginary parts of $\mathbf{z}$ form a fundamental set of solutions that lead to the general solution

$$\mathbf{y}(t) = e^{-t} \left(C_1 \begin{pmatrix} \cos 5t \\ -\sin 5t \end{pmatrix} + C_2 \begin{pmatrix} \sin 5t \\ \cos 5t \end{pmatrix} \right).$$

49. From Exercise 41, the general solution is

$$\mathbf{y}(t) = e^{4t} \left[(C_1 + C_2 t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right].$$

Because $\mathbf{y}(0) = (3, 1)^T$,

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Reduce the augmented matrix.

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Thus, $C_1 = 1$ and $C_2 = -1$ and the particular solution is

$$\begin{aligned}\mathbf{y}(t) &= e^{4t} \left[(1-t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= e^{4t} \begin{pmatrix} 3-2t \\ 1-t \end{pmatrix}.\end{aligned}$$

50. From Exercise 42, the general solution is

$$\mathbf{y}(t) = C_1 e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Because $\mathbf{y}(0) = (3, 1)^T$,

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Reduce the augmented matrix

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \end{pmatrix}.$$

Thus, $C_1 = -4$ and $C_2 = -5$ and the particular solution is

$$\begin{aligned}\mathbf{y}(t) &= -4e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} - 5e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 8e^{-3t} - 5e^{2t} \\ -4e^{-3t} + 5e^{2t} \end{pmatrix}.\end{aligned}$$

51. From Exercise 43, the general solution is

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}.$$

Because $\mathbf{y}(0) = (1, 0)^T$,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}.$$

Reduce the augmented matrix.

$$\begin{pmatrix} -2 & -3/2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

Thus, $C_1 = -2$ and $C_2 = 2$ and the particular solution is

$$\begin{aligned}\mathbf{y}(t) &= -2e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 2e^{-3t} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 4e^{-t} - 3e^{-3t} \\ -2e^{-t} + 2e^{-3t} \end{pmatrix}.\end{aligned}$$

52. From Exercise 44, the general solution is

$$\mathbf{y}(t) = e^{-6t} \left[(C_1 + C_2 t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

Because $\mathbf{y}(0) = (1, 0)^T$,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is easy to see that $C_1 = 1$ and $C_2 = 0$ and the particular solution is

$$\mathbf{y}(t) = e^{-6t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-6t} \\ 0 \end{pmatrix}.$$

53. From Exercise 45, the general solution is

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} 5 \cos t \\ -3 \cos t + \sin t \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 5 \sin t \\ -\cos t - 3 \sin t \end{pmatrix}.$$

Because $\mathbf{y}(0) = (-3, 2)^T$,

$$\begin{pmatrix} -3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 5 \\ -3 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Reduce the augmented matrix.

$$\left(\begin{array}{cc|c} 5 & 0 & -3 \\ -3 & -1 & 2 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & -3/5 \\ 0 & 1 & -1/5 \end{array} \right)$$

Thus, $C_1 = -3/5$ and $C_2 = -1/5$ and the particular solution is

$$\begin{aligned} \mathbf{y}(t) &= -\frac{3}{5} e^{-t} \begin{pmatrix} 5 \cos t \\ -3 \cos t + \sin t \end{pmatrix} - \frac{1}{5} e^{-t} \begin{pmatrix} 5 \sin t \\ -\cos t - 3 \sin t \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} -3 \cos t - \sin t \\ 2 \cos t \end{pmatrix}. \end{aligned}$$

54. From Exercise 46, the general solution is

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} \cos 4t \\ \cos 4t - \sin 4t \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} \sin 4t \\ \cos 4t + \sin 4t \end{pmatrix}.$$

Because $\mathbf{y}(0) = (4, 0)^T$,

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Reduce the augmented matrix.

$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 1 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -4 \end{array} \right)$$

Thus, $C_1 = 4$ and $C_2 = -4$ and the particular solution is

$$\begin{aligned} \mathbf{y}(t) &= 4e^{-2t} \begin{pmatrix} \cos 4t \\ \cos 4t - \sin 4t \end{pmatrix} - 4e^{-2t} \begin{pmatrix} \sin 4t \\ \cos 4t + \sin 4t \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} 4 \cos 4t - 4 \sin 4t \\ -8 \sin 4t \end{pmatrix}. \end{aligned}$$

55. From Exercise 47, the general solution is

$$\mathbf{y}(t) = C_1 e^{-4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Because $\mathbf{y}(0) = (2, 1)^T$,

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Reduce the augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 1 & 2 \\ 3 & 2 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \end{array} \right)$$

Thus, $C_1 = 3$ and $C_2 = -4$ and the particular solution is

$$\begin{aligned} \mathbf{y}(t) &= 3e^{-4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 4e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 6e^{-4t} - 4e^{-2t} \\ 9e^{-4t} - 8e^{-2t} \end{pmatrix} \end{aligned}$$

56. From Exercise 48, the general solution is

$$\mathbf{y}(t) = e^{-t} \left[C_1 \begin{pmatrix} \cos 5t \\ -\sin 5t \end{pmatrix} + C_2 \begin{pmatrix} \sin 5t \\ \cos 5t \end{pmatrix} \right].$$

Because $\mathbf{y}(0) = (5, 5)^T$,

$$\begin{pmatrix} 5 \\ 5 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, $C_1 = 5$ and $C_2 = 5$ and the particular solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{-t} \left[5 \begin{pmatrix} \cos 5t \\ -\sin 5t \end{pmatrix} + 5 \begin{pmatrix} \sin 5t \\ \cos 5t \end{pmatrix} \right] \\ &= 5e^{-t} \begin{pmatrix} \cos 5t + \sin 5t \\ \cos 5t - \sin 5t \end{pmatrix}. \end{aligned}$$

57. (a) Let

$$(A - \lambda I)^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and assume that $(A - \lambda I)^2 \mathbf{v} = \mathbf{0}$ for all $\mathbf{v}$ in $\mathbf{R}$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so $(A - \lambda I)^2 = 0I$.

- (b) Let $\mathbf{v}_1$ be an eigenvector of A associated with the eigenvalue λ . Note that this means that $A\mathbf{v}_1 - \lambda\mathbf{v}_1 = \mathbf{0}$. Let $\mathbf{v} = \alpha\mathbf{v}_1$ be a multiple of the eigenvector $\mathbf{v}_1$. Then

$$\begin{aligned} (A - \lambda I)^2 \mathbf{v} &= (A - \lambda I)^2 (\alpha\mathbf{v}_1) \\ &= \alpha(A - \lambda I)^2 \mathbf{v}_1 \\ &= \alpha(A - \lambda I)(A\mathbf{v}_1 - \lambda\mathbf{v}_1) \\ &= \alpha(A - \lambda I)\mathbf{0} \\ &= \alpha\mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

- (c) Now choose $\mathbf{v}$ in $\mathbf{R}^2$ such that $\mathbf{v}$ is **not** a multiple of the eigenvector $\mathbf{v}_1$. Note that this means that $\mathbf{v}$ is not an eigenvector associated with the eigenvalue λ . The set $B = \{\mathbf{v}, \mathbf{v}_1\}$ is independent with dimension two. Therefore, it must span all of $\mathbf{R}^2$ and is a basis for $\mathbf{R}^2$.
- (d) Set $\mathbf{w} = (A - \lambda I)\mathbf{v}$. Note that this means that $\mathbf{w}$ is nonzero, for otherwise $\mathbf{v}$ would be an eigenvector associated with the eigenvalue λ . In part (c), we saw that $B = \{\mathbf{v}, \mathbf{v}_1\}$ was a basis for $\mathbf{R}^2$. Thus, B spans $\mathbf{R}^2$ and we can find a and b such that

$$\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}.$$

- (e) From (d), $\mathbf{w} = (A - \lambda I)\mathbf{v}$ and $\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}$. Thus,

$$\begin{aligned} (A - \lambda I)\mathbf{w} &= (A - \lambda I)(a\mathbf{v}_1 + b\mathbf{v}) \\ &= a(A - \lambda I)\mathbf{v}_1 + b(A - \lambda I)\mathbf{v} \\ &= \mathbf{0} + b\mathbf{w} \\ &= b\mathbf{w}, \end{aligned}$$

Hence,

$$\begin{aligned} (A - \lambda I)\mathbf{w} &= b\mathbf{w} \\ A\mathbf{w} - \lambda\mathbf{w} &= b\mathbf{w} \\ A\mathbf{w} &= (\lambda + b)\mathbf{w}. \end{aligned}$$

Thus, $\mathbf{w}$, being nonzero, is an eigenvector of A with eigenvalue $\lambda + b$. But, λ is the only eigenvalue, so b must equal zero and $\mathbf{w}$ must be a multiple of $\mathbf{v}_1$.

- (f) Finally, because $b = 0$ and $(A - \lambda I)\mathbf{v} = \mathbf{w}$,

$$\begin{aligned} (A - \lambda I)^2 \mathbf{v} &= (A - \lambda I)(A - \lambda I)\mathbf{v} \\ &= (A - \lambda I)\mathbf{w} \\ &= b\mathbf{w} \\ &= \mathbf{0}. \end{aligned}$$

Consequently, whether $\mathbf{v}$ is a multiple of $\mathbf{v}_1$ or not, $(A - \lambda I)^2 \mathbf{v} = \mathbf{0}$. Since this is true for any arbitrary $\mathbf{v}$ in $\mathbf{R}^2$, by part (a), $(A - \lambda I)^2 = 0I$.

58. (a) Let $x_1(t)$ and $x_2(t)$ represent the number of pounds of salt as a function of time in the upper and lower tanks, respectively. The rate at which salt is changing the first tank is

$$\frac{dx_1}{dt} = \text{Rate In} - \text{Rate out} = -\frac{x_1}{100}.$$

The rate at which salt content is changing in the lower tank is

$$\frac{dx_2}{dt} = \text{Rate In} - \text{Rate out} = \frac{x_1}{100} - \frac{x_2}{100}.$$

In matrix-vector form, this becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1/100 & 0 \\ 1/100 & -1/100 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The initial condition is $\mathbf{x}(0) = (x_1(0), x_2(0))^T = (100, 0)^T$, as there is initially 100 pounds of salt in the first tank, but none in the lower tank.

- (b) With the trace $T = -2/100 = -1/50$ and the determinant $D = (-1/100)^2$, the characteristic polynomial becomes

$$\begin{aligned} p(\lambda) &= \lambda^2 + \frac{2}{100}\lambda + \left(-\frac{1}{100}\right)^2 \\ &= \left(\lambda + \frac{1}{100}\right)^2. \end{aligned}$$

Thus, we have a single repeated eigenvalue, $\lambda = -1/100$, of algebraic multiplicity 2. To find the eigenvector, we examine the nullspace of

$$A + \lambda I = A + \frac{1}{100}I = \begin{pmatrix} 0 & 0 \\ 1/100 & 0 \end{pmatrix}.$$

Clearly, $\mathbf{v}_1 = (0, 1)^T$ will work as an eigenvector. Consequently, one solution is $\mathbf{x}_1(t) = e^{-t/100}(1, 0)^T$. To find a second solution, start with any vector independent of $\mathbf{v}_1$, say $\mathbf{w} = (1, 0)^T$. Then,

$$\begin{aligned} (A - \lambda I)\mathbf{w} &= \left(A + \frac{1}{100}I\right)\mathbf{w} \\ &= \begin{pmatrix} 0 & 0 \\ 1/100 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.1) \\ &= \frac{1}{100} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{100}\mathbf{v}_1. \end{aligned}$$

Thus, choose $\mathbf{v}_2 = 100\mathbf{w} = (100, 0)^T$, and a second independent solution is

$$\begin{aligned} \mathbf{x}_2 &= e^{-\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1] \\ &= e^{-t/100} \left[\begin{pmatrix} 100 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-t/100} \begin{pmatrix} 100 \\ t \end{pmatrix}. \end{aligned}$$

Thus, the general solution is

$$\mathbf{x}(t) = e^{-t/100} \left[C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 100 \\ t \end{pmatrix} \right]. \quad (2.2)$$

With the initial condition $\mathbf{x}(0) = (x_1(0), x_2(0))^T = (100, 0)^T$,

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \mathbf{x}(0) = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 100 \\ 0 \end{pmatrix},$$

which gives $C_1 = 0$ and $C_2 = 1$. Thus, the solution of the initial value problem is

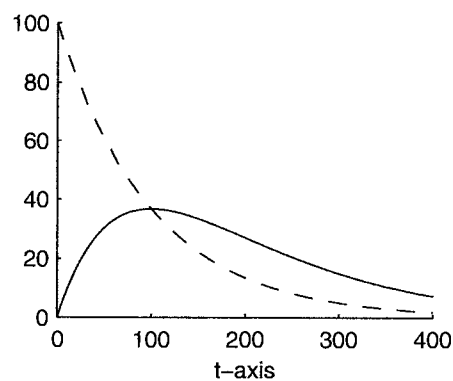
$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= \mathbf{x}(t) \\ &= e^{-t/100} \begin{pmatrix} 100 \\ t \end{pmatrix} \\ &= \begin{pmatrix} 100e^{-t/100} \\ te^{-t/100} \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= \mathbf{x}(t) = e^{-t/100} \begin{pmatrix} 100 \\ t \end{pmatrix} \\ &= \begin{pmatrix} 100e^{-t/100} \\ te^{-t/100} \end{pmatrix}. \end{aligned}$$

- (c) Thus, the component solutions are

$$\begin{aligned} x_1 &= 100e^{-t/100} \\ x_2 &= te^{-t/100}, \end{aligned}$$

and the plot of the component solutions (the salt content in each tank) is shown below. Note that the time constant is the amount of time required for the salt content to be e^{-1} of what it was at the beginning of the time period. Thus, $T_c/100 = 1$, or $T_c = 100$. The plot of the salt content is shown over four time constants, enough time for the salt content to decay to e^{-4} of what it was in the first tank.



Note that the salt content in each tank decays to zero. This makes physical sense, because if only pure water enters the first tank, eventually, all salt will be flushed out of both tanks. Mathematically, both component functions

$$100e^{-t/100} \rightarrow 0$$

$$te^{-t/100} \rightarrow 0,$$

as $t \rightarrow +\infty$. The second fact can be shown by L'Hopital's rule.

59. (a) Let $x_A(t)$ and $x_B(t)$ represent the number of pounds of salt as a function of time in the left and right tanks, respectively. The rate at which salt is changing the left tank is

$$\frac{dx_A}{dt} = \text{Rate In} - \text{Rate out} = -\frac{x_A}{40} + \frac{x_B}{90}.$$

The rate at which salt content is changing in the right tank is

$$\frac{dx_B}{dt} = \text{Rate In} - \text{Rate out} = \frac{x_A}{40} - \frac{x_B}{40}.$$

In matrix-vector form, this becomes

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix}' = \begin{pmatrix} -1/40 & 1/90 \\ 1/40 & -1/40 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix}.$$

The initial condition is $\mathbf{x}(0) = (x_A(0), x_B(0))^T = (60, 0)^T$, as there is initially 60 pounds of salt in the left tank, but none in the right tank.

- (b) The trace of the coefficient matrix is $T = -2/40 = -1/20$ and the determinant is $D = 5/14400$. Thus, the characteristic polynomial is

$$p(\lambda) = \lambda^2 + \frac{1}{20}\lambda + \frac{5}{14400}$$

$$= \left(\lambda + \frac{1}{120}\right)\left(\lambda + \frac{1}{24}\right),$$

so the eigenvalues are $\lambda = -1/120$ and $\lambda = -1/24$. With the first eigenvalue $\lambda = -1/120$,

$$A + \frac{1}{120}I = \begin{pmatrix} -1/60 & 1/90 \\ 1/40 & -1/60 \end{pmatrix}$$

$$\xrightarrow{\text{ref}} \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix}.$$

Thus, $(2/3, 1)^T$ will serve as an eigenvector. As any multiple of an eigenvalue is also an eigenvalue, let's choose $\mathbf{v}_1 = (2, 3)^T$. This leads to the solution $\mathbf{x}_1 = e^{-t/120}(2, 3)^T$. Next, with the second eigenvalue $\lambda = -1/24$,

$$A + \frac{1}{24}I = \begin{pmatrix} 1/60 & 1/90 \\ 1/40 & 1/60 \end{pmatrix}$$

$$\xrightarrow{\text{ref}} \begin{pmatrix} 1 & 2/3 \\ 0 & 0 \end{pmatrix}.$$

Thus, $(-2/3, 1)^T$ is an eigenvector, but we again choose a multiple, $\mathbf{v}_2 = (-2, 3)^T$. Thus, a second independent solution (distinct eigenvalues) is $\mathbf{x}_2 = e^{-t/24}(-2, 3)^T$. Thus, the general solution is

$$\mathbf{x} = C_1 e^{-t/120} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-t/24} \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

The initial condition provides

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix} = \mathbf{x}(0) = C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Solving, $C_1 = 15$ and $C_2 = -15$, so the solution of the initial value problem is

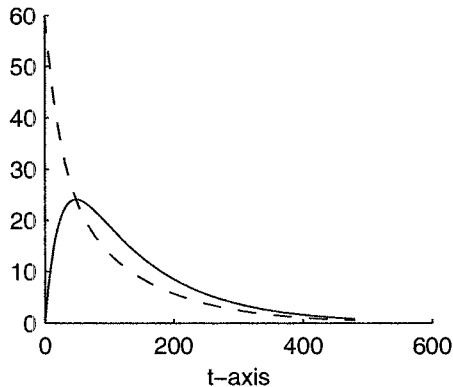
$$\mathbf{x} = 15e^{-t/120} \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 15e^{-t/24} \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

- (c) The component solutions are

$$x_A = 30e^{-t/120} + 30e^{-t/24}$$

$$x_B = 45e^{-t/120} - 45e^{-t/24}.$$

The plot of the salt content in each tank is shown below.



Note that the time constant on the exponential $e^{-t/24}$ is $T_c = 24$, but on the exponential $e^{-t/120}$, the time constant is $T_c = 120$. We chose the larger of these two time constants and made our plot over four time constants $[0, T_c] = [0, 480]$. This allows enough time to show both components decaying to zero. Physically, our intuition tells us that if we keep pouring pure water into the leftmost tank, eventually the system will purge itself of all salt content. Mathematically, both

$$\begin{aligned} 30e^{-t/120} + 30e^{-t/24} &\rightarrow 0 \\ 45e^{-t/120} - 45e^{-t/24} &\rightarrow 0, \end{aligned}$$

as $t \rightarrow \infty$.

60. Substitute $R = 1/2$, $C = 1$, and $L = 1/2$ in

$$\begin{aligned} V' &= -\frac{V}{RC} - \frac{I}{C} \\ I' &= \frac{V}{L} \end{aligned}$$

to obtain, in matrix-vector form,

$$\begin{pmatrix} V \\ I \end{pmatrix}' = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

The trace is $T = -2$, the determinant is $D = 2$, leading to the characteristic polynomial

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 2\lambda + 2.$$

The quadratic formula provides the roots (eigenvalues) $\lambda = -1 \pm i$. Consider,

$$A + \lambda I = \begin{pmatrix} -2 + \lambda & -1 \\ 2 & \lambda \end{pmatrix}.$$

Trusting that the matrix $A + \lambda I$ is singular, we see that the vector $(\lambda, -2)^T$ will zero out the second row of $A + \lambda I$. Thus, using $\lambda = -1 + i$, an associated eigenvector is $\mathbf{v} = (-1 + i, 2)^T$, leading to the complex solution $\mathbf{z} = e^{(-1+i)t}(-1+i, 2)^T$. Using Euler's identity, this can be broken into real and imaginary parts.

$$\begin{aligned} \mathbf{z}(t) &= e^{-t}(\cos t + i \sin t) \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &= e^{-t} \left[\cos t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &\quad + i e^{-t} \left[\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right] \end{aligned}$$

The general solution is a linear combination of the real and imaginary parts.

$$\begin{aligned} \mathbf{x}(t) &= C_1 e^{-t} \begin{pmatrix} -\cos t - \sin t \\ 2 \cos t \end{pmatrix} \\ &\quad + C_2 e^{-t} \begin{pmatrix} \cos t - \sin t \\ 2 \sin t \end{pmatrix}. \end{aligned}$$

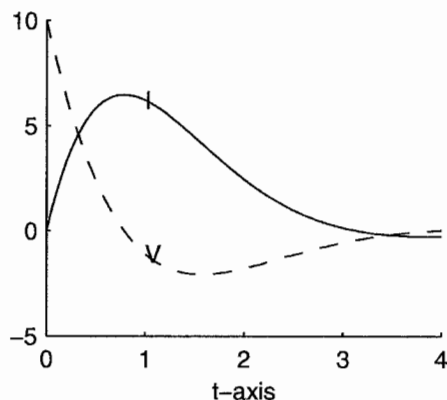
Using the initial condition,

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = \mathbf{x}(0) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

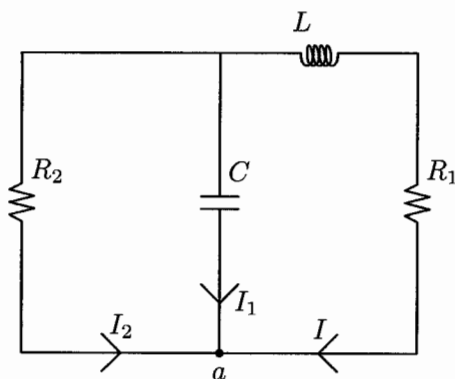
obtaining $C_1 = 0$ and $C_2 = 10$. Thus, the solution is

$$\begin{pmatrix} V(t) \\ I(t) \end{pmatrix} = \mathbf{x}(t) = \begin{pmatrix} 10e^{-t}(\cos t - \sin t) \\ 20e^{-t} \sin t \end{pmatrix}.$$

The time constant is $T_c = 1$, so we provide a solution over $[0, 4T_c] = [0, 4]$ in the following plot.



61. We've labeled current and added a note at a in the figure below.



The current coming into the node at a must equal the current coming out, so

$$I + I_1 + I_2 = 0.$$

Moving counterclockwise around the leftmost loop containing the resistor R_2 and capacitor,

$$-R_2 I_2 + V = 0.$$

However, $I_2 = -I - I_1$, so

$$-R_2(-I - I_1) + V = 0.$$

Now, the voltage across the capacitor follows the law $V = q_1/C$, where q_1 is the charge on the capacitor. Differentiating, $CV' = q_1' = I_1$. Substituting this in the last equation,

$$R_2 I + R_2 C V' + V = 0$$

$$V' = -\frac{1}{R_2 C} V - \frac{1}{C} I.$$

Passing around the loop (counterclockwise) at the far right, containing capacitor, inductor, and resistor R_1 ,

$$-V + LI' + R_1 I = 0$$

$$I' = \frac{1}{L} V - \frac{R_1}{L} I.$$

In matrix form, and using $C = 1$, $L = 1$, $R_1 = 5$, and $R_2 = 1$,

$$\begin{pmatrix} V \\ I \end{pmatrix}' = \begin{pmatrix} -1/(R_2 C) & -1/C \\ 1/L & -R_1/L \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

Now, the trace is $T = -6$, the determinant is $D = 6$, and the characteristic polynomial is

$$p(\lambda) = \lambda^2 + 6\lambda + 6.$$

The quadratic formula provides eigenvalues $\lambda = -3 \pm \sqrt{3}$, and

$$A - \lambda I = \begin{pmatrix} -1 - \lambda & -1 \\ 1 & -5 - \lambda \end{pmatrix}.$$

Trusting that $A - \lambda I$ is singular, we look for a vector that will zero the second row, so $\mathbf{v} = (5 + \lambda, 1)^T$. With $\lambda = -3 + \sqrt{3}$, this gives the solution $\mathbf{x}_1 = e^{(-3+\sqrt{3})t} (2 + \sqrt{3}, 1)^T$. With $\lambda = -3 - \sqrt{3}$, this gives the solution $\mathbf{x}_2 = e^{(-3-\sqrt{3})t} (2 - \sqrt{3}, 1)^T$. Thus, the general solution is

$$\mathbf{x} = C_1 e^{(-3+\sqrt{3})t} \begin{pmatrix} 2 + \sqrt{3} \\ 1 \end{pmatrix} + C_2 e^{(-3-\sqrt{3})t} \begin{pmatrix} 2 - \sqrt{3} \\ 1 \end{pmatrix}.$$

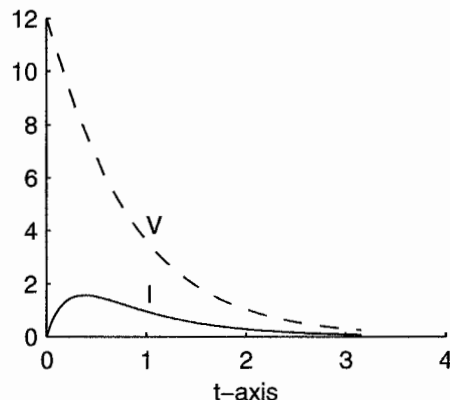
The initial voltage is 12 volts, the initial current is 0 amps, so

$$\begin{pmatrix} 12 \\ 0 \end{pmatrix} = \begin{pmatrix} V(0) \\ I(0) \end{pmatrix} = \mathbf{x}(0) \\ = C_1 \begin{pmatrix} 2 + \sqrt{3} \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 - \sqrt{3} \\ 1 \end{pmatrix}.$$

Solving, $C_1 = 2\sqrt{3}$ and $C_2 = -2\sqrt{3}$, giving the final solution $\mathbf{x}(t) = 2\sqrt{3}e^{(-3+\sqrt{3})t}(2 + \sqrt{3}, 1)^T - 2\sqrt{3}e^{(-3-\sqrt{3})t}(2 - \sqrt{3}, 1)^T$, which leads to the component solutions

$$V = (4\sqrt{3} + 6)e^{(-3+\sqrt{3})t} - (4\sqrt{3} - 6)e^{(-3-\sqrt{3})t} \\ I = 2\sqrt{3}e^{(-3+\sqrt{3})t} - 2\sqrt{3}e^{(-3-\sqrt{3})t}.$$

The solution is plotted below.



Section 9.3. Phase Plane Portraits

1. If

$$A = \begin{pmatrix} -10 & -25 \\ 5 & 10 \end{pmatrix},$$

then $T = 0$ and $D = 25$, leading to

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 25.$$

On the other hand,

$$p(\lambda) = \det(A - \lambda I) \\ = \begin{vmatrix} -10 - \lambda & -25 \\ 5 & 10 - \lambda \end{vmatrix} \\ = (-10 - \lambda)(10 - \lambda) + 125 \\ = \lambda^2 + 25.$$

2. If

$$A = \begin{pmatrix} 0 & 5 \\ -1 & 4 \end{pmatrix},$$

then $T = 4$ and $D = 5$, leading to

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 4\lambda + 5.$$

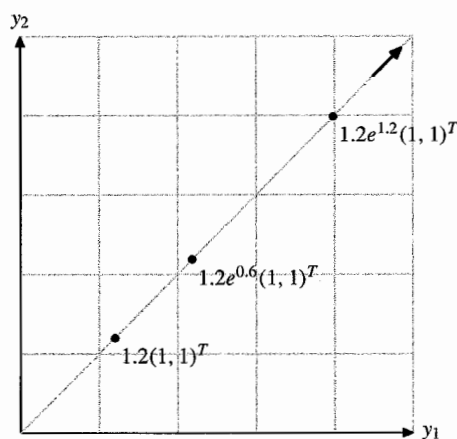
On the other hand,

$$p(\lambda) = \det(A - \lambda I) \\ = \begin{vmatrix} -\lambda & 5 \\ -1 & 4 - \lambda \end{vmatrix} \\ = -\lambda(4 - \lambda) + 5 \\ = \lambda^2 - 4\lambda + 5.$$

3. Position of

$$\mathbf{y}(t) = 1.2e^{0.6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

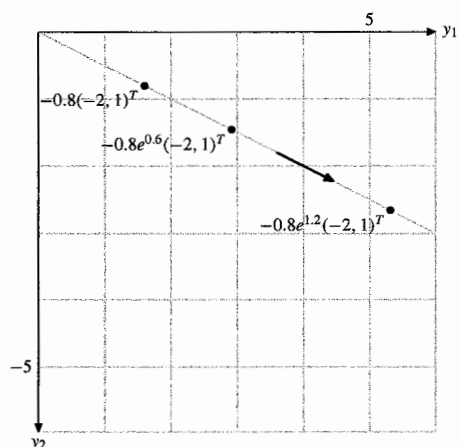
on halfline solution at times $t = 0, 1$, and 2 .



4. Position of

$$\mathbf{y}(t) = -0.8e^{0.6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

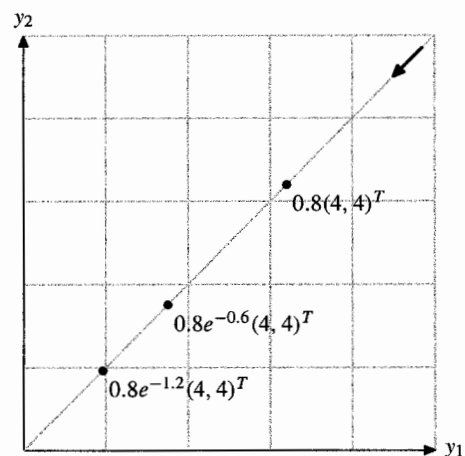
on halfline solution at times $t = 0, 1$, and 2 .



5. Position of

$$\mathbf{y}(t) = 0.8e^{-0.6t} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

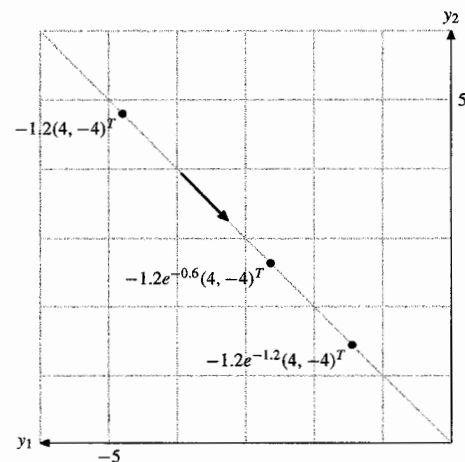
on halfline solution at times $t = 0, 1$, and 2 .



6. Position of

$$\mathbf{y}(t) = -1.2e^{-0.6t} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

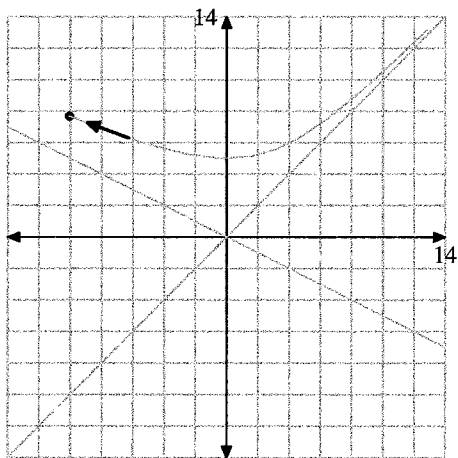
on halfline solution at times $t = 0, 1$, and 2 .



7. Position of

$$\mathbf{y}(t) = 0.8e^{0.4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 1.2e^{-0.2t} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

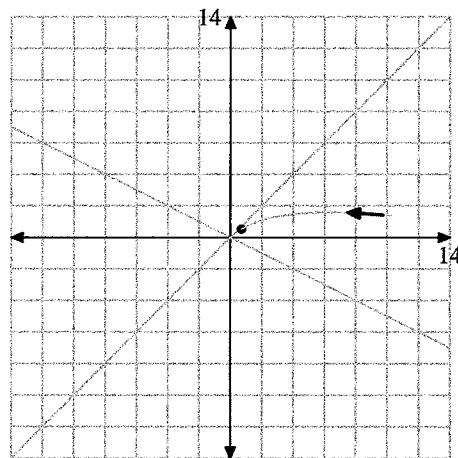
at times $t = -5, -4, \dots, 5$.



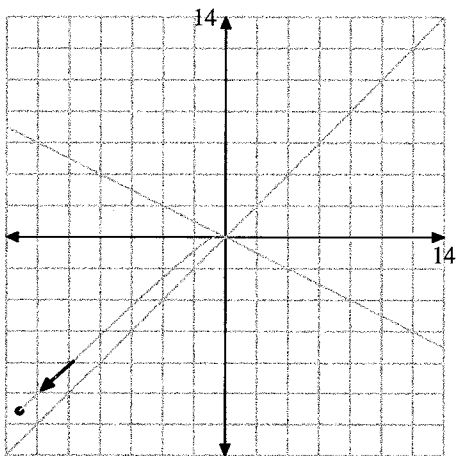
8. Position of

$$\mathbf{y}(t) = 0.4e^{0.1t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} - 0.4e^{0.4t} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

at times $t = -5, -4, \dots, 5$.



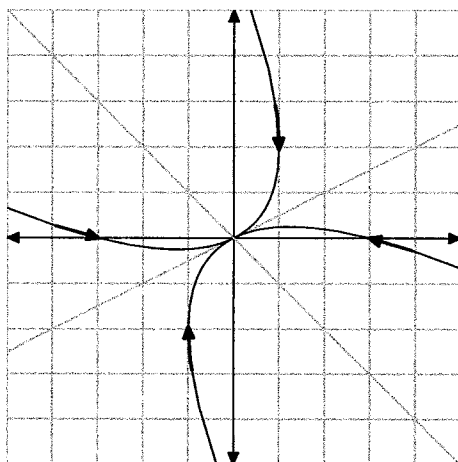
10. Both eigenvalues are negative, so the equilibrium point at the origin is a sink. Solutions dive toward the origin tangent to the slow exponential solution, $e^{-t}(2, 1)^T$. As solutions move backward in time, the eventually parallel the fast exponential solution, $e^{-2t}(-1, 1)^T$.



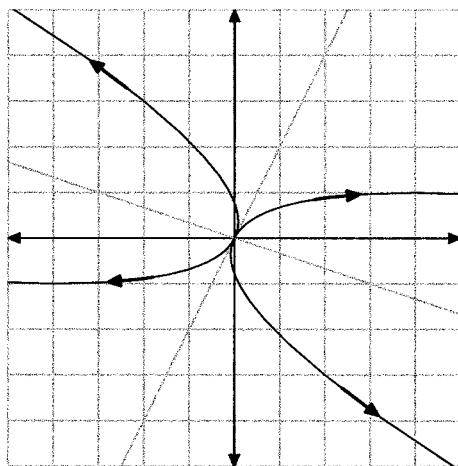
9. Position of

$$\mathbf{y}(t) = -0.4e^{-0.4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 0.4e^{-0.2t} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

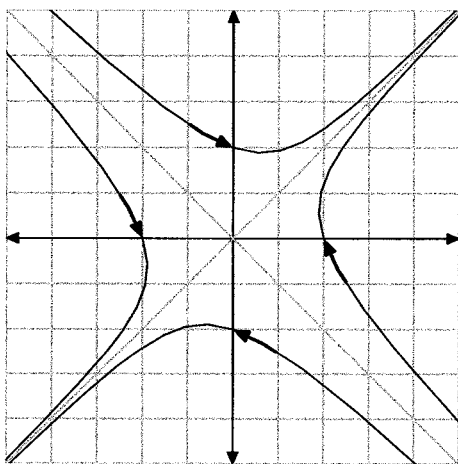
at times $t = -5, -4, \dots, 5$.



11. Both eigenvalues are positive, so the equilibrium point at the origin is a source. Solutions emanate from the origin tangent to the slow exponential solution, $e^t(-1, -2)^T$, eventually paralleling the fast exponential solution, $e^{2t}(3, -1)^T$.

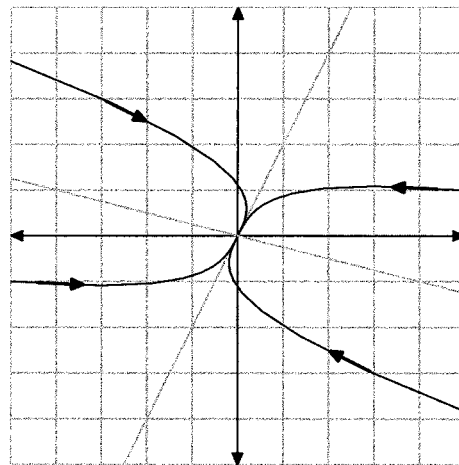


12. One eigenvalue is positive, the other negative, so the equilibrium point at the origin is a saddle. As $t \rightarrow +\infty$, solutions parallel the exponential solution $e^t(1, 1)^T$. As $t \rightarrow -\infty$, solutions parallel the exponential solution $e^{-2t}(1, -1)^T$.

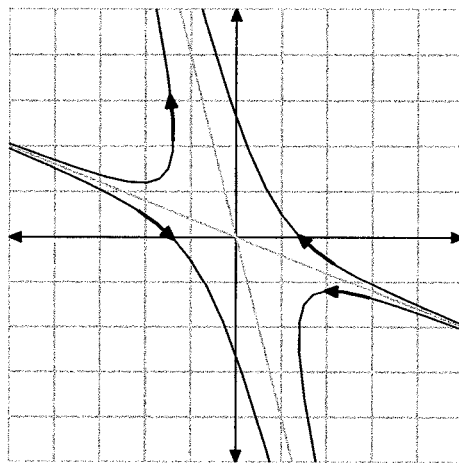


13. Both eigenvalues are negative, so the equilibrium point at the origin is a sink. Solutions dive toward the origin tangent to the slow exponential solution, $e^{-t}(1, 2)^T$. As solutions move backward in time,

they eventually parallel the fast exponential solution, $e^{-3t}(-4, 1)^T$.

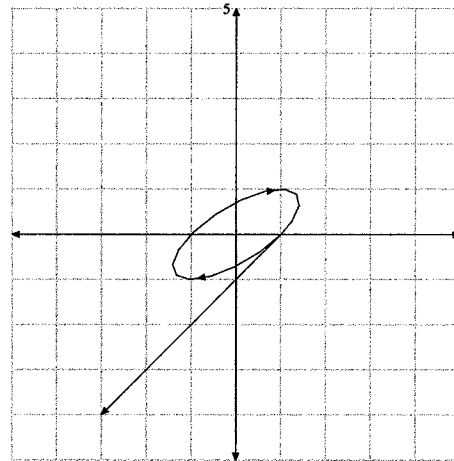
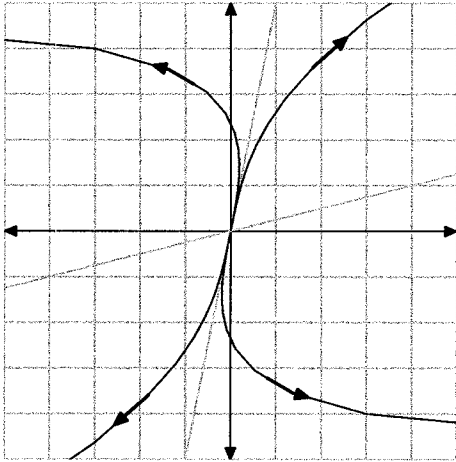


14. One eigenvalue is positive, the other negative, so the equilibrium point at the origin is a saddle. As $t \rightarrow +\infty$, solutions parallel the exponential solution $e^{2t}(-1, 4)^T$. As $t \rightarrow -\infty$, solutions parallel the exponential solution $e^{-t}(-5, 2)^T$.



15. Both eigenvalues are positive, so the equilibrium point at the origin is a source. Solutions emanate

from the origin tangent to the slow exponential solution, $e^t(1, 5)^T$, eventually paralleling the fast exponential solution, $e^{3t}(4, 1)^T$.



The phase portrait, drawn using a numeric solver, follows.

16. Matrix

$$A = \begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix}$$

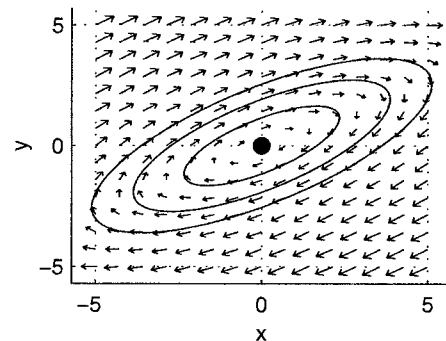
has trace $T = 0$ and determinant $D = 16$. Thus, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 16,$$

which produces eigenvalues $\lambda_1 = -4i$ and $\lambda_2 = 4i$. Because the real part of these eigenvalues is zero, the equilibrium point at the origin is a center. At $(1, 0)$,

$$\begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}.$$

Thus, the motion is clockwise. A hand sketch follows.



17. Matrix

$$A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

has trace $T = 0$ and determinant $D = 9$. Thus, the characteristic polynomial is

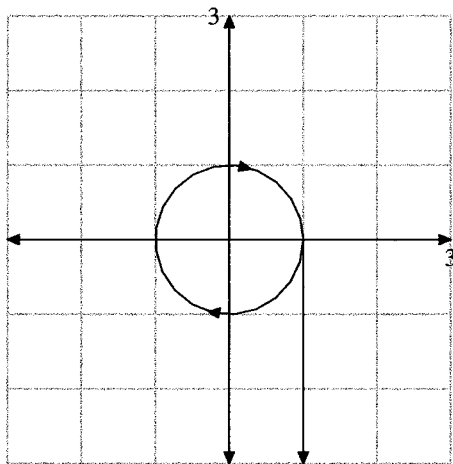
$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 9,$$

which produces eigenvalues $\lambda_1 = 3i$ and $\lambda_2 = -3i$. Because the real part of these eigenvalues is zero, the

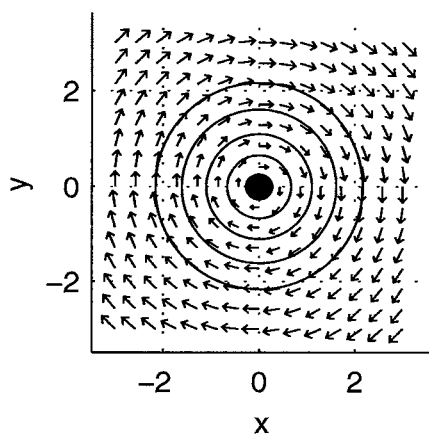
equilibrium point at the origin is a center. At $(1, 0)$, 18. Matrix

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

Thus, the motion is clockwise. A hand sketch follows.



The phase portrait, drawn using a numerical solver, follows.



$$A = \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix}$$

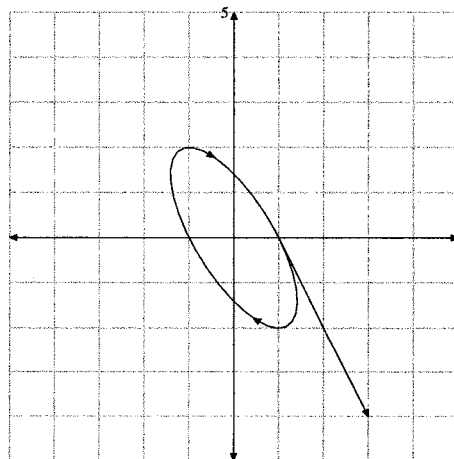
has trace $T = 0$ and determinant $D = 4$. Thus, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 4,$$

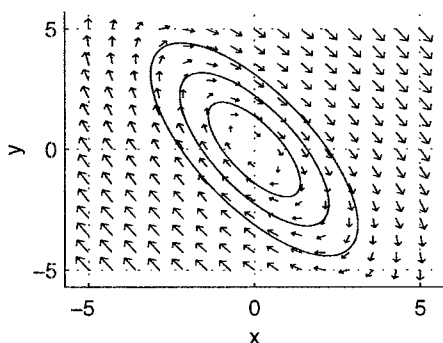
which produces eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Because the real part of these eigenvalues is zero, the equilibrium point at the origin is a center. At $(1, 0)$,

$$\begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

Thus, the motion is clockwise. A hand sketch follows.



The phase portrait, drawn using a numeric solver, follows.



19. Matrix

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

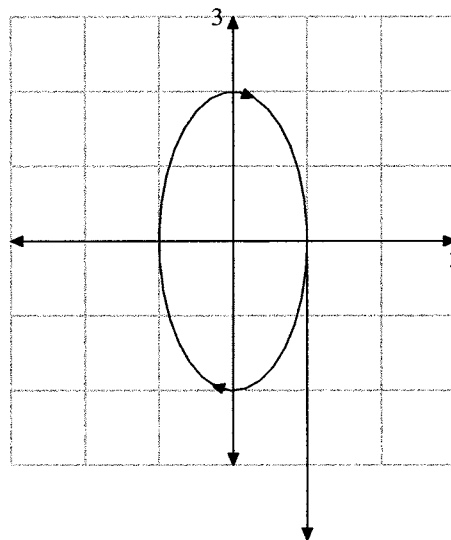
has trace $T = 0$ and determinant $D = 4$. Thus, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 4,$$

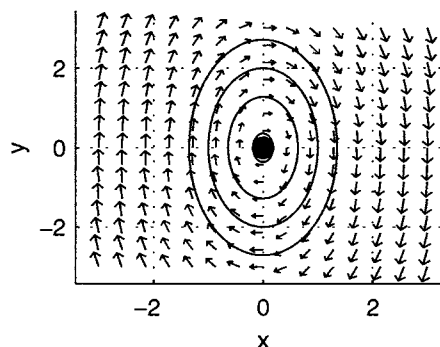
which produces eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Because the real part of these eigenvalues is zero, the equilibrium point at the origin is a center. At $(1, 0)$,

$$\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

Thus, the motion is clockwise. A hand sketch follows.



The phase portrait, drawn using a numerical solver, follows.



20. Matrix

$$A = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}$$

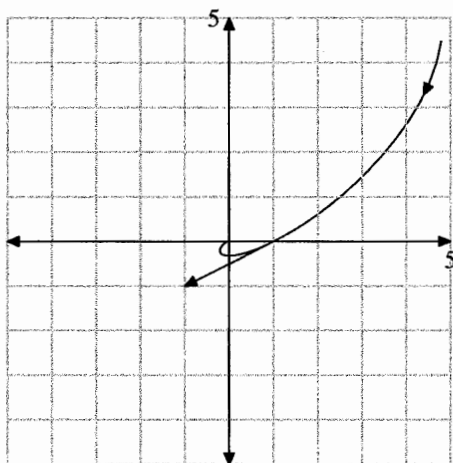
has trace $T = -2$ and determinant $D = 2$. thus, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 2\lambda + 2,$$

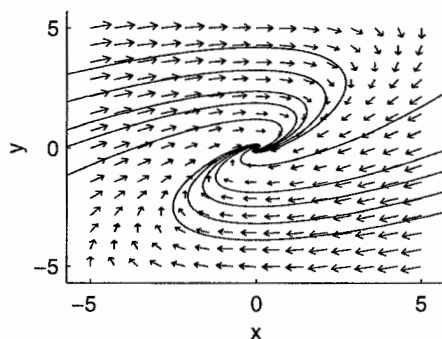
which produces eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Because the real part of the eigenvalues is negative, the equilibrium point at the origin is a spiral sink. At $(1, 0)$,

$$\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix},$$

so the motion is clockwise. A hand sketch follows.



The phase portrait, drawn in a numeric solver, follows.



21. Matrix

$$A = \begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix}$$

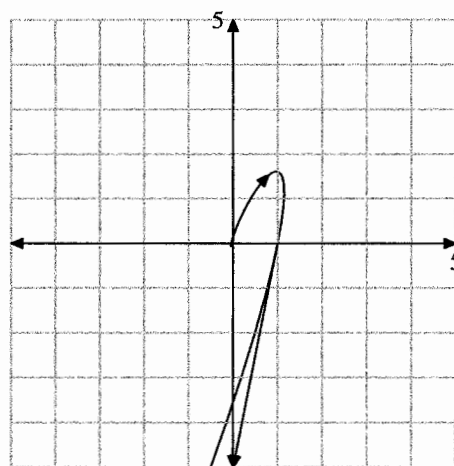
has trace $T = 2$ and determinant $D = 2$. Thus, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 2\lambda + 2,$$

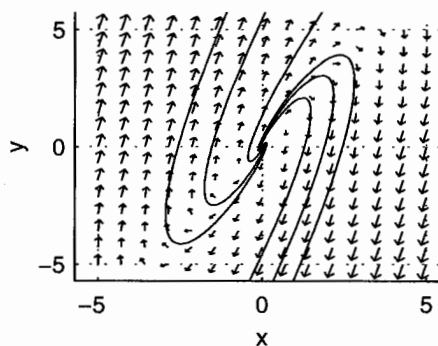
which produces eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. Because the real part of the eigenvalues is positive, the equilibrium point at the origin is a spiral source. At $(1, 0)$,

$$\begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix},$$

so the motion is clockwise. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



22. Matrix

$$A = \begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix}$$

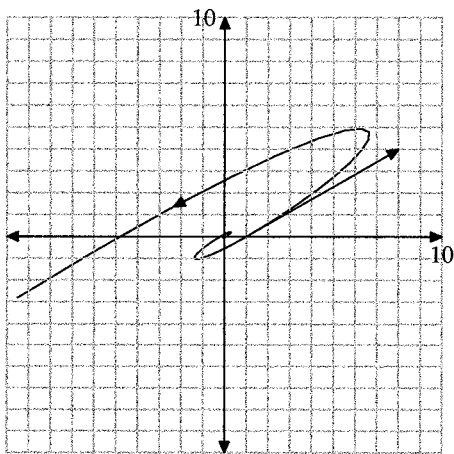
has trace $T = 2$ and determinant $D = 5$. Thus, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 2\lambda + 5,$$

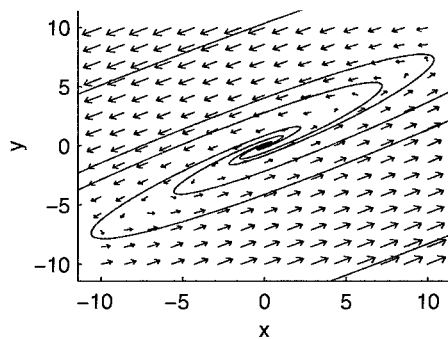
which produces eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Because the real part of the eigenvalues is positive, the equilibrium point at the origin is a spiral source. At $(1, 0)$,

$$\begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix},$$

so the motion is counterclockwise. A hand sketch follows.



The phase portrait, draw with a numerical solver, follows.



23. Matrix

$$A = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}$$

has trace $T = -2$ and determinant $D = 5$. Thus, the characteristic polynomial is

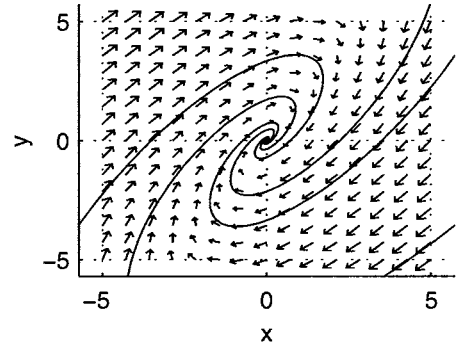
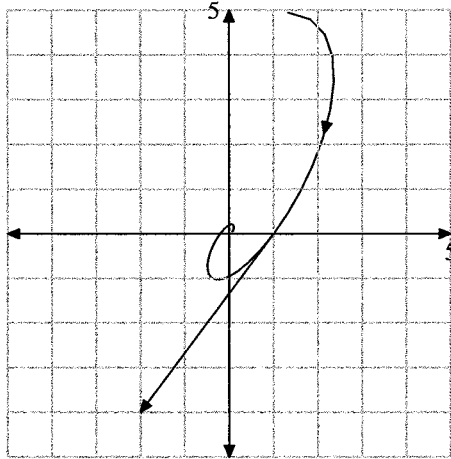
$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 2\lambda + 5,$$

which produces eigenvalues $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$. Because the real part of the eigenvalues is negative, the equilibrium point at the origin is a spiral sink. At $(1, 0)$,

$$\begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix},$$

so the motion is clockwise. A hand sketch follows.

The phase portrait, drawn in a numerical solver, follows.



Section 9.4. The Trace-Determinant Plane

1. If

$$A = \begin{pmatrix} 8 & 20 \\ -4 & -8 \end{pmatrix},$$

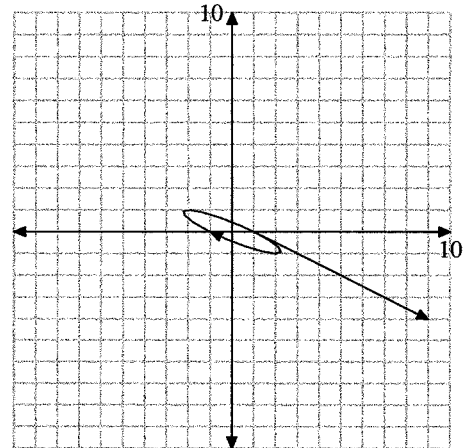
then the trace is $T = 0$ and the determinant is $D = 16$. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 16,$$

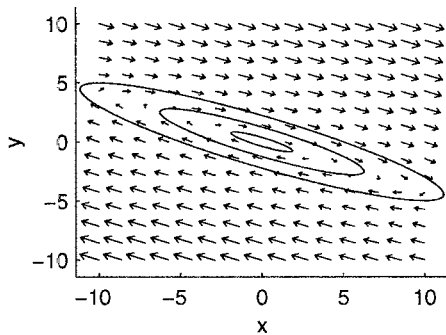
which produces eigenvalues $\lambda_1 = 4i$ and $\lambda_2 = -4i$. Therefore, the equilibrium point at the origin is a center. At $(1, 0)$,

$$\begin{pmatrix} 8 & 20 \\ -4 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix},$$

so the motion is clockwise. A hand sketch follows.



The phase portrait, drawn with a numerical solver, follows.



2. If

$$A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix},$$

then the trace is $T = -5$ and the determinant is $D = -14 < 0$. Hence, the equilibrium point at the origin is a saddle. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 5\lambda - 14,$$

which produces eigenvalues $\lambda_1 = -7$ and $\lambda_2 = 2$. Because

$$A + 7I = \begin{pmatrix} -9 & 9 \\ -18 & 18 \end{pmatrix},$$

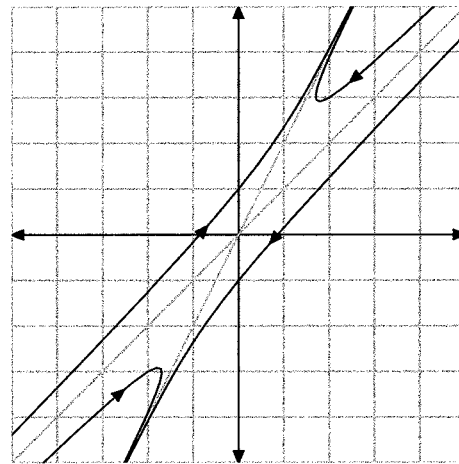
$\mathbf{v}_1 = (1, 1)^T$, leading to the exponential solution $e^{-7t}(1, 1)^T$. Because

$$A - 2I = \begin{pmatrix} -18 & 9 \\ -18 & 9 \end{pmatrix},$$

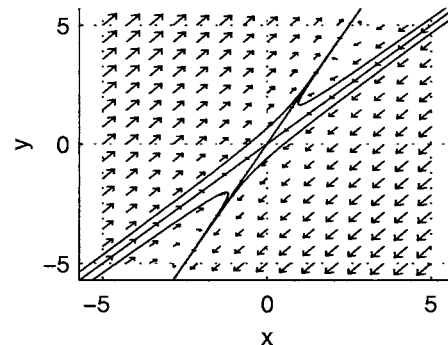
$\mathbf{v}_2 = (1, 2)^T$, leading to the exponential solution $e^{2t}(1, 2)^T$. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{-7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solutions approach the halfline generated by $C_2(1, 2)^T$ as they move forward in time, but they approach the halfline generated by $C_1(1, 1)^T$ as they move backward in time. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



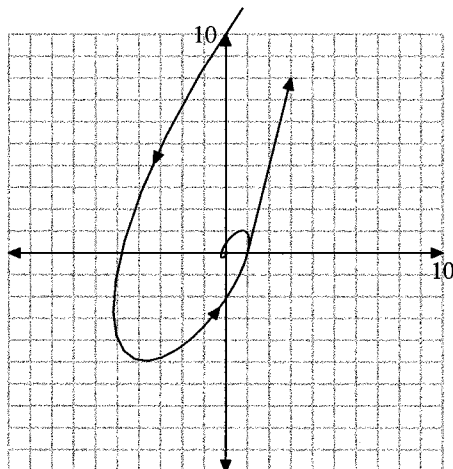
3. If

$$A = \begin{pmatrix} 2 & -4 \\ 8 & -6 \end{pmatrix}$$

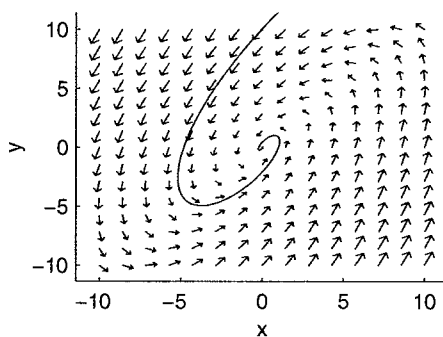
then the trace is $T = -4$ and the determinant is $D = 20$. Further, $T^2 - 4D = (-4)^2 - 4(20) = -64 < 0$, so the equilibrium point at the origin is a spiral sink. At $(1, 0)$,

$$\begin{pmatrix} 2 & -4 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix},$$

so the motion is counterclockwise. A hand sketch follows.



The phase portrait, draw with a numerical solver, follows.



4. If

$$A = \begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix},$$

then the trace is $T = 7$ and the determinant is $D = 10 > 0$. Further, $T^2 - 4D = (7)^2 - 4(10) = 9 > 0$,

so the equilibrium point at the origin is a nodal source. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 7\lambda + 10,$$

which produces eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. Because

$$A - 2I = \begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix},$$

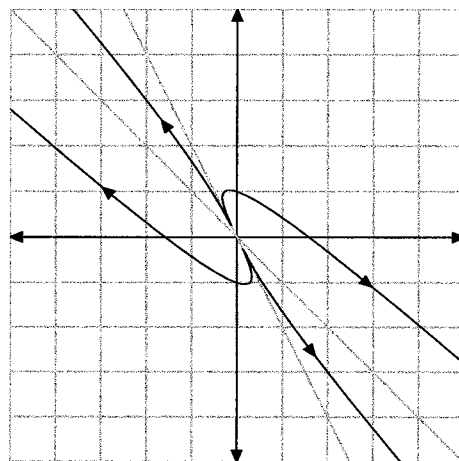
$\mathbf{v}_1 = (1, -2)^T$, leading to the exponential solution $e^{2t}(1, -2)^T$. Because

$$A - 5I = \begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix},$$

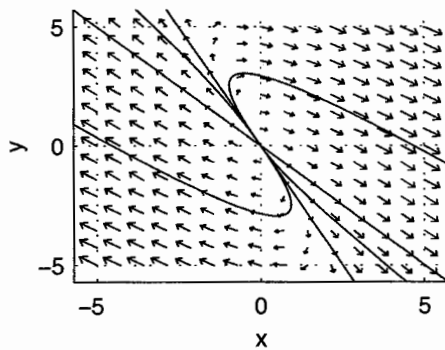
$\mathbf{v}_2 = (1, -1)^T$, leading to the exponential solution $e^{5t}(1, -1)^T$. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solutions emanate from the source tangent to the “slow” halfline solution generated by $C_1(1, -2)^T$ and eventually parallel the “fast” halfline generated by $C_2(1, -1)^T$ as they move forward in time. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



5. If

$$A = \begin{pmatrix} -11 & -5 \\ 10 & 4 \end{pmatrix},$$

then the trace is $T = -7$ and the determinant is $D = 6$. Further, $T^2 - 4D = (-7)^2 - 4(6) = 25 > 0$, so the equilibrium point at the origin is a nodal sink. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 7\lambda + 6,$$

which produces eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. Because

$$A + I = \begin{pmatrix} -10 & -5 \\ 10 & 5 \end{pmatrix},$$

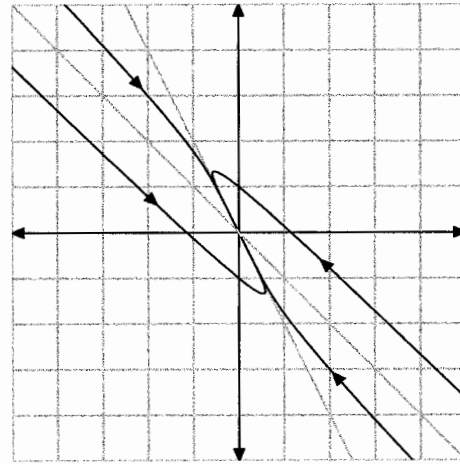
$\mathbf{v}_1 = (1, -2)^T$, leading to the exponential solution $e^{-t}(1, -2)^T$. Because

$$A + 6I = \begin{pmatrix} -5 & -5 \\ 10 & 10 \end{pmatrix},$$

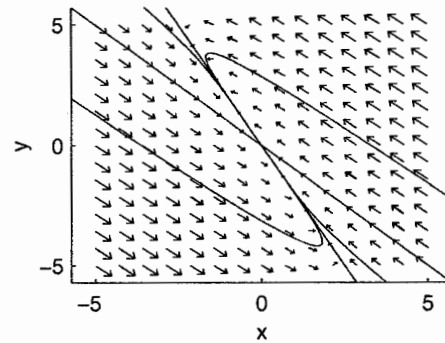
$\mathbf{v}_2 = (1, -1)^T$, leading to the exponential solution $e^{-6t}(1, -1)^T$. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solutions approach the origin tangent to the “slow” halfline solution generated by $C_1(1, -2)^T$. As time moves backwards, solutions eventually parallel the halfline generated by $C_2(1, -1)^T$, the “fast” solution. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



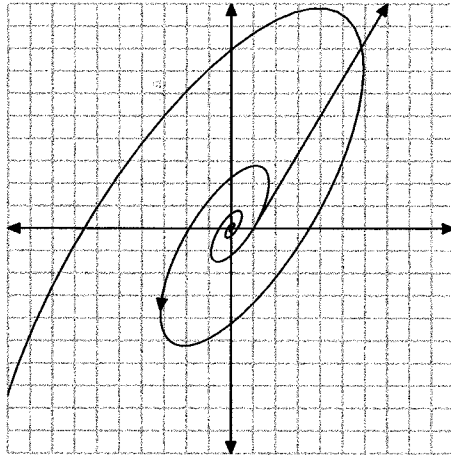
6. If

$$A = \begin{pmatrix} 6 & -5 \\ 10 & -4 \end{pmatrix},$$

then the trace is $T = 2$ and the determinant is $D = 26 > 0$. Further, $T^2 - 4D = (2)^2 - 4(26) = -100 < 0$, so the equilibrium point at the origin is a spiral source. Further,

$$\begin{pmatrix} 6 & -5 \\ 10 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix},$$

so the motion is counterclockwise. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.

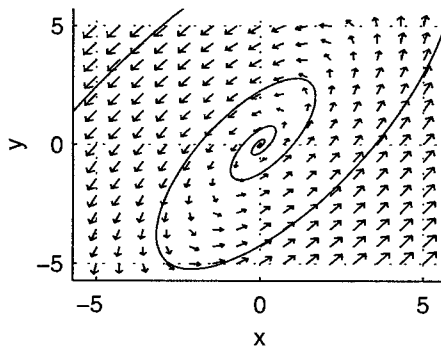
$\mathbf{v}_1 = (2, 1)^T$, leading to the exponential solution $e^{-2t}(2, 1)^T$. Because

$$A - 3I = \begin{pmatrix} -10 & 10 \\ -5 & 5 \end{pmatrix},$$

$\mathbf{v}_2 = (1, 1)^T$, leading to the exponential solution $e^{3t}(1, 1)^T$. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solutions approach the halfline $C_2(1, 1)$ as they move forward in time, but they approach the halfline generated by $C_1(2, 1)$ as they move backward in time. A hand sketch follows.



7. If

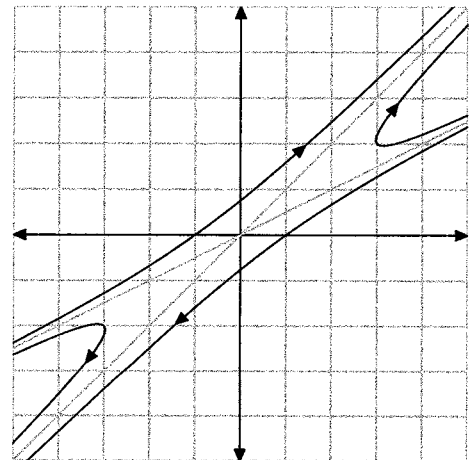
$$A = \begin{pmatrix} -7 & 10 \\ -5 & 8 \end{pmatrix},$$

then the trace is $T = 1$ and the determinant is $D = -6$, so the origin is a saddle point. Further, the characteristic polynomial is

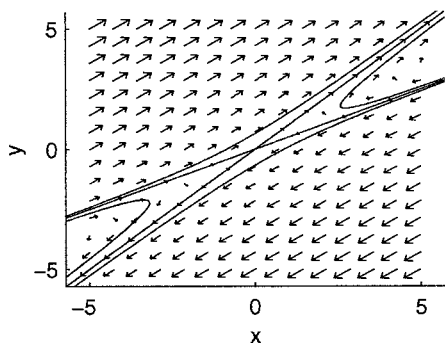
$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - \lambda - 6.$$

which produces eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 3$. Because

$$A + 2I = \begin{pmatrix} -5 & 10 \\ -5 & 10 \end{pmatrix}$$



The phase portrait, drawn in a numerical solver, follows.



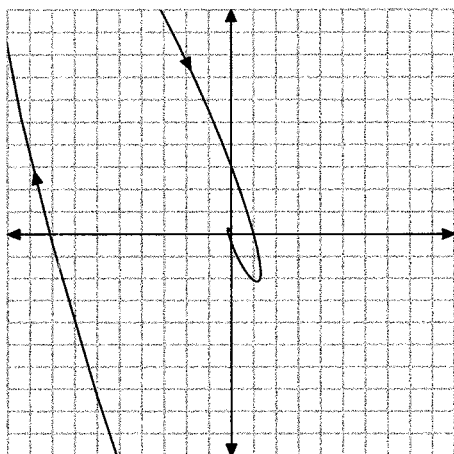
8. If

$$A = \begin{pmatrix} 4 & 3 \\ -15 & -8 \end{pmatrix},$$

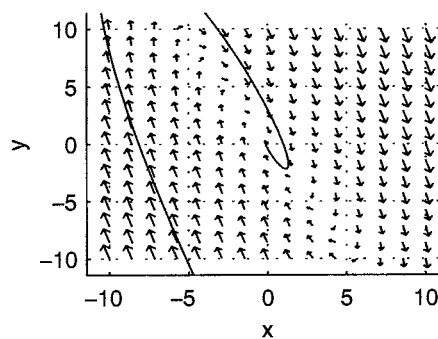
then the trace is $T = -4$ and the determinant is $D = 13 > 0$. Further, $T^2 - 4D = (-4)^2 - 4(13) = -36 < 0$, so the equilibrium point at the origin is a spiral sink. Further,

$$\begin{pmatrix} 4 & 3 \\ -15 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -15 \end{pmatrix},$$

so the motion is clockwise. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



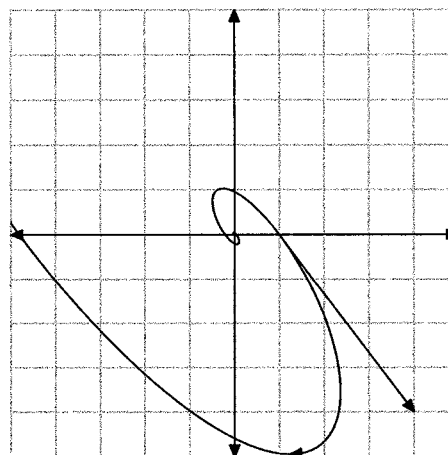
9. If

$$A = \begin{pmatrix} 3 & 2 \\ -4 & -1 \end{pmatrix},$$

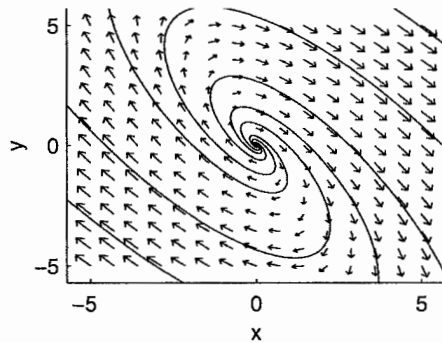
then the trace $T = 2$ and the determinant is $D = 5$, and the discriminant is $T^2 - 4D = (2)^2 - 4(5) = -16 < 0$. Thus, the origin is a spiral source. At $(1, 0)$,

$$\begin{pmatrix} 3 & 2 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix},$$

so the motion is clockwise. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



10. If

$$A = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix},$$

then the trace is $T = -3$ and the determinant is $D = 2 > 0$. Further, $T^2 - 4D = (-3)^2 - 4(2) = 1 > 0$, so the equilibrium point at the origin is a nodal sink. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 3\lambda + 2,$$

which produces eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Because

$$A + I = \begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix},$$

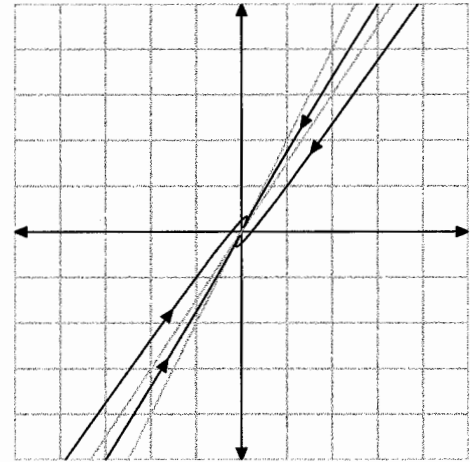
$\mathbf{v}_1 = (1, 2)^T$, leading to the exponential solution $e^{-t}(1, 2)^T$. Because

$$A + 2I = \begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix},$$

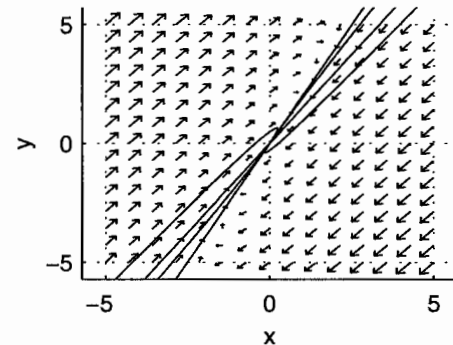
$\mathbf{v}_2 = (2, 3)^T$, leading to the exponential solution $e^{-2t}(2, 3)^T$. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Solutions sink into the origin tangent to the “slow” halfline solution generated by $C_1(1, 2)^T$ and eventually parallel the “fast” halfline generated by $C_2(2, 3)^T$ as they move backward in time. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



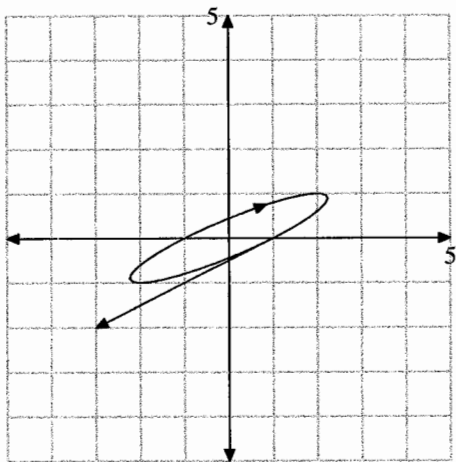
11. If

$$A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$$

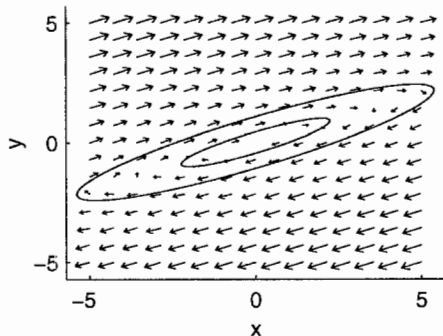
then the trace is $T = 0$ and the determinant is $D = 4$, so the origin is a center. At $(1, 0)$,

$$\begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix},$$

so the rotation is clockwise. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



12. If

$$A = \begin{pmatrix} -2 & -6 \\ 4 & 8 \end{pmatrix},$$

then the trace is $T = 6$ and the determinant is $D = 8 > 0$. Further, $T^2 - 4D = (6)^2 - 4(8) = 4 > 0$, so the equilibrium point at the origin is a nodal source. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 6\lambda + 8,$$

which produces eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$. Because

$$A - 2I = \begin{pmatrix} -4 & -6 \\ 4 & 6 \end{pmatrix},$$

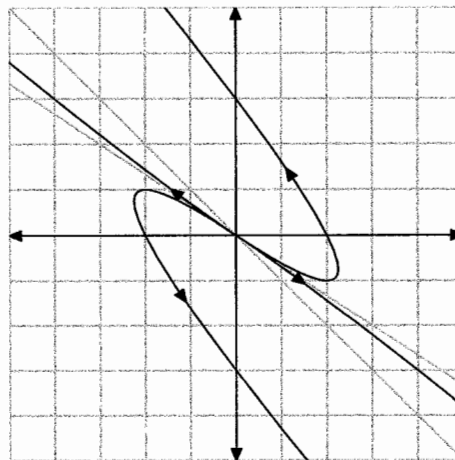
$\mathbf{v}_1 = (3, -2)^T$, leading to the exponential solution $e^{2t}(3, -2)^T$. Because

$$A - 4I = \begin{pmatrix} -6 & -6 \\ 4 & 4 \end{pmatrix},$$

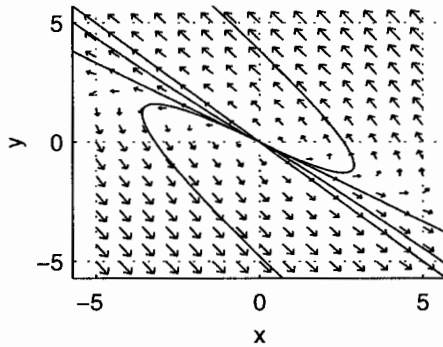
$\mathbf{v}_2 = (1, -1)^T$, leading to the exponential solution $e^{4t}(1, -1)^T$. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 3 \\ -2 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solutions emanate from the source tangent to the “slow” halfline solution generated by $C_1(3, -2)^T$ and eventually parallel the “fast” halfline generated by $C_2(1, -1)^T$ as they move forward in time. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



13. (a) For

$$A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}$$

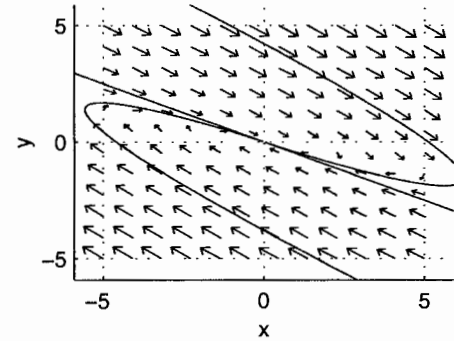
we have $T = \text{tr}(A) = -2$ and $D = \det(A) = 1$. Since the discriminant $T^2 - 4D = 0$ the point (T, D) lies on the parabola that divides nodal sinks from spiral sinks in the trace-determinant plane.

- (b) The general solution can be written

$$\mathbf{y}(t) = e^{-t} \left((C_1 + C_2 t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right)$$

Because $te^{-t} \rightarrow 0$ as $t \rightarrow \infty$ (use l'Hôpital's rule), both $e^{-t}(C_1 + C_2 t)(2, -1)^T \rightarrow 0$ and $C_2 e^{-t}(0, 1/2)^T \rightarrow 0$ as $t \rightarrow \infty$. However, the first term is larger for large values of t . Thus, as $t \rightarrow \infty$, $\mathbf{y}(t) \approx e^{-t}(C_1 + C_2 t)(2, -1)^T$, which implies that solutions approach the origin tangent to the halfline generated by $(2, -1)^T$. In a similar manner, as $t \rightarrow -\infty$, the term $e^{-t}(C_1 + C_2 t)(2, -1)^T$ is larger than the term $C_2 e^{-t}(0, 1/2)^T$, so solutions eventually parallel the halfline generated by $(2, -1)^T$ as time moves backwards.

- (c) The following figure shows the half-line solutions and one other in each sector. The solutions clearly exhibit the behavior predicted in part (a).



14. (a) There is one exponential solution, $e^{\lambda t} \mathbf{v}_1$. Because $\lambda < 0$, this solution decays to the equilibrium point at the origin along the half line generated by $C \mathbf{v}_1$.

- (b) The general solution is

$$\mathbf{y}(t) = e^{\lambda t} [(C_1 + C_2 t) \mathbf{v}_1 + C_2 \mathbf{v}_2].$$

Because $\lambda < 0$, the terms $e^{\lambda t}(C_1 + C_2 t) \mathbf{v}_1$ and $C_2 e^{\lambda t} \mathbf{v}_2$ both decay to zero. However, the first term is larger for large values of t . Thus, as $t \rightarrow \infty$, $\mathbf{y}(t) \approx e^{\lambda t}(C_1 + C_2 t) \mathbf{v}_1$, which implies that the solution approaches zero tangent to the halfline generated by $C_2 \mathbf{v}_1$.

- (c) Because $\lambda < 0$, the terms $e^{\lambda t}(C_1 + C_2 t) \mathbf{v}_1$ and $C_2 e^{\lambda t} \mathbf{v}_2$ get infinitely large in magnitude as $t \rightarrow -\infty$. However, the first term is larger in magnitude for negative values of t that are large in magnitude. Thus, as $t \rightarrow -\infty$, $\mathbf{y}(t) \approx e^{\lambda t}(C_1 + C_2 t) \mathbf{v}_1$, which implies that the solution eventually parallels the halfline generated by $-C_2 \mathbf{v}_1$.

- (d) Degenerate nodal sink.

15. In general everything moves in the opposite direction in comparison to the situation in Exercise 14.

- (a) As $t \rightarrow \infty$ the exponential solutions tends to ∞ along the half-line generated by $C_1 \mathbf{v}_1$.
- (b) As $t \rightarrow \infty$ the general solution tends to ∞ and becomes parallel to the half-line generated by $C_2 \mathbf{v}_1$.
- (c) "As $t \rightarrow -\infty$ the general solution tends to 0 tangent to the half-line generated by $-C_2 \mathbf{v}_1$.

(d) The origin is a degenerate nodal source.

16. Because the linear degenerate nodal sources and sinks have only one eigenvalue, and because the eigenvalues are given by

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2},$$

we must have $T^2 - 4D = 0$. Therefore, the degenerate nodal sources and sinks lie on the parabola $T^2 - 4D = 0$ in the trace-determinant plane. This positioning on the boundary between the nodal sink and sources and the spiral sinks and sources is significant. The solutions attempt to spiral, but they cannot. The present of the halfline solutions prevents them from spiralling (solutions cannot cross).

17. If $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix},$$

then the trace is $T = 8$ and the determinant is $D = 16$. Further, $T^2 - 4D = 8^2 - 4(16) = 0$, so this system lies on the parabola $T^2 - 4D = 0$ that separates spiral sources and sinks from nodal sources and sinks in the trace determinant plane. Thus, the equilibrium point at the origin is a degenerate nodal source ($T = 8$).

The characteristic equation is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 8\lambda + 16,$$

which produces a single eigenvalue $\lambda = 4$. Because

$$A - 4I = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix},$$

$\mathbf{v}_1 = (2, -1)^T$ and we have the exponential solution $e^{4t}(2, -1)^T$. To find another solution, we must solve $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$. Start with any vector that is not a multiple of $\mathbf{v}_1$, say $\mathbf{w} = (1, 0)^T$. Then

$$(A - 4I)\mathbf{w} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \mathbf{v}_1.$$

Thus, let $\mathbf{v}_2 = \mathbf{w} = (1, 0)^T$. Thus, a second, independent solution is

$$e^{4t}(\mathbf{v}_2 + t\mathbf{v}_1) = e^{4t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right],$$

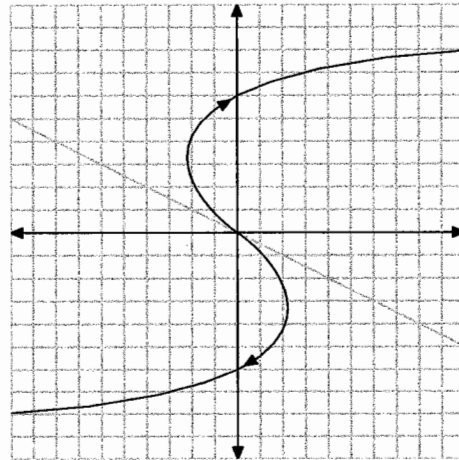
and the general solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{4t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 e^{4t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right] \\ &= e^{4t} \left[(C_1 + C_2 t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]. \end{aligned}$$

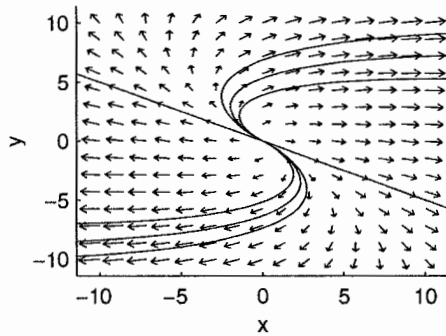
We know that solutions must emanate from the origin parallel to the halflines generated by $C_1(2, -1)$. Not only that, the solutions must also turn parallel to the halflines as time marches forward. At $(1, 0)$,

$$\begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix},$$

so the rotation is clockwise. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



18. If $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -4 & -4 \\ 1 & 0 \end{pmatrix},$$

then the trace is $T = -4$ and the determinant is $D = 4$. Further, $T^2 - 4D = (-4)^2 - 4(4) = 0$, so this system lies on the parabola $T^2 - 4D = 0$ that separates the spiral sources and sinks from the nodal sources and sinks in the trace determinant plane. Thus, the equilibrium point at the origin is a degenerate nodal sink ($T = -4$).

The characteristic equation is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 4\lambda + 4,$$

which produces the single eigenvalue $\lambda = -2$. Because

$$A + 2I = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix},$$

$\mathbf{v}_1 = (2, -1)^T$ and we have the exponential solution $e^{-2t}(2, -1)^T$. To find another solution, we must solve $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$. Start with any vector that is not a multiple of $\mathbf{v}_1$, say $\mathbf{w} = (1, 0)^T$. Then,

$$\begin{aligned} (A + 2I)\mathbf{w} &= \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ &= -1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= -1\mathbf{v}_1. \end{aligned}$$

Thus, let $\mathbf{v}_2 = -\mathbf{w} = (-1, 0)^T$. Thus, a second, independent solution is

$$e^{-2t}(\mathbf{v}_2 + t\mathbf{v}_1) = e^{-2t} \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right],$$

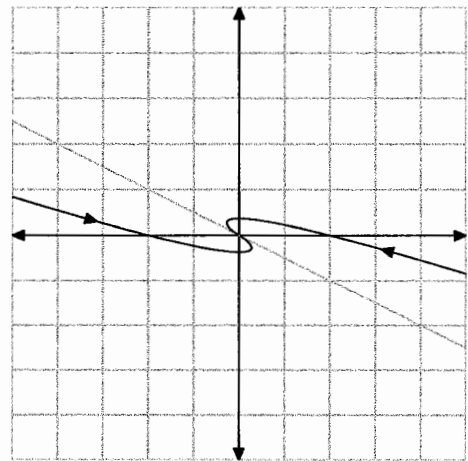
and the general solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{-2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &\quad + C_2 e^{-2t} \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right] \\ &= e^{-2t} \left[(C_1 + C_2 t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]. \end{aligned}$$

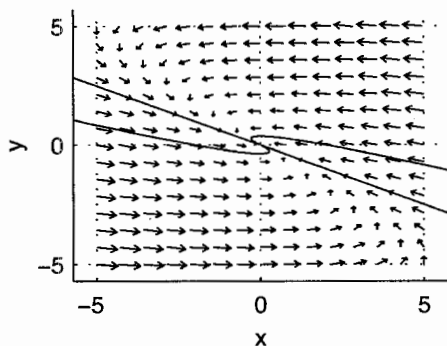
We know that solution must decay to the origin in a manner parallel to the halflines generated by $C_1(2, -1)^T$. Not only that, the solutions must also turn parallel to the half lines as time marches backward. We need only find whether the rotation is clockwise or counterclockwise. But,

$$\begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

so the rotation is counterclockwise. A hand sketch follows.



The phase portrait, drawn in a numerical solver, follows.



19. (a) In matrix form, the system

$$\begin{aligned}x' &= x + ay \\ y' &= x + y\end{aligned}$$

is written

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The trace of the coefficient matrix is $T = 2$ and the determinant is $D = 1 - a$. The discriminant is

$$T^2 - 4D = (2)^2 - 4(1 - a) = 4a.$$

If the origin is a nodal source, then we must have $D > 0$ and $T^2 - 4D > 0$. Thus,

$$1 - a > 0 \quad \text{and} \quad 4a > 0.$$

This leads to the requirement $0 < a < 1$.

- (b) Let

$$A = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}.$$

In the case that $0 < a < 1$,

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 2\lambda + (1 - a).$$

The quadratic formula reveals the eigenvalues, $\lambda_1 = 1 + \sqrt{a}$ and $\lambda_2 = 1 - \sqrt{a}$. Because

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & a \\ 1 & 1 - \lambda \end{pmatrix},$$

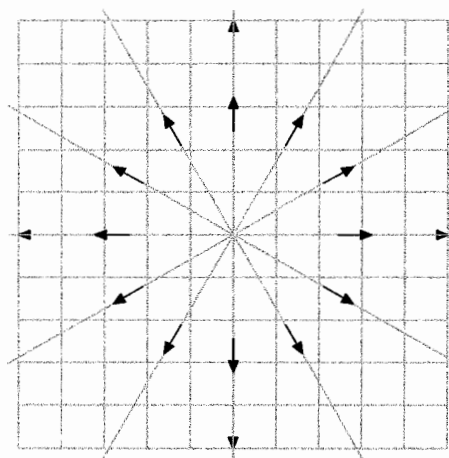
$\mathbf{v} = (\lambda - 1, 1)^T$ is the eigenvector associated with λ . If $\lambda_1 = 1 + \sqrt{a}$, then $\mathbf{v}_1 = (\sqrt{a}, 1)^T$ is its associated eigenvector. If $\lambda_2 = 1 - \sqrt{a}$, then $\mathbf{v}_2 = (-\sqrt{a}, 1)^T$ is its associated eigenvector. Thus, the equations of the halfline solutions are $y = \pm x/\sqrt{a}$. As $a \rightarrow 0$, the halfline solutions coalesce into one halfline solution, which lies on the y -axis with equation $x = 0$.

- (c) When $a = 0$, $T = 2$ and $D = 1$. Moreover, $T^2 - 4D = (2)^2 - 4(1) = 0$, and we lie on the parabola $T^2 - 4D = 0$ in the trace-determinant plane. By part (b), the eigenvalues and eigenvectors coalesce, and we have a degenerate nodal source. If $a < 0$, then $T^2 - 4D = 4a < 0$, and we move above the parabola $T^2 - 4D = 0$ into the land of spiral sources.

20. (a) If $\mathbf{y}' = B\mathbf{y}$, where

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

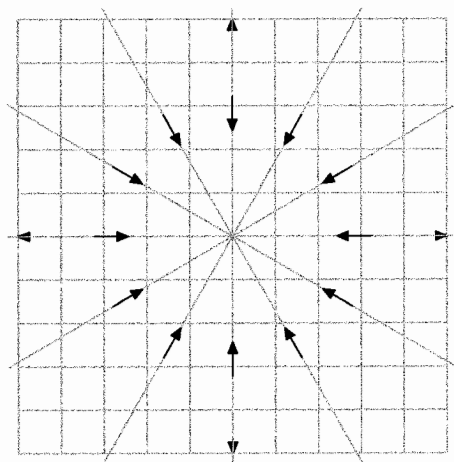
then B has a single eigenvalue $\lambda = 2$ and all vectors in $\mathbf{R}^2$ are eigenvectors. Thus, $e^{2t}(a, b)^T$ is an exponential solution for all $(a, b)^T \in \mathbf{R}^2$. Moreover, these solutions will have to increase to infinity along the halflines generated by $C(a, b)^T$.



If $\mathbf{y}' = C\mathbf{y}$, where

$$C = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

then the eigenvalue is -2 , making $e^{-2t}(a, b)^T$ an exponential solution for all $(a, b)^T \in \mathbb{R}^2$. Thus, the phase portrait is identical to the first graph, only with time reversed. Solutions now decay to zero along the halflines generated by $C(a, b)^T$.



- (b) The system $\mathbf{y}' = B\mathbf{y}$ has a star source at the origin, but the system $\mathbf{y}' = C\mathbf{y}$ has a star sink.
 (c) Because

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

the trace is $T = 4$ and the determinant is $D = 4$. Further, $T^2 - 4D = (4)^2 - 4(4) = 0$, so this case lives on the parabola $T^2 - 4D = 0$ in the trace determinant plane. Moreover, because $T = 4$, it lives on the right half of the parabola, nestled in the land of sources. In the case of

$$C = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

this case also lives on the parabola $T^2 - 4D = 0$, but because $T = -4$, it lives on the left half, in the land of the sinks.

(d) If $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

then the trace is $T = 2a$ and the determinant is a^2 . Further, $T^2 - 4D = (2a)^2 - 4(a)^2 = 0$, placing the star sinks and sources on the parabola $T^2 - 4D = 0$ in the trace-determinant plane. If $a > 0$, then $T = 2a > 0$, placing it on the right half of the parabola, making the equilibrium point at the origin a star source. A similar argument shows that if $a < 0$, then the equilibrium point is a star sink.

21. Let A be a 2×2 matrix with real entries. If $D = \det(A) = 0$, then the characteristic polynomial becomes

$$\begin{aligned} p(\lambda) &= \lambda^2 - T\lambda + D \\ &= \lambda^2 - T\lambda \\ &= \lambda(\lambda - T). \end{aligned}$$

Thus, $\lambda = 0$ is an eigenvalue. On the other hand, if one eigenvalue is $\lambda = 0$, then λ must be a factor of the characteristic equation $\lambda^2 - T\lambda + D$. This can only happen if $D = 0$.

22. (i) If

$$A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix},$$

then the trace is $T = -3$ and the determinant is $D = 0$. Thus, this degenerate case lies on the horizontal axis, separating the saddles from the nodal sinks.

- (ii) To find the equilibrium points, we set the right hand side of $\mathbf{y}' = A\mathbf{y}$ equal to zero, as in $A\mathbf{y} = \mathbf{0}$. Consequently, the equilibrium points are simply the nullspace of A , which is generated by a single vector, $\mathbf{v}_1 = (1, -2)^T$. Thus, we have a whole line of equilibrium points. Everything on the line $y = -2x$ is an equilibrium point.

- (iii) The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 3\lambda,$$

which produces eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -3$. Because

$$A - 0I = A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix},$$

the eigenvector is $\mathbf{v}_1 = (1, -2)^T$, the same vector that produces a line of equilibrium points. Because

$$A + 3I = \begin{pmatrix} 5 & 1 \\ -10 & -2 \end{pmatrix},$$

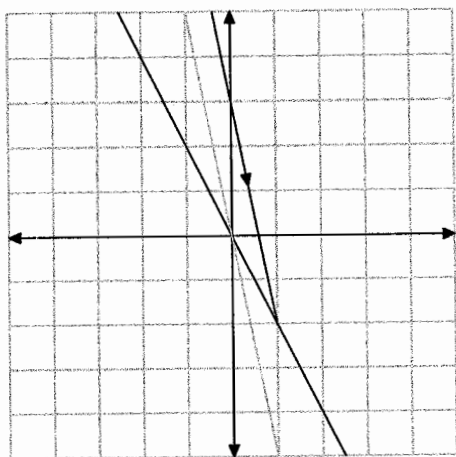
$\mathbf{v}_2 = (1, -5)^T$. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{0t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -5 \end{pmatrix},$$

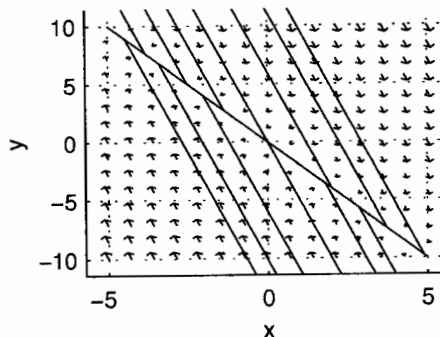
or

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -5 \end{pmatrix}.$$

Note that each solution in this family is the sum of a fixed multiple of $(1, -2)^T$ and a decaying multiple of $(1, -5)^T$. Thus, as $t \rightarrow \infty$, solutions move in lines parallel to $(1, -5)^T$, decaying into the line of equilibrium points as shown in the following figure.



Our numerical solver provides further evidence of this behavior.



23. (i) If

$$A = \begin{pmatrix} 8 & 4 \\ -10 & -5 \end{pmatrix}$$

then the trace is $T = 3$ and the determinant is $D = 0$. Thus, this degenerate case lies on the horizontal axis, separating saddles from the nodal sources.

(ii) To find the equilibrium points, we set the right-hand side of $\mathbf{y}' = A\mathbf{y}$ equal to zero, as in $A\mathbf{y} = \mathbf{0}$. Consequently, the equilibrium points are simply the nullspace of A , which is generated by a single vector, $\mathbf{v}_1 = (1, -2)^T$. Thus, everything on the line $y = -2x$ is an equilibrium point.

(iii) The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 3\lambda,$$

which produces eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 3$. Because

$$A + 0I = \begin{pmatrix} 8 & 4 \\ -10 & -5 \end{pmatrix},$$

the eigenvector is $\mathbf{v}_1 = (1, -2)^T$, the same vector that produces a line of equilibrium points. Because

$$A - 3I = \begin{pmatrix} 5 & 4 \\ -10 & -8 \end{pmatrix},$$

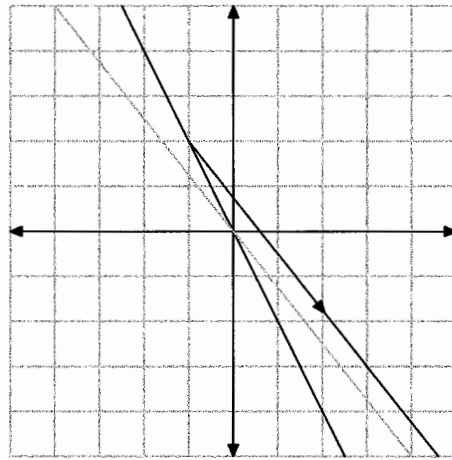
$\mathbf{v}_2 = (4, -5)^T$. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{0t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 4 \\ -5 \end{pmatrix},$$

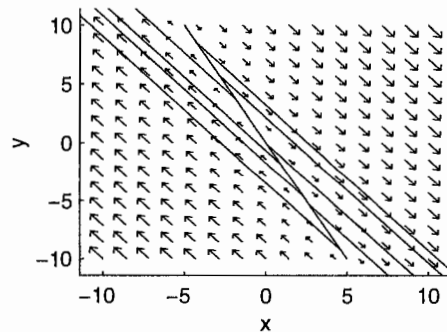
or

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 4 \\ -5 \end{pmatrix}.$$

Note that each solutions in this family is the sum of a fixed multiple of $(1, -2)^T$ and an increasing multiple of $(4, -5)^T$. Thus, as $t \rightarrow \infty$, solutions move away from the line of equilibrium points along lines parallel to $(4, -5)^T$, as shown in the following figure.



Our numerical solver provides further evidence of this behavior.



24. The solutions emanate from a line of equilibrium points, rather than decaying into the line of equilibrium points.

25. Kirchoff's Current Law (KCL) tells us that the current entering the node at "a" equals the current leaving the node at "a". Thus,

$$i_R = i_L + i_C. \quad (4.3)$$

Traversing (clockwise) rightmost loop containing capacitor and inductor, Kirchoff's Voltage Law (KVL) tells us that

$$v_C - Li'_L = 0. \quad (4.4)$$

Traversing (clockwise) the outermost loop containing resistor and capacitor, KVL provides us with

$$Ri_R + v_C = 0. \quad (4.5)$$

First, substitute equation (4.3) in (4.5) to obtain

$$Ri_L + Ri_C + v_C = 0. \quad (4.6)$$

The charge on the capacitor is given by $q_C = Cv_C$, which when differentiated, provides us with the current across the capacitor, $i_C = Cv'_C$. With this substitution, equation (4.6) becomes

$$Ri_L + RCv'_C + v_C = 0. \quad (4.7)$$

Solving equations (4.4) and (4.7) for i'_L and v'_C gives us the requested system.

$$\begin{aligned} v'_C &= -\frac{1}{RC}v_C - \frac{1}{C}i_L \\ i'_L &= \frac{1}{L}v_C. \end{aligned} \quad (4.8)$$

In matrix form,

$$\begin{bmatrix} v_C \\ i_L \end{bmatrix}' = \begin{bmatrix} -1/RC & -1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}. \quad (4.9)$$

Note that the trace and determinant of the coefficient matrix are

$$T = -\frac{1}{RC} < 0 \quad \text{and} \quad D = \frac{1}{LC} > 0,$$

respectively, indicating that the equilibrium point at $(0, 0)$ is either a nodal sink, a degenerate sink, or a spiral sink. Moreover, the characteristic polynomial is

$$\lambda^2 - T\lambda + D = \lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC}. \quad (4.10)$$

The discriminant

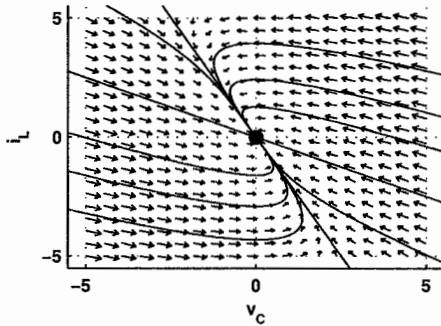
$$T^2 - 4D = \frac{1}{R^2 C^2} - \frac{4}{LC} = \frac{L - 4R^2 C}{LR^2 C^2}$$

has sign completely determined by the sign of its numerator (the denominator $LR^2 C^2$ is always a positive number in this application).

- If $L - 4R^2 C > 0$, then the system has two real, negative roots, and the equilibrium point at $(0, 0)$ is a nodal sink. As an example, consider the case where $L = 5$ H, $R = 2/5$ Ω , and $C = 5$ F. Then,

$$L - 4R^2 C = (5) - 4 \left(\frac{2}{5} \right)^2 (5) = \frac{9}{5} > 0,$$

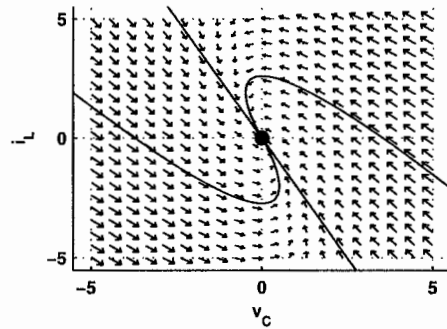
and the system is *overdamped*, as shown below.



- If $L - 4R^2 C = 0$, then the system has one real, negative root, and the equilibrium point lies in the trace-determinant plane precisely on the parabola that divides the region of spiral sinks from nodal sinks. We say that the node is a degenerate sink. As an example, consider $L = 1$ H, $R = 1/4$ Ω , and $C = 4$ F, then

$$L - 4R^2 C = 1 - 4 \left(\frac{1}{4} \right)^2 (4) = 0,$$

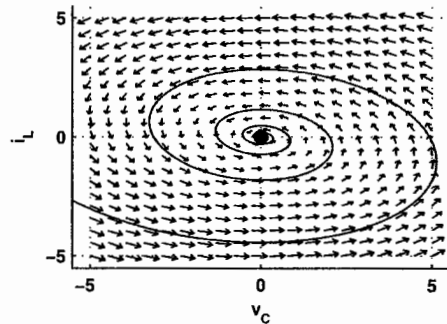
and the system is *critically damped*, as shown below.



- If $L - 4R^2 C < 0$, then the system has two complex roots, and the equilibrium point at $(0, 0)$ is a spiral sink. As an example, let $L = 2$ H, $R = 5$ Ω , and $C = 1$ F, then

$$L - 4R^2 C = 2 - 4(5)^2(1) = -98 < 0,$$

and the system is *underdamped*, as shown below.



26. (a) First, the rate at which the salt content $x_A(t)$ is changing in Tank A is

$$\frac{dx_A}{dt} = \text{Rate in} - \text{Rate out} = -\frac{r_A}{V} x_A.$$

Next, the rate at which the salt content $x_B(t)$ is changing in Tank B is

$$\begin{aligned} \frac{dx_B}{dt} &= \text{Rate in} - \text{Rate out} \\ &= \frac{r_A}{V} x_A - \frac{r_B + r_D}{V} x_B. \end{aligned}$$

- (b) In matrix-vector form, the system of part (a) becomes

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} -r_A/V & 0 \\ r_A/V & -(r_B + r_D)/V \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix}.$$

The trace and determinant of the coefficient matrix are

$$T = -\frac{r_A + r_B + r_D}{V}$$

and

$$D = \frac{r_A r_B + r_A r_D}{V^2}.$$

Clearly, the trace is negative and the determinant is positive, so the equilibrium point is some sort of sink: nodal, degenerate, or spiral. We

need to check $T^2 - 4D$ to make a determination.

$$\begin{aligned} T^2 - 4D &= \left(-\frac{r_A + r_B + r_D}{V} \right)^2 \\ &\quad - 4 \left(\frac{r_A r_B + r_A r_D}{V^2} \right) \\ &= \frac{1}{V^2} [r_A^2 + 2r_A r_B + r_B^2 + 2r_A r_D \\ &\quad + 2r_B r_D + r_D^2 - 4r_A r_B - 4r_A r_D] \\ &= \frac{1}{V^2} [r_A^2 - 2r_A r_B + r_B^2 + 2r_B r_D \\ &\quad - 2r_A r_D + r_D^2] \\ &= \left(\frac{r_A - r_B + r_D}{V} \right)^2 \end{aligned}$$

Thus, $T^2 - 4D \geq 0$, regardless of the values of r_A , r_B , r_I , and r_D , making the sink either a nodal or degenerate sink. It is impossible for the solution to spiral (oscillate).

————— × —————

Section 9.5. Higher Dimensional Systems

1. If

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 6 & 10 & -1 \end{pmatrix},$$

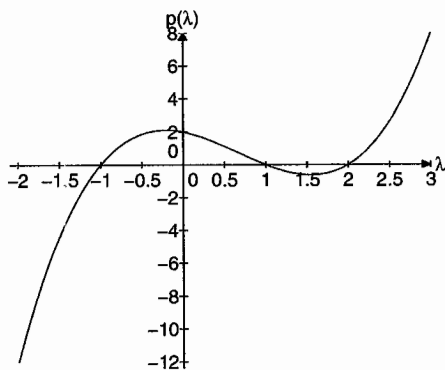
then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= - \begin{vmatrix} 2 - \lambda & 1 & 0 \\ 0 & 1 - \lambda & 0 \\ 6 & 10 & -1 - \lambda \end{vmatrix} \end{aligned}$$

Expanding down the third column,

$$\begin{aligned} p(\lambda) &= -(-1 - \lambda) \begin{vmatrix} 2 - \lambda & 1 \\ 0 & 1 - \lambda \end{vmatrix} \\ &= (\lambda + 1)(2 - \lambda)(1 - \lambda) \\ &= (\lambda + 1)(\lambda - 2)(\lambda - 1). \end{aligned}$$

Thus, the eigenvalues are -1 , 2 , and 1 , respectively. The graph of the characteristic polynomial follows. Note that the graph crosses the horizontal axis at the eigenvalues -1 , 2 , and 1 .



2. If

$$A = \begin{pmatrix} -1 & 6 & 2 \\ 0 & -1 & 0 \\ -1 & 11 & 2 \end{pmatrix},$$

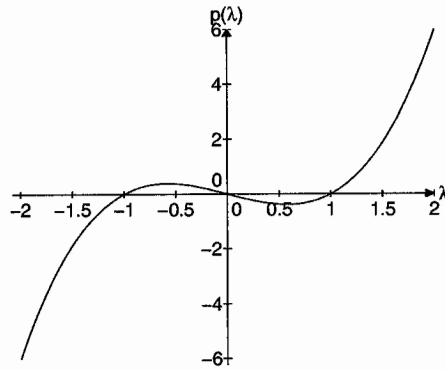
then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= - \begin{vmatrix} -1-\lambda & 6 & 2 \\ 0 & -1-\lambda & 0 \\ -1 & 11 & 2-\lambda \end{vmatrix}. \end{aligned}$$

Expanding across the second row,

$$\begin{aligned} p(\lambda) &= -(-1-\lambda) \begin{vmatrix} -1-\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} \\ &= (\lambda+1)((-1-\lambda)(2-\lambda)+2) \\ &= (\lambda+1)(\lambda^2-\lambda) \\ &= \lambda(\lambda+1)(\lambda-1). \end{aligned}$$

Thus, the eigenvalues are 0, -1, and 1, respectively. The graph of the characteristic polynomial follows. Note that the graph crosses the horizontal axis at the eigenvalues 0, -1, and 1.



3. If

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -6 & 1 & -4 \\ -3 & 0 & -1 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= - \begin{vmatrix} 2-\lambda & 0 & 0 \\ -6 & 1-\lambda & -4 \\ -3 & 0 & -1-\lambda \end{vmatrix} \end{aligned}$$

Expanding across the first row,

$$\begin{aligned} p(\lambda) &= -(2-\lambda) \begin{vmatrix} 1-\lambda & -4 \\ 0 & -1-\lambda \end{vmatrix} \\ &= (\lambda-2)(1-\lambda)(-1-\lambda) \\ &= (\lambda-2)(\lambda-1)(\lambda+1). \end{aligned}$$

Thus, the eigenvalues are 2, 1, and -1, respectively. Because $A - 2I$ reduces,

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ -6 & -1 & -4 \\ -3 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix},$$

it is easily seen that the nullspace of $A - 2I$ is generated by the eigenvector $\mathbf{v}_1 = (-1, 2, 1)^T$. In a similar manner, we arrive at the following eigenvalue-eigenvector pairs.

$$1 \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad -1 \rightarrow \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

Because

$$\det \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = -1,$$

the eigenvectors are independent.

4. If

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & 1 \\ 5 & -5 & 2 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= - \begin{vmatrix} 1-\lambda & 0 & 0 \\ 3 & -2-\lambda & 1 \\ 5 & -5 & 2-\lambda \end{vmatrix}. \end{aligned}$$

Expanding across the first row,

$$\begin{aligned} p(\lambda) &= -(1-\lambda) \begin{vmatrix} -2-\lambda & 1 \\ -5 & 2-\lambda \end{vmatrix} \\ &= (\lambda-1)((-2-\lambda)(2-\lambda) + 5) \\ &= (\lambda-1)(\lambda^2 + 1) \end{aligned}$$

Thus, the eigenvalues are 1, i , and $-i$, respectively. Because $A - iI$ reduces,

$$\begin{aligned} A - iI &= \begin{pmatrix} 1-i & 0 & 0 \\ 3 & -2-i & 1 \\ 5 & -5 & 2-i \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/5 + 1/5i \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

it is easily seen that the nullspaces of $A - iI$ is generated by the eigenvector $\mathbf{v} = (0, 2-i, 5)^T$. In a similar manner, we arrive at the following eigenvalue-eigenvector pairs.

$$-i \rightarrow \begin{pmatrix} 0 \\ 2+i \\ 5 \end{pmatrix} \quad \text{and} \quad 1 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Because

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 2-i & 2+i & 1 \\ 5 & 5 & 0 \end{pmatrix} = -10i,$$

the eigenvectors are independent.

5. If

$$A = \begin{pmatrix} -4 & 0 & 2 \\ 12 & 2 & -6 \\ -6 & 0 & 3 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= - \begin{vmatrix} -4-\lambda & 0 & 2 \\ 12 & 2-\lambda & -6 \\ -6 & 0 & 3-\lambda \end{vmatrix} \end{aligned}$$

Expanding down the second column,

$$\begin{aligned} p(\lambda) &= -(2-\lambda) \begin{vmatrix} -4-\lambda & 2 \\ -6 & 3-\lambda \end{vmatrix} \\ &= (\lambda-2)(\lambda^2 + \lambda) \\ &= \lambda(\lambda-2)(\lambda+1). \end{aligned}$$

Thus, the eigenvalues are 0, 2, and -1 , respectively. Because $A - 0I$ reduces,

$$A - 0I = \begin{pmatrix} -4 & 0 & 2 \\ 12 & 2 & -6 \\ -6 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it is easily seen that the nullspace of $A - 0I$ is generated by the eigenvector $\mathbf{v}_1 = (1, 0, 2)^T$. In a similar manner, we arrive at the following eigenvalue-eigenvector pairs.

$$2 \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad -1 \rightarrow \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}$$

Because

$$\det \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 2 & 0 & -3 \end{pmatrix} = 1,$$

the eigenvectors are independent.

6. If

$$A = \begin{pmatrix} -5 & -2 & 0 \\ 4 & 1 & 0 \\ -3 & -1 & -2 \end{pmatrix}.$$

then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -\begin{vmatrix} -5-\lambda & -2 & 0 \\ 4 & 1-\lambda & 0 \\ -3 & -1 & -2-\lambda \end{vmatrix}. \end{aligned}$$

Expanding down the third column,

$$\begin{aligned} p(\lambda) &= -(-2-\lambda) \begin{vmatrix} -5-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} \\ &= (\lambda+2)((-5-\lambda)(1-\lambda) + 8) \\ &= (\lambda+2)(\lambda^2 + 4\lambda + 3) \\ &= (\lambda+2)(\lambda+1)(\lambda+3). \end{aligned}$$

Thus, the eigenvalues are -2 , -1 , and -3 , respectively. Because

$$A + 2I = \begin{pmatrix} -3 & -2 & 0 \\ 4 & 3 & 0 \\ -3 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it is easily seen that the nullspace of $A + 2I$ is generated by the eigenvector $\mathbf{v} = (0, 0, 1)^T$. In a similar manner, we arrive at the following eigenvalue-eigenvector pairs.

$$-1 \rightarrow \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad -3 \rightarrow \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

Because

$$\det \begin{pmatrix} 0 & -1 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = 1,$$

the eigenvectors are independent.

7. The system in matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 4 & -5 & 4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

reveals that the matrix

$$A = \begin{pmatrix} 4 & -5 & 4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

is upper triangular. Thus, the eigenvalues are located on the main diagonal and are -1 , 4 , and 1 . Because

$$A + I = \begin{pmatrix} 5 & -5 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

it is easily seen that $-1 \rightarrow (1, 1, 0)^T$ is an eigenvalue-eigenvector pair. Similarly,

$$4 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad 1 \rightarrow \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

are the remaining eigenvalue-eigenvector pairs. These lead to the general solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^t \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

8. For

$$A = \begin{pmatrix} -3 & 0 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix}$$

we have

$$A - \lambda I = \begin{pmatrix} -3-\lambda & 0 & -1 \\ 3 & 2-\lambda & 3 \\ 2 & 0 & -\lambda \end{pmatrix}$$

We can compute the characteristic polynomial $p(\lambda) = -\det(A - \lambda I)$ by expanding along the second column to get

$$\begin{aligned} p(\lambda) &= -(2-\lambda) \det \begin{pmatrix} -3-\lambda & -1 \\ 2 & -\lambda \end{pmatrix} \\ &= (\lambda-2)(\lambda^2 + 3\lambda + 2) \\ &= (\lambda-2)(\lambda+1)(\lambda+2). \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -1$, and $\lambda_3 = 2$. For $\lambda_1 = -2$ we have

$$A - \lambda_1 I = A + 2I = \begin{pmatrix} -1 & 0 & -1 \\ 3 & 4 & 3 \\ 2 & 0 & 2 \end{pmatrix}$$

The nullspace is generated by the vector $\mathbf{v}_1 = (-1, 0, 1)^T$. For $\lambda_2 = -1$ we have

$$A - \lambda_2 I = A + I = \begin{pmatrix} -2 & 0 & -1 \\ 3 & 3 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

The nullspace is generated by the vector $\mathbf{v}_2 = (1, 1, -2)^T$. For $\lambda_3 = 2$ we have

$$A - \lambda_3 I = A - 2I = \begin{pmatrix} -5 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & 0 & -2 \end{pmatrix}$$

The nullspace is generated by the vector $\mathbf{v}_3 = (0, 1, 0)^T$. Thus we have three exponential solutions:

$$\mathbf{y}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-2t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{y}_2(t) = e^{\lambda_2 t} \mathbf{v}_2 = e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\mathbf{y}_3(t) = e^{\lambda_3 t} \mathbf{v}_3 = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Since the three eigenvalues are distinct, these solutions are linearly independent and form a fundamental set of solutions. The general solution is

$$\mathbf{y}(t) = C_1 \mathbf{y}_1(t) + C_2 \mathbf{y}_2(t) + C_3 \mathbf{y}_3(t).$$

9. In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

the characteristic polynomial of matrix

$$A = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix},$$

is found by calculating

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -\begin{vmatrix} -3-\lambda & 0 & 0 \\ -5 & 6-\lambda & -4 \\ -5 & 2 & -\lambda \end{vmatrix}. \end{aligned}$$

Expanding across the first row,

$$\begin{aligned} p(\lambda) &= -(-3-\lambda) \begin{vmatrix} 6-\lambda & -4 \\ 2 & -\lambda \end{vmatrix} \\ &= (\lambda+3)(\lambda^2-6\lambda+8) \\ &= (\lambda+3)(\lambda-4)(\lambda-2). \end{aligned}$$

Thus, the eigenvalues are 4, -3, and 2. Because $A - 4I$ reduces,

$$A - 4I = \begin{pmatrix} -7 & 0 & 0 \\ -5 & 2 & -4 \\ -5 & 2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix},$$

it is easily seen that the nullspace of $A - 4I$ is generated by the eigenvector $\mathbf{v}_1 = (0, 2, 1)^T$. In a similar manner, we arrive at the following eigenvalue-eigenvector pairs.

$$-3 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad 2 \rightarrow \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Thus, the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

10. For

$$A = \begin{pmatrix} -3 & -6 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix},$$

we have

$$A - \lambda I = \begin{pmatrix} -3-\lambda & -6 & -2 \\ 0 & 1-\lambda & 0 \\ 0 & -2 & -1-\lambda \end{pmatrix}.$$

We can compute the characteristic polynomial $p(\lambda) = -\det(A - \lambda I)$ by expanding across the second row to get

$$\begin{aligned} p(\lambda) &= -(1-\lambda) \begin{vmatrix} -3-\lambda & -2 \\ 0 & -1-\lambda \end{vmatrix} \\ &= (\lambda-1)(\lambda+3)(\lambda+1) \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -3$, and $\lambda_3 = -1$, respectively. For $\lambda_1 = 1$, we have

$$A - I = \begin{pmatrix} -4 & -6 & -2 \\ 0 & 0 & 0 \\ 0 & -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nullspace is generated by the vector $\mathbf{v}_1 = (1, -1, 1)^T$. For $\lambda_2 = -3$,

$$A + 3I = \begin{pmatrix} 0 & -6 & -2 \\ 0 & 4 & 0 \\ 0 & -2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The nullspace is generated by the vector $\mathbf{v}_2 = (1, 0, 0)^T$. For $\lambda_3 = -1$

$$A + I = \begin{pmatrix} -2 & -6 & -2 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The nullspace is generated by $\mathbf{v}_3 = (-1, 0, 1)^T$. Thus, we have three exponential solutions.

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Since the three eigenvalues are distinct, these solutions are linearly independent and form a fundamental set of solutions. The general solution is

$$\mathbf{y}(t) = C_1 e^t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

11. The characteristic polynomial of matrix

$$A = \begin{pmatrix} -3 & 4 & 8 \\ -2 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

is found by calculating

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= - \begin{vmatrix} -3-\lambda & 4 & 8 \\ -2 & 3-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix}. \end{aligned}$$

Expanding across the third row,

$$\begin{aligned} p(\lambda) &= -(2-\lambda) \begin{vmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} \\ &= (\lambda-2)(\lambda^2-1) \\ &= (\lambda-2)(\lambda+1)(\lambda-1). \end{aligned}$$

Thus, the eigenvalues are -1 , 1 , and 2 . Because $A + I$ reduces,

$$A + I = \begin{pmatrix} -2 & 4 & 8 \\ -2 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

it is easily seen that the nullspace of $A + I$ is generated by the eigenvector $\mathbf{v}_1 = (2, 1, 0)^T$. In a similar manner, we arrive at the following eigenvalue-eigenvector pairs.

$$1 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad 2 \rightarrow \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

12. In matrix form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 2 & 4 & 0 & -4 \\ 3 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & -2 & -3 \end{pmatrix}.$$

Using a computer, we find the following eigenvalue-eigenvector pairs.

$$\begin{aligned} -2 &\rightarrow \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix}, & 2 &\rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \\ -1 &\rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, & 1 &\rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

Because the eigenvalues are distinct, the eigenvectors are independent and the exponential solutions

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{y}_4(t) = e^t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

form a fundamental set of solutions. Thus,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$+ C_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + C_4 e^t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

is the general solution.

13. The general solution in Exercise 7 was

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ C_3 e^t \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

If $x(0) = 1$, $y(0) = -1$, and $z(0) = 2$, then

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Thus, $C_1 = -5$, $C_2 = 2$, and $C_3 = 2$, and the

particular solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -5e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2e^t \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5e^{-t} + 2e^{4t} + 4e^t \\ -5e^{-t} + 4e^t \\ 2e^t \end{pmatrix}$$

14. The solution has the form

$$\mathbf{y}(t) = C_1 \mathbf{y}_1(t) + C_2 \mathbf{y}_2(t) + C_3 \mathbf{y}_3(t).$$

where $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are the fundamental set of solutions found in Exercise 9.4.8. Hence we must have

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \mathbf{y}(0)$$

$$= C_1 \mathbf{y}_1(0) + C_2 \mathbf{y}_2(0) + C_3 \mathbf{y}_3(0)$$

$$= C_1 \mathbf{v}_1 + C_2 \mathbf{v}_2 + C_3 \mathbf{v}_3$$

$$= [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix},$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are the eigenvectors of A found in Exercise 9.4.8. To solve the system we form the augmented matrix

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{y}(0)] = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 2 \end{pmatrix}$$

This is reduced to the row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Backsolving, we find that $C_1 = -4$, $C_2 = -3$, and $C_3 = 2$. Hence the solution is

$$\mathbf{y}(t) = -4e^{-2t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - 3e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + 2e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4e^{-2t} - 3e^{-t} \\ -3e^{-t} + 2e^{2t} \\ -4e^{-2t} + 6e^{-t} \end{pmatrix}.$$

15. The general solution in Exercise 9 was

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

If $x(0) = -2$, $y(0) = 0$, and $z(0) = 2$, then

$$\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & -2 & -2 \\ 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 6 & 6 \end{array} \right)$$

Thus, $C_1 = -2$, $C_2 = -2$, and $C_3 = 6$, and the particular solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= -2e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - 2e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 6e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -2e^{-3t} \\ -4e^{4t} - 2e^{-3t} + 6e^{2t} \\ -2e^{4t} - 2e^{-3t} + 6e^{2t} \end{pmatrix} \end{aligned}$$

16. The general solution in Exercise 10 was

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &\quad + C_3 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Because $\mathbf{y}(0) = (-3, -3, 0)$,

$$\begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -3 & -3 \\ -1 & 0 & 0 & -3 & -3 \\ 1 & 0 & 1 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -9 \\ 0 & 1 & 0 & -9 & -9 \\ 0 & 0 & 1 & -3 & 9 \end{array} \right)$$

Thus, $C_1 = 3$, $C_2 = -9$, and $C_3 = -3$, and the particular solution is

$$\mathbf{y}(t) = 3e^t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 9e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

17. The general solution in Exercise 11 was

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

If $\mathbf{y}(0) = (1, -2, 1)^T$, then

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right)$$

Thus, $C_1 = 1$, $C_2 = -1$, and $C_3 = 1$, and the particular solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{-t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-t} - e^t \\ e^{-t} - e^t - 2e^{2t} \\ e^{2t} \end{pmatrix} \end{aligned}$$

18. The general solution in Exercise 12 was

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} &= C_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ &\quad + C_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + C_4 e^t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

Because $x_1(0) = 1$, $x_2(0) = -1$, $x_3(0) = 0$, and $x_4(0) = 2$,

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + C_4 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -2 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus, $C_1 = 3$, $C_2 = -2$, $C_3 = 7$, and $C_4 = 0$, and the particular solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = 3e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix} - 2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 7e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

19. Using Euler's formula

$$\begin{aligned} \mathbf{y}(t) &= e^{2it} \begin{pmatrix} 1 \\ 1+2i \\ -3i \end{pmatrix} \\ &= (\cos 2t + i \sin 2t) \left[\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \right] \\ &= \cos 2t \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \\ &\quad + i \cos 2t \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + i \sin 2t \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} \cos 2t \\ \cos 2t - 2 \sin 2t \\ -3 \cos 2t + 3 \sin 2t \end{pmatrix} \\ &\quad + i \begin{pmatrix} \sin 2t \\ 2 \cos 2t + \sin 2t \\ -3 \cos 2t - 3 \sin 2t \end{pmatrix}. \end{aligned}$$

Thus, the real and imaginary parts of the complex solution $\mathbf{y}(t) = \mathbf{y}_1(t) + i\mathbf{y}_2(t)$ are

$$\begin{aligned} \mathbf{y}_1(t) &= \begin{pmatrix} \cos 2t \\ \cos 2t - 2 \sin 2t \\ -3 \cos 2t + 3 \sin 2t \end{pmatrix} \quad \text{and} \\ \mathbf{y}_2(t) &= \begin{pmatrix} \sin 2t \\ 2 \cos 2t + \sin 2t \\ -3 \cos 2t - 3 \sin 2t \end{pmatrix}. \end{aligned}$$

20. Using Euler's formula,

$$\begin{aligned} \mathbf{y}(t) &= e^{(1+i)t} \begin{pmatrix} 1 \\ 1+i \\ 1-i \\ 0 \end{pmatrix} \\ &= e^t (\cos t + i \sin t) \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right) \\ &= e^t \left(\cos t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right) \\ &\quad + i e^t \left(\cos t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right) \end{aligned}$$

Thus, the real and imaginary parts of the complex solution $\mathbf{y}(t) = \mathbf{y}_1(t) + i\mathbf{y}_2(t)$ are

$$\begin{aligned} \mathbf{y}_1(t) &= e^t \begin{pmatrix} \cos t \\ \cos t - \sin t \\ \cos t + \sin t \\ 0 \end{pmatrix} \quad \text{and} \\ \mathbf{y}_2(t) &= e^t \begin{pmatrix} \sin t \\ \cos t + \sin t \\ -\cos t + \sin t \\ 0 \end{pmatrix}. \end{aligned}$$

21. In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Using a computer, matrix

$$A = \begin{pmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

has eigenvalues 2, $4i$, and $-4i$. For the eigenvalue 2, we look for a vector in the nullspace (eigenspace) of

$$A - 2I = \begin{pmatrix} -6 & 8 & 8 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The computer tells us that $(0, -1, 1)^T$ is in the nullspace of $A - 2I$. Thus, one solution is $\mathbf{y}_1(t) = e^{2t}(0, -1, 1)^T$. In a similar vein, our computer tells us that $(1 - i, 1, 0)^T$ is in the nullspace of $A - (4i)I$. Thus, we have conjugate solutions

$$\mathbf{z}(t) = e^{4it} \begin{pmatrix} 1 - i \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{\mathbf{z}}(t) = e^{-4it} \begin{pmatrix} 1 + i \\ 1 \\ 0 \end{pmatrix}.$$

Using Euler's formula, we find the real and imaginary parts of the solution $\mathbf{z}(t)$.

$$\begin{aligned} \mathbf{z}(t) &= e^{4it} \begin{pmatrix} 1 - i \\ 1 \\ 0 \end{pmatrix} \\ &= (\cos 4t + i \sin 4t) \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{pmatrix} \\ &\quad + i \begin{pmatrix} -\cos 4t + \sin 4t \\ \sin 4t \\ 0 \end{pmatrix}. \end{aligned}$$

The real and imaginary parts of $\mathbf{z}$ are solutions and we can write the general solution

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -\cos 4t + \sin 4t \\ \sin 4t \\ 0 \end{pmatrix}.$$

22. Using a computer, matrix

$$A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 2 & 3 \\ -3 & -4 & -5 \end{pmatrix}$$

has the following eigenvalue-eigenvector pairs.

$$\begin{aligned} -1 &\rightarrow \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, & 2i &\rightarrow \begin{pmatrix} -2 \\ -1 - i \\ 2 \end{pmatrix}, \\ -2i &\rightarrow \begin{pmatrix} -2 \\ -1 + i \\ 2 \end{pmatrix}. \end{aligned}$$

Using Euler's formula,

$$\begin{aligned} \mathbf{z}(t) &= e^{2it} \begin{pmatrix} -2 \\ -1 - i \\ 2 \end{pmatrix} \\ &= (\cos 2t + i \sin 2t) \\ &\quad \times \left(\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right) \\ &= \left(\cos 2t \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right) \\ &\quad + i \left(\cos 2t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \right). \end{aligned}$$

The real and imaginary parts of $\mathbf{z}$ are solutions and we can write the general solution.

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -2 \cos 2t \\ -\cos 2t + \sin 2t \\ 2 \cos 2t \end{pmatrix} \\ &\quad + C_3 \begin{pmatrix} -2 \sin 2t \\ -\cos 2t - \sin 2t \\ 2 \sin 2t \end{pmatrix} \end{aligned}$$

23. In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Using a computer, matrix

$$A = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix}$$

has eigenvalues -2 , $2 + 4i$, and $2 - 4i$. For the eigenvalue -2 , we look for a vector in the nullspace (eigenspace) of

$$A + 2I = \begin{pmatrix} 8 & 0 & -4 \\ 8 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix}.$$

The computer tells us that $(0, 1, 0)^T$ is in the nullspace of $A + 2I$. Thus, one solution is $\mathbf{y}_1(t) = e^{-2t}(0, 1, 0)^T$. In a similar vein, our computer tells us that $(1 + i, 2, 2)^T$ is in the nullspace of $A - (2 + 4i)I$. Thus, we have conjugate solutions

$$\begin{aligned} \mathbf{z}(t) &= e^{(2+4i)t} \begin{pmatrix} 1+i \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \\ \bar{\mathbf{z}}(t) &= e^{(2-4i)t} \begin{pmatrix} 1-i \\ 2 \\ 2 \end{pmatrix}. \end{aligned}$$

Using Euler's formula, we find the real and imaginary parts of the solution $\mathbf{z}(t)$.

$$\begin{aligned} \mathbf{z}(t) &= e^{2t} e^{4it} \begin{pmatrix} 1+i \\ 2 \\ 2 \end{pmatrix} \\ &= e^{2t} (\cos 4t + i \sin 4t) \left[\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \\ &= e^{2t} \begin{pmatrix} \cos 4t - \sin 4t \\ 2 \cos 4t \\ 2 \cos 4t \end{pmatrix} \\ &\quad + i e^{2t} \begin{pmatrix} \cos 4t + \sin 4t \\ 2 \sin 4t \\ 2 \sin 4t \end{pmatrix} \end{aligned}$$

The real and imaginary parts of $\mathbf{z}$ are solutions and

we can write the general solution

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \cos 4t - \sin 4t \\ 2 \cos 4t \\ 2 \cos 4t \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} \cos 4t + \sin 4t \\ 2 \sin 4t \\ 2 \sin 4t \end{pmatrix}.$$

24. Using a computer, matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -52 & -11 & 26 \\ -20 & -4 & 9 \end{pmatrix}$$

has the following eigenvalue-eigenvector pairs.

$$\begin{aligned} -1 + 2i &\rightarrow \begin{pmatrix} 0 \\ 5 - i \\ 2 \end{pmatrix}, & -1 - 2i &\rightarrow \begin{pmatrix} 0 \\ 5 + i \\ 2 \end{pmatrix}, \\ -1 &\rightarrow \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

Using Euler's formula,

$$\begin{aligned} \mathbf{z}(t) &= e^{(-1+2i)t} \begin{pmatrix} 0 \\ 5 - i \\ 2 \end{pmatrix} \\ &= e^{-t} (\cos 2t + i \sin 2t) \left(\begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right) \\ &= e^{-t} \left(\cos 2t \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right) \\ &\quad + i e^{-t} \left(\cos 2t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} \right). \end{aligned}$$

The real and imaginary parts of $\mathbf{z}$ are solutions and

we can write the general solution.

$$\begin{aligned} \mathbf{y}(t) = & C_1 e^{-t} \begin{pmatrix} 0 \\ 5 \cos 2t + \sin 2t \\ 2 \cos 2t \end{pmatrix} \\ & + C_2 e^{-t} \begin{pmatrix} 0 \\ -\cos 2t + 5 \sin 2t \\ 2 \sin 2t \end{pmatrix} \\ & + C_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

25. In system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -7 & -13 & 0 \\ 2 & 3 & 0 \\ 3 & 8 & -2 \end{pmatrix},$$

we have

$$A - \lambda I = \begin{pmatrix} -7 - \lambda & -13 & 0 \\ 2 & 3 - \lambda & 0 \\ 3 & 8 & -2 - \lambda \end{pmatrix}.$$

We can compute the characteristic polynomial by expanding along the third column. We get

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -(-2 - \lambda) \det \begin{pmatrix} -7 - \lambda & -13 \\ 2 & 3 - \lambda \end{pmatrix} \\ &= (\lambda + 2)(\lambda^2 + 4\lambda + 5). \end{aligned}$$

Hence we have one real eigenvalue $\lambda_1 = -2$, and the quadratic $\lambda^2 + 4\lambda + 5$ has complex roots $\lambda_2 = -2 + i$, and $\bar{\lambda}_2 = -2 - i$. For the eigenvalue $\lambda_1 = -2$, we look for a vector in the nullspace (eigenspace) of

$$A - \lambda_1 I = A + 2I = \begin{pmatrix} -5 & -13 & 0 \\ 2 & 5 & 0 \\ 3 & 8 & 0 \end{pmatrix}.$$

The eigenspace is generated by $\mathbf{v}_1 = (0, 0, 1)^T$. Thus, one solution is

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For the eigenvalue $\lambda_2 = -2 + i$, we look for an vector in the nullspace (eigenspace) of

$$A - \lambda_2 I = A + (2 - i)I = \begin{pmatrix} -5 - i & -13 & 0 \\ 2 & 5 - i & 0 \\ 3 & 8 & -i \end{pmatrix}.$$

The eigenspace is generated by $(-5 + i, 2, 3 - i)^T$. Thus, we have the complex conjugate solutions

$$\mathbf{z}(t) = e^{(-2+i)t} \begin{pmatrix} -5 + i \\ 2 \\ 3 - i \end{pmatrix} \quad \text{and}$$

$$\bar{\mathbf{z}}(t) = e^{(-2-i)t} \begin{pmatrix} -5 - i \\ 2 \\ 3 + i \end{pmatrix}.$$

Using Euler's formula, we find the real and imaginary parts of the solution $\mathbf{z}(t)$.

$$\begin{aligned} \mathbf{z}(t) &= e^{-2t} e^{it} \begin{pmatrix} -5 + i \\ 2 \\ 3 - i \end{pmatrix} \\ &= e^{-2t} (\cos t + i \sin t) \left[\begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} -5 \cos t - \sin t \\ 2 \cos t \\ 3 \cos t + \sin t \end{pmatrix} \\ &\quad + i e^{-2t} \begin{pmatrix} \cos t - 5 \sin t \\ 2 \sin t \\ -\cos t + 3 \sin t \end{pmatrix} \end{aligned}$$

Thus we have the solutions

$$\mathbf{y}_2(t) = \operatorname{Re}(\mathbf{z}(t)) = e^{-2t} \begin{pmatrix} -5 \cos t - \sin t \\ 2 \cos t \\ 3 \cos t + \sin t \end{pmatrix}$$

and

$$\mathbf{y}_3(t) = \operatorname{Im}(\mathbf{z}(t)) = e^{-2t} \begin{pmatrix} \cos t - 5 \sin t \\ 2 \sin t \\ -\cos t + 3 \sin t \end{pmatrix}.$$

The general solution is

$$\mathbf{y}(t) = C_1 \mathbf{y}_1(t) + C_2 \mathbf{y}_2(t) + C_3 \mathbf{y}_3(t).$$

26. If

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 10 & 5 & -10 \\ 1 & 2 & -3 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= - \begin{vmatrix} -2-\lambda & 0 & 0 \\ 10 & 5-\lambda & -10 \\ 1 & 2 & -3-\lambda \end{vmatrix}. \end{aligned}$$

Expanding across the first row,

$$\begin{aligned} p(\lambda) &= -(-2-\lambda) \begin{vmatrix} 5-\lambda & -10 \\ 2 & -3-\lambda \end{vmatrix} \\ &= (\lambda+2)(\lambda^2 - 2\lambda + 5), \end{aligned}$$

which has roots -2 and $1 \pm 2i$. For $\lambda_1 = -2$,

$$A + 2I = \begin{pmatrix} 0 & 0 & 0 \\ 10 & 7 & -10 \\ 1 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\mathbf{v}_1 = (1, 0, 1)^T$ is its associated eigenvector and

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is an exponential solution. For $\lambda_2 = 1 + 2i$,

$$\begin{aligned} A - (1 + 2i)I &= \begin{pmatrix} -3-2i & 0 & 0 \\ 10 & 4-2i & -10 \\ 1 & 2 & -4-2i \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2-i \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and $\mathbf{v}_2 = (0, 2 + i, 1)^T$ is its associated eigenvector.

Using Euler's formula,

$$\begin{aligned} \mathbf{z}(t) &= e^{(1+2i)t} \begin{pmatrix} 0 \\ 2+i \\ 1 \end{pmatrix} \\ &= e^t (\cos 2t + i \sin 2t) \left(\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \\ &= e^t \left(\cos 2t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \\ &\quad + i e^t \left(\cos 2t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right). \end{aligned}$$

The real and imaginary parts of $\mathbf{z}(t)$ give two more independent solutions

$$\begin{aligned} \mathbf{y}_2(t) &= e^t \begin{pmatrix} 0 \\ 2 \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} \quad \text{and} \\ \mathbf{y}_3(t) &= e^t \begin{pmatrix} 0 \\ \cos 2t + 2 \sin 2t \\ \sin 2t \end{pmatrix} \end{aligned}$$

and the general solution is

$$\mathbf{y}(t) = C_1 \mathbf{y}_1(t) + C_2 \mathbf{y}_2(t) + C_3 \mathbf{y}_3(t).$$

27. In Exercise 21, the general solution was

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= C_1 e^{2t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \\ &\quad + C_2 \begin{pmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{pmatrix} \\ &\quad + C_3 \begin{pmatrix} -\cos 4t + \sin 4t \\ \sin 4t \\ 0 \end{pmatrix}. \end{aligned}$$

If $x(0) = 1$, $y(0) = 0$, and $z(0) = 0$, then

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Thus, $C_1 = C_2 = 0$ and $C_3 = -1$, giving the particular solution

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos 4t - \sin 4t \\ -\sin 4t \\ 0 \end{pmatrix}.$$

28. In Exercise 22, the general solution was

$$\begin{aligned} \mathbf{y}(t) = & C_1 e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \\ & + C_2 \begin{pmatrix} -2 \cos 2t \\ -\cos 2t + \sin 2t \\ 2 \cos 2t \end{pmatrix} \\ & + C_3 \begin{pmatrix} -2 \sin 2t \\ -\cos 2t - \sin 2t \\ 2 \sin 2t \end{pmatrix}. \end{aligned}$$

If $\mathbf{y}(0) = (1, -1, 0)^T$, then

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 0 & -2 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 2 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}.$$

Thus, $C_1 = 1$, $C_2 = -1/2$, $C_3 = 1/2$ and the solution is

$$\begin{aligned} \mathbf{y}(t) = & e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -2 \cos 2t \\ -\cos 2t + \sin 2t \\ 2 \cos 2t \end{pmatrix} \\ & + \frac{1}{2} \begin{pmatrix} -2 \sin 2t \\ -\cos 2t - \sin 2t \\ 2 \sin 2t \end{pmatrix} \\ = & \begin{pmatrix} \cos 2t - \sin 2t \\ -e^{-t} - \sin 2t \\ e^{-t} - \cos 2t + \sin 2t \end{pmatrix}. \end{aligned}$$

29. In Exercise 23, the general solution was

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \cos 4t - \sin 4t \\ 2 \cos 4t \\ 2 \cos 4t \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} \cos 4t + \sin 4t \\ 2 \sin 4t \\ 2 \sin 4t \end{pmatrix}.$$

If $x(0) = -2$, $y(0) = -1$, and $z(0) = 0$, then

$$\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 0 & 1 & 1 & -2 \\ 1 & 2 & 0 & -1 \\ 0 & 2 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

Thus, $C_1 = -1$, $C_2 = 0$ and $C_3 = -2$, giving the particular solution

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} e^{2t}(-2 \cos 4t - 2 \sin 4t) \\ -e^{-2t} - 4e^{2t} \sin 4t \\ -4e^{2t} \sin 4t \end{pmatrix}.$$

30. In Exercise 24, the general solution was

$$\begin{aligned} \mathbf{y}(t) = & C_1 e^{-t} \begin{pmatrix} 0 \\ 5 \cos 2t + \sin 2t \\ 2 \cos 2t \end{pmatrix} \\ & + C_2 e^{-t} \begin{pmatrix} 0 \\ -\cos 2t + 5 \sin 2t \\ 2 \sin 2t \end{pmatrix} \\ & + C_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}. \end{aligned}$$

If $\mathbf{y}(0) = (-2, 4, -2)^T$, then

$$\begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 0 & 0 & 1 & -2 \\ 5 & -1 & 0 & 4 \\ 2 & 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

Thus, $C_1 = 1$, $C_2 = 1$, $C_3 = -2$, and the solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{-t} \begin{pmatrix} 0 \\ 5 \cos 2t + \sin 2t \\ 2 \cos 2t \end{pmatrix} \\ &\quad + e^{-t} \begin{pmatrix} 0 \\ -\cos 2t + 5 \sin 2t \\ 2 \sin 2t \end{pmatrix} + -2e^{-t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} -2 \\ 4 \cos 2t + 6 \sin 2t \\ 2 \cos 2t + 2 \sin 2t - 4 \end{pmatrix}. \end{aligned}$$

31. In Exercise 25, the general solution was

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= C_1 e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &\quad + C_2 e^{-2t} \begin{pmatrix} -5 \cos t - \sin t \\ 2 \cos t \\ 3 \cos t + \sin t \end{pmatrix} \\ &\quad + C_3 e^{-2t} \begin{pmatrix} \cos t - 5 \sin t \\ 2 \sin t \\ -\cos t + 3 \sin t \end{pmatrix}. \end{aligned}$$

If $\mathbf{y}(0) = (-1, 1, 1)^T$, then

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 0 & -5 & 1 & -1 \\ 0 & 2 & 0 & 1 \\ 1 & 3 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 3/2 \end{pmatrix}$$

Thus, $C_1 = 1$, $C_2 = 1/2$ and $C_3 = 3/2$, giving the particular solution

$$\mathbf{y}(t) = e^{-2t} \begin{pmatrix} -\cos t - \sin t \\ \cos t + 3 \sin t \\ 1 + 5 \sin t \end{pmatrix}.$$

32. In Exercise 26, the general solution was

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ 2 \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} \\ &\quad + C_3 e^t \begin{pmatrix} 0 \\ \cos 2t + 2 \sin 2t \\ \sin 2t \end{pmatrix} \end{aligned}$$

If $\mathbf{y}(0) = (-1, 1, -1)^T$, then

$$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Thus, $C_1 = -1$, $C_2 = 0$, $C_3 = 1$, and the solution is

$$\begin{aligned} \mathbf{y}(t) &= -e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ \cos 2t + 2 \sin 2t \\ \sin 2t \end{pmatrix} \\ &= \begin{pmatrix} -e^{-2t} \\ e^t (\cos 2t + 2 \sin 2t) \\ -e^{-2t} + e^t \sin 2t \end{pmatrix}. \end{aligned}$$

33. In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -10 & 8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Using a computer, matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -10 & 8 & 5 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda - 1)^2(\lambda - 5).$$

Thus, A has eigenvalues 1 and 5 with algebraic multiplicities 2 and 1, respectively. For the eigenvalue

1, we look for a vector in the nullspace (eigenspace) of

$$A - I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -10 & 8 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that there is one free variable and the eigenspace is generated by the single eigenvector $(0, 1, -2)^T$. Therefore, the eigenvalue 1 has geometric multiplicity 1. For the eigenvalue 5, we look for a vector in the nullspace (eigenspace) of

$$A - 5I = \begin{pmatrix} -4 & 0 & 0 \\ 1 & -4 & 0 \\ -10 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note again that there is only one free variable and the eigenspace is generated by the single eigenvector $(0, 0, 1)^T$. Therefore, the eigenvalue 5 has geometric multiplicity 1. Consequently, there are not enough independent eigenvectors to form a fundamental solution set.

34. Using a computer, matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -6 & 2 & 3 \\ 6 & 0 & -1 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda - 2)^2(\lambda + 1).$$

Thus, A has eigenvalues 2 and -1 with algebraic multiplicities 2 and 1, respectively. For $\lambda_1 = 2$,

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ -6 & 0 & 3 \\ 6 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that there are two free variables, so $\lambda_1 = 2$ has geometric multiplicity 2. Thus, there are two independent eigenvectors and independent exponential solutions

$$\mathbf{y}_1(t) = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = -1$,

$$A + I = \begin{pmatrix} 3 & 0 & 0 \\ -6 & 3 & 3 \\ 6 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that there is one free variable so $\lambda_2 = -1$ has geometric multiplicity 1. Thus, the nullspace is generated by the single eigenvector $\mathbf{v} = (0, -1, 1)^T$, and

$$\mathbf{y}_3(t) = e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

is another independent solution. Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

35. In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 4 & 0 & 0 \\ -6 & -2 & 0 \\ 7 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Using a computer, matrix

$$A = \begin{pmatrix} 4 & 0 & 0 \\ -6 & -2 & 0 \\ 7 & 1 & -2 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda - 4)(\lambda + 2)^2.$$

Thus, A has eigenvalues 4 and -2 with algebraic multiplicities 1 and 2, respectively. For the eigenvalue 4, we look for a vector in the nullspace (eigenspace) of

$$A - 4I = \begin{pmatrix} 0 & 0 & 0 \\ -6 & -6 & 0 \\ 7 & 1 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that there is one free variable and the eigenspace is generated by the single eigenvector $(1, -1, 1)^T$. Therefore, the eigenvalue 4 has geometric multiplicity 1. For the eigenvalue -2 , we look for a vector in the nullspace (eigenspace) of

$$A + 2I = \begin{pmatrix} 6 & 0 & 0 \\ -6 & 0 & 0 \\ 7 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note again that there is only one free variable and the eigenspace is generated by the single eigenvector $(0, 0, 1)^T$. Therefore, the eigenvalue -4 has geometric multiplicity 1. Consequently, there are not enough independent eigenvectors to form a fundamental solution set.

36. Using a computer, matrix

$$A = \begin{pmatrix} 6 & -5 & 10 \\ -1 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda - 5)(\lambda - 1)^2.$$

Thus, A has eigenvalues 5 and 1, with algebraic multiplicities 1 and 2, respectively. For $\lambda_1 = 5$,

$$A - 5I = \begin{pmatrix} 1 & -5 & 10 \\ -1 & -3 & -2 \\ -1 & 1 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that there is one free variable, so $\lambda_1 = 5$ has geometric multiplicity 1. The eigenvector $\mathbf{v} = (-5, 1, 1)^T$ gives the exponential solution

$$\mathbf{y}_1(t) = e^{5t} \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$,

$$A - I = \begin{pmatrix} 5 & -5 & 10 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that there are two free variables, so $\lambda_1 = 1$ has geometric multiplicity 2. The eigenvectors $(1, 1, 0)^T$ and $(-2, 0, 1)^T$ produce two independent exponential solutions

$$\mathbf{y}_2(t) = e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_3(t) = e^t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 e^{5t} \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

37. Using a computer, matrix

$$A = \begin{pmatrix} -6 & 2 & -3 \\ -1 & -1 & -1 \\ 4 & -2 & 1 \end{pmatrix}$$

has eigenvalue–eigenvector pairs

$$-2 \rightarrow \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad -3 \rightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

and

$$-1 \rightarrow \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore,

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-3t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \\ \mathbf{y}_3(t) = e^{-t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

form a fundamental set of solutions.

38. Using a computer we find that the eigenvalues of

$$A = \begin{pmatrix} -7 & 2 & -4 \\ 42 & -11 & 18 \\ 38 & -10 & 18 \end{pmatrix}$$

are -2 and the complex conjugate pair $1 \pm 2i$. For the eigenvalue -2 we look for a basis of the eigenspace, which is the nullspace of $A - \lambda I = A + 2I$. Using a computer we find that the eigenspace has dimension 1 and is spanned by the vector $\mathbf{v} = (0, 2, 1)^T$. Hence we have the solution

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Next we look at the eigenspace for the eigenvalue $1 + 2i$. The computer tells us that it has dimension 1

and is spanned by $\mathbf{w} = (-1 + i, 3 + 3i, 4)^T$. Therefore we have the complex valued solution.

$$\mathbf{z}(t) = e^{(1+2i)t} \begin{pmatrix} -1 + i \\ 3 + 3i \\ 4 \end{pmatrix}.$$

Expanding using Euler's formula, we get

$$\begin{aligned} \mathbf{z}(t) &= e^t [\cos 2t + i \sin 2t] \left[\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} + i \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right] \\ &= e^t \left[\cos 2t \cdot \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} - \sin 2t \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right] \\ &\quad + i e^t \left[\cos 2t \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \sin 2t \cdot \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \right] \end{aligned}$$

Since the real and imaginary parts of $\mathbf{z}(t)$ are solutions we get two real solutions

$$\begin{aligned} \mathbf{y}_2(t) &= \operatorname{Re}(\mathbf{z}(t)) = e^t \begin{pmatrix} -\cos 2t - \sin 2t \\ 3 \cos 2t - 3 \sin 2t \\ 4 \cos 2t \end{pmatrix} \\ \mathbf{y}_3(t) &= \operatorname{Im}(\mathbf{z}(t)) = e^t \begin{pmatrix} \cos 2t - \sin 2t \\ 3 \cos 2t + 3 \sin 2t \\ 4 \sin 2t \end{pmatrix}. \end{aligned}$$

The functions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ form a fundamental set of solutions.

39. Using a computer, matrix

$$A = \begin{pmatrix} 8 & 12 & -4 \\ -9 & -13 & 4 \\ -1 & -3 & 0 \end{pmatrix}$$

has eigenvalue-eigenvector pairs

$$\begin{aligned} -1 &\rightarrow \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad -2 + 2i \rightarrow \begin{pmatrix} -2 \\ 2 \\ 1 + i \end{pmatrix}, \quad \text{and} \\ -2 - 2i &\rightarrow \begin{pmatrix} -2 \\ 2 \\ 1 - i \end{pmatrix}. \end{aligned}$$

Therefore,

$$\mathbf{y}_1(t) = e^{-t} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

is a solution. Because

$$\begin{aligned} \mathbf{z}(t) &= e^{(-2+2i)t} \begin{pmatrix} -2 \\ 2 \\ 1 + i \end{pmatrix} \\ &= e^{-2t} (\cos 2t + i \sin 2t) \left[\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} -2 \cos 2t \\ 2 \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} \\ &\quad + i e^{-2t} \begin{pmatrix} -2 \sin 2t \\ 2 \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}, \end{aligned}$$

the set $\mathbf{y}_1(t) = e^{-t}(0, 1, 3)^T$,

$$\mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} -2 \cos 2t \\ 2 \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix},$$

and

$$\mathbf{y}_3(t) = e^{-2t} \begin{pmatrix} -2 \sin 2t \\ 2 \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

forms a fundamental set of solutions.

40. Using a computer, matrix

$$A = \begin{pmatrix} -1 & -2 & 4 \\ -1 & 0 & -4 \\ -1 & 2 & -6 \end{pmatrix}$$

has eigenvalue-eigenvector pairs

$$\begin{aligned} -2 &\rightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad -2 \rightarrow \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \\ -3 &\rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Note that the eigenvalue -2 has algebraic multiplicity 2 and geometric multiplicity 2, so we have enough eigenvectors to form a fundamental set of solutions.

$$\begin{aligned} \mathbf{y}_1(t) &= e^{-2t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, & \mathbf{y}_2(t) &= e^{-2t} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \\ \mathbf{y}_3(t) &= e^{-3t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

41. Using a computer, matrix

$$A = \begin{pmatrix} -18 & -18 & 10 \\ 18 & 17 & -10 \\ 10 & 10 & -7 \end{pmatrix}$$

has eigenvalue-eigenvector pairs

$$\begin{aligned} -2 &\rightarrow \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, & -3 + 2i &\rightarrow \begin{pmatrix} -6 + 2i \\ 8 - i \\ 5 \end{pmatrix}, \\ -3 - 2i &\rightarrow \begin{pmatrix} -6 - 2i \\ 8 + i \\ 5 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

is a solution. Because,

$$\begin{aligned} \mathbf{z}(t) &= e^{(-3+2i)t} \begin{pmatrix} -6 + 2i \\ 8 - i \\ 5 \end{pmatrix} \\ &= e^{-3t} (\cos 2t + i \sin 2t) \\ &\quad \times \left[\begin{pmatrix} -6 \\ 8 \\ 5 \end{pmatrix} + i \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right] \\ &= e^{-3t} \begin{pmatrix} -6 \cos 2t - 2 \sin 2t \\ 8 \cos 2t + \sin 2t \\ 5 \cos 2t \end{pmatrix} \\ &\quad + i e^{-3t} \begin{pmatrix} 2 \cos 2t - 6 \sin 2t \\ -\cos 2t + 8 \sin 2t \\ 5 \sin 2t \end{pmatrix}. \end{aligned}$$

the set

$$\begin{aligned} \mathbf{y}_1(t) &= e^{-2t} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \\ \mathbf{y}_2(t) &= e^{-3t} \begin{pmatrix} -6 \cos 2t - 2 \sin 2t \\ 8 \cos 2t + \sin 2t \\ 5 \cos 2t \end{pmatrix}, & \text{and} \\ \mathbf{y}_3(t) &= e^{-3t} \begin{pmatrix} 2 \cos 2t - 6 \sin 2t \\ -\cos 2t + 8 \sin 2t \\ 5 \sin 2t \end{pmatrix} \end{aligned}$$

forms a fundamental set of solutions.

42. The matrix

$$A = \begin{pmatrix} -6 & 6 & 8 \\ -12 & 16 & 24 \\ 8 & -12 & -18 \end{pmatrix}$$

has eigenvalue-eigenvector pairs

$$\begin{aligned} -2 &\rightarrow \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, & -2 &\rightarrow \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \\ -4 &\rightarrow \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}. \end{aligned}$$

Note that the eigenvalue -2 has algebraic multiplicity 2 and geometric multiplicity 2, so we have enough eigenvectors to form a fundamental set of solutions.

$$\begin{aligned} \mathbf{y}_1(t) &= e^{-2t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, & \mathbf{y}_2(t) &= e^{-2t} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \\ \mathbf{y}_3(t) &= e^{-4t} \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}. \end{aligned}$$

43. The matrix

$$A = \begin{pmatrix} 1 & 4 & 1 & -5 \\ -6 & -10 & -2 & 10 \\ 3 & 4 & -1 & -5 \\ -3 & -4 & -1 & 3 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda + 1)(\lambda + 2)^3,$$

indicating eigenvalues -1 and -2 , with algebraic multiplicities 1 and 3, respectively. The matrix

$$A + I = \begin{pmatrix} 2 & 4 & 1 & -5 \\ -6 & -9 & -2 & 10 \\ 3 & 4 & 0 & -5 \\ -3 & -4 & -1 & 4 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has one free variable, generating a single eigenvector and the solution

$$\mathbf{y}_1(t) = e^{-t} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix}.$$

The matrix

$$A + 2I = \begin{pmatrix} 3 & 4 & 1 & -5 \\ -6 & -8 & -2 & 10 \\ 3 & 4 & 1 & -5 \\ -3 & -4 & -1 & 5 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & 4/3 & 1/3 & -5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has three free variables. A basis for the nullspace (eigenspace) of $A + 2I$ contains the vectors

$$\begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 5 \\ 0 \\ 0 \\ 3 \end{pmatrix},$$

so, together with $\mathbf{y}_1(t) = e^{-t}(-1, 2, -1, 1)^T$,

$$\mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{y}_3(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix},$$

$$\mathbf{y}_4(t) = e^{-2t} \begin{pmatrix} 5 \\ 0 \\ 0 \\ 3 \end{pmatrix},$$

complete a fundamental set of solutions.

44. The matrix

$$A = \begin{pmatrix} 6 & -6 & -6 & -8 \\ 8 & -8 & -6 & -8 \\ -1 & 7 & -10 & -9 \\ -8 & 6 & 6 & 6 \end{pmatrix}$$

has eigenvalue-eigenvector pairs

$$-2 \rightarrow \begin{pmatrix} 11 \\ 8 \\ 0 \\ 5 \end{pmatrix} \quad \text{and} \quad -2 \rightarrow \begin{pmatrix} 9 \\ 7 \\ 5 \\ 0 \end{pmatrix}.$$

Note that the eigenvalue -2 has algebraic multiplicity 2 and geometric multiplicity 2, so there are sufficient independent eigenvectors to form the independent solutions

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 11 \\ 8 \\ 0 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 9 \\ 7 \\ 5 \\ 0 \end{pmatrix}.$$

Further, the eigenvalue-eigenvector pair

$$-1 + 3i \rightarrow \begin{pmatrix} 3 + i \\ 3 + i \\ 5 \\ -3 - i \end{pmatrix}$$

allows us to form the complex solution

$$\mathbf{z}(t) = e^{(-1+3i)t} \begin{pmatrix} 3 + i \\ 3 + i \\ 5 \\ -3 - i \end{pmatrix} \\ = e^{-t}(\cos 3t + i \sin 3t) \\ \times \left(\begin{pmatrix} 3 \\ 3 \\ 5 \\ -3 \end{pmatrix} + i \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right) \\ = e^{-t} \left(\cos 3t \begin{pmatrix} 3 \\ 3 \\ 5 \\ -3 \end{pmatrix} - \sin 3t \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right) \\ + i \left(\cos 3t \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \sin 3t \begin{pmatrix} 3 \\ 3 \\ 5 \\ -3 \end{pmatrix} \right).$$

The real and imaginary parts provide two additional independent solutions

$$\mathbf{y}_3(t) = e^{-t} \begin{pmatrix} 3 \cos 3t - \sin 3t \\ 3 \cos 3t - \sin 3t \\ 5 \cos 3t \\ -3 \cos 3t + \sin 3t \end{pmatrix} \quad \text{and}$$

$$\mathbf{y}_4(t) = e^{-t} \begin{pmatrix} \cos 3t + 3 \sin 3t \\ \cos 3t + 3 \sin 3t \\ 5 \sin 3t \\ -\cos 3t - 3 \sin 3t \end{pmatrix}.$$

Thus, $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, $\mathbf{y}_3(t)$, and $\mathbf{y}_4(t)$ form a fundamental set of solutions.

45. In Exercise 37, the fundamental set of solutions found there lead to the general solution

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

The initial condition $\mathbf{y}(0) = (-6, 2, 9)^T$ provides

$$\begin{pmatrix} -6 \\ 2 \\ 9 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & -1 & -1 & -6 \\ -1 & 0 & -1 & 2 \\ -2 & 1 & 1 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Thus, $C_1 = -3$, $C_2 = 2$ and $C_3 = 1$, leading to

$$\mathbf{y}(t) = \begin{pmatrix} -3e^{-2t} - 2e^{-3t} - e^{-t} \\ 3e^{-2t} - e^{-t} \\ 6e^{-2t} + 2e^{-3t} + e^{-t} \end{pmatrix}.$$

46. In Exercise 38 we found the fundamental set of so-

lutions

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{y}_2(t) = e^t \begin{pmatrix} -\cos 2t - \sin 2t \\ 3 \cos 2t - 3 \sin 2t \\ 4 \cos 2t \end{pmatrix}$$

$$\mathbf{y}_3(t) = e^t \begin{pmatrix} \cos 2t - \sin 2t \\ 3 \cos 2t + 3 \sin 2t \\ 4 \sin 2t \end{pmatrix}$$

Our solution has the form $\mathbf{y}(t) = C_1 \mathbf{y}_1(t) + C_2 \mathbf{y}_2(t) + C_3 \mathbf{y}_3(t)$. At $t = 0$ we have

$$\begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 1 \\ 2 & 3 & 3 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

We can allow our computer to solve this system of equations, obtaining $C_1 = 1$, $C_2 = 1$, and $C_3 = -1$. Thus our solution is

$$\mathbf{y}(t) = \mathbf{y}_1(t) + \mathbf{y}_2(t) - \mathbf{y}_3(t)$$

$$= \begin{pmatrix} -2e^t \cos 2t \\ -6e^t \sin 2t + 2e^{-2t} \\ 4e^t \cos 2t - 4e^t \sin 2t + e^{-2t} \end{pmatrix}.$$

47. In Exercise 39, the fundamental set of solutions found there lead to the general solution

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -2 \cos 2t \\ 2 \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix}$$

$$+ C_3 e^{-2t} \begin{pmatrix} -2 \sin 2t \\ 2 \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

The initial condition $\mathbf{y}(0) = (0, 8, 5)^T$ provides

$$\begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 0 & -2 & 0 & 0 \\ 1 & 2 & 0 & 8 \\ 3 & 1 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -19 \end{pmatrix}$$

Thus, $C_1 = 8$, $C_2 = 0$ and $C_3 = -19$, leading to

$$\mathbf{y}(t) = \begin{pmatrix} 38e^{-2t} \sin 2t \\ 8e^{-t} - 38e^{-2t} \sin 2t \\ 24e^{-t} - 19e^{-2t} \cos 2t - 19e^{-2t} \sin 2t \end{pmatrix}.$$

48. In Exercise 40, the fundamental set of solutions found there lead to the general solution

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-3t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

The initial condition $\mathbf{y}(0) = (1, 0, 0)^T$ provides

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

The augmented matrix reduces

$$\begin{pmatrix} 2 & -4 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Thus, $C_1 = -1$, $C_2 = -1$, and $C_3 = 1$, leading to

$$\mathbf{y}(t) = \begin{pmatrix} 2e^{-2t} - e^{-3t} \\ -e^{-2t} + e^{-3t} \\ -e^{-2t} + e^{-3t} \end{pmatrix}$$

49. In Exercise 41, the fundamental set of solutions found there lead to the general solution

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} -6 \cos 2t - 2 \sin 2t \\ 8 \cos 2t + \sin 2t \\ 5 \cos 2t \end{pmatrix} + C_3 e^{-3t} \begin{pmatrix} 2 \cos 2t - 6 \sin 2t \\ -\cos 2t + 8 \sin 2t \\ 5 \sin 2t \end{pmatrix}.$$

The initial condition $\mathbf{y}(0) = (-1, 7, 3)^T$ provides

$$\begin{pmatrix} -1 \\ 7 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} -6 \\ 8 \\ 5 \end{pmatrix} + C_3 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 1 & -6 & 2 & -1 \\ -2 & 8 & -1 & 7 \\ -2 & 5 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 17/5 \\ 0 & 0 & 1 & 31/5 \end{pmatrix}$$

Thus, $C_1 = 7$, $C_2 = 17/5$ and $C_3 = 31/5$, leading to

$$\mathbf{y}(t) = \begin{pmatrix} 7e^{-2t} - e^{-3t}(8 \cos 2t - 44 \sin 2t) \\ -14e^{-2t} + e^{-3t}(21 \cos 2t + 53 \sin 2t) \\ -14e^{-2t} + e^{-3t}(17 \cos 2t + 31 \sin 2t) \end{pmatrix}.$$

50. In Exercise 42, the fundamental set of solutions found there lead to the general solution.

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + C_3 e^{-4t} \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$$

The initial condition $\mathbf{y}(0) = (-1, -4, 1)^T$ provides

$$\begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} 2 & 3 & -1 & -1 \\ 0 & 2 & -3 & -4 \\ 1 & 0 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & -6 \end{pmatrix}$$

Thus, $C_1 = 13$, $C_2 = -11$, and $C_3 = -6$, leading to

$$\mathbf{y}(t) = \begin{pmatrix} -7e^{-2t} + 6e^{-4t} \\ -22e^{-2t} + 18e^{-4t} \\ 13e^{-2t} - 12e^{-4t} \end{pmatrix}.$$

51. In Exercise 43, the fundamental set of solutions found there lead to the general solution

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix} + C_4 e^{-2t} \begin{pmatrix} 5 \\ 0 \\ 0 \\ 3 \end{pmatrix}.$$

The initial condition $\mathbf{y}(0) = (-1, 5, 2, 4)^T$ provides

$$\begin{pmatrix} -1 \\ 5 \\ 2 \\ 4 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix} + C_4 \begin{pmatrix} 5 \\ 0 \\ 0 \\ 3 \end{pmatrix}.$$

The augmented matrix reduces.

$$\begin{pmatrix} -1 & 4 & 1 & 5 & -1 \\ 2 & -3 & 0 & 0 & 5 \\ -1 & 0 & -3 & 0 & 2 \\ 1 & 0 & 0 & 3 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Thus, $C_1 = 1$, $C_2 = -1$, $C_3 = -1$ and $C_4 = 1$, leading to

$$\mathbf{y}(t) = \begin{pmatrix} -e^{-t} \\ 2e^{-t} + 3e^{-2t} \\ -e^{-t} + 3e^{-2t} \\ e^{-t} + 3e^{-2t} \end{pmatrix}.$$

52. In Exercise 44, the fundamental set of solutions

found there lead to the general solution

$$\mathbf{y}(t) = C_1 e^{-2t} \begin{pmatrix} 11 \\ 8 \\ 0 \\ 5 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 9 \\ 7 \\ 5 \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 3 \cos 3t - \sin 3t \\ 3 \cos 3t - \sin 3t \\ 5 \cos 3t \\ -3 \cos 3t + \sin 3t \end{pmatrix} + C_4 e^{-t} \begin{pmatrix} \cos 3t + 3 \sin 3t \\ \cos 3t + 3 \sin 3t \\ 5 \sin 3t \\ -\cos 3t - 3 \sin 3t \end{pmatrix}.$$

The initial condition $\mathbf{y}(0) = (-2, -1, 6, -5)^T$ provides

$$\begin{pmatrix} -2 \\ -1 \\ 6 \\ -5 \end{pmatrix} = C_1 \begin{pmatrix} 11 \\ 8 \\ 0 \\ 5 \end{pmatrix} + C_2 \begin{pmatrix} 9 \\ 7 \\ 5 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 3 \\ 3 \\ 5 \\ -3 \end{pmatrix} + C_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

The augmented matrix reduces

$$\begin{pmatrix} 11 & 9 & 3 & 1 & -2 \\ 8 & 7 & 3 & 1 & -1 \\ 0 & 5 & 5 & 0 & 6 \\ 5 & 0 & -3 & -1 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1/5 \\ 0 & 0 & 0 & 1 & -3/5 \end{pmatrix}$$

Thus, $C_1 = -1$, $C_2 = 1$, $C_3 = 1/5$, and $C_4 =$

$-3/5$, leading to

$$\begin{aligned} \mathbf{y}(t) &= -e^{-2t} \begin{pmatrix} 11 \\ 8 \\ 0 \\ 5 \end{pmatrix} + e^{-2t} \begin{pmatrix} 9 \\ 7 \\ 5 \\ 0 \end{pmatrix} \\ &\quad + \frac{1}{5}e^{-t} \begin{pmatrix} 3 \cos 3t - \sin 3t \\ 3 \cos 3t - \sin 3t \\ 5 \cos 3t \\ -3 \cos 3t + \sin 3t \end{pmatrix} \\ &\quad - \frac{3}{5}e^{-t} \begin{pmatrix} \cos 3t + 3 \sin 3t \\ \cos 3t + 3 \sin 3t \\ 5 \sin 3t \\ -\cos 3t - 3 \sin 3t \end{pmatrix} \\ &= \begin{pmatrix} -2e^{-2t} - 2e^{-t} \sin 3t \\ -e^{-2t} - 2e^{-t} \sin 3t \\ 5e^{-2t} + e^{-t} \cos 3t - 3e^{-t} \sin 3t \\ -5e^{-2t} + 2e^{-t} \sin 3t \end{pmatrix}. \end{aligned}$$

53. The rate at which the salt content is changing in the topmost tank is

$$\frac{dx_1}{dt} = \text{Rate in} - \text{Rate out} = -\frac{x_1}{20}.$$

The rate at which the salt content is changing in the second tank is

$$\frac{dx_2}{dt} = \text{Rate in} - \text{Rate out} = \frac{x_1}{20} - \frac{x_2}{15}.$$

The rate at which the salt content is changing in the bottom tank is

$$\frac{dx_3}{dt} = \text{Rate in} - \text{Rate out} = \frac{x_2}{15} - \frac{x_3}{10}.$$

In matrix-vector form, the system becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -1/20 & 0 & 0 \\ 1/20 & -1/15 & 0 \\ 0 & 1/15 & -1/10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

As the coefficient matrix is lower triangular, the eigenvalues lie on the diagonal: $-1/20$, $-1/15$, and

$-1/10$. With $\lambda = -1/20$,

$$\begin{aligned} A - \lambda I &= A + \frac{1}{25}I = \begin{pmatrix} 0 & 0 & 0 \\ 1/20 & -1/60 & 0 \\ 0 & 1/15 & -1/20 \end{pmatrix} \\ &\xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -3/4 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, $(1/4, 3/4, 1)^T$ is an eigenvector, but we will use a multiple of this to form the solution $\mathbf{x}_1(t) = e^{-t/20}(1, 3, 4)^T$. In similar fashion, we can use the remaining eigenvalues and eigenvectors to form the general solution

$$\begin{aligned} \mathbf{x}(t) &= C_1 e^{-t/20} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + C_2 e^{-t/15} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &\quad + C_3 e^{-t/10} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Now, with the initial condition, we get

$$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \mathbf{x}(0) = C_1 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

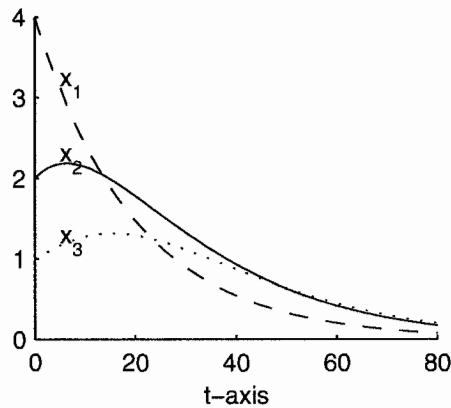
with solutions $C_1 = 4$, $C_2 = -10$, and $C_3 = 5$. Thus, the solution is:

$$\mathbf{x}(t) = 4e^{-t/20} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} - 10e^{-t/15} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 5e^{-t/10} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The components are

$$\begin{aligned} x_1 &= 4e^{-t/20} \\ x_2 &= 12e^{-t/20} - 10e^{-t/15} \\ x_3 &= 16e^{-t/20} - 20e^{-t/15} + 5e^{-t/10}. \end{aligned}$$

and the plot of each component is shown below.



54. As we've seen before, the first spring exerts a force $-k_1x$ on the first mass, while the middle spring exerts a force $k_2(y - x)$ on the right side of the first mass. Hence, by Newton's law,

$$\begin{aligned} m_1 x'' &= -k_1 x + k_2(y - x) \\ x'' &= -\frac{k_1 + k_2}{m_1} x + \frac{k_2}{m_1} y. \end{aligned}$$

The middle spring must exert an equal, but opposite force $-k_2(y - x)$ on the second mass, and the third spring exerts a force $-k_3 y$ on the second mass. Hence,

$$\begin{aligned} m_2 y'' &= -k_2(y - x) - k_3 y \\ y'' &= \frac{k_2}{m_2} x - \frac{k_2 + k_3}{m_2} y. \end{aligned}$$

Now, set $x_1 = x$, $x_2 = x'$, $x_3 = y$, and $x_4 = y'$, to obtain

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_3 \\ x'_3 &= x_4 \\ x'_4 &= \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_3. \end{aligned}$$

With $m_1 = m_2 = 1$ kg, $k_1 = k_3 = 1$ N/m, and $k_2 = 2$ N/m, in matrix-vector form, this system be-

comes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

A computer provides an eigenvalue $\lambda = i$ and associated eigenvector $\mathbf{v} = (-i, 1, -i, 1)^T$. This leads to the complex solution

$$\begin{aligned} \mathbf{x}(t) &= e^{it} \begin{pmatrix} -i \\ 1 \\ -i \\ 1 \end{pmatrix} \\ &= (\cos t + i \sin t) \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right] \\ &= \left[\cos t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right] \\ &\quad + i \left[\sin t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \cos t \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right]. \end{aligned}$$

The real and imaginary parts of this complex solution,

$$\mathbf{y}_1 = \begin{pmatrix} \sin t \\ \cos t \\ \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{pmatrix} -\cos t \\ \sin t \\ -\cos t \\ \sin t \end{pmatrix},$$

form two independent real solutions. We need 2 more independent solutions for this system of dimension 4. A computer provides another eigenvalue-eigenvector pair and a complex solution

$$\mathbf{z}(t) = e^{i\sqrt{5}t} \begin{pmatrix} \sqrt{5}i \\ -5 \\ -\sqrt{5}i \\ 5 \end{pmatrix}.$$

Breaking this into real and imaginary parts as above, the real and imaginary parts provide two more inde-

pendent solutions

$$\mathbf{y}_3 = \begin{pmatrix} -\sqrt{5} \sin \sqrt{5}t \\ -5 \cos \sqrt{5}t \\ \sqrt{5} \sin \sqrt{5}t \\ 5 \cos \sqrt{5}t \end{pmatrix} \quad \text{and}$$

$$\mathbf{y}_4 = \begin{pmatrix} \sqrt{5} \cos \sqrt{5}t \\ -5 \sin \sqrt{5}t \\ -\sqrt{5} \cos \sqrt{5}t \\ 5 \sin \sqrt{5}t \end{pmatrix}.$$

Thus, the general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} \sin t \\ \cos t \\ \sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} -\cos t \\ \sin t \\ -\cos t \\ \sin t \end{pmatrix}$$

$$+ C_3 \begin{pmatrix} -\sqrt{5} \sin \sqrt{5}t \\ -5 \cos \sqrt{5}t \\ \sqrt{5} \sin \sqrt{5}t \\ 5 \cos \sqrt{5}t \end{pmatrix}$$

$$+ C_4 \begin{pmatrix} \sqrt{5} \cos \sqrt{5}t \\ -5 \sin \sqrt{5}t \\ -\sqrt{5} \cos \sqrt{5}t \\ 5 \sin \sqrt{5}t \end{pmatrix}.$$

With the initial condition,

$$\begin{pmatrix} -2 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \mathbf{y}(0) = C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$+ C_3 \begin{pmatrix} 0 \\ -5 \\ 0 \\ 5 \end{pmatrix} + C_4 \begin{pmatrix} \sqrt{5} \\ 0 \\ -\sqrt{5} \\ 0 \end{pmatrix}$$

Hence, $C_1 = C_2 = C_3 = 0$ and $C_4 = -2\sqrt{5}/5$, and the solution we seek is

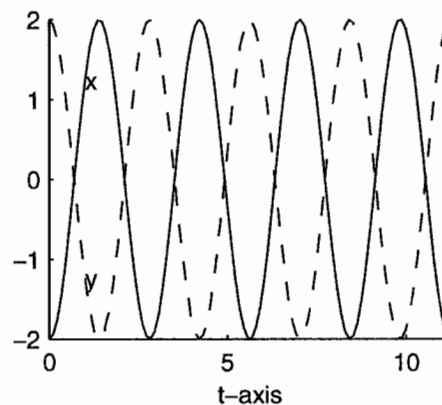
$$\mathbf{y}(t) = \begin{pmatrix} -2 \cos \sqrt{5}t \\ 2\sqrt{5} \sin \sqrt{5}t \\ 2 \cos \sqrt{5}t \\ -2\sqrt{5} \sin \sqrt{5}t \end{pmatrix}.$$

Because we are asked to plot the position of each mass versus time, and because $x_1 = x$ and $x_3 = y$, it's the first and third components of the solution that we wish to plot; i.e.,

$$x = -2 \cos \sqrt{5}t$$

$$y = 2 \cos \sqrt{5}t.$$

The plot follows.



Section 9.6. The Exponential of a Matrix

1. It is easily checked that

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the series

$$e^A = I + A + \frac{1}{2!}A^2 + \dots$$

truncates and

$$\begin{aligned} e^A &= I + A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}. \end{aligned}$$

2. It is easily checked that

$$A^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the series

$$e^A = I + A + \frac{1}{2!}A^2 + \dots$$

truncates and

$$\begin{aligned} e^A &= I + A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

3. It is easily checked that

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the series

$$e^A = I + A + \frac{1}{2!}A^2 + \dots$$

truncates and

$$\begin{aligned} e^A &= I + A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

4. If

$$A = \begin{pmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

use a computer to check that

$$\begin{aligned} A^3 &= AA^2 \\ &= \begin{pmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, the series

$$e^A = I + A + \frac{1}{2!}A^2 + \dots$$

truncates and

$$\begin{aligned} e^A &= I + A + \frac{1}{2}A^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \\ &\quad + \frac{1}{2} \begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 & -2 \\ -1 & 5/2 & -1/2 \\ 1 & -3/2 & 3/2 \end{pmatrix}. \end{aligned}$$

5. (a) If $A^2 = \alpha A$, $\alpha \neq 0$, then

$$\begin{aligned} A^3 &= AA^2 = A(\alpha A) = \alpha A^2 = \alpha(\alpha A) \\ &= \alpha^2 A. \end{aligned}$$

Similarly,

$$\begin{aligned} A^4 &= AA^3 = A(\alpha^2 A) = \alpha^2 A^2 \\ &= \alpha^2(\alpha A) = \alpha^3 A. \end{aligned}$$

Proceeding inductively,

$$A^k = \alpha^{k-1} A.$$

Now,

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots \\ &= I + tA + \frac{t^2}{2!}(\alpha A) + \frac{t^3}{3!}(\alpha^2 A) + \cdots \\ &= I + \left(t + \frac{\alpha t^2}{2!} + \frac{\alpha^2 t^3}{3!} + \cdots\right)A \\ &= I + \frac{e^{\alpha t} - 1}{\alpha}A. \end{aligned}$$

(b) One can easily show that

$$A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3A.$$

Thus, we can apply the formula developed in part (a). With $\alpha = 3$:

$$\begin{aligned} e^{tA} &= I + \frac{e^{3t} - 1}{3}A \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{e^{3t} - 1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (e^{3t} + 2)/3 & (e^{3t} - 1)/3 & (e^{3t} - 1)/3 \\ (e^{3t} - 1)/3 & (e^{3t} + 2)/3 & (e^{3t} - 1)/3 \\ (e^{3t} - 1)/3 & (e^{3t} - 1)/3 & (e^{3t} + 2)/3 \end{pmatrix} \end{aligned}$$

6. First, if

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

note that

$$\begin{aligned} A^2 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ A^4 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

after which $A^5 = A$ and the sequence repeats with

period 4. Thus,

$$\begin{aligned} e^{tA} &= I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\quad + \frac{t^3}{3!} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{t^4}{4!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 - t^2/2! + \cdots & -t + t^3/3! - \cdots \\ t - t^3/3! + \cdots & 1 - t^2/2! + \cdots \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \end{aligned}$$

7. Note that

$$\begin{aligned} A &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \\ &= aI + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Thus, by the result shown in Exercise 6,

$$\begin{aligned} e^{tA} &= e^{atI + bt \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \\ &= e^{at} I e^{bt \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \\ &= e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}. \end{aligned}$$

8. We can write

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = aI + bB,$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that B commutes with I , so

$$\begin{aligned} e^{tA} &= e^{t(aI+bB)} \\ &= e^{at} I e^{btB} \\ &= e^{at} \left(I + btB + \frac{b^2 t^2}{2!} B^2 + \frac{b^3 t^3}{3!} B^3 + \cdots \right) \\ &= e^{at} (I + btB) \\ &= e^{at} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & bt \\ 0 & 0 \end{pmatrix} \right) \\ &= e^{at} \begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

9. (a) On the one hand

$$AB = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix},$$

but

$$BA = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}.$$

- (b) Note that if $t = 1$, the result from Exercise 7 becomes

$$e^{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}.$$

Thus,

$$\begin{aligned} e^{A+B} &= e^{\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}} \\ &= e^{\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}} \\ &= \begin{pmatrix} \cos 2 & -\sin 2 \\ \sin 2 & \cos 2 \end{pmatrix}. \end{aligned}$$

- (c) Both A^2 and B^2 equal the zero matrix, so the series expansions for e^A and e^B truncate.

$$\begin{aligned} e^A &= I + A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \\ e^B &= I + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

Thus,

$$e^A e^B = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix},$$

which is not the same as e^{A+B} calculated in part (b). The problem arises because $AB \neq BA$, as was shown in part (a).

10. If $A = PDP^{-1}$, then

$$e^{tA} = I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \cdots$$

However, note that

$$A^2 = (PDP^{-1})^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}.$$

In a similar manner,

$$A^k = PD^kP^{-1},$$

for $k = 3, 4, 5, \dots$. Thus,

$$\begin{aligned} e^{tA} &= I + P(tD)P^{-1} + P(t^2D^2/2!)P^{-1} \\ &\quad + P(t^3D^3/3!)P^{-1} + \cdots \\ &= P[I + tD + t^2D^2/2! \\ &\quad + t^3D^3/3! + \cdots]P^{-1} \\ &= Pe^{tD}P^{-1}. \end{aligned}$$

11. If

$$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix},$$

then the characteristic polynomial is $p(\lambda) = (\lambda - 2)(\lambda + 1)$, giving eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. Set

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

The nullspace (eigenspace) of

$$A - 2I = \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix}$$

is generated by the single eigenvector $\mathbf{v}_1 = (1, 0)^T$. The nullspace of

$$A + I = \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix}$$

is generated by the single eigenvector $\mathbf{v}_1 = (2, -1)^T$. Set

$$P = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

It is easily checked that

$$P^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad A = PDP^{-1}.$$

Now,

$$\begin{aligned} e^{tA} &= Pe^{tD}P^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} e^{\begin{pmatrix} 2t & 0 \\ 0 & -t \end{pmatrix}} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 2e^{2t} \\ 0 & -e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} & 2e^{2t} - 2e^{-t} \\ 0 & e^{-t} \end{pmatrix}. \end{aligned}$$

12. Matrix

$$A = \begin{pmatrix} -2 & 0 \\ -3 & -3 \end{pmatrix}$$

is lower triangular, so -2 and -3 (diagonal elements) are eigenvalues. For $\lambda = -2$,

$$A + 2I = \begin{pmatrix} 0 & 0 \\ -3 & -1 \end{pmatrix},$$

so $\mathbf{v} = (1, -3)^T$ is its eigenvector. For $\lambda = -3$,

$$A + 3I = \begin{pmatrix} 1 & 0 \\ -3 & 0 \end{pmatrix},$$

so $\mathbf{v} = (0, 1)^T$ is its eigenvector. Set

$$P = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

It is easily checked that

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad A = PDP^{-1}.$$

Thus,

$$\begin{aligned} e^{tA} &= Pe^{tD}P^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} e^{\begin{pmatrix} -2t & 0 \\ 0 & -3t \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 3e^{-3t} & e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} e^{-2t} & 0 \\ -3e^{-2t} + 3e^{-3t} & e^{-3t} \end{pmatrix}. \end{aligned}$$

13. If a 2×2 matrix A has a single eigenvalue λ_1 , then the characteristic polynomial must be $p(\lambda) = (\lambda - \lambda_1)^2$. According to Cayley's Theorem, any matrix must satisfy its characteristic equation, so $(A - \lambda_1 I)^2 = 0$. Further, we know that

$$\begin{aligned} e^{At} &= e^{\lambda_1 It + (A - \lambda_1 I)t} \\ &= e^{\lambda_1 It} e^{(A - \lambda_1 I)t} \\ &= e^{\lambda_1 t} I \left[I + (A - \lambda_1 I)t \right. \\ &\quad \left. + \frac{1}{2}(A - \lambda_1 I)^2 t^2 + \dots \right]. \end{aligned}$$

But, from the above, $(A - \lambda_1 I)^k = 0$, for $k \geq 2$. Hence,

$$e^{At} = e^{\lambda_1 t} [I + (A - \lambda_1 I)t].$$

14. Matrix

$$A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix},$$

has characteristic polynomial $p(\lambda) = (\lambda + 1)^2$ and repeated eigenvalue $\lambda = -1$. We can write

$$\begin{aligned} e^{tA} &= e^{t(-I + (A + I))} \\ &= e^{-tI} e^{t(A + I)} \\ &= e^{-t} \left(I + t(A + I) + \frac{t^2}{2!}(A + I)^2 + \dots \right). \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A + I)^2 = 0$ and $(A + I)^k = 0$ for $k \geq 2$. Thus,

the series truncates.

$$\begin{aligned} e^{tA} &= e^{-t}(I + t(A + I)) \\ &= e^{-t} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right] \\ &= e^{-t} \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix} \end{aligned}$$

15. Matrix

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda + 1)^2$ and repeated eigenvalue $\lambda = -1$. We can write

$$\begin{aligned} e^{tA} &= e^{t(-I+(A+I))} \\ &= e^{-tI} e^{t(A+I)} \\ &= e^{-t} \left(I + t(A + I) + \frac{t^2}{2!}(A + I)^2 + \cdots \right). \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A + I)^2 = 0$ and $(A + I)^k = 0$ for $k \geq 2$. Thus, the series truncates.

$$\begin{aligned} e^{tA} &= e^{-t}(I + t(A + I)) \\ &= e^{-t} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \\ &= e^{-t} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \end{aligned}$$

16. Matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},$$

has characteristic polynomial $p(\lambda) = (\lambda - 1)^2$ and repeated eigenvalue $\lambda = 1$. We can write

$$\begin{aligned} e^{tA} &= e^{t(I+(A-I))} \\ &= e^{tI} e^{t(A-I)} \\ &= e^t \left(I + t(A - I) + \frac{t^2}{2!}(A - I)^2 + \cdots \right) \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A - I)^2 = 0$ and $(A - I)^k = 0$ for $k \geq 2$. Thus,

the series truncates.

$$\begin{aligned} e^{tA} &= e^t(I + t(A - I)) \\ &= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix} \end{aligned}$$

17. Matrix

$$A = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda + 1)^2$ and repeated eigenvalue $\lambda = -1$. We can write

$$\begin{aligned} e^{tA} &= e^{t(-I+(A+I))} \\ &= e^{-tI} e^{t(A+I)} \\ &= e^{-t} \left(I + t(A + I) + \frac{t^2}{2!}(A + I)^2 + \cdots \right). \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A + I)^2 = 0$ and $(A + I)^k = 0$ for $k \geq 2$. Thus the series truncates.

$$\begin{aligned} e^{tA} &= e^{-t}(I + t(A + I)) \\ &= e^{-t} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \right) \\ &= e^{-t} \begin{pmatrix} 1-2t & -t \\ 4t & 1+2t \end{pmatrix}. \end{aligned}$$

18. Using a computer, matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ -2 & 4 & -3 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda + 1)^3$ and repeated eigenvalue $\lambda = -1$. We can write

$$\begin{aligned} e^{tA} &= e^{t(-I+(A+I))} \\ &= e^{-tI} e^{t(A+I)} \\ &= e^{-t} \left(I + t(A + I) + \frac{t^2}{2!}(A + I)^2 + \cdots \right) \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A + I)^3 = 0$ and $(A + I)^k = 0$ for $k \geq 3$. But,

$$A + I = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -2 & 4 & -2 \end{pmatrix} \quad \text{and}$$

$$(A + I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so $(A + I)^k = 0$ for $k \geq 2$ and the series will truncate earlier.

$$\begin{aligned} e^{tA} &= e^{-t} (I + t(A + I)) \\ &= e^{-t} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -2 & 4 & -2 \end{pmatrix} \right] \\ &= e^{-t} \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 + 2t & -t \\ -2t & 4t & 1 - 2t \end{pmatrix} \end{aligned}$$

19. Using a computer we find that A has eigenvalue -1 with algebraic multiplicity 3. We also find that

$$A + I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix},$$

$$(A + I)^2 = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix},$$

and

$$(A + I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} e^{tA} &= e^{\lambda t} e^{t(A - \lambda I)} \\ &= e^{-t} e^{t(A + I)} \\ &= e^{-t} \left[I + t(A + I) + \frac{t^2}{2} (A + I)^2 \right] \\ &= e^{-t} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \right. \\ &\quad \left. + \frac{t^2}{2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} \right] \\ &= e^{-t} \begin{pmatrix} 1 + t^2/2 & -t - t^2/2 & t^2/2 \\ -t & 1 + t & -t \\ -t - t^2/2 & 2t + t^2/2 & 1 - t - t^2/2 \end{pmatrix} \end{aligned}$$

20. Using a computer, matrix

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -4 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda + 2)^3$ and repeated eigenvalue $\lambda = -2$. We can write

$$\begin{aligned} e^{tA} &= e^{t(-2I + (A + 2I))} \\ &= e^{-2tI} e^{t(A + 2I)} \\ &= e^{-2t} \left(I + t(A + 2I) + \frac{t^2}{2!} (A + 2I)^2 + \dots \right) \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A + 2I)^3 = 0$, but

$$A + 2I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -4 & -2 \end{pmatrix} \quad \text{and}$$

$$(A + 2I)^2 = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so $(A + I)^k = 0$ for $k \geq 3$ and the series truncates

at this point.

$$\begin{aligned}
 e^{tA} &= e^{-2t} \left(I + t(A + 2I) + \frac{t^2}{2!}(A + 2I)^2 \right) \\
 &= e^{-2t} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -4 & -2 \end{pmatrix} \right. \\
 &\quad \left. + \frac{t^2}{2} \begin{pmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\
 &= e^{-2t} \begin{pmatrix} 1 & -t - t^2 & -t^2/2 \\ 0 & 1 + 2t & t \\ 0 & -4t & 1 - 2t \end{pmatrix}
 \end{aligned}$$

21. Using a computer, matrix

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & -2 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda + 2)^3$ and repeated eigenvalue -2 . We can write

$$\begin{aligned}
 e^{tA} &= e^{t(-2I + (A + 2I))} \\
 &= e^{-2tI} e^{t(A + 2I)} \\
 &= e^{-2t} \left(I + t(A + 2I) + \frac{t^2}{2!}(A + 2I)^2 + \cdots \right)
 \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A + 2I)^3 = 0$, but

$$\begin{aligned}
 A + 2I &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \quad \text{and} \\
 (A + 2I)^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

so $(A + 2I)^2 = 0$ for $k \geq 2$ and the series truncates

at this point.

$$\begin{aligned}
 e^{tA} &= e^{-2t} (I + t(A + 2I)) \\
 &= e^{-2t} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \right) \\
 &= e^{-2t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & t & 1 \end{pmatrix}.
 \end{aligned}$$

22. Using a computer, matrix

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda - 1)^4$ and repeated eigenvalue $\lambda = 1$. We can write

$$\begin{aligned}
 e^{tA} &= e^{t(I + (A - I))} \\
 &= e^{tI} e^{t(A - I)} \\
 &= e^t \left(I + t(A - I) + \frac{t^2}{2!}(A - I)^2 + \cdots \right)
 \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A - I)^4 = 0$, but

$$\begin{aligned}
 A - I &= \begin{pmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \quad \text{and} \\
 (A - I)^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

so $(A - I)^k = 0$ for $k \geq 2$ and the series truncates at this point.

$$\begin{aligned}
 e^{tA} &= e^t (I + t(A - I)) \\
 &= e^t \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \right] \\
 &= e^t \begin{pmatrix} 1 & -t & 2t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -t & 2t & 1 \end{pmatrix}
 \end{aligned}$$

23. Using a computer, matrix

$$A = \begin{pmatrix} -5 & 0 & -1 & 4 \\ -4 & 0 & 1 & 5 \\ 4 & -4 & -5 & -4 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda + 3)^4$ and repeated eigenvalue $\lambda = -3$. We can write

$$\begin{aligned} e^{tA} &= e^{t(-3I + (A+3I))} \\ &= e^{-3tI} e^{t(A+3I)} \\ &= e^{-3t} \left(I + t(A+3I) + \frac{t^2}{2!}(A+3I)^2 + \cdots \right) \end{aligned}$$

Matrix A must satisfy its characteristic polynomial, so $(A + 3I)^4 = 0$, but

$$A + 3I = \begin{pmatrix} -2 & 0 & -1 & 4 \\ -4 & 3 & 1 & 5 \\ 4 & -4 & -2 & -4 \\ 0 & -1 & -1 & 1 \end{pmatrix} \quad \text{and}$$

$$(A + 3I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

So $(A + 3I)^k = 0$ for $k \geq 2$ and the series truncates at this point.

$$\begin{aligned} e^{tA} &= e^{-3t} (I + t(A + 3I)) \\ &= e^{-3t} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\quad + t e^{-3t} \begin{pmatrix} -2 & 0 & -1 & 4 \\ -4 & 3 & 1 & 5 \\ 4 & -4 & -2 & -4 \\ 0 & -1 & -1 & 1 \end{pmatrix} \\ &= e^{-3t} \begin{pmatrix} 1-2t & 0 & -t & 4t \\ -4t & 1+3t & t & 5t \\ 4t & -4t & 1-2t & -4t \\ 0 & -t & -t & 1+t \end{pmatrix}. \end{aligned}$$

24. Using a computer, matrix

$$A = \begin{pmatrix} 0 & 4 & 5 & -2 \\ 1 & -5 & -7 & 3 \\ 0 & 2 & 3 & -1 \\ 3 & -10 & -13 & 6 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda - 1)^4$ and repeated eigenvalue $\lambda = 1$. We can write

$$e^{tA} = e^{t(I + (A-I))} = e^{tI} e^{t(A-I)} = e^t \left(I + t(A-I) + \frac{t^2}{2!}(A-I)^2 + \cdots \right)$$

Matrix A must satisfy its characteristic polynomial, so $(A - I)^4 = 0$, but

$$\begin{aligned} A - I &= \begin{pmatrix} -1 & 4 & 5 & -2 \\ 1 & -6 & -7 & 3 \\ 0 & 2 & 2 & -1 \\ 3 & -10 & -13 & 5 \end{pmatrix}, & (A - I)^2 &= \begin{pmatrix} -1 & 2 & 3 & -1 \\ 2 & -4 & -6 & 2 \\ -1 & 2 & 3 & -1 \\ 2 & -4 & -6 & 2 \end{pmatrix}, \quad \text{and} \\ (A - I)^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

so $(A - I)^k = 0$ for $k \geq 3$ and the series truncates at this point.

$$\begin{aligned} e^{tA} &= e^t \left(I + t(A - I) + \frac{t^2}{2}(A - I)^2 \right) \\ &= e^t \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} -1 & 4 & 5 & -2 \\ 1 & -6 & -7 & 3 \\ 0 & 2 & 2 & -1 \\ 3 & -10 & -13 & 5 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -1 & 2 & 3 & -1 \\ 2 & -4 & -6 & 2 \\ -1 & 2 & 3 & -1 \\ 2 & -4 & -6 & 2 \end{pmatrix} \right] \\ &= \frac{1}{2} e^t \begin{pmatrix} 2 - 2t - t^2 & 8t + 2t^2 & 10t + 3t^2 & -4t - t^2 \\ 2t + 2t^2 & 2 - 12t - 4t^2 & -14t - 6t^2 & 6t + 2t^2 \\ -t^2 & 4t + 2t^2 & 2 + 4t + 3t^2 & -2t - t^2 \\ 6t + 2t^2 & -20t - 4t^2 & -26t - 6t^2 & 2 + 10t + 2t^2 \end{pmatrix} \end{aligned}$$

25. Using a computer, matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -9 & 4 & 1 & 4 \\ 13 & -3 & -1 & -5 \\ 2 & -1 & 0 & 0 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda - 1)^4$ and repeated eigenvalue $\lambda = 1$. We can write

$$e^{tA} = e^{t(I + (A - I))} = e^{tI} e^{t(A - I)} = e^t \left(I + t(A - I) + \frac{t^2}{2!}(A - I)^2 + \cdots \right).$$

Matrix A must satisfy its characteristic polynomial, so $(A - I)^4 = 0$. But

$$\begin{aligned} A - I &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -9 & 3 & 1 & 4 \\ 13 & -3 & -2 & -5 \\ 2 & -1 & 0 & -1 \end{pmatrix}, \quad (A - I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -6 & 2 & 1 & 3 \\ -9 & 2 & 1 & 3 \\ 7 & -2 & -1 & -3 \end{pmatrix}, \quad \text{and} \\ (A - I)^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

so $(A - I)^k = 0$ for $k \geq 4$ and the series truncates at this point.

$$\begin{aligned}
 e^{tA} &= \left(I + t(A - I) + \frac{t^2}{2!}(A - I)^2 + \frac{t^3}{3!}(A - I)^3 \right) \\
 &= e^t \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 & 0 \\ -9 & 3 & 1 & 4 \\ 13 & -3 & -2 & -5 \\ 2 & -1 & 0 & -1 \end{pmatrix} \right. \\
 &\quad \left. + \frac{t^2}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -6 & 2 & 1 & 3 \\ -9 & 2 & 1 & 3 \\ 7 & -2 & -1 & -3 \end{pmatrix} + \frac{t^3}{6} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \right] \\
 &= e^t \begin{pmatrix} 1 & 0 & 0 & 0 \\ -9t - 3t^2 + t^3/6 & 1 + 3t + t^2 & t + t^2/2 & 4t + 3t^2/2 \\ 13t - 9t^2/2 + t^3/6 & -3t + t^2 & 1 - 2t + t^2/2 & -5t + 3t^2/2 \\ 2t + 7t^2/2 - t^3/6 & -t - t^2 & -t^2/2 & 1 - t - 3t^2/2 \end{pmatrix}.
 \end{aligned}$$

26. If

$$A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix},$$

then

$$\begin{aligned}
 p(\lambda) &= -\det(A - \lambda I) \\
 &= - \begin{vmatrix} -2 - \lambda & 1 & -1 \\ 1 & -3 - \lambda & 0 \\ 3 & -5 & -\lambda \end{vmatrix}
 \end{aligned}$$

Expanding down the third column,

$$\begin{aligned}
 p(\lambda) &= 1 \begin{vmatrix} 1 & -3 - \lambda \\ 3 & -5 \end{vmatrix} + \lambda \begin{vmatrix} -2 - \lambda & 1 \\ 1 & -3 - \lambda \end{vmatrix} \\
 &= (4 + 3\lambda) + \lambda(\lambda^2 + 5\lambda + 5) \\
 &= \lambda^3 - 5\lambda^2 - 8\lambda - 4.
 \end{aligned}$$

Zeros must be factors of the constant term, so -1 is a possibility. Dividing by $\lambda + 1$ leads to the following factorization

$$p(\lambda) = (\lambda + 1)(\lambda + 2)^2$$

and eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Because

$$A + I = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -2 & 0 \\ 3 & -5 & 1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

the geometric multiplicity of $\lambda_1 = -1$ is one, and an eigenvector is $\mathbf{v}_1 = (-2, -1, 1)^T$, leading to the solution

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

Next,

$$A + 2I = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

the geometric multiplicity of $\lambda_2 = -2$ is one, and an eigenvector is $\mathbf{v}_2(t) = (1, 1, 1)^T$, leading to the solution

$$\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Notice that

$$(A + 2I)^2 = \begin{pmatrix} -2 & 4 & -2 \\ -1 & 2 & -1 \\ 1 & -2 & 1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has dimension two, equalling the algebraic multiplicity of λ_2 . Thus, we can pick a vector in the nullspace of $(A + 2I)^2$ that is not in the nullspace of $A + 2I$. Choose $\mathbf{v}_3 = (-1, 0, 1)^T$, which is not a multiple of

$\mathbf{v}_2$, making the set $\{\mathbf{v}_2, \mathbf{v}_3\}$ independent, and giving a third solution,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA} \mathbf{v}_3 \\ &= e^{-2t} [\mathbf{v}_3 + t(A + 2I)\mathbf{v}_3] \\ &= e^{-2t} \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-2t} \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} -1-t \\ -t \\ 1-t \end{pmatrix}. \end{aligned}$$

Because

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \begin{vmatrix} -2 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1,$$

the solutions are independent for all t and form a fundamental set of solutions.

27. If

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -\begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & 2-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{vmatrix}. \end{aligned}$$

Expanding across the third row,

$$p(\lambda) = (\lambda - 2) \begin{vmatrix} 1-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = (\lambda - 2)^2(\lambda - 1),$$

so the eigenvalues are 2 and 1, with algebraic multiplicities 2 and 1, respectively. For $\lambda_1 = 1$,

$$A - I = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so the geometric multiplicity of λ_1 is 1 and an eigenvector is $\mathbf{v}_1 = (-1, 2, 0)^T$, providing exponential solution

$$\mathbf{y}_1(t) = e^{tA} \mathbf{v}_1 = e^t \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 2$,

$$A - 2I = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so there are two free variables and the geometric multiplicity is 2. Thus, $\mathbf{v}_2 = (0, 1, 0)^T$ and $\mathbf{v}_3 = (1, 0, 1)^T$ are independent eigenvectors and

$$\begin{aligned} \mathbf{y}_2(t) &= e^{tA} \mathbf{v}_2 = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \\ \mathbf{y}_3(t) &= e^{tA} \mathbf{v}_3 = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

are independent solutions. Because

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1,$$

the solutions are independent for all t and form a fundamental set of solutions.

28. If

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 0 & 1 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -\begin{vmatrix} -\lambda & 1 & 0 \\ -4 & 4-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} \end{aligned}$$

Expanding down the third column,

$$\begin{aligned} p(\lambda) &= (\lambda - 1) \begin{vmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{vmatrix} \\ &= (\lambda - 1)(\lambda^2 - 4\lambda + 4) \\ &= (\lambda - 1)(\lambda - 2)^2, \end{aligned}$$

providing eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, with algebraic multiplicities 1 and 2, respectively. Because

$$A - I = \begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ -2 & 0 & 0 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the geometric multiplicity of $\lambda_1 = 1$ is one, and an eigenvector is $\mathbf{v}_1 = (0, 0, 1)^T$, leading to the solution

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next,

$$A - 2I = \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 0 & -1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

the geometric multiplicity of $\lambda_2 = 2$ is one, and an eigenvector is $\mathbf{v}_2(t) = (-1, -2, 2)^T$, leading to the solution

$$\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{2t} \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}.$$

Next

$$(A - 2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & -2 & 1 \end{pmatrix}$$

has dimension two, equaling the algebraic multiplicity of λ_2 . Thus, we can pick a vector in the nullspace of $(A - 2I)^2$ that is not in the nullspace of $A - 2I$. Choose $\mathbf{v}_3 = (1, 0, -6)^T$, which is not a multiple of $\mathbf{v}_2$, making the set $\{\mathbf{v}_2, \mathbf{v}_3\}$ independent, and giving a third solution,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA}\mathbf{v}_3 \\ &= e^{2t} [\mathbf{v}_3 + t(A - 2I)\mathbf{v}_3] \\ &= e^{2t} \left[\begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} + t \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} \right] \\ &= e^{2t} \left[\begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} + t \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} \right] \\ &= e^{2t} \begin{pmatrix} 1 - 2t \\ -4t \\ -6 + 4t \end{pmatrix}. \end{aligned}$$

Because

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \begin{vmatrix} 0 & -1 & 1 \\ 0 & -2 & 0 \\ 1 & 2 & -6 \end{vmatrix} = 2,$$

the solutions are independent for all t and form a fundamental set of solutions.

29. If

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -1 \\ 0 & 4 & -1 \end{pmatrix},$$

then

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -\begin{vmatrix} -1 - \lambda & 0 & 0 \\ 2 & -5 - \lambda & -1 \\ 0 & 4 & -1 - \lambda \end{vmatrix}. \end{aligned}$$

$$\begin{aligned} p(\lambda) &= (\lambda + 1) \begin{vmatrix} -5 - \lambda & -1 \\ 4 & -1 - \lambda \end{vmatrix} \\ &= (\lambda + 1)(\lambda^2 + 6\lambda + 9) \\ &= (\lambda + 1)(\lambda + 3)^2, \end{aligned}$$

providing eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$, with algebraic multiplicities 1 and 2, respectively. Because

$$A + I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -1 \\ 0 & 4 & 0 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the geometric multiplicity of $\lambda_1 = -1$ is 1, and an eigenvector is $\mathbf{v}_1 = (1, 0, 2)^T$, providing the exponential solution

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -3$

$$A + 3I = \begin{pmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ 0 & 4 & 2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

has one free variable, so the geometric multiplicity of $\lambda_2 = -3$ is 1. An eigenvector is $\mathbf{v}_2 = (0, -1, 2)^T$, giving a second exponential solution,

$$\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{-3t} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

Next,

$$(A + 3I)^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has dimension 2, equaling the algebraic multiplicity of λ_2 . Thus, we can pick a vector in the nullspace of $(A + 3I)^2$ that is not in the nullspace of $A + 3I$. Choose $\mathbf{v}_3 = (0, 1, 0)^T$, which is not a multiple of $\mathbf{v}_2$, making the set $\{\mathbf{v}_2, \mathbf{v}_3\}$ independent and giving a third solution,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA}\mathbf{v}_3 \\ &= e^{-3t} [\mathbf{v}_3 + t(A + 3I)\mathbf{v}_3] \\ &= e^{-3t} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \\ &= e^{-3t} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \right] \\ &= e^{-3t} \begin{pmatrix} 0 \\ 1 - 2t \\ 4t \end{pmatrix}. \end{aligned}$$

Because

$$\det [\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 2 & 0 \end{vmatrix} = -2,$$

the solutions are independent for all t and form a fundamental set of solutions.

30. Using a computer, matrix

$$A = \begin{pmatrix} 11 & -42 & 4 & 28 \\ -12 & 39 & -4 & -28 \\ 0 & 0 & -1 & 0 \\ -24 & 81 & -8 & -57 \end{pmatrix},$$

has characteristic polynomial,

$$p(\lambda) = (\lambda + 3)^2(\lambda + 1)^2,$$

providing eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$, with algebraic multiplicities 2 and 2, respectively. Because

$$\begin{aligned} A + 3I &= \begin{pmatrix} 14 & -42 & 4 & 28 \\ -12 & 42 & -4 & -28 \\ 0 & 0 & 2 & 0 \\ -24 & 81 & -8 & -54 \end{pmatrix} \\ &\xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

the geometric multiplicity of $\lambda_1 = -3$ is one, and an eigenvector is $\mathbf{v}_1 = (0, -2, 0, -3)^T$, leading to the solution

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{-3t} \begin{pmatrix} 0 \\ -2 \\ 0 \\ -3 \end{pmatrix}.$$

Next, the nullspace of

$$\begin{aligned} (A + 3I)^2 &= \begin{pmatrix} 28 & -84 & 8 & 56 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ -12 & 36 & -4 & -24 \end{pmatrix} \\ &\xrightarrow{\text{ref}} \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

has dimension two, equaling the algebraic multiplicity of λ_1 . Thus, we can pick a vector in the nullspace of $(A + 3I)^2$ that is not in the nullspace of $A + 3I$. Choose $\mathbf{v}_2 = (-2, 0, 0, 1)^T$, which is not a multiple of $\mathbf{v}_1$, making the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ independent, and

giving a second solution,

$$\begin{aligned} \mathbf{y}_2(t) &= e^{tA} \mathbf{v}_2 \\ &= e^{-3t} [\mathbf{v}_2 + t(A + 3I)\mathbf{v}_2] \\ &= e^{-3t} \left[\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -4 \\ 0 \\ -6 \end{pmatrix} \right] \\ &= e^{-3t} \begin{pmatrix} -2 \\ -4t \\ 0 \\ 1 - 6t \end{pmatrix}. \end{aligned}$$

Because the nullspace of

$$\begin{aligned} A + I &= \begin{pmatrix} 12 & -42 & 4 & 28 \\ -12 & 40 & -4 & -28 \\ 0 & 0 & 0 & 0 \\ -24 & 81 & -8 & -56 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1/3 & 7/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

has dimension two, equaling the algebraic multiplicity of λ_2 , we can pick two independent eigenvectors in the nullspace of $A + I$. Choose $\mathbf{v}_3 = (-1, 0, 3, 0)^T$ and $\mathbf{v}_4 = (-7, 0, 0, 3)^T$. Note that they are not multiples of one another, making the set $\{\mathbf{v}_3, \mathbf{v}_4\}$ independent, and giving a third and fourth solution,

$$\mathbf{v}_3(t) = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_4(t) = e^{-t} \begin{pmatrix} -7 \\ 0 \\ 0 \\ 3 \end{pmatrix}.$$

Because

$$\begin{aligned} \det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0), \mathbf{y}_4(0)] &= \begin{vmatrix} 0 & -2 & -1 & -7 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ -3 & 1 & 0 & 3 \end{vmatrix} \\ &= 6, \end{aligned}$$

the solutions are independent for all t and form a fundamental set of solutions.

31. Using a computer, matrix

$$A = \begin{pmatrix} 18 & -7 & 24 & 24 \\ 15 & -8 & 20 & 16 \\ 0 & 0 & -1 & 0 \\ -12 & 4 & -15 & -17 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = (\lambda + 3)^2(\lambda + 1)^2$, providing eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$, with algebraic multiplicities 2 and 2, respectively. Because

$$\begin{aligned} A + 3I &= \begin{pmatrix} 21 & -7 & 24 & 24 \\ 15 & -5 & 20 & 16 \\ 0 & 0 & 2 & 0 \\ -12 & 4 & -15 & -14 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

the geometric multiplicity of $\lambda_1 = -3$ is 1, and an eigenvector is $\mathbf{v}_1 = (1, 3, 0, 0)^T$, giving solution

$$\mathbf{y}_1(t) = e^{tA} \mathbf{v}_1 = e^{-3t} \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

Next, the nullspace of

$$\begin{aligned} (A + 3I)^2 &= \begin{pmatrix} 48 & -16 & 52 & 56 \\ 48 & -16 & 60 & 56 \\ 0 & 0 & 4 & 0 \\ -24 & 8 & -28 & -28 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1/3 & 0 & 7/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

has dimension 2, equaling the algebraic multiplicity of λ_1 . Thus, we can pick a vector in the nullspace of $(A + 3I)^2$ that is not in the nullspace of $A + 3I$. Choose $\mathbf{v}_2 = (-7, 0, 0, 6)^T$, which is not a multiple of $\mathbf{v}_1$, making the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ independent, and

giving a second solution

$$\begin{aligned} \mathbf{y}_2(t) &= e^{tA} \mathbf{v}_2 \\ &= e^{-3t} [\mathbf{v}_2 + t(A + 3I)\mathbf{v}_2] \\ &= e^{-3t} \left[\begin{pmatrix} -7 \\ 0 \\ 0 \\ 6 \end{pmatrix} + t \begin{pmatrix} -3 \\ -9 \\ 0 \\ 0 \end{pmatrix} \right] \\ &= e^{-3t} \begin{pmatrix} -7 - 3t \\ -9t \\ 0 \\ 6 \end{pmatrix}. \end{aligned}$$

For $\lambda_2 = -1$,

$$\begin{aligned} A + I &= \begin{pmatrix} 19 & -7 & 24 & 24 \\ 15 & -7 & 20 & 16 \\ 0 & 0 & 0 & 0 \\ -12 & 4 & -15 & -16 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

the geometric multiplicity of $\lambda_2 = -1$ is 1, and an eigenvector is $\mathbf{v}_3 = (-2, -2, 0, 1)^T$, giving the exponential solution

$$\mathbf{y}_3(t) = e^{tA} \mathbf{v}_3 = e^{-t} \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

Next, the nullspace of

$$\begin{aligned} (A + I)^2 &= \begin{pmatrix} -32 & 12 & -44 & -40 \\ -12 & 8 & -20 & -8 \\ 0 & 0 & 0 & 0 \\ 24 & -8 & 32 & 32 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

has dimension 2, equaling the algebraic multiplicity of λ_2 , so we can pick a vector in the nullspace of $(A + I)^2$ that is not in the nullspace of $A + I$.

Choose $\mathbf{v}_4 = (-1, 1, 1, 0)^T$, which is not a multiple of $\mathbf{v}_3$, making the set $\{\mathbf{v}_3, \mathbf{v}_4\}$ independent, and giving a fourth solution.

$$\begin{aligned} \mathbf{y}_4(t) &= e^{tA} \mathbf{v}_4 \\ &= e^{-t} [\mathbf{v}_4 + t(A + I)\mathbf{v}_4] \\ &= e^{-t} \left[\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-t} \begin{pmatrix} -1 - 2t \\ 1 - 2t \\ 1 \\ t \end{pmatrix}. \end{aligned}$$

Because

$$\begin{aligned} \det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0), \mathbf{y}_4(0)] &= \begin{vmatrix} 1 & -7 & -2 & -1 \\ 3 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 6 & 1 & 0 \end{vmatrix} \\ &= 3, \end{aligned}$$

the solutions are independent for all t and form a fundamental set of solutions.

32. Using a computer, matrix

$$A = \begin{pmatrix} 0 & -30 & -42 & 40 & -48 & 14 \\ 1 & 7 & 9 & -9 & 10 & -2 \\ -1 & 5 & 8 & -6 & 6 & -2 \\ 2 & 45 & 64 & -60 & 72 & -20 \\ 2 & 33 & 47 & -45 & 55 & -15 \\ 0 & 7 & 11 & -10 & 10 & -1 \end{pmatrix},$$

has characteristic polynomial,

$$p(\lambda) = (\lambda - 1)^3(\lambda - 2)^3,$$

providing eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, with al-

gebraic multiplicities 3 and 3, respectively. Because

$$A - I = \begin{pmatrix} -1 & -30 & -42 & 40 & -48 & 14 \\ 1 & 6 & 9 & -9 & 10 & -2 \\ -1 & 5 & 7 & -6 & 6 & -2 \\ 2 & 45 & 64 & -61 & 72 & -20 \\ 2 & 33 & 47 & -45 & 54 & -15 \\ 0 & 7 & 11 & -10 & 10 & -2 \end{pmatrix}$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & 4 & -2 \\ 0 & 1 & 0 & 0 & 4 & -3 \\ 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the geometric multiplicity of $\lambda_1 = 1$ is two, and we can choose two independent eigenvectors from the nullspace of $A - I$, $\vec{v}_1 = (-4, -4, -2, -4, 1, 0)^T$ and $\vec{v}_2 = (2, 3, 1, 3, 0, 1)^T$. These eigenvectors provide the solutions

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^t \begin{pmatrix} -4 \\ -4 \\ -2 \\ -4 \\ 1 \\ 0 \end{pmatrix} \quad \text{and}$$

$$\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^t \begin{pmatrix} 2 \\ 3 \\ 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$

Next, the nullspace of

$$(A - I)^2 = \begin{pmatrix} -3 & -46 & -64 & 62 & -76 & 22 \\ -2 & -38 & -53 & 51 & -62 & 18 \\ -1 & 9 & 12 & -11 & 12 & -4 \\ 1 & 21 & 29 & -28 & 34 & -10 \\ 2 & 25 & 35 & -34 & 42 & -12 \\ -4 & -37 & -52 & 51 & -64 & 18 \end{pmatrix}$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -4/7 & 12/7 & -2/7 \\ 0 & 1 & 0 & -5/7 & 8/7 & -6/7 \\ 0 & 0 & 1 & -3/7 & 2/7 & 2/7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has dimension three, equaling the algebraic multiplicity of λ_1 . Thus, we can pick a vector in the nullspace of $(A - I)^2$ that is not in the nullspace of $A - I$. We will try $\mathbf{v}_3 = (4, 5, 3, 7, 0, 0)^T$, but we'll need to check independence before proceeding. However,

$$\begin{pmatrix} -4 & 2 & 4 \\ -4 & 3 & 5 \\ -2 & 1 & 3 \\ -4 & 3 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and a pivot in each column tells us that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an independent set. A third solution is

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA}\mathbf{v}_3 \\ &= e^t [\mathbf{v}_3 + t(A - I)\mathbf{v}_3] \\ &= e^t \left[\begin{pmatrix} 4 \\ 5 \\ 3 \\ 7 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 14 \\ -4 \\ -2 \\ -22 \\ -16 \\ -4 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} 4 + 14t \\ 5 - 4t \\ 3 - 2t \\ -16t \\ 1 - 4t \end{pmatrix}. \end{aligned}$$

Next, the nullspace of

$$A - 2I = \begin{pmatrix} -2 & -30 & -42 & 40 & -48 & 14 \\ 1 & 5 & 9 & -9 & 10 & -2 \\ -1 & 5 & 6 & -6 & 6 & -2 \\ 2 & 45 & 64 & -62 & 72 & -20 \\ 2 & 33 & 47 & -45 & 53 & -15 \\ 0 & 7 & 11 & -10 & 10 & -3 \end{pmatrix}$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

has dimension one, giving a single eigenvalue $\vec{v}_4 = (2, 2, -1, -1, -1, 1)^T$ and a fourth solution,

$$\mathbf{y}_4(t) = e^{tA} \mathbf{v}_4 = e^{2t} \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Next,

$$\begin{aligned} (A - 2I)^2 &= \begin{pmatrix} 0 & 14 & 20 & -18 & 20 & -6 \\ -4 & -49 & -71 & 69 & -82 & 22 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ -3 & -69 & -99 & 95 & -110 & 30 \\ -2 & -41 & -59 & 56 & -65 & 18 \\ -4 & -51 & -74 & 71 & -84 & 23 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

which has dimension one. Pick $\mathbf{v}_5 = (0, -1, 1, 0, 0, 1)^T$ in the nullspace of $(A - 2I)^2$. Note that it is not a multiple of $\mathbf{v}_4$ and is therefore independent of $\mathbf{v}_4$. Now,

$$\begin{aligned} \mathbf{y}_5(t) &= e^{tA} \mathbf{v}_5 \\ &= e^{2t} [\mathbf{v}_5 + t(A - 2I)\mathbf{v}_5] \\ &= e^{2t} \left[\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right] \\ &= e^{2t} \begin{pmatrix} 2t \\ -1 + 2t \\ 1 - t \\ -t \\ -t \\ 1 + t \end{pmatrix}. \end{aligned}$$

Finally, examine the nullspace of

$$\begin{aligned} (A - 2I)^3 &= \begin{pmatrix} -2 & -22 & -32 & 30 & -36 & 10 \\ 4 & 73 & 105 & -101 & 118 & -32 \\ 0 & 5 & 7 & -7 & 8 & -2 \\ 6 & 105 & 151 & -145 & 170 & -46 \\ 4 & 61 & 88 & -84 & 99 & -27 \\ 5 & 79 & 114 & -109 & 128 & -35 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

which has dimension three. Pick $\mathbf{v}_6 = (-1, -3, 1, 0, 1, 0)^T$ in the nullspace of $(A - 2I)^3$. We will need to check independence. Since

$$\begin{pmatrix} 2 & 0 & -1 \\ 2 & -1 & -3 \\ -1 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has a pivot in each column, the set $\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is independent. A sixth solution is formed as follows.

$$\begin{aligned} \mathbf{y}_6(t) &= e^{tA} \mathbf{v}_6 \\ &= e^{2t} \left[\mathbf{v}_6 + t(A - 2I)\mathbf{v}_6 + \frac{t^2}{2}(A - 2I)^2\mathbf{v}_6 \right] \\ &= e^{2t} \left[\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right] \\ &= e^{2t} \begin{pmatrix} -1 + 2t - t^2 \\ -3 + 3t - t^2 \\ 1 - 2t + t^2/2 \\ -t + t^2/2 \\ 1 - t + t^2/2 \\ -t^2/2 \end{pmatrix} \end{aligned}$$

Because

$$\begin{aligned} & \det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0), \mathbf{y}_4(0), \mathbf{y}_5(0), \mathbf{y}_6(0)] \\ &= \begin{vmatrix} -4 & 2 & 4 & 2 & 0 & -1 \\ -4 & 3 & 5 & 2 & -1 & -3 \\ -2 & 1 & 3 & -1 & 1 & 1 \\ -4 & 3 & 7 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{vmatrix} \\ &= 1, \end{aligned}$$

the solutions are independent for all t and form a fundamental set of solutions.

33. Using a computer, matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 \\ -14 & -2 & -7 & 11 & -9 & -8 \\ -9 & -3 & -3 & 7 & -6 & -4 \\ -19 & -5 & -9 & 17 & -12 & -9 \\ -29 & -7 & -13 & 23 & -16 & -15 \\ 19 & 5 & 9 & -15 & 12 & 11 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda - 1)^3(\lambda - 2)^3,$$

providing eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, with algebraic multiplicities 3 and 3, respectively. Using a computer, the nullspace of $A - I$ has dimension one, as $A - I$ reduces to

$$A - I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\lambda_1 = 1$ has geometric multiplicity 1. Using a computer to reduce $(A - I)^2$, you can check that the nullspace of $(A - I)^2$ has dimension 2. The key here is that $(A - I)^3$ reduces to

$$(A - I)^3 \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 0 & -3/2 & -5/2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A basis for the nullspace is provided by the vectors.

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 3/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \\ \mathbf{v}_3 &= \begin{pmatrix} -2 \\ 5/2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Because $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are in the nullspace of $(A - I)^3$ we know that

$$\mathbf{y}(t) = e^{At}\mathbf{v} = \left[\mathbf{v} + t(A - I)\mathbf{v} + \frac{t^2}{2}(A - I)^2\mathbf{v} \right]$$

for each $\mathbf{v} = \mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. This fact, and a computer, provide the following solutions.

$$\begin{aligned} \mathbf{y}_1(t) &= e^{tA}\mathbf{v}_1 = e^t \begin{pmatrix} -1 - t - t^2/2 \\ 2 + t \\ 1 - t - t^2 \\ -t^2/2 \\ 2t + t^2/2 \\ t^2/2 \end{pmatrix}, \\ \mathbf{y}_2(t) &= e^{tA}\mathbf{v}_2 = e^t \begin{pmatrix} -1 - t - t^2/4 \\ 3/2 + t/2 \\ -3t/2 - t^2/2 \\ -t/2 - t^2/4 \\ 1 + 3t/2 + t^2/4 \\ t/2 + t^2/4 \end{pmatrix}, \end{aligned}$$

and

$$\mathbf{y}_3(t) = e^{tA}\mathbf{v}_3 = e^t \begin{pmatrix} -2 - t - 3t^2/4 \\ 5/2 + 3t/2 \\ -t/2 - 3t^2/2 \\ -1 + t/2 - 3t^2/4 \\ 10t/4 + 3t^2/4 \\ 1 - t/2 + 3t^2/4 \end{pmatrix}.$$

On the other hand, $A - 2I$ reduces to

$$A - 2I \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 1/6 & -5/6 & 1/2 & 0 \\ 0 & 1 & 7/6 & 1/6 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so the nullspace of $A - 2I$ has dimension 3, and the geometric multiplicity of $\lambda_2 = 2$ is 3. A basis for the eigenspace contains the vectors

$$\mathbf{v}_4 = \begin{pmatrix} -1/6 \\ -7/6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 5/6 \\ -1/6 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$\mathbf{v}_6 = \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The corresponding solutions are

$$\mathbf{y}_4(t) = e^{tA}\mathbf{v}_4 = e^{2t} \begin{pmatrix} -1/6 \\ -7/6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{y}_5(t) = e^{tA}\mathbf{v}_5 = e^{2t} \begin{pmatrix} 5/6 \\ -1/6 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\mathbf{y}_6(t) = e^{tA}\mathbf{v}_6 = e^{2t} \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Because

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0), \mathbf{y}_4(0), \mathbf{y}_5(0), \mathbf{y}_6(0)]$$

$$= \begin{vmatrix} -1 & -1 & -2 & -1/6 & 5/6 & -1/2 \\ 2 & 3/2 & 5/2 & -7/6 & -1/6 & -1/2 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{12},$$

the solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, $\mathbf{y}_3(t)$, $\mathbf{y}_4(t)$, $\mathbf{y}_5(t)$, and $\mathbf{y}_6(t)$ are independent for all t and form a fundamental set of solutions.

34. Consider the first student's solution. If $\mathbf{y}_1(t) = e^{2t}(1, 4, 4)$, then

$$\mathbf{y}'_1(t) = e^{2t} \begin{pmatrix} 2 \\ 8 \\ 8 \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} -2 & 2 & -1 \\ -4 & 3 & 0 \\ 0 & -1 & 3 \end{pmatrix} \mathbf{y}_1(t) = e^{2t} \begin{pmatrix} 2 \\ 8 \\ 8 \end{pmatrix},$$

so $\mathbf{y}_1$ is a solution. Similarly, $\mathbf{y}_2$ and $\mathbf{y}_3$ are seen to be solutions. Further,

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \begin{vmatrix} 1 & 1 & -1 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{vmatrix} = 1,$$

so the solutions are independent and form a fundamental solution set. Looking at the second student's solution,

$$\mathbf{y}'_2(t) = e^t \begin{pmatrix} -5 \\ -10 \\ -5 \end{pmatrix} + e^t \begin{pmatrix} 3 - 5t \\ 1 - 10t \\ -2 - 5t \end{pmatrix}$$

$$= e^t \begin{pmatrix} -2 - 5t \\ -9 - 10t \\ -7 - 5t \end{pmatrix},$$

and

$$\begin{pmatrix} -2 & 2 & -1 \\ -4 & 3 & 0 \\ 0 & -1 & 3 \end{pmatrix} \mathbf{y}_2(t) = e^t \begin{pmatrix} -2 - 5t \\ -9 - 10t \\ -7 - 5t \end{pmatrix},$$

so $\mathbf{y}_2$ is a solution. In a similar manner, you can check that the second student's $\mathbf{y}_1$ and $\mathbf{y}_3$ are solutions. Moreover,

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \begin{vmatrix} 1 & 3 & 3 \\ 4 & 1 & -1 \\ 4 & -2 & -4 \end{vmatrix} = -6,$$

so the solutions are independent and form a fundamental solution set. Thus, both students are correct. They both have fundamental solution sets. They are just using different bases.

35. If

$$A = \begin{pmatrix} 6 & 0 & -4 \\ -2 & 4 & 5 \\ 1 & 0 & 2 \end{pmatrix},$$

then the characteristic polynomial is found with the following computation

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -\begin{vmatrix} 6-\lambda & 0 & -4 \\ -2 & 4-\lambda & 5 \\ 1 & 0 & 2-\lambda \end{vmatrix}. \end{aligned}$$

Expanding down the second column,

$$\begin{aligned} p(\lambda) &= (\lambda - 4) \begin{vmatrix} 6-\lambda & -4 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (\lambda - 4)(\lambda^2 - 8\lambda + 16) \\ &= (\lambda - 4)^3. \end{aligned}$$

Because a matrix must satisfy its characteristic, the series

$$\begin{aligned} e^{tA} &= e^{t[4I + (A - 4I)]} \\ &= e^{4tI} e^{t(A - 4I)} \\ &= e^{4t} \left[I + t(A - 4I) + \frac{t^2}{2!}(A - 4I)^2 + \dots \right] \end{aligned}$$

truncates, with $(A - 4I)^k = 0$ for $k = 3, 4, \dots$

Thus,

$$\begin{aligned} e^{tA} &= e^{4t} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & 0 & -4 \\ -2 & 0 & 5 \\ 1 & 0 & -2 \end{pmatrix} \right. \\ &\quad \left. + \frac{t^2}{2!} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \right] \\ &= e^{4t} \begin{pmatrix} 1+2t & 0 & -4t \\ -2t+t^2 & 1 & 5t-t^2 \\ t & 0 & 1-2t \end{pmatrix}. \end{aligned}$$

Choose

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{y}_1(t) &= e^{tA} \mathbf{e}_1 = e^{4t} \begin{pmatrix} 1+2t \\ -2t+t^2/2 \\ t \end{pmatrix}, \\ \mathbf{y}_2(t) &= e^{tA} \mathbf{e}_2 = e^{4t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \\ \mathbf{y}_3(t) &= e^{tA} \mathbf{e}_3 = e^{4t} \begin{pmatrix} -4t \\ 5t-t^2 \\ 1-2t \end{pmatrix} \end{aligned}$$

form a fundamental set of solutions.

36. If

$$A = \begin{pmatrix} 8 & 3 & 2 \\ 0 & 4 & 0 \\ -8 & -6 & 0 \end{pmatrix},$$

then the characteristic polynomial is found with the following computation.

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -\begin{vmatrix} 8-\lambda & 3 & 2 \\ 0 & 4-\lambda & 0 \\ -8 & -6 & -\lambda \end{vmatrix}. \end{aligned}$$

Expanding across the second row,

$$\begin{aligned} p(\lambda) &= (\lambda - 4) \begin{vmatrix} 8-\lambda & 2 \\ -8 & -\lambda \end{vmatrix} \\ &= (\lambda - 4)(\lambda^2 - 8\lambda + 16) \\ &= (\lambda - 4)^3. \end{aligned}$$

Thus, $\lambda = 4$ is a repeated eigenvalue having algebraic multiplicity 3. Because

$$A - 4I = \begin{pmatrix} 4 & 3 & 2 \\ 0 & 0 & 0 \\ -8 & -6 & -4 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 3/4 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has dimension two, we can select two eigenvectors from the nullspace of $A - 4I$, $\mathbf{v}_1 = (-3, 4, 0)^T$ and $\mathbf{v}_2 = (-1, 0, 2)$. Of course, these lead to the independent solutions

$$\begin{aligned} \mathbf{y}_1(t) &= e^{tA} \mathbf{v}_1 = e^{4t} \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} \\ \mathbf{y}_2(t) &= e^{tA} \mathbf{v}_2 = e^{4t} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

Examining

$$(A - 4I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we note that $(A - 4I)^k = 0$ for $k \geq 2$. We can write

$$\begin{aligned} e^{tA} &= e^{t(4I + (A - 4I))} \\ &= e^{4tI} e^{t(A - 4I)} \\ &= e^{4t} [I + t(A - 4I)], \end{aligned}$$

knowing that the series truncates. Choose any vector independent from $\mathbf{v}_1$ and $\mathbf{v}_2$, such as $\mathbf{v}_3 = (1, 0, 0)$ (check this), then

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA} \mathbf{v}_3 \\ &= e^{4t} [\mathbf{v}_3 + t(A - 4I)\mathbf{v}_3] \\ &= e^{4t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t(A - 4I) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \\ &= e^{4t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix} \right] \\ &= e^{4t} \begin{pmatrix} 1 + 4t \\ 0 \\ -8t \end{pmatrix} \end{aligned}$$

provides the remaining solution.

37. In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -2 & -4 & 13 \\ 0 & 5 & -4 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which leads to the characteristic polynomial

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= - \begin{vmatrix} -2 - \lambda & -4 & 13 \\ 0 & 5 - \lambda & -4 \\ 0 & 1 & 1 - \lambda \end{vmatrix}. \end{aligned}$$

Expanding down the first column,

$$\begin{aligned} p(\lambda) &= (\lambda + 2) \begin{vmatrix} 5 - \lambda & -4 \\ 1 & 1 - \lambda \end{vmatrix} \\ &= (\lambda + 2)(\lambda^2 - 6\lambda + 9) \\ &= (\lambda + 2)(\lambda - 3)^2. \end{aligned}$$

for $\lambda_1 = -2$,

$$A + 2I = \begin{pmatrix} 0 & -4 & 13 \\ 0 & 7 & -4 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the eigenvector $\mathbf{v}_1 = (1, 0, 0)^T$ provides the solution

$$\mathbf{y}_1(t) = e^{tA} \mathbf{v}_1 = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 3$,

$$A - 3I = \begin{pmatrix} -5 & -4 & 13 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

and the eigenvector $\mathbf{v}_2 = (1, 2, 1)^T$ provides the solution

$$\mathbf{y}_2(t) = e^{tA} \mathbf{v}_2 = e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Because

$$(A - 3I)^2 = \begin{pmatrix} 25 & 25 & -75 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the nullspace of $(A - 3I)^2$ has dimension 2. Pick $\mathbf{v}_3 = (3, 0, 1)^T$ in the nullspace of $(A - 3I)^2$. Note that $\mathbf{v}_2$ and $\mathbf{v}_3$ are independent. This gives a third solution,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA} \mathbf{v}_3 \\ &= e^{3t} [\mathbf{v}_3 + t(A - 3I)\mathbf{v}_3] \\ &= e^{3t} \left[\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -5 & -4 & 13 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= e^{3t} \left[\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix} \right] \\ &= e^{3t} \begin{pmatrix} 3 - 2t \\ -4t \\ 1 - 2t \end{pmatrix}. \end{aligned}$$

Because

$$\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2,$$

the solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, and $\mathbf{y}_3(t)$ are independent for all t and form a fundamental set of solutions.

38. In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -1 & 5 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which leads to the characteristic polynomial

$$\begin{aligned} p(\lambda) &= -\det(A - \lambda I) \\ &= -\begin{vmatrix} -1 - \lambda & 5 & 3 \\ 0 & 1 - \lambda & 1 \\ 0 & -2 & -2 - \lambda \end{vmatrix}. \end{aligned}$$

Expanding down the first column,

$$\begin{aligned} p(\lambda) &= (\lambda + 1) \begin{vmatrix} 1 - \lambda & 1 \\ -2 & -2 - \lambda \end{vmatrix} \\ &= (\lambda + 1)(\lambda^2 + \lambda) \\ &= \lambda(\lambda + 1)^2. \end{aligned}$$

Thus, $\lambda_1 = 0$ and $\lambda_2 = -1$ are eigenvalues having algebraic multiplicities 1 and 2, respectively. Because

$$A - 0I = \begin{pmatrix} -1 & 5 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$\mathbf{v}_1 = (-2, -1, 1)^T$ is an eigenvector, providing the solution

$$\mathbf{y}_1(t) = e^{tA} \mathbf{v}_1 = e^{0t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

Because

$$A + I = \begin{pmatrix} 0 & 5 & 3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

has dimension one, we can choose an eigenvector $\mathbf{v}_2 = (1, 0, 0)^T$ to produce a second solution,

$$\mathbf{y}_2(t) = e^{tA} \mathbf{v}_2 = e^{-t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Examining

$$(A + I)^2 = \begin{pmatrix} 0 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we note that the nullspace of $(A + I)^2$ has dimension two, so we can choose $\mathbf{v}_3 = (0, 1, -2)$ in the nullspace $(A + I)^2$ independent of $\mathbf{v}_2$ (it's not a multiple of $\mathbf{v}_2$). Then,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA} \mathbf{v}_3 \\ &= e^{t(-I + (A + I))} \mathbf{v}_3 \\ &= e^{-tI} e^{t(A + I)} \mathbf{v}_3 \\ &= e^{-t} I \left[I + t(A + I) + \frac{t^2}{2} (A + I)^2 \right. \\ &\quad \left. + \cdots \right] \mathbf{v}_3 \\ &= e^{-t} [\mathbf{v}_3 + t(A + I)\mathbf{v}_3], \end{aligned}$$

because $\mathbf{v}_3$ is in the nullspace of $(A + I)^2$ and $(A + I)^k \mathbf{v}_3 = 0$ for all $k \geq 2$. Thus,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{-t} \left[\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + t(A + I) \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right] \\ &= e^{-t} \left[\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right] \\ &= e^{-t} \begin{pmatrix} -t \\ 1 \\ -2 \end{pmatrix}. \end{aligned}$$

39. If

$$A = \begin{pmatrix} 5 & -1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 1 & 1 & -1 & -3 \\ 0 & -1 & 0 & 7 \end{pmatrix},$$

a computer reveals the characteristic equation

$$p(\lambda) = (\lambda + 1)(\lambda - 5)^3,$$

so $\lambda_1 = 1$ and $\lambda_2 = 5$ are eigenvalues, with algebraic multiplicities 1 and 3 respectively. Because

$$A + I = \begin{pmatrix} 6 & -1 & 0 & 2 \\ 0 & 4 & 0 & 4 \\ 1 & 1 & 0 & -3 \\ 0 & -1 & 0 & 8 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the eigenvector $\mathbf{v}_1 = (0, 0, 1, 0)^T$ provides the solution

$$\mathbf{y}_1(t) = e^{At} \mathbf{v}_1 = e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Because

$$\begin{aligned} A - 5I &= \begin{pmatrix} 0 & -1 & 0 & 2 \\ 0 & -2 & 0 & 4 \\ 1 & 1 & -6 & -3 \\ 0 & -1 & 0 & 2 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -6 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

the nullspace of $A - 5I$ has dimension 2 and the eigenvector $\mathbf{v}_2 = (6, 0, 1, 0)^T$ and $\mathbf{v}_3 = (1, 2, 0, 1)^T$ provide two more solutions.

$$\begin{aligned} \mathbf{y}_2(t) &= e^{At} \mathbf{v}_2 = e^{5t} \begin{pmatrix} 6 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \\ \mathbf{y}_3(t) &= e^{At} \mathbf{v}_3 = e^{5t} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Because

$$\begin{aligned} (A - 5I)^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & -6 & 36 & 18 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & -6 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

the nullspace of $(A - 5I)^2$ has dimension 3 and we can pick $\mathbf{v}_4 = (3, 0, 0, 1)^T$ independent of $\mathbf{v}_2$ and $\mathbf{v}_3$. This gives solution

$$\begin{aligned} \mathbf{y}_4(t) &= e^{At} \mathbf{v}_4 \\ &= e^{5t} [\mathbf{v}_4 + t(A - 5I)\mathbf{v}_4] \\ &= e^{5t} \left[\begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 0 \\ 2 \end{pmatrix} \right] \\ &= e^{5t} \begin{pmatrix} 3 + 2t \\ 4t \\ 0 \\ 1 + 2t \end{pmatrix}. \end{aligned}$$

Because

$$\begin{aligned} \det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0), \mathbf{y}_4(0)] &= \begin{vmatrix} 0 & 6 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} \\ &= 12, \end{aligned}$$

the solutions are independent for all t and form a fundamental set of solutions.

40. If

$$A = \begin{pmatrix} -12 & -1 & 8 & 10 \\ -8 & 0 & -1 & 9 \\ 0 & 0 & 5 & 0 \\ -17 & -1 & 8 & 15 \end{pmatrix},$$

a computer reveals the characteristic equation

$$p(\lambda) = (\lambda + 1)^2(\lambda - 5)^2,$$

so $\lambda_1 = -1$ and $\lambda_2 = 5$ are eigenvalues, each having algebraic multiplicities 2. Because

$$A - 5I = \begin{pmatrix} -17 & -1 & 8 & 10 \\ -8 & -5 & -1 & 9 \\ 0 & 0 & 0 & 0 \\ -17 & -1 & 8 & 10 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -41/77 & -41/77 \\ 0 & 1 & 81/77 & -73/77 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the nullspace of $A - 5I$ provides two eigenvectors, $\mathbf{v}_1 = (41, -81, 77, 0)^T$ and $\mathbf{v}_2 = (43, 73, 0, 77)^T$, and solutions

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{5t} \begin{pmatrix} 41 \\ -81 \\ 77 \\ 0 \end{pmatrix}$$

$$\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{5t} \begin{pmatrix} 41 \\ 73 \\ 0 \\ 77 \end{pmatrix}.$$

Examining

$$A + I = \begin{pmatrix} -11 & -1 & 8 & 10 \\ -8 & 1 & -1 & 9 \\ 0 & 0 & 6 & 0 \\ -17 & -1 & 8 & 16 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

showing that $A + I$ has dimension one, giving up only one eigenvector, $\mathbf{v}_3 = (1, -1, 0, 1)^T$, and one

solution

$$\mathbf{y}_3(t) = e^{tA}\mathbf{v}_3 = e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

But, the nullspace of

$$(A + I)^2 = \begin{pmatrix} -41 & 0 & 41 & 41 \\ -73 & 0 & 1 & 73 \\ 0 & 0 & 36 & 0 \\ -77 & 0 & 41 & 77 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has dimension two, so $\mathbf{v}_4 = (1, 0, 0, 1)^T$ is in the nullspace of $(A + I)^2$, independent of $\mathbf{v}_3$ (it's not a multiple of $\mathbf{v}_3$), and $(A + I)^k\mathbf{v}_4 = 0$ for $k \geq 2$. Thus,

$$\begin{aligned} \mathbf{y}_4(t) &= e^{tA}\mathbf{v}_4 \\ &= e^{t(-I+(A+I))}\mathbf{v}_4 \\ &= e^{-tI}e^{t(A+I)}\mathbf{v}_4 \\ &= e^{-tI} \left[I + t(A + I) + \frac{t^2}{2}(A + I)^2 \right. \\ &\quad \left. + \cdots \right] \mathbf{v}_4 \\ &= e^{-t} [\mathbf{v}_4 + t(A + I)\mathbf{v}_4]. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{y}_4(t) &= e^{-t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t(A + I) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right] \\ &= e^{-t} \begin{pmatrix} 1-t \\ t \\ 0 \\ 1-t \end{pmatrix}. \end{aligned}$$

41. If

$$A = \begin{pmatrix} -1 & 0 & 0 & 2 \\ -6 & 13 & 0 & -42 \\ 0 & -6 & -2 & 13 \\ -2 & 5 & 0 & -16 \end{pmatrix},$$

a computer reveals the characteristic polynomial

$$p(\lambda) = (\lambda + 2)^2(\lambda^2 + 2\lambda + 5).$$

For $\lambda = -2$,

$$(A + 2I)^2 = \begin{pmatrix} -3 & 10 & 0 & -26 \\ -12 & 15 & 0 & -54 \\ 10 & -25 & 0 & 70 \\ -4 & 5 & 0 & -18 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so we can pick $\mathbf{v}_1 = (0, 0, 1, 0)^T$ and $\mathbf{v}_2 = (-2, 2, 0, 1)^T$. Thus,

$$\begin{aligned} \mathbf{y}_1(t) &= e^{tA}\mathbf{v}_1 = e^{-2t}[\mathbf{v}_1 + t(A + 2I)\mathbf{v}_1] \\ &= e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{y}_2(t) &= e^{tA}\mathbf{v}_2 = e^{-2t}[\mathbf{v}_2 + t(A + 2I)\mathbf{v}_2] \\ &= e^{-2t} \begin{pmatrix} -2 \\ 2 \\ t \\ 1 \end{pmatrix}. \end{aligned}$$

The remaining eigenvalues are $-1 \pm 2i$. A computer reveals an eigenvector $\mathbf{w} = (1, 3i, -2 - i, i)^T$ associated with $\lambda = -1 + 2i$. Thus,

$$\mathbf{z}(t) = e^{tA}\mathbf{w} = e^{(-1+2i)t}\mathbf{w} = e^{-t}e^{2it} \begin{pmatrix} 1 \\ 3i \\ -2-i \\ i \end{pmatrix}.$$

Using Euler's identity,

$$\begin{aligned} \mathbf{z}(t) &= e^{-t}(\cos 2t + i \sin 2t) \\ &\times \left[\begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \\ -1 \\ 1 \end{pmatrix} \right] \\ &= e^{-t} \left(\cos 2t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 3 \\ -1 \\ 1 \end{pmatrix} \right) \\ &\quad + i e^{-t} \left(\cos 2t \begin{pmatrix} 0 \\ 3 \\ -1 \\ 1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} \right). \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{-t} \begin{pmatrix} \cos 2t \\ -3 \sin 2t \\ -2 \cos 2t + \sin 2t \\ -\sin 2t \end{pmatrix} \quad \text{and} \\ \mathbf{y}_4(t) &= e^{-t} \begin{pmatrix} \sin 2t \\ 3 \cos 2t \\ -\cos 2t - 2 \sin 2t \\ \cos 2t \end{pmatrix} \end{aligned}$$

are solutions. Because

$$\begin{aligned} \det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0), \mathbf{y}_4(0)] &= \begin{vmatrix} 0 & -2 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \end{vmatrix} \\ &= 1, \end{aligned}$$

the solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, $\mathbf{y}_3(t)$, and $\mathbf{y}_4(t)$ are independent for all t and form a fundamental set of solutions.

42. If

$$A = \begin{pmatrix} -8 & -2 & 3 & 12 \\ -3 & -2 & 2 & 6 \\ 2 & 0 & -3 & -4 \\ -4 & -1 & 2 & 6 \end{pmatrix},$$

a computer reveals the characteristic equation

$$p(\lambda) = (\lambda + 1)(\lambda + 2)^3,$$

so $\lambda_1 = -1$ and $\lambda_2 = -2$ are eigenvalues, having algebraic multiplicities 1 and 3, respectively. Because

$$A + I = \begin{pmatrix} -7 & -2 & 3 & 12 \\ -3 & -1 & 2 & 6 \\ 2 & 0 & -2 & -4 \\ -4 & -1 & 2 & 7 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the nullspace of $A + I$ provides one eigenvector, $\vec{v}_1 = (1, 1, -1, 1)^T$ and one solution,

$$\mathbf{y}_1(t) = e^{tA} \mathbf{v}_1 = e^{-t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Examining

$$A + 2I = \begin{pmatrix} -6 & -2 & 3 & 12 \\ -3 & 0 & 2 & 6 \\ 2 & 0 & -1 & -4 \\ -4 & -1 & 2 & 8 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

showing that the nullspace of $A + 2I$ has dimension one, giving up only one eigenvector, $\mathbf{v}_2 = (2, 0, 0, 1)^T$, and one solution

$$\mathbf{y}_2(t) = e^{tA} \mathbf{v}_2 = e^{-2t} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

But, the nullspace of

$$(A + 2I)^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ -2 & 0 & 1 & 4 \\ 2 & 0 & -1 & -4 \\ -1 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has dimension two, so $\mathbf{v}_3 = (0, 1, 0, 0)^T$ is in the nullspace of $(A + 2I)^2$, independent of $\mathbf{v}_2$ (it's not a multiple of $\mathbf{v}_2$), and $(A + 2I)^k \mathbf{v}_3 = 0$ for $k \geq 2$. Thus,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA} \mathbf{v}_3 \\ &= e^{t(-2I + (A + 2I))} \mathbf{v}_3 \\ &= e^{-2tI} e^{t(A + 2I)} \mathbf{v}_3 \\ &= e^{-2t} I \left[I + t(A + 2I) + \frac{t^2}{2}(A + 2I)^2 \right. \\ &\quad \left. + \cdots \right] \mathbf{v}_3 \\ &= e^{-2t} [\mathbf{v}_3 + t(A + 2I)\mathbf{v}_3]. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{-2t} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t(A + 2I) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] \\ &= e^{-2t} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} -2t \\ 1 \\ 0 \\ -t \end{pmatrix}. \end{aligned}$$

Examining

$$\begin{aligned} (A + 2I)^3 &= \begin{pmatrix} -2 & 0 & 1 & 4 \\ -2 & 0 & 1 & 4 \\ 2 & 0 & -1 & -4 \\ -2 & 0 & 1 & 4 \end{pmatrix} \\ &\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1/2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

we see that the nullspace of $(A + 2I)^3$ has dimension three. Thus, we pick $\mathbf{v}_4 = (1, 0, 2, 0)^T$ from the nullspace of $(A + 2I)^3$ independent of $\mathbf{v}_2$ and $\mathbf{v}_3$ (check this), having the property that $(A + 2I)^k = 0$

for $k \geq 3$. Thus,

$$\begin{aligned}
 \mathbf{y}_4(t) &= e^{tA} \mathbf{v}_4 \\
 &= e^{t(-2I + (A+2I))} \mathbf{v}_4 \\
 &= e^{-2tI} e^{t(A+2I)} \mathbf{v}_4 \\
 &= e^{-2t} \left[I + t(A+2I) + \frac{t^2}{2!} (A+2I)^2 \right. \\
 &\quad \left. + \cdots \right] \mathbf{v}_4 \\
 &= e^{-2t} \left[\mathbf{v}_4 + t(A+2I)\mathbf{v}_4 \right. \\
 &\quad \left. + \frac{t^2}{2} (A+2I)^2 \mathbf{v}_4 \right] \\
 &= e^{-2t} \left[\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -2 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right] \\
 &= e^{-2t} \begin{pmatrix} 1-t^2 \\ t \\ 2 \\ -t^2/2 \end{pmatrix}.
 \end{aligned}$$

43. If

$$A = \begin{pmatrix} -2 & 2 & -2 & 0 & -3 \\ -1 & 0 & -1 & 0 & -3 \\ 15 & -16 & -1 & 10 & 33 \\ 12 & -13 & 1 & 6 & 26 \\ -5 & 5 & 0 & -3 & -12 \end{pmatrix},$$

a computer reveals the characteristic equation

$$p(\lambda) = (\lambda + 1)(\lambda + 2)^4.$$

The eigenvalue-eigenvector pair $\lambda_1 = -1$, $\mathbf{v}_1 = (-3, -2, -2, -2, -1)^T$ give the solution

$$\mathbf{y}_1(t) = e^{tA} \mathbf{v}_1 = e^{-t} \begin{pmatrix} -3 \\ -2 \\ -2 \\ -2 \\ 1 \end{pmatrix}.$$

Because $(A + 2I)^4$ reduces

$$(A + 2I)^4 \xrightarrow{\text{ref}} \begin{pmatrix} 1 & -4/3 & 1/3 & 2/3 & 8/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the dimension of the nullspace of $(A + 2I)^4$ is 4 and the collection

$$\begin{aligned}
 \mathbf{v}_2 &= \begin{pmatrix} 4 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & \mathbf{v}_3 &= \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \\
 \mathbf{v}_4 &= \begin{pmatrix} -2 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}, & \mathbf{v}_5 &= \begin{pmatrix} -8 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}
 \end{aligned}$$

is a basis for the nullspace of $(A + 2I)^4$. Moreover, for $i = 2, 3, 4$, and 5,

$$\begin{aligned}
 \mathbf{y}_i(t) &= e^{tA} \mathbf{v}_i \\
 &= e^{-2t} \left[\mathbf{v}_i + t(A+2I)\mathbf{v}_i + \frac{t^2}{2!} (A+2I)^2 \mathbf{v}_i \right. \\
 &\quad \left. + \frac{t^3}{3!} (A+2I)^3 \mathbf{v}_i \right].
 \end{aligned}$$

Using this result and a computer,

$$\begin{aligned}
 \mathbf{y}_2(t) &= e^{tA} \mathbf{v}_2 = \frac{1}{2} e^{-2t} \begin{pmatrix} 8 + 12t - 5t^2 + t^3 \\ 6 + 4t + t^2 + t^3 \\ 24t - 5t^2 + t^3 \\ 18t \\ -10t + 3t^2 \end{pmatrix} \\
 \mathbf{y}_3(t) &= e^{tA} \mathbf{v}_3 = -\frac{1}{2} e^{-2t} \begin{pmatrix} 2 + 12t - 5t^2 + t^3 \\ 4t + t^2 + t^3 \\ -6 + 24t - 5t^2 + t^3 \\ 18t \\ 10t + 3t^2 \end{pmatrix} \\
 \mathbf{y}_4(t) &= e^{tA} \mathbf{v}_4 = \frac{1}{2} e^{-2t} \begin{pmatrix} -4 + t^2 \\ 4 + t \\ t^2 \\ 6 \\ 2t \end{pmatrix} \\
 \mathbf{y}_5(t) &= e^{tA} \mathbf{v}_5 = -e^{-2t} \begin{pmatrix} 8 + 9t - 5t^2 + t^3 \\ t + t^2 + t^3 \\ 21t - 5t^2 + t^3 \\ 18t \\ -3 - 10t + 3t^2 \end{pmatrix}
 \end{aligned}$$

Because

$$\begin{aligned} & \det[y_1(t), y_2(0), y_3(t), y_4(0), y_5(0)] \\ &= \begin{vmatrix} -3 & 4 & -1 & -2 & -8 \\ -2 & 3 & 0 & 0 & 0 \\ -2 & 0 & 3 & 0 & 0 \\ -2 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 3 \end{vmatrix} \\ &= 27, \end{aligned}$$

the solutions are independent for all t and form a fundamental set of solutions.

44. If

$$A = \begin{pmatrix} -4 & 3 & 6 & 4 & 2 \\ 0 & -8 & -10 & -8 & 2 \\ -1 & 7 & 10 & 9 & -1 \\ 1 & -4 & -7 & -7 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix},$$

a computer reveals the characteristic equation

$$p(\lambda) = (\lambda + 1)(\lambda + 2)^4,$$

so $\lambda_1 = -1$ and $\lambda_2 = -2$ are eigenvalues, having algebraic multiplicities 1 and 4, respectively. Because

$$\begin{aligned} A + I &= \begin{pmatrix} -3 & 3 & 6 & 4 & 2 \\ 0 & -7 & -10 & -8 & 2 \\ -1 & 7 & 11 & 9 & -1 \\ 1 & -4 & -7 & -6 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \\ &\xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

the nullspace of $A + I$ provides one eigenvector, $\mathbf{v}_1 = (0, 2, -2, 1, 1)^T$ and one solution,

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t} \begin{pmatrix} 0 \\ 2 \\ -2 \\ 1 \\ 1 \end{pmatrix}.$$

Examining

$$\begin{aligned} A + 2I &= \begin{pmatrix} -2 & 3 & 6 & 4 & 2 \\ 0 & -6 & -10 & -8 & 2 \\ -1 & 7 & 12 & 9 & -1 \\ 1 & -4 & -7 & -5 & 0 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \\ &\xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

showing that the nullspace of $A + 2I$ has dimension two, giving two eigenvectors, $\mathbf{v}_2 = (-1, 2, -2, 1, 0)^T$ and $\mathbf{v}_3 = (1, 2, -1, 0, 1)^T$, and two solutions.

$$\begin{aligned} \mathbf{y}_2(t) &= e^{tA}\mathbf{v}_2 = e^{-2t} \begin{pmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{y}_3(t) &= e^{tA}\mathbf{v}_3 = e^{-2t} \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

But, the nullspace of

$$\begin{aligned} (A + 2I)^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -6 & -4 & 2 \\ 0 & 4 & 6 & 4 & -2 \\ 0 & -2 & -3 & -2 & 1 \\ 0 & -2 & -3 & -2 & 1 \end{pmatrix} \\ &\xrightarrow{\text{ref}} \begin{pmatrix} 0 & 1 & 3/2 & 1 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

has dimension four, so $\mathbf{v}_4 = (1, 0, 0, 0, 0)^T$ and $\mathbf{v}_5 = (0, -1, 0, 1, 0)^T$ are in the nullspace of $(A + 2I)^2$. Also,

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & 2 & 0 & -1 \\ -2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so each column is a pivot column and the vectors $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and $\mathbf{v}_5$ are independent. Furthermore, $(A + 2I)^k \mathbf{v}_4 = 0$ and $(A + 2I)^k \mathbf{v}_5 = 0$ for $k \geq 2$. Thus,

$$\begin{aligned} \mathbf{y}_4(t) &= e^{-2t} [\mathbf{v}_4 + t(A + 2I)\mathbf{v}_4] \\ &= e^{-2t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} 1 - 2t \\ 0 \\ -t \\ t \\ -t \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{y}_5(t) &= e^{-2t} [\mathbf{v}_5 + t(A + 2I)\mathbf{v}_5] \\ &= e^{-2t} \left[\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \\ 0 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} t \\ -1 - 2t \\ 2t \\ 1 - t \\ 0 \end{pmatrix}, \end{aligned}$$

45. In matrix form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}' = \begin{pmatrix} 5 & 7 & 1 & 1 & 8 \\ 3 & 6 & 5 & 4 & 5 \\ -3 & -8 & -2 & -5 & -12 \\ 3 & 14 & 8 & 10 & 18 \\ -4 & -9 & -6 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

A computer reveals the characteristic equation

$$p(\lambda) = -(\lambda + 1)^2(\lambda - 4)^3.$$

For $\lambda = -1$, $(A + I)^2$ reduces

$$(A + I)^2 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\mathbf{v}_1 = (1, -1, 0, 1, 0)^T$ and $\mathbf{v}_2 = (1, -2, 1, 0, 1)^T$ form a basis for the nullspace of $(A + I)^2$. Moreover,

$$\mathbf{y}_i(t) = e^{tA} \mathbf{v}_i = e^{-t} [\mathbf{v}_i + t(A + I)\mathbf{v}_i]$$

for $i = 1, 2$. Using this result and a computer,

$$\begin{aligned} \mathbf{y}_1(t) &= e^{tA} \mathbf{v}_1 = e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ \mathbf{y}_2(t) &= e^{tA} \mathbf{v}_2 = e^{-t} \begin{pmatrix} 1 + t \\ -2 - t \\ 1 \\ t \\ 1 \end{pmatrix}. \end{aligned}$$

For $\lambda = 4$, $(A - 4I)^3$ reduces

$$(A - 4I)^3 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\mathbf{v}_3 = (-1, 0, 1, 0, 0)^T$, $\mathbf{v}_4 = (-1, 0, 0, 1, 0)^T$, and $\mathbf{v}_5 = (-1, -1, 0, 0, 1)^T$ form a basis for the nullspace of $(A - 4I)^3$. Moreover,

$$\begin{aligned} \mathbf{y}_i(t) &= e^{tA} \mathbf{v}_i = e^{4t} \left[\mathbf{v}_i + t(A - 4I)\mathbf{v}_i \right. \\ &\quad \left. + \frac{t^2}{2!}(A - 4I)^2 \mathbf{v}_i \right] \end{aligned}$$

for $i = 3, 4$, and 5 . Using this result and a computer,

$$\begin{aligned} \mathbf{y}_3(t) &= e^{At} \mathbf{v}_3 = \frac{1}{2} e^{4t} \begin{pmatrix} -2 \\ 4t - t^2 \\ 2 - 6t + t^2 \\ 10t - 2t^2 \\ -4t + t^2 \end{pmatrix} \\ \mathbf{y}_4(t) &= e^{At} \mathbf{v}_4 = \frac{1}{2} e^{4t} \begin{pmatrix} -2 \\ 2t - t^2 \\ -4t + t^2 \\ 2 + 6t - 2t^2 \\ -2t + t^2 \end{pmatrix} \\ \mathbf{y}_5(t) &= e^{At} \mathbf{v}_5 = \frac{1}{2} e^{4t} \begin{pmatrix} -2 \\ -2 - t^2 \\ -2t + t^2 \\ 2t - 2t^2 \\ 2 + t^2 \end{pmatrix} \end{aligned}$$

Because

$$\begin{aligned} &\det[\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0), \mathbf{y}_4(0), \mathbf{y}_5(0)] \\ &= \begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix} \\ &= 1, \end{aligned}$$

the solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, $\mathbf{y}_3(t)$, $\mathbf{y}_4(t)$, and $\mathbf{y}_5(t)$ are independent for all t and form a fundamental set of solutions.

46. Let $x_1(t)$ and $x_2(t)$ represent the salt content (in pounds) of the upper and lower tanks, respectively. Because pure water flows into the upper tank, no salt is coming into the tank. However, the rate at which salt is leaving the tank is

$$\begin{aligned} \text{Rate Out} &= 4 \text{ gal/min} \times \frac{x_1(t)}{100} \text{ lb/gal} \\ &= \frac{x_1(t)}{25} \text{ lb/min.} \end{aligned}$$

Because $dx_1/dt = \text{Rate In} - \text{Rate Out}$, we write

$$\frac{dx_1}{dt} = -\frac{x_1}{25}. \quad (6.11)$$

Salt enters the lower tank at the same rate as it is leaving the upper tank. Further, the rate at which salt leaves the lower tank is

$$\begin{aligned} \text{Rate Out} &= 4 \text{ gal/min} \times \frac{x_2(t)}{100} \text{ lb/gal} \\ &= \frac{x_2(t)}{25} \text{ lb/min.} \end{aligned}$$

Again, because $dx_2/dt = \text{Rate In} - \text{Rate Out}$, we write

$$\frac{dx_2}{dt} = \frac{x_1}{25} - \frac{x_2}{25}. \quad (6.12)$$

In matrix form, the system defined by equations (6.11) and (*) is

$$\mathbf{x}' = A\mathbf{x}, \quad (6.13)$$

where

$$A = \begin{bmatrix} -1/25 & 0 \\ 1/25 & -1/25 \end{bmatrix}.$$

Because A is lower triangular, the eigenvalues are located on the main diagonal. Note that we have a repeated eigenvalue $-1/25$. Hence,

$$\begin{aligned} e^{tA} &= e^{-t/25} (I + t(A + (1/25)I)) \\ &= e^{-t/25} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 1/25 & 0 \end{bmatrix} \right) \\ &= e^{-t/25} \begin{bmatrix} 1 & 0 \\ t/25 & 1 \end{bmatrix} \end{aligned} \quad (6.14)$$

Recall that the initial salt content in the upper and lower tanks is 10 and 20 pounds, respectively. Thus, the initial condition is $\mathbf{x}(0) = (x_1(0), x_2(0)) = (10, 20)$, giving a solution $\mathbf{x}(t) = e^{tA}\mathbf{x}(0)$, or

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= e^{-t/25} \begin{bmatrix} 1 & 0 \\ t/25 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} \\ &= \begin{bmatrix} 10e^{-t/25} \\ e^{-t/25}(10t/25 + 20) \end{bmatrix}. \end{aligned} \quad (6.15)$$

47. The salt content in the topmost tank is changing at a rate

$$\frac{dx_1}{dt} = \text{Rate in} - \text{Rate out} = -\frac{x_1}{25}.$$

The salt content in the second tank is changing at a rate

$$\frac{dx_2}{dt} = \text{Rate in} - \text{Rate out} = \frac{x_1}{25} - \frac{x_2}{25}.$$

The salt content in the third tank is changing at a rate

$$\frac{dx_3}{dt} = \text{Rate in} - \text{Rate out} = \frac{x_2}{25} - \frac{x_3}{25}.$$

In matrix-vector form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -1/25 & 0 & 0 \\ 1/25 & -1/25 & 0 \\ 0 & 1/25 & -1/25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Initially, $\mathbf{x}(0) = (x_1(0), x_2(0), x_3(0))^T = (10, 8, 4)^T$. Because the coefficient matrix is lower triangular, the main diagonal contains the eigenvalues. Note that there is a single eigenvalue of algebraic multiplicity 3, namely, $\lambda = -1/25$. Hence,

$$e^{tA} = e^{-t/25} \left[I + t \left(A + \frac{1}{25} I \right) + \frac{t^2}{2} \left(A + \frac{1}{25} I \right)^2 \right].$$

However,

$$A + \frac{1}{25} I = \begin{pmatrix} 0 & 0 & 0 \\ 1/25 & 0 & 0 \\ 0 & 1/25 & 0 \end{pmatrix} \quad \text{and}$$

$$\left(A + \frac{1}{25} I \right)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/625 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} e^{tA} &= e^{-t/25} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &+ t e^{-t/25} \begin{pmatrix} 0 & 0 & 0 \\ 1/25 & 0 & 0 \\ 0 & 1/25 & 0 \end{pmatrix} \\ &+ \frac{t^2}{2} e^{-t/25} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/625 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Simplifying,

$$e^{tA} = e^{-t/25} \begin{pmatrix} 1 & 0 & 0 \\ t/25 & 1 & 0 \\ t^2/1250 & t/25 & 1 \end{pmatrix}.$$

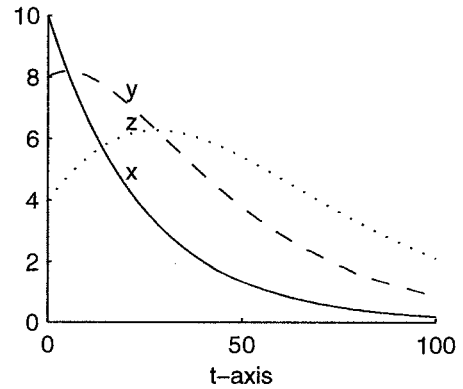
Finally, the solution with initial condition $\mathbf{x}_0 = (10, 8, 4)^T$ is

$$\begin{aligned} \mathbf{x}(t) &= e^{tA} \mathbf{x}_0 \\ &= e^{-t/25} \begin{pmatrix} 1 & 0 & 0 \\ t/25 & 1 & 0 \\ t^2/1250 & t/25 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \\ 4 \end{pmatrix}, \\ &= e^{-t/25} \begin{pmatrix} 10 \\ 2t/5 + 8 \\ t^2/125 + 8t/25 + 4 \end{pmatrix}. \end{aligned}$$

At $t = 8$ minutes, the amount of salt in each tank is

$$\begin{pmatrix} x_1(8) \\ x_2(8) \\ x_3(8) \end{pmatrix} = \mathbf{x}(8) = \begin{pmatrix} 7.2614 \text{ lb} \\ 8.1328 \text{ lb} \\ 5.1353 \text{ lb} \end{pmatrix}.$$

The plot of the salt in each tank over time is shown in the following figure.



Section 9.7. Qualitative Analysis of Linear Systems

1. In matrix form,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -0.2 & 2.0 \\ -2.0 & -0.2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

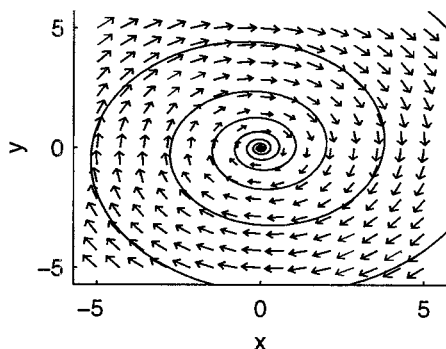
the coefficient matrix

$$A = \begin{pmatrix} -0.2 & 2.0 \\ -2.0 & -0.2 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^2 + 0.4\lambda + 4.04,$$

producing eigenvalues $\lambda = -0.2 \pm 2i$. Because the real part of each eigenvalue is negative, the equilibrium point at the origin is asymptotically stable.



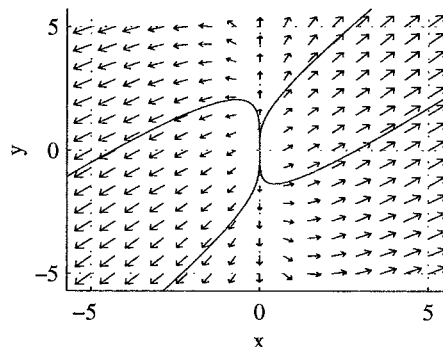
2. In matrix form,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The coefficient matrix,

$$A = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}$$

is lower triangular, so the eigenvalues lie on the diagonal, $\lambda_1 = 4$ and $\lambda_2 = 1$. Because both eigenvalues are positive, the equilibrium point at the origin is a source and unstable.



3. In matrix form,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -6 & -15 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

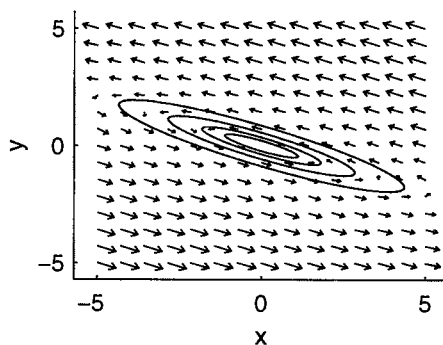
the coefficient matrix

$$A = \begin{pmatrix} -6 & -15 \\ 3 & 6 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^2 + 9,$$

producing eigenvalues $\lambda = \pm 3i$. Therefore, the equilibrium point at the origin is a stable center.



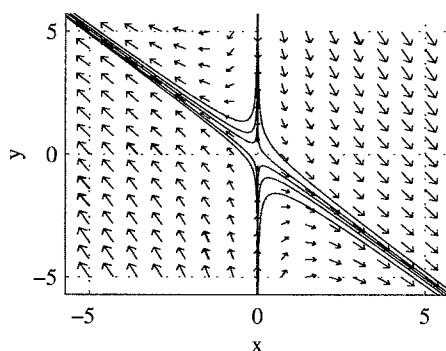
4. In matrix form,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The coefficient matrix

$$A = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix}$$

is lower triangular, so the eigenvalues lie on the diagonal, $\lambda_1 = 2$ and $\lambda_2 = -1$. Because there is at least one positive eigenvalue, the equilibrium point at the origin is unstable. Indeed, with one positive and one negative eigenvalue, the origin is a saddle.



5. In system

$$\mathbf{y}' = \begin{pmatrix} 0.1 & 2.0 \\ -2.0 & 0.1 \end{pmatrix} \mathbf{y},$$

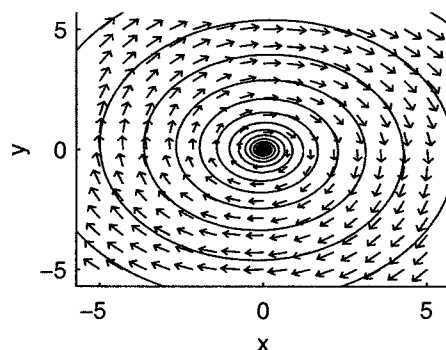
the coefficient matrix

$$A = \begin{pmatrix} 0.1 & 2.0 \\ -2.0 & 0.1 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^2 - 0.2\lambda + 4.01,$$

producing eigenvalues $\lambda = 0.1 \pm 2i$. Therefore, the equilibrium point at the origin is unstable. Indeed, the equilibrium point is a spiral source.



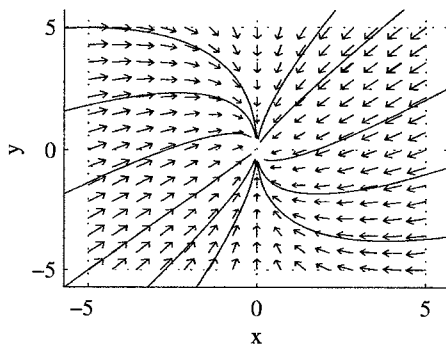
6. In system

$$\mathbf{y}' = \begin{pmatrix} -0.2 & 0.0 \\ -0.1 & -0.1 \end{pmatrix} \mathbf{y},$$

the coefficient matrix

$$A = \begin{pmatrix} -0.2 & 0.0 \\ -0.1 & -0.1 \end{pmatrix}$$

is lower triangular. The eigenvalues lie on the diagonal, $\lambda_1 = -0.2$ and $\lambda_2 = -0.1$. Since both eigenvalues are negative, the equilibrium point at the origin is asymptotically stable. Indeed, the origin is a sink.



7. In system

$$\mathbf{y}' = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix} \mathbf{y},$$

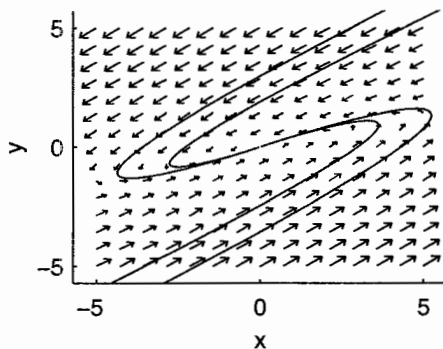
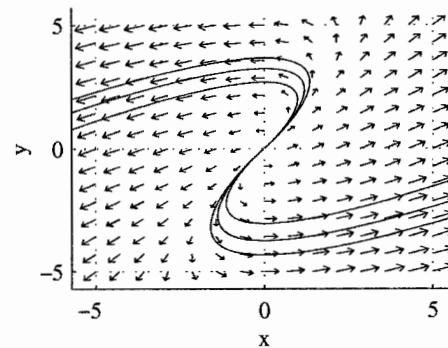
the coefficient matrix

$$A = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^2 + 2\lambda + 1,$$

producing the repeated eigenvalue $\lambda = -1$. Because the real part of every eigenvalue is negative, the equilibrium point at the origin is asymptotically stable. Indeed, the equilibrium point is a degenerate sink.



8. In system

$$\mathbf{y}' = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{y},$$

the coefficient matrix

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2,$$

producing a repeated eigenvalue, $\lambda = 1$. Because the eigenvalue is positive, the equilibrium point at the origin is unstable.

9. Consider the system

$$\mathbf{y}' = \begin{pmatrix} -3 & -4 & 2 \\ -2 & -7 & 4 \\ -3 & -8 & 4 \end{pmatrix} \mathbf{y}.$$

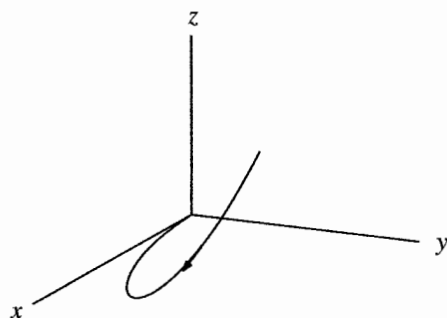
Using a computer, matrix

$$A = \begin{pmatrix} -3 & -4 & 2 \\ -2 & -7 & 4 \\ -3 & -8 & 4 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6,$$

and eigenvalues $\lambda_1 = -3$, $\lambda_2 = -2$, and $\lambda_3 = -1$. Because the real parts of all eigenvalues are negative, the equilibrium point at the origin is asymptotically stable. One such solution, with initial condition $(1, 1, 1)^T$, is shown in the following figure.



10. Consider the system

$$\mathbf{y}' = \begin{pmatrix} -3 & -1 & 0 \\ 2 & 0 & 0 \\ -6 & -1 & 3 \end{pmatrix} \mathbf{y}.$$

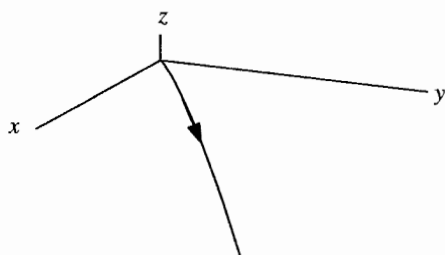
Using a computer, matrix

$$A = \begin{pmatrix} -3 & -1 & 0 \\ 2 & 0 & 0 \\ -6 & -1 & 3 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

and eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Because there is a positive eigenvalue, the equilibrium point at the origin is unstable. One solution, starting at the point $(0.01, 0.01, 0.01)$ is shown in the following figure.



11. In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 6 \\ 0 & -3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

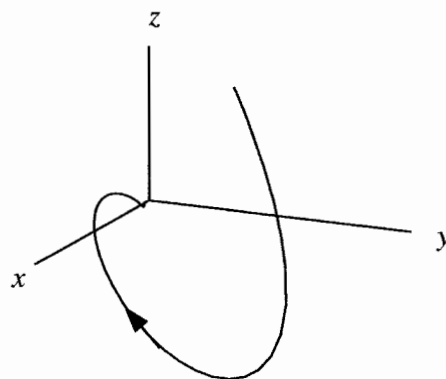
Using a computer, matrix

$$A = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 6 \\ 0 & -3 & -5 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^3 + 5\lambda^2 + 1\lambda + 13,$$

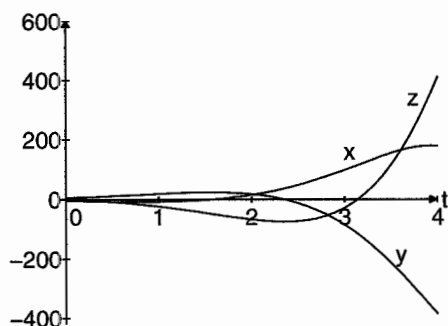
and eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2 + 3i$, and $\lambda_3 = -2 - 3i$. Because all the real parts of the eigenvalues are negative, the equilibrium point at the origin is asymptotically stable. One such solution, with initial condition $(1, 1, 1)^T$, is shown in the following figure.



12. In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 0 \\ -4 & -6 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The matrix has eigenvalues -2 and $1 \pm i$. Since the complex eigenvalues have positive real part, the origin is an unstable equilibrium point. This is illustrated by the solution plotted in the accompanying figure.



13. If

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 4 & -2 & -3 \end{pmatrix} \mathbf{y},$$

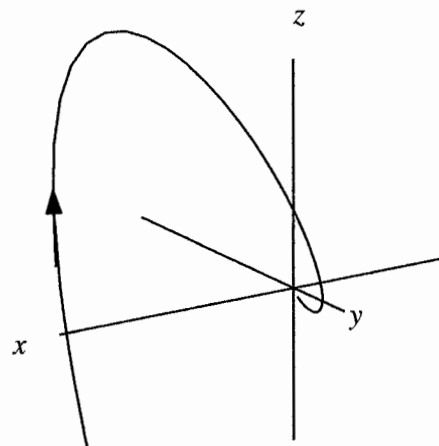
then matrix

$$A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 4 & -2 & -3 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda + 1)(\lambda^2 + 2\lambda + 2)$$

and eigenvalues -1 , $-1 + i$, and $-1 - i$. Therefore, the real part of eigenvalue is negative, so the hypotheses of Theorem 7.4 are satisfied and the equilibrium point at the origin is asymptotically stable. One such solution, with initial condition $(2, -1, -2)^T$, is shown in the image that follows.



14. If

$$\mathbf{y}' = \begin{pmatrix} 3 & -3 & -5 \\ 0 & 1 & 0 \\ 0 & -3 & -2 \end{pmatrix} \mathbf{y},$$

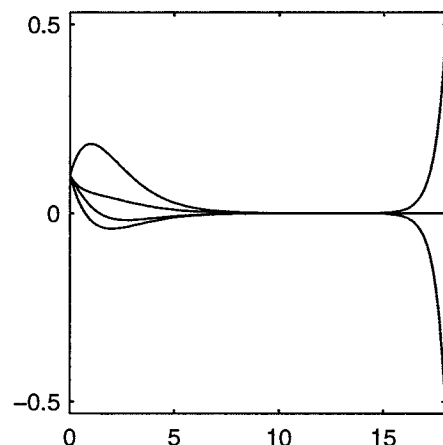
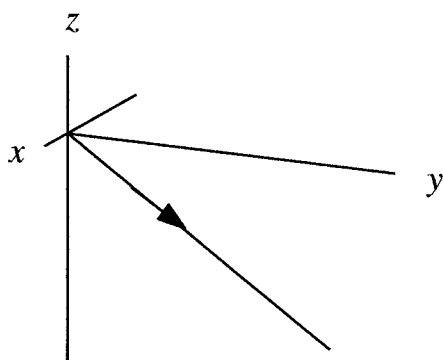
then a computer reveals that matrix

$$A = \begin{pmatrix} 3 & -3 & -5 \\ 0 & 1 & 0 \\ 0 & -3 & -2 \end{pmatrix}$$

has characteristic equation

$$p(\lambda) = (\lambda - 3)(\lambda - 1)(\lambda + 2)$$

and eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_3 = -2$. Thus, at least one eigenvalue has positive real part and Theorem 7.4 predicts that the equilibrium point at the origin is unstable. One solution, starting at $(0.01, 0.01, 0.01)$ is shown in the following figure.



15. If

$$\mathbf{y}' = \begin{pmatrix} 3 & -2 & -5 & 3 \\ 16 & -6 & -17 & 9 \\ -14 & 5 & 15 & -8 \\ -19 & 8 & 23 & -13 \end{pmatrix} \mathbf{y},$$

then a computer reveals that matrix

$$A = \begin{pmatrix} 3 & -2 & -5 & 3 \\ 16 & -6 & -17 & 9 \\ -14 & 5 & 15 & -8 \\ -19 & 8 & 23 & -13 \end{pmatrix}$$

has characteristic equation

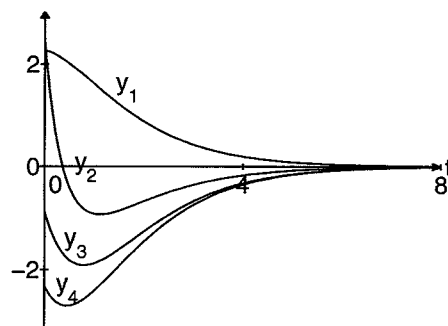
$$p(\lambda) = (\lambda - 2)(\lambda + 1)^3$$

and eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, the latter having algebraic multiplicity 3. Thus, one eigenvalue has positive real part and Theorem 7.4 predicts that the equilibrium point at the origin is unstable. One such solution, with initial condition $(0.1, 0.1, 0.1, 0.1)^T$, seems to approach the origin, only to veer away with the passage of time, much like a saddle point solution in the phase plane. This behavior is indicated in the following plot of each component of the solution versus time.

16. With a computer we find that the eigenvalues of the matrix

$$A = \begin{pmatrix} -3 & 3 & 0 & -4 \\ 4 & -7 & 0 & 8 \\ 0 & -1 & -3 & 2 \\ 4 & -6 & 0 & 7 \end{pmatrix}$$

are -3 and -1 , the latter having algebraic multiplicity 3. Since all of the eigenvalues are negative, Theorem 6.2 tells us that the origin is an asymptotically stable equilibrium point. This is verified by the solution plotted in the accompanying figure.



17. (a) In matrix form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -3 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Using a computer, matrix

$$A = \begin{pmatrix} -3 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

has characteristic polynomial

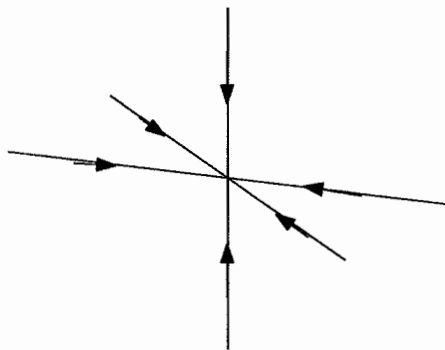
$$p(\lambda) = (\lambda + 3)(\lambda + 2)(\lambda + 1)$$

and eigenvalues -3 , -2 and -1 . A computer also reveals the associated eigenvectors which lead to the following exponential solutions.

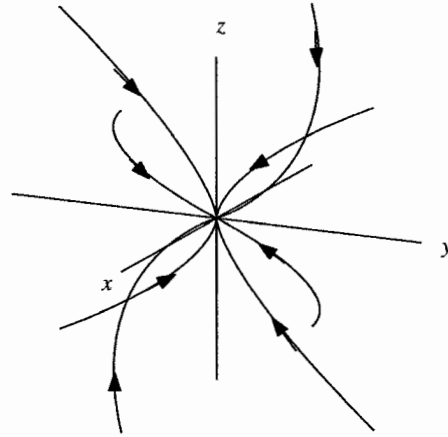
$$\mathbf{y}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and}$$

$$\mathbf{y}_3(t) = e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

These exponential solutions generate the half-line solutions shown in the following figure. Each of the half-line solutions decay to the origin with the passage of time.



- (b) We selected initial conditions $(1, 0, 1)^T$, $(-1, 0, 1)^T$, $(1/2, 1, 1)^T$, $(-1/2, -1, 1)^T$, $(1, 0, -1)^T$, $(-1, 0, -1)^T$, $(1/2, 1, -1)^T$, and $(-1/2, -1, -1)^T$ to craft the portrait in the following figure.



- (c) Nodal sink

18. (a) If

$$\mathbf{y}' = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{y},$$

then the coefficient matrix

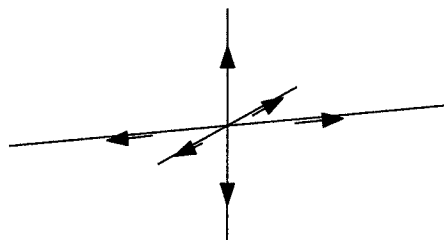
$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

is upper triangular, so the eigenvalues lie on the diagonal, $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. A computer reveals the associated eigenvectors, and consequently, the exponential solutions

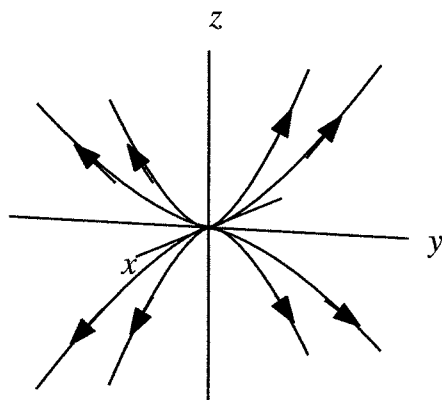
$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{2t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These exponential solutions generate the half-line solutions shown in the following figure.



- (b) We selected initial conditions $(1, 2, 1)^T$, $(1, 2, -1)^T$, $(2, -1, 1)^T$, $(2, -1, -1)^T$, $(-1, -2, 1)^T$, $(-1, -2, -1)^T$, $(-2, 1, 1)^T$, and $(-2, 1, -1)^T$ to craft the portrait in the following figure. Each was scaled by a factor of 1×10^{-3} .



- (c) Nodal source.

19. (a) If

$$\mathbf{y}' = \begin{pmatrix} -1 & -10 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{y},$$

then, using a computer, matrix

$$A = \begin{pmatrix} -1 & -10 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda + 1)(\lambda^2 + 2\lambda + 101)$$

and eigenvalues -1 , $-1 + 10i$ and $-1 - 10i$. A computer also generates associated eigenvectors, leading to the real solution

$$\mathbf{y}_1(t) = e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the complex solution

$$\begin{aligned} \mathbf{z}(t) &= e^{(-1+10i)t} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \\ &= e^{-t} (\cos 10t + i \sin 10t) \\ &\quad \times \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right] \\ &= e^{-t} \begin{pmatrix} \cos 10t \\ \sin 10t \\ 0 \end{pmatrix} + i e^{-t} \begin{pmatrix} \sin 10t \\ -\cos 10t \\ 0 \end{pmatrix}. \end{aligned}$$

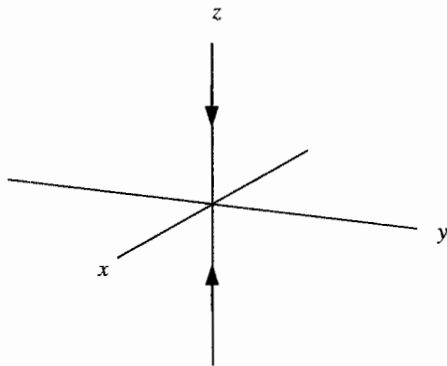
This leads to the real solutions

$$\begin{aligned} \mathbf{y}_2(t) &= e^{-t} \begin{pmatrix} \cos 10t \\ \sin 10t \\ 0 \end{pmatrix}, \quad \text{and} \\ \mathbf{y}_3(t) &= e^{-t} \begin{pmatrix} \sin 10t \\ -\cos 10t \\ 0 \end{pmatrix}. \end{aligned}$$

- (b) Any solution starting on the z -axis lies on the half-lines generated by the exponential solution

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, the solution will remain on the z -axis as it decays to the equilibrium point at the origin. In the following image, solutions with initial conditions $(0, 0, 1)^T$ and $(0, 0, -1)^T$ remain on the z -axis and decay to the origin.



(c) The general solution is

$$\mathbf{y}(t) = C_1 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} \cos 10t \\ \sin 10t \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} \sin 10t \\ -\cos 10t \\ 0 \end{pmatrix}.$$

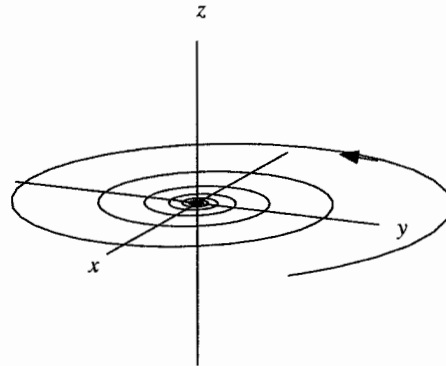
If a solution starts in the xy -plane with initial condition $\mathbf{y}(0) = (a, b, 0)^T$, then

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

leading to $C_1 = 0$, $C_2 = a$, and $C_3 = -b$. Thus, the particular solution is

$$\mathbf{y}(t) = a e^{-t} \begin{pmatrix} \cos 10t \\ \sin 10t \\ 0 \end{pmatrix} + b e^{-t} \begin{pmatrix} \sin 10t \\ -\cos 10t \\ 0 \end{pmatrix},$$

so these solutions will remain in the xy -plane and spiral inward to the equilibrium point at the origin. This is shown in the following figure, where we have plotted the solution with initial condition $(1, 1, 0)^T$.



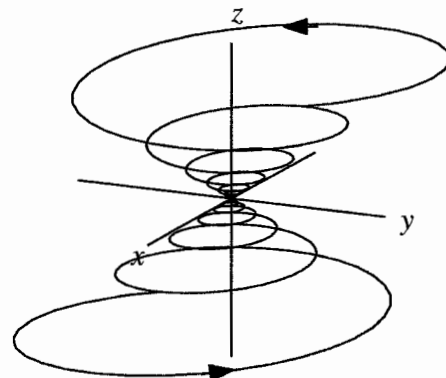
(d) A solution having initial condition $\mathbf{y}(0) = (a, b, c)^T$, where $c \neq 0$, would lead to

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

and $C_1 = c$, $C_2 = a$, and $C_3 = -b$. Thus, the particular solution is

$$\mathbf{y}(t) = \begin{pmatrix} e^{-t}(a \cos 10t - b \sin 10t) \\ e^{-t}(a \sin 10t + b \cos 10t) \\ c e^{-t} \end{pmatrix}.$$

We saw in part (c) that if $c = 0$, solutions spiral into the origin while remaining in the xy -plane. In this case, the z -coordinate decays to zero, so it is reasonable to assume that solutions will spiral into the origin, staying above or below the xy -plane. Solutions with initial conditions $(1, 1, 1)^T$ and $(-1, -1, -1)^T$ are shown in the following figure.



Section 9.8. Higher-Order Linear Systems

1. (a) If

$$\mathbf{x}_1(t) = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix},$$

then

$$\mathbf{x}'_1(t) = \begin{pmatrix} 3e^{3t} \\ 9e^{3t} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \mathbf{x}_1(t) = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 9e^{3t} \end{pmatrix},$$

so $\mathbf{x}_1$ is a solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \mathbf{x}.$$

Similarly, if

$$\mathbf{x}_2(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix},$$

then

$$\mathbf{x}'_2(t) = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \mathbf{x}_2(t) = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix},$$

so $\mathbf{x}_2$ is a solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \mathbf{x}.$$

To show independence, we need only show that the functions are independent at one value of t . However,

$$\mathbf{x}_1(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are clearly independent ($\mathbf{x}_2(0)$ is not a multiple of $\mathbf{x}_1(0)$).

(b) Because

$$\begin{aligned} \mathbf{x}(t) &= C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) \\ &= C_1 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}, \end{aligned}$$

the first component of $\mathbf{x}(t)$ is $y(t) = C_1 e^{3t} + C_2 e^{-t}$. Thus,

$$\begin{aligned} y' &= 3C_1 e^{3t} - C_2 e^{-t} \\ y'' &= 9C_1 e^{3t} + C_2 e^{-t}, \end{aligned}$$

and

$$\begin{aligned} y'' - 2y' - 3y &= (9C_1 e^{3t} + C_2 e^{-t}) \\ &\quad - 2(3C_1 e^{3t} - C_2 e^{-t}) \\ &\quad - 3(C_1 e^{3t} + C_2 e^{-t}) \\ &= 0 \end{aligned}$$

2. (a) If $\mathbf{x}_1(t) = (\sin 2t, 2 \cos 2t)^T$, then $\mathbf{x}'_1(t) = (2 \cos 2t, -4 \sin 2t)^T$ and

$$\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}_1(t) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix} = \begin{pmatrix} 2 \cos 2t \\ -4 \sin 2t \end{pmatrix},$$

so $\mathbf{x}_1$ is a solution of

$$\mathbf{x}'_1 = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}.$$

Similarly, if $\mathbf{x}_2(t) = (\cos 2t, -2 \sin 2t)^T$, then $\mathbf{x}'_2(t) = (-2 \sin 2t, -4 \cos 2t)^T$ and

$$\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}_2(t) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix} = \begin{pmatrix} -2 \sin 2t \\ -4 \cos 2t \end{pmatrix},$$

so $\mathbf{x}_2$ is also a solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}.$$

To show independence, we need only show that the functions are independent at one value of t . However,

$$\mathbf{x}_1(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are clearly independent ($\mathbf{x}_2(0)$ is not a multiple of $\mathbf{x}_1(0)$).

(b) Because

$$\begin{aligned} \mathbf{x}(t) &= C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) \\ &= C_1 \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix} + C_2 \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix}, \end{aligned}$$

The first component of $\mathbf{x}(t)$ is $y(t) = C_1 \sin 2t + C_2 \cos 2t$. Thus,

$$\begin{aligned} y' &= 2C_1 \cos 2t - 2C_2 \sin 2t, \quad \text{and} \\ y'' &= -4C_1 \sin 2t - 4C_2 \cos 2t, \end{aligned}$$

and

$$\begin{aligned} y'' + 4y &= (-4C_1 \sin 2t - 4C_2 \cos 2t) \\ &\quad + 4(C_1 \sin 2t + C_2 \cos 2t) \\ &= 0. \end{aligned}$$

3. If $y_1(t) = e^t$ and $y_2(t) = e^{2t}$, suppose that there exists constants c_1 and c_2 such that

$$c_1 e^t + c_2 e^{2t} = 0$$

for all t . Then,

$$\begin{aligned} t = 0 &\Rightarrow c_1 + c_2 = 0 \\ t = 1 &\Rightarrow c_1 e + c_2 e^2 = 0. \end{aligned}$$

Solving the first equation, $c_1 = -c_2$, and substituting into the second equation gives

$$\begin{aligned} -c_2 e + c_2 e^2 &= 0 \\ c_2(e^2 - e) &= 0. \end{aligned}$$

Because $e^2 - e \neq 0$, this gives $c_2 = 0$, whence $c_1 = -c_2 = 0$. Hence, y_1 and y_2 are independent.

4. Suppose $y_1(t) = e^t \cos t$ and $y_2(t) = e^t \sin t$, and there are constants C_1 and C_2 such that $C_1 y_1(t) + C_2 y_2(t) = e^t [C_1 \cos t + C_2 \sin t] = 0$ for all t . Then at $t = 0$ we have $C_1 = 0$, and at $t = \pi/2$ we have $C_2 e^{\pi/2} = 0$. Hence both constants are 0, so the functions are linearly independent.

5. If $y_1(t) = \cos t$, $y_2(t) = \sin t$, and $y_3(t) = e^t$, suppose that there exists constants c_1 , c_2 , and c_3 such that

$$c_1 \cos t + c_2 \sin t + c_3 e^t = 0$$

for all t . Then,

$$\begin{aligned} t = 0 &\Rightarrow c_1 + c_3 = 0 \\ t = \pi/2 &\Rightarrow c_2 + c_3 e^{\pi/2} = 0 \\ t = \pi &\Rightarrow -c_1 + c_3 e^\pi = 0. \end{aligned}$$

Solving the first equation, $c_1 = -c_3$, and substituting this into the third equation gives

$$0 = c_3 + c_3 e^\pi = c_3(1 + e^\pi).$$

Because $e^\pi + 1 \neq 0$, this gives $c_3 = 0$, whence $c_1 = -c_3 = 0$. Substituting $c_3 = 0$ into the second equation gives

$$0 = c_2 + 0e^\pi = c_2.$$

Therefore, y_1 , y_2 , and y_3 are linearly independent.

6. If $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = t^2 e^t$, suppose that there exists constants C_1 , C_2 , and C_3 such that

$$C_1 e^t + C_2 t e^t + C_3 t^2 e^t = 0$$

for all t . If $t = 1$, then

$$\begin{aligned} C_1 e + C_2 e + C_3 e &= 0 \\ (C_1 + C_2 + C_3)e &= 0 \\ C_1 + C_2 + C_3 &= 0. \end{aligned}$$

If $t = -1$, then

$$\begin{aligned} C_1 e^{-1} - C_2 e^{-1} + C_3 e^{-1} &= 0 \\ (C_1 - C_2 + C_3)e^{-1} &= 0 \\ C_1 - C_2 + C_3 &= 0 \end{aligned}$$

Finally, if $t = 0$, then $C_1 = 0$ and the last two equations become

$$\begin{aligned} C_2 + C_3 &= 0 \\ -C_2 + C_3 &= 0. \end{aligned}$$

Because the coefficient matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has determinant $D = 2$, the coefficient matrix is nonsingular and this last system has unique solution $C_2 = C_3 = 0$. Hence, $C_1 = C_2 = C_3 = 0$ and the solutions $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = t^2 e^t$ are linearly independent.

7. If $y_1(t) = \cos 3t$, then

$$\begin{aligned}y'_1(t) &= -3 \sin 3t \\y''_1(t) &= -9 \cos 3t\end{aligned}$$

and

$$y''_1 + 9y_1 = -9 \cos 3t + 9 \cos 3t = 0.$$

Similarly, if $y_2(t) = \sin 3t$, then

$$\begin{aligned}y'_2(t) &= 3 \cos 3t \\y''_2(t) &= -9 \sin 3t,\end{aligned}$$

and

$$y''_2 + 9y_2 = -9 \sin 3t + 9 \sin 3t = 0.$$

Thus, both y_1 and y_2 are solutions of $y'' + 9y = 0$. Finally, the Wronskian is

$$\begin{aligned}W(t) &= \det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \\&= \det \begin{pmatrix} \cos 3t & \sin 3t \\ -3 \sin 3t & 3 \cos 3t \end{pmatrix} \\&= 3 \cos^2 3t - 3 \sin^2 3t \\&= 3,\end{aligned}$$

which is nonzero for all t . Hence, the solutions y_1 and y_2 are linearly independent.

8. If $y_1(t) = e^{-10t}$ and $y_2(t) = e^t$, we have $y'_1(t) = -10e^{-10t}$ and $y''_1(t) = 100e^{-10t}$. Hence $y''_1 + 9y'_1 - 10y_1 = (100 - 90 - 10)e^{-10t} = 0$, so y_1 is a solution. Similarly, $y'_2(t) = y''_2(t) = e^t$, so $y''_2 + 9y'_2 - 10y_2 = (1 + 9 - 10)e^t = 0$, so y_2 is a solution. We have $y_1(t) = e^{-10t} = e^{-11t}e^t = e^{-11t}y_2(t)$. Since e^{-11t} is not constant, the functions are linearly independent, and therefore form a fundamental set of solutions.

9. If $y_1(t) = e^{2t}$, then

$$\begin{aligned}y'_1(t) &= 2e^{2t} \\y''_1(t) &= 4e^{2t}.\end{aligned}$$

and

$$y''_1 - 4y'_1 + 4y_1 = 4e^{2t} - 8e^{2t} + 4e^{2t} = 0.$$

Similarly, if $y_2(t) = e^{2t}$, then

$$\begin{aligned}y'_2(t) &= e^{2t}(2t + 1) \\y''_2(t) &= e^{2t}(4t + 4),\end{aligned}$$

and

$$\begin{aligned}y''_2 - 4y'_2 + 4y_2 &= e^{2t}(4t + 4) - 4e^{2t}(2t + 1) \\&\quad + 4te^{2t} \\&= e^{2t}(4t + 4 - 8t - 4 + 4t) \\&= 0.\end{aligned}$$

Thus, both y_1 and y_2 are solutions of $y'' - 4y' + 4y = 0$. Finally, the Wronskian is

$$\begin{aligned}W(t) &= \det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \\&= \det \begin{pmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}(2t + 1) \end{pmatrix} \\&= e^{4t}(2t + 1) - 2te^{4t} \\&= e^{4t},\end{aligned}$$

which is nonzero for all t . Hence, the solutions y_1 and y_2 are linearly independent.

10. If $y_1(t) = \cos 3t$, then

$$\begin{aligned}y'_1 &= -3 \sin 3t, \\y''_1 &= -9 \cos 3t, \text{ and} \\y'''_1 &= 27 \sin 3t.\end{aligned}$$

Thus,

$$\begin{aligned}y'''_1 - 3y''_1 + 9y'_1 - 27y_1 &= 27 \sin 3t - 3(-9 \cos 3t) \\&\quad + 9(-3 \sin 3t) - 27(\cos 3t) \\&= 0\end{aligned}$$

and y_1 is a solution of $y''' - 3y'' + 9y' - 27y = 0$. In a similar manner, $y_2(t) = \sin 3t$ and $y_3(t) = e^{3t}$ are also solutions. Finally, the Wronskian is

$$\begin{aligned}W(t) &= \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{pmatrix} \\&= \begin{vmatrix} \cos 3t & \sin 3t & e^{3t} \\ -3 \sin 3t & 3 \cos 3t & 3e^{3t} \\ -9 \cos 3t & -9 \sin 3t & 9e^{3t} \end{vmatrix}.\end{aligned}$$

Using a computer, $W(t) = 54e^{3t}$, which is never zero. Therefore, the solutions y_1 , y_2 , and y_3 are linearly independent.

11. If $y_1(t) = e^t$, then

$$y_1''' - 3y_1'' + 3y_1' - y_1 = e^t - 3e^t + 3e^t - e^t = 0.$$

If $y_2(t) = te^t$, then

$$y_2' = (t+1)e^t$$

$$y_2'' = (t+2)e^t$$

$$y_2''' = (t+3)e^t$$

and

$$\begin{aligned} y_2''' - 3y_2'' + 3y_2' - y_2 &= (t+3)e^t - 3(t+2)e^t \\ &\quad + 3(t+1)e^t - te^t \\ &= e^t(t+3-3t-6+3t+3-t) \\ &= 0. \end{aligned}$$

If $y_3(t) = t^2e^t$, then

$$y_3' = (t^2 + 2t)e^t$$

$$y_3'' = (t^2 + 4t + 2)e^t$$

$$y_3''' = (t^2 + 6t + 6)e^t,$$

and

$$\begin{aligned} y_3''' - 3y_3'' + 3y_3' - y_3 &= (t^2 + 6t + 6)e^t - 3(t^2 + 4t + 2)e^t \\ &\quad + 3(t^2 + 2t)e^t - t^2e^t \\ &= e^t(t^2 + 6t + 6 \\ &\quad - 3t^2 - 12t - 6 + 3t^2 + 6t - t^2) \\ &= 0. \end{aligned}$$

Thus, y_1 , y_2 , and y_3 are solutions of the equation $y''' - 3y'' + 3y' - y = 0$. Finally, the Wronskian is

$$\begin{aligned} W(t) &= \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} \\ &= \det \begin{pmatrix} e^t & te^t & t^2e^t \\ e^t & (t+1)e^t & (t^2+2t)e^t \\ e^t & (t+2)e^t & (t^2+4t+2)e^t \end{pmatrix}. \end{aligned}$$

Using a computer, $W(t) = 2e^{3t}$, which is never zero. Therefore, the solutions y_1 , y_2 and y_3 are linearly independent.

————— × —————

12. If $y_1 = \cos 3t$, then

$$y_1' = -3 \sin 3t, \quad y_1'' = -9 \cos 3t, \quad y_1''' = 27 \sin 3t, \quad \text{and} \quad y_1^{(4)} = 81 \cos 3t.$$

Thus,

$$y_1^{(4)} + 13y_1'' + 36y_1 = 81 \cos 3t + 13(-9 \cos 3t) + 36 \cos 3t = 0$$

and y_1 is a solution of $y^{(4)} + 13y'' + 36y = 0$. In a similar manner, $y_2 = \sin 3t$, $y_3 = \cos 2t$, and $y_4 = \sin 2t$ are also solutions. Finally, the Wronskian is

$$W(t) = \det \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{pmatrix} = \det \begin{pmatrix} \cos 3t & \sin 3t & \cos 2t & \sin 2t \\ -3 \sin 3t & 3 \cos 3t & -2 \sin 2t & 2 \cos 2t \\ -9 \cos 3t & -9 \sin 3t & -4 \cos 2t & -4 \sin 2t \\ 27 \sin 3t & -27 \cos 3t & 8 \sin 2t & -8 \cos 2t \end{pmatrix}.$$

Using a computer, $W(t) = 150$, so the solutions y_1 , y_2 , y_3 , and y_4 are linearly independent.

13. (a) If $y = e^{\lambda t}$, then

$$y' = \lambda e^{\lambda t}, \quad y'' = \lambda^2 e^{\lambda t}, \quad \text{and} \quad y''' = \lambda^3 e^{\lambda t}.$$

Subbing these results into $y''' + ay'' + by' + cy = 0$ gives

$$\begin{aligned}\lambda^3 e^{\lambda t} + a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^3 + a\lambda^2 + b\lambda + c) &= 0.\end{aligned}$$

Because $e^{\lambda t}$ can never equal zero, we must have

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0.$$

(b) If

$$y''' = -ay'' - by' - cy,$$

let $x_1 = y$, $x_2 = y'$, and $x_3 = y''$. Then

$$\begin{aligned}x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -ax_3 - bx_2 - cx_1.\end{aligned}$$

In matrix form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and if

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix},$$

the characteristic polynomial is

$$p(\lambda) = -\det(A - \lambda I) = -\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -c & -b & -a - \lambda \end{vmatrix}.$$

Expanding across the first row,

$$\begin{aligned}p(\lambda) &= \lambda \begin{vmatrix} -\lambda & 1 \\ -b & -a - \lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ -c & -a - \lambda \end{vmatrix} \\ &= \lambda(a\lambda + \lambda^2 + b) + 1(c) \\ &= \lambda^3 + a\lambda^2 + b\lambda + c.\end{aligned}$$

14. The characteristic polynomial of the equation $y''' - 2y'' - y' + 2y = 0$ is $p(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$. Notice that 2 is a root. Hence the polynomial factors as $p(\lambda) = (\lambda - 2)(\lambda^2 - 1) = (\lambda - 2)(\lambda - 1)(\lambda + 1)$. Consequently, the roots are -1, 1, and 2. We have the exponential solutions $y_1(t) = e^{-t}$, $y_2(t) = e^t$, and $y_3(t) = e^{2t}$. Since the roots are distinct, these solu-

tions are linearly independent, and therefore form a fundamental set of solutions.

15. If $y''' - 3y'' - 4y' + 12y = 0$, then the characteristic

equation factors

$$\begin{aligned}\lambda^3 - 3\lambda^2 - 4\lambda + 12 &= 0 \\ \lambda^2(\lambda - 3) - 4(\lambda - 3) &= 0 \\ (\lambda + 2)(\lambda - 2)(\lambda - 3) &= 0.\end{aligned}$$

Thus, the characteristic equation has roots -2 , 2 , and 3 , leading to the general solution

$$y(t) = C_1 e^{-2t} + C_2 e^{2t} + C_3 e^{3t}.$$

16. If $y^{(4)} - 5y'' + 4y = 0$, then the characteristic equation factors

$$\begin{aligned}\lambda^4 - 5\lambda^2 + 4 &= 0 \\ (\lambda^2 - 4)(\lambda^2 - 1) &= 0 \\ (\lambda + 2)(\lambda - 2)(\lambda + 1)(\lambda - 1) &= 0.\end{aligned}$$

Thus, the characteristic equation has roots -2 , 2 , -1 , and 1 , leading to the general solution

$$y(t) = C_1 e^{-2t} + C_2 e^{2t} + C_3 e^{-t} + C_4 e^t.$$

17. If $y^{(4)} - 13y'' + 36y = 0$, then the characteristic equation factors

$$\begin{aligned}\lambda^4 - 13\lambda^2 + 36 &= 0 \\ (\lambda^2 - 4)(\lambda^2 - 9) &= 0 \\ (\lambda + 2)(\lambda - 2)(\lambda + 3)(\lambda - 3) &= 0.\end{aligned}$$

Thus, the characteristic equation has roots -3 , -2 , 2 , and 3 , leading to the general solution

$$y(t) = C_1 e^{-3t} + C_2 e^{-2t} + C_3 e^{2t} + C_4 e^{3t}.$$

18. If $y''' + 2y'' - 5y' - 6y = 0$, then the characteristic equation is $p(\lambda) = \lambda^3 + 2\lambda^2 - 5\lambda - 6$. Note that -1 is a root of p , so

$$\begin{aligned}p(\lambda) &= (\lambda + 1)(\lambda^2 + \lambda - 6) \\ &= (\lambda + 1)(\lambda + 3)(\lambda - 2).\end{aligned}$$

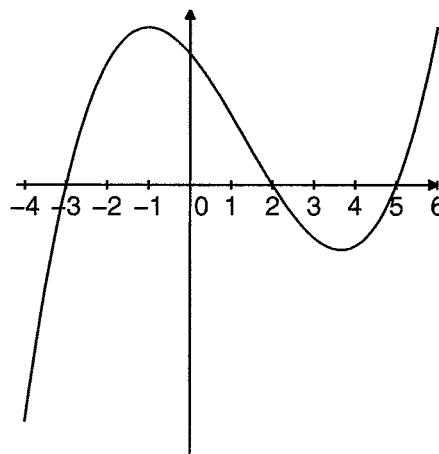
Thus, the characteristic polynomial has roots -1 , -3 , and 2 , leading to the general solution

$$y(t) = C_1 e^{-t} + C_2 e^{-3t} + C_3 e^{2t}.$$

19. If $y''' - 4y'' - 11y' + 30y = 0$, then the characteristic equation is

$$\lambda^3 - 4\lambda^2 - 11\lambda + 30 = 0.$$

A plot of the characteristic equation (computer or calculator) reveals possible roots.



The plot suggests that -3 is a root, but division by $\lambda + 3$ guarantees that -3 is a root and $\lambda + 3$ is a factor.

$$\begin{aligned}(\lambda + 3)(\lambda^2 - 7\lambda + 10) &= 0 \\ (\lambda + 3)(\lambda - 2)(\lambda - 5) &= 0\end{aligned}$$

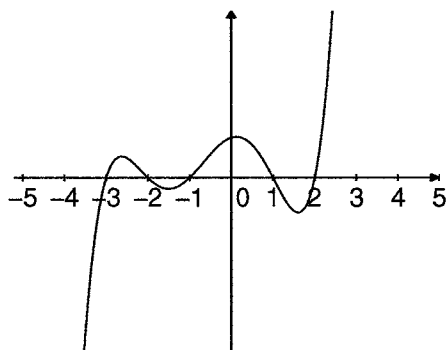
Thus, the roots are -3 , 2 , and 5 , and the general solution is

$$y(t) = C_1 e^{-3t} + C_2 e^{2t} + C_3 e^{5t}.$$

20. If $y^{(5)} + 3y^{(4)} - 5y''' - 15y'' + 4y' + 12y = 0$, then the characteristic equation is

$$\lambda^5 + 3\lambda^4 - 5\lambda^3 - 15\lambda^2 + 4\lambda + 12 = 0.$$

A plot of the characteristic equation (computer or calculator) reveals possible roots.



The plot suggests a root at -3 . Long (or synthetic) division reveals

$$\begin{aligned}(\lambda + 3)(\lambda^4 - 5\lambda^2 + 4) &= 0 \\(\lambda + 3)(\lambda^2 - 4)(\lambda^2 - 1) &= 0 \\(\lambda + 3)(\lambda + 2)(\lambda - 2)(\lambda + 1)(\lambda - 1) &= 0.\end{aligned}$$

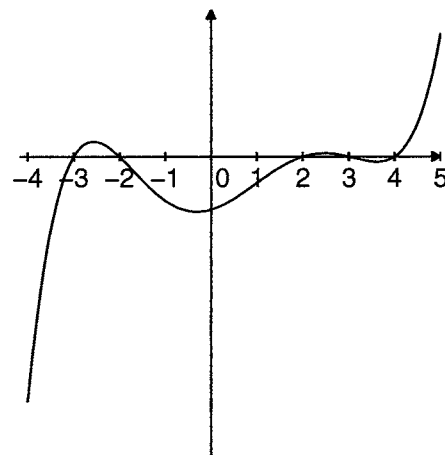
Thus, the roots of the characteristic equation are -3 , -2 , -1 , 1 , and 2 , and the general solution is

$$y(t) = C_1 e^{-3t} + C_2 e^{-2t} + C_3 e^{-t} + C_4 e^t + C_5 e^{2t}.$$

21. If $y^{(5)} - 4y^{(4)} - 13y''' + 52y'' + 36y' - 144y = 0$, then the characteristic equation is

$$\lambda^5 - 4\lambda^4 - 13\lambda^3 + 52\lambda^2 + 36\lambda - 144 = 0.$$

A plot of the characteristic equation (computer or calculator) reveals possible roots.



The plot suggests a root at -3 . Long (or synthetic) division reveals

$$(\lambda + 3)(\lambda^4 - 7\lambda^3 + 8\lambda^2 + 28\lambda - 48) = 0.$$

The plot suggests a root at -2 . Again, division reveals

$$(\lambda + 3)(\lambda + 2)(\lambda^3 - 9\lambda^2 + 26\lambda - 24) = 0.$$

The plot suggest a root at 2 . Again, division reveals

$$\begin{aligned}(\lambda + 3)(\lambda + 2)(\lambda - 2)(\lambda^2 - 7\lambda + 12) &= 0 \\(\lambda + 3)(\lambda + 2)(\lambda - 2)(\lambda - 3)(\lambda - 4) &= 0.\end{aligned}$$

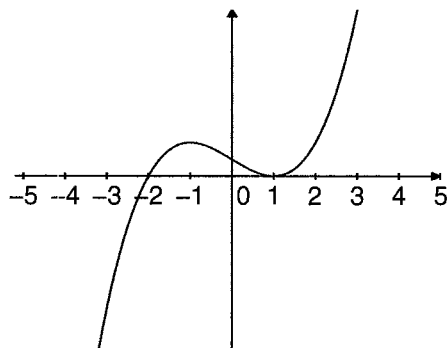
Thus, the roots of the characteristic equation are -3 , -2 , 2 , 3 , and 4 , and the general solution is

$$y(t) = C_1 e^{-3t} + C_2 e^{-2t} + C_3 e^{2t} + C_4 e^{3t} + C_5 e^{4t}.$$

22. If $y''' - 3y' + 2y = 0$, the characteristic equation is

$$\lambda^3 - 3\lambda + 2 = 0.$$

The plot of the characteristic equation



suggests a root at $\lambda = -2$. Division by $\lambda + 2$ reveals

$$(\lambda + 2)(\lambda^2 - 2\lambda + 1) = 0$$

$$(\lambda + 2)(\lambda - 1)^2 = 0.$$

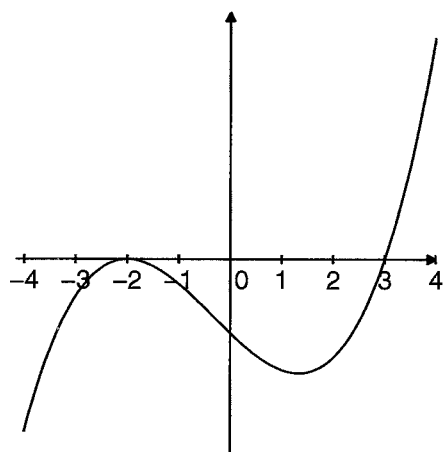
Thus, the roots are -2 and 1 , with the latter having multiplicity of 2 . Therefore, the general solution is

$$y(t) = C_1 e^{-2t} + C_2 e^t + C_3 t e^t.$$

23. If $y''' + y'' - 8y' - 12y = 0$, then the characteristic equation is

$$\lambda^3 + \lambda^2 - 8\lambda - 12 = 0.$$

The plot of the characteristic equation



suggests a root at -2 . Division shows that

$$(\lambda + 2)(\lambda^2 - \lambda - 6) = 0$$

$$(\lambda + 2)(\lambda + 2)(\lambda - 3) = 0.$$

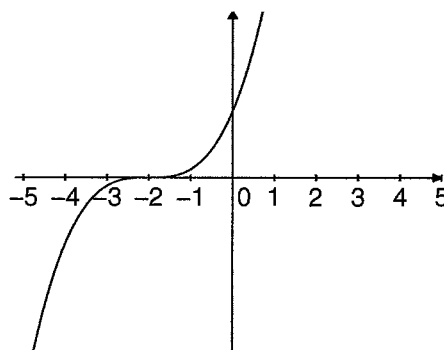
Hence there are two roots, -2 and 3 , with the former having algebraic multiplicity 2 . Thus, the general solution is

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t} + C_3 e^{3t}.$$

24. If $y''' + 6y'' + 12y' + 8y = 0$, then the characteristic equation is

$$\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0.$$

A plot of the characteristic equation



suggests a multiple root at -2 . Division by $\lambda + 2$ reveals that

$$(\lambda + 2)(\lambda^2 + 4\lambda + 4) = 0$$

$$(\lambda + 2)^3 = 0.$$

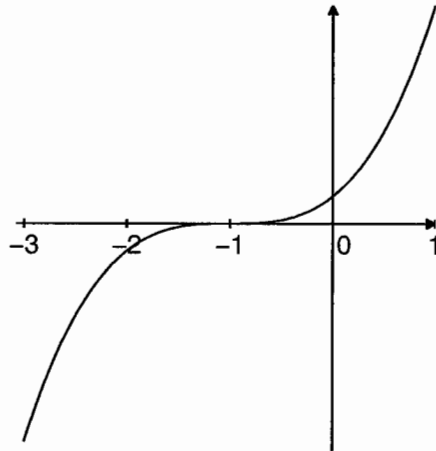
Thus, $\lambda = -2$ is a root of algebraic multiplicity 3 . Therefore, the general solution is

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t} + C_3 t^2 e^{-2t}.$$

25. If $y''' + 3y'' + 3y' + y = 0$, then the characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0.$$

The plot of the characteristic equation



suggests a root at -1 . Division shows that

$$(\lambda + 1)(\lambda^2 + 2\lambda + 1) = 0$$

$$(\lambda + 1)^3 = 0.$$

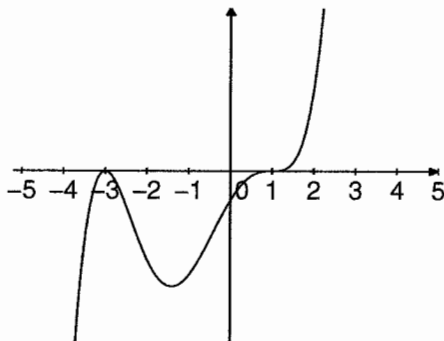
Thus, -1 is a root of algebraic multiplicity 3. Therefore, the general solution is

$$y(t) = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}.$$

26. If $y^{(5)} + 3y^{(4)} - 6y''' - 10y'' + 21y' - 9y = 0$, then the characteristic equation is

$$\lambda^5 + 3\lambda^4 - 6\lambda^3 - 10\lambda^2 + 21\lambda - 9 = 0.$$

A plot of the characteristic equation



suggests multiple roots at -3 and 1 . Repeated division by $\lambda - 1$ reveals

$$(\lambda - 1)^3(\lambda^2 + 6\lambda + 9) = 0$$

$$(\lambda - 1)^3(\lambda + 3)^2 = 0.$$

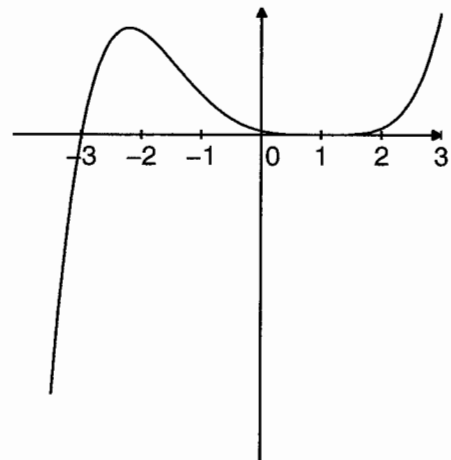
Thus, -3 and 1 are roots, with algebraic multiplicities 2 and 3, respectively. Therefore, the general solution is

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t} + C_3 e^t + C_4 t e^t + C_5 t^2 e^t.$$

27. If $y^{(5)} - y^{(4)} - 6y''' + 14y'' - 11y' + 3y = 0$, then the characteristic equation is

$$\lambda^5 - \lambda^4 - 6\lambda^3 + 14\lambda^2 - 11\lambda + 3 = 0.$$

The plot of the characteristic equation



suggests a multiple root at 1 . Repeated division by $\lambda - 1$ reveals that

$$(\lambda - 1)^4(\lambda + 3) = 0.$$

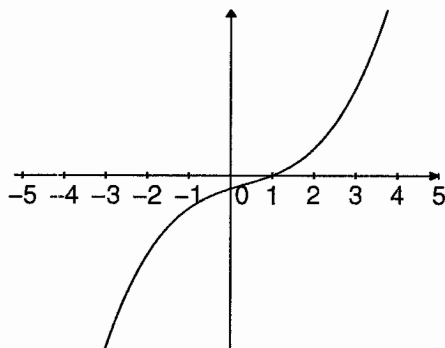
Thus, the roots are 1 and -3 , with the former having algebraic multiplicity 4. Therefore, the general solution is

$$y(t) = C_1 e^t + C_2 t e^t + C_3 t^2 e^t + C_4 t^3 e^t + C_5 e^{-3t}.$$

28. If $y''' - y'' + 4y' - 4y = 0$, then the characteristic equation is

$$\lambda^3 - \lambda^2 + 4\lambda - 4 = 0.$$

A plot of the characteristic equation



reveals a possible root at $\lambda = 1$. Division by $\lambda - 1$ reveals

$$(\lambda - 1)(\lambda^2 + 4) = 0.$$

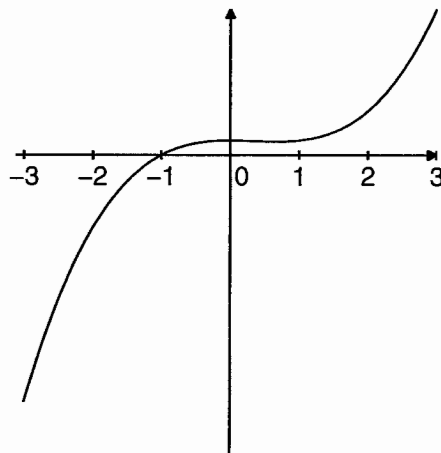
Thus, the zero are 1, $-2i$, and $2i$, and the general solution is

$$y(t) = C_1 e^t + C_2 \cos 2t + C_3 \sin 2t.$$

29. If $y''' - y'' + 2y = 0$, then the characteristic equation is

$$\lambda^3 - \lambda^2 + 2 = 0.$$

A plot of the characteristic equation



suggests a root at -1 . Division reveals

$$(\lambda + 1)(\lambda^2 - 2\lambda + 2) = 0.$$

The quadratic formula provides the remaining roots, $1 \pm i$. Thus, the general solution is

$$y(t) = C_1 e^{-t} + C_2 e^t \cos t + C_3 e^t \sin t.$$

30. If $y^{(4)} + 17y'' + 16y = 0$, then the characteristic equation is

$$\lambda^4 + 17\lambda^2 + 16 = 0.$$

This factors

$$(\lambda^2 + 16)(\lambda^2 + 1) = 0,$$

so we have zeros $\pm 4i$ and $\pm i$. Consequently, the general solution is

$$y(t) = C_1 \cos 4t + C_2 \sin 4t + C_3 \cos t + C_4 \sin t.$$

31. If $y^{(4)} + 2y'' + y = 0$, then the characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = 0,$$

which easily factors as

$$(\lambda^2 + 1)^2 = 0.$$

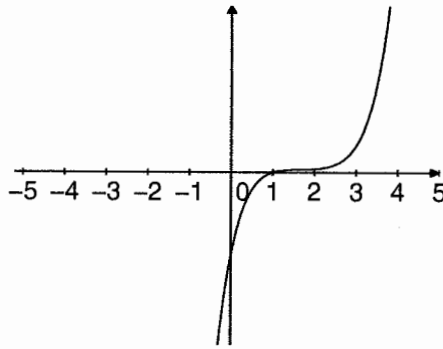
Thus, both i and $-i$ are roots of multiplicity 2. Therefore, the general solution is

$$y(t) = C_1 \cos t + C_2 t \cos t + C_3 \sin t + C_4 t \sin t.$$

32. If $y^{(5)} - 9y^{(4)} + 34y''' - 66y'' + 65y' - 25y = 0$, then the characteristic equation is

$$\lambda^5 - 9\lambda^4 + 34\lambda^3 - 66\lambda^2 + 65\lambda - 25 = 0.$$

A plot of the characteristic equation



reveals a possible zero at 1. Division by $\lambda - 1$ reveals

$$(\lambda - 1)(\lambda^4 - 8\lambda^3 + 26\lambda^2 - 40\lambda + 25) = 0.$$

A computer is used to find that $2+i$ and $2-i$ are zeros of the second factor, each with algebraic multiplicity 2. Thus,

$$y(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t + C_3 t e^{2t} \cos t + C_4 t e^{2t} \sin t + C_5 e^t$$

is the general solution.

33. If $y^{(6)} + 3y^{(4)} + 3y'' + y = 0$, then the characteristic equation is

$$\lambda^6 + 3\lambda^4 + 3\lambda^2 + 1 = 0.$$

The form of the characteristic equation suggests the binomial theorem and

$$(\lambda^2 + 1)^3 = 0.$$

Thus, i and $-i$ are roots, each having algebraic multiplicity 3. Therefore, the general solution is

$$y(t) = C_1 \cos t + C_2 t \cos t + C_3 t^2 \cos t + C_4 \sin t + C_5 t \sin t + C_6 t^2 \sin t.$$

34. If $y'' - 2y' - 3y = 0$, then the characteristic equation is

$$\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0.$$

Thus, the zeros are 3 and -1 and the general solution is

$$y(t) = C_1 e^{3t} + C_2 e^{-t}.$$

The initial condition $y(0) = 4$ provides

$$4 = C_1 + C_2.$$

Differentiating,

$$y'(t) = 3C_1 e^{3t} - C_2 e^{-t}.$$

The initial condition $y'(0) = 0$ provides

$$0 = 3C_1 - C_2.$$

Thus, $C_1 = 1$ and $C_2 = 3$ and

$$y(t) = e^{3t} + 3e^{-t}.$$

35. If $y'' + 2y' + 5y = 0$, then the characteristic equation is

$$\lambda^2 + 2\lambda + 5 = 0.$$

The quadratic formula provides the roots, $-1 \pm 2i$. Thus, the general solution is

$$y(t) = C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t.$$

Substituting the initial condition $y(0) = 2$ provides $C_1 = 2$. The derivative of the general solution is

$$y'(t) = C_1 e^{-t} (-\cos 2t - 2 \sin 2t) + C_2 e^{-t} (-\sin 2t + 2 \cos 2t).$$

The initial condition $y'(0) = 0$ provides

$$0 = -C_1 + 2C_2,$$

which in turn, because $C_1 = 2$, generates $C_2 = 1$. Thus, the solution of the initial value problem is

$$y(t) = 2e^{-t} \cos 2t + e^{-t} \sin 2t.$$

36. If $y'' + 4y' + 4y = 0$, then the characteristic equation is

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.$$

Thus, $\lambda = -2$ is a zero of algebraic multiplicity 2 and the general solution is

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t} = (C_1 + C_2 t) e^{-2t}.$$

The initial condition $y(0) = 2$ provides $C_1 = 2$. Differentiating,

$$y'(t) = C_2 e^{-2t} - 2(C_1 + C_2 t) e^{-2t}.$$

The initial condition $y'(0) = -1$ provides

$$-1 = C_2 - 2C_1.$$

Thus, $C_2 = 3$ and the solution is

$$y(t) = (2 + 3t) e^{-2t}.$$

37. If $y'' - 2y' + y = 0$, then the characteristic equation is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0.$$

Thus, 1 is a single root of multiplicity 2, and the general solution is

$$y(t) = C_1 e^t + C_2 t e^t.$$

The initial condition $y(0) = 1$ provides $C_1 = 1$. The derivative of the general solution is

$$y'(t) = C_1 e^t + C_2(t + 1)e^t.$$

The initial condition $y'(0) = 0$ provides

$$0 = C_1 + C_2,$$

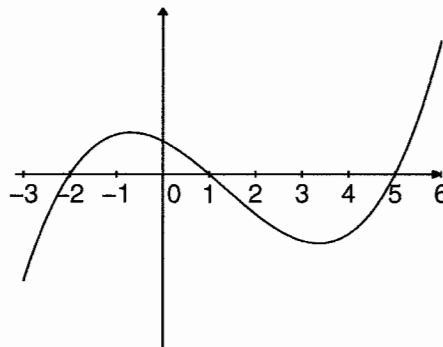
which in turn, because $C_1 = 1$, generates $C_2 = -1$. Therefore, the solution of the initial value problem is

$$y(t) = e^t - t e^t.$$

38. If $y''' - 4y'' - 7y' + 10y = 0$, then the characteristic equation is

$$\lambda^3 - 4\lambda^2 - 7\lambda + 10 = 0.$$

The plot of the characteristic equation



suggests a zero at -2 . Division by $\lambda + 2$ reveals

$$(\lambda + 2)(\lambda^2 - 6\lambda + 5) = 0$$

$$(\lambda + 2)(\lambda - 5)(\lambda - 1) = 0.$$

Thus, the zeros are -2 , 1 , and 5 and the general solution and its derivatives are

$$y(t) = C_1 e^{-2t} + C_2 e^t + C_3 e^{5t}$$

$$y'(t) = -2C_1 e^{-2t} + C_2 e^t + 5C_3 e^{5t}$$

$$y''(t) = 4C_1 e^{-2t} + C_2 e^t + 25C_3 e^{5t}.$$

The initial conditions $y(0) = 1$, $y'(0) = 0$, and $y''(0) = -1$ provide

$$1 = C_1 + C_2 + C_3$$

$$0 = -2C_1 + C_2 + 5C_3$$

$$-1 = 4C_1 + C_2 + 25C_3$$

The augmented matrix reduces

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & 1 & 5 & 0 \\ 4 & 1 & 25 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 4/21 \\ 0 & 1 & 0 & 11/12 \\ 0 & 0 & 1 & -3/28 \end{pmatrix},$$

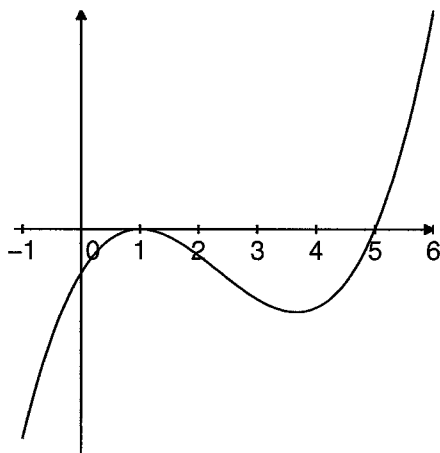
revealing $C_1 = 4/21$, $C_2 = 11/12$, and $C_3 = -3/28$. Thus, the solution is

$$y(t) = \frac{4}{21} e^{-2t} + \frac{11}{12} e^t - \frac{3}{28} e^{5t}.$$

39. If $y''' - 7y'' + 11y' - 5y = 0$, then the characteristic equation is

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

The plot of the characteristic equation



suggest a root at 1. Division reveals

$$\begin{aligned}(\lambda - 1)(\lambda^2 - 6\lambda + 5) &= 0 \\(\lambda - 1)^2(\lambda - 5) &= 0.\end{aligned}$$

Thus, the roots are 1 and 5, the former having algebraic multiplicity 2. Thus, the general solution is

$$y(t) = C_1 e^t + C_2 t e^t + C_3 e^{5t}.$$

The initial condition $y(0) = -1$ provides

$$C_1 + C_3 = -1.$$

The derivative of the general solution is

$$y'(t) = C_1 e^t + C_2 e^t(t + 1) + 5C_3 e^{5t}.$$

The initial condition $y'(0) = 1$ provides

$$C_1 + C_2 + 5C_3 = 1.$$

The second derivative of the general solution is

$$y''(t) = C_1 e^t + C_2 e^t(t + 2) + 25C_3 e^{5t}.$$

The initial condition $y''(0) = 0$ provides

$$C_1 + 2C_2 + 25C_3 = 0.$$

The augmented matrix

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 5 & 1 \\ 1 & 2 & 25 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -13/16 \\ 0 & 1 & 0 & 11/4 \\ 0 & 0 & 1 & -3/16 \end{pmatrix}$$

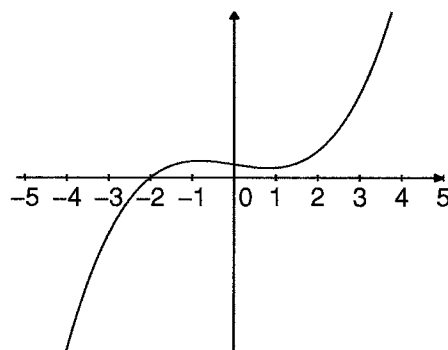
provides $C_1 = -13/16$, $C_2 = 11/4$, and $C_3 = -3/16$. Therefore, the solution of the initial value problem is

$$y(t) = -\frac{13}{16}e^t + \frac{11}{4}te^t - \frac{3}{16}e^{5t}.$$

40. If $y''' - 2y' + 4y = 0$, then the characteristic equation is

$$\lambda^3 - 2\lambda + 4 = 0.$$

A plot of the characteristic equation



suggests a zero at -2 . Division by $\lambda + 2$ reveals

$$(\lambda + 2)(\lambda^2 - 2\lambda + 2) = 0,$$

and the quadratic formula produces the zeros of the second factor, $1 \pm i$. Therefore, the general solution is

$$y(t) = C_1 e^{-2t} + C_2 e^t \cos t + C_3 e^t \sin t.$$

The initial condition $y(0) = 1$ provides

$$1 = C_1 + C_2.$$

Differentiating,

$$\begin{aligned}y'(t) &= -2C_1 e^{-2t} + C_2 e^t \cos t - C_2 e^t \sin t \\&\quad + C_3 e^t \sin t + C_3 e^t \cos t.\end{aligned}$$

The initial condition $y'(0) = -1$ provides

$$-1 = -2C_1 + C_2 + C_3.$$

Differentiating again,

$$\begin{aligned} y''(t) &= 4C_1 e^{-2t} + C_2 e^t (\cos t - \sin t) \\ &\quad + C_2 e^t (-\sin t - \cos t) \\ &\quad + C_3 e^t (\sin t + \cos t) \\ &\quad + C_3 e^t (\cos t - \sin t). \end{aligned}$$

The initial condition $y''(0) = 0$ provides

$$0 = 4C_1 + 2C_3.$$

The augmented matrix reduces

$$\left(\begin{array}{cccc} 1 & 1 & 0 & 1 \\ -2 & 1 & 1 & -1 \\ 4 & 0 & 2 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cccc} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -4/5 \end{array} \right),$$

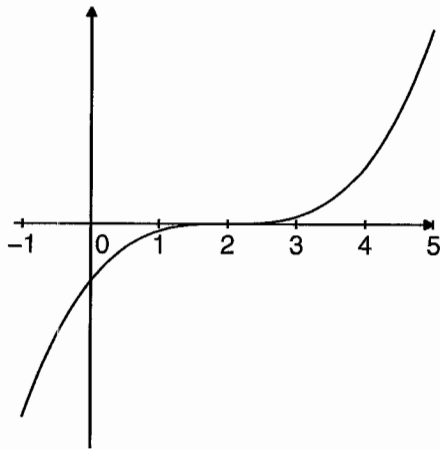
so $C_1 = 2/5$, $C_2 = 3/5$, $C_3 = -4/5$, and the solution is

$$y(t) = \frac{2}{5}e^{-2t} + \frac{3}{5}e^t \cos t - \frac{4}{5}e^t \sin t.$$

41. If $y''' - 6y'' + 12y' - 8y = 0$, then the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0.$$

The plot of the characteristic equation



suggest a multiple root at 2. Repeated division by $\lambda - 2$ reveals

$$(\lambda - 2)^3 = 0.$$

Thus, the characteristic polynomial has a single root, 2, with algebraic multiplicity 3. Thus, the general solution is

$$y(t) = C_1 e^{2t} + C_2 t e^{2t} + C_3 t^2 e^{2t}.$$

The initial condition $y(0) = -2$ provides $C_1 = -2$. The derivative of the general solution is

$$y'(t) = 2C_1 e^{2t} + C_2 e^{2t}(2t + 1) + C_3 e^{2t}(2t^2 + 2t).$$

The initial condition $y'(0) = 0$ provides

$$2C_1 + C_2 = 0,$$

which in turn, because $C_1 = -2$, provides $C_2 = 4$. The second derivative of the general solution is

$$y''(t) = 4C_1 e^{2t} + C_2 e^{2t}(4 + 4t) + C_3 e^{2t}(4t^2 + 8t + 2).$$

The initial condition $y''(0) = 2$ provides

$$4C_1 + 4C_2 + 2C_3 = 0,$$

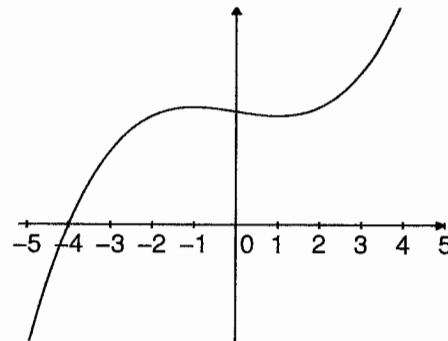
which in turn, because $C_1 = -2$ and $C_2 = 4$, provides $C_3 = -3$. Therefore, the solution of the initial value problem is

$$y(t) = -2e^{2t} + 4te^{2t} - 3t^2e^{2t}.$$

42. If $y''' - 3y' + 52y = 0$, then the characteristic equation is

$$\lambda^3 - 3\lambda + 52 = 0.$$

The plot of the characteristic equation



suggests a zero at -4 . Division by $\lambda + 4$ reveals

$$(\lambda + 4)(\lambda^2 - 4\lambda + 13) = 0.$$

The quadratic formula reveals the zeros of the second factor, $2 \pm 3i$. Thus, the general solution is

$$y(t) = C_1 e^{-4t} + e^{2t}(C_2 \cos 3t + C_3 \sin 3t).$$

The initial condition $y(0) = 0$ provides

$$0 = C_1 + C_2.$$

Differentiate.

$$\begin{aligned} y'(t) &= -4C_1 e^{-4t} + e^{2t}(-3C_2 \sin 3t + 3C_3 \cos 3t) \\ &\quad + 2e^{2t}(C_2 \cos 3t + C_3 \sin 3t) \\ &= -4C_1 e^{-4t} + e^{2t}((3C_3 + 2C_2) \cos 3t \\ &\quad + (2C_3 - 3C_2) \sin 3t) \end{aligned}$$

The initial condition $y'(0) = -1$ provides

$$-1 = -4C_1 + 2C_2 + 3C_3.$$

Differentiate again.

$$\begin{aligned} y''(t) &= 16C_1 e^{-4t} + e^{2t}[(-9C_3 - 6C_2) \sin 3t \\ &\quad + (6C_3 - 9C_2) \cos 3t] \\ &\quad + 2e^{2t}[(3C_3 + 2C_2) \cos 3t \\ &\quad + (2C_3 - 3C_2) \sin 3t] \end{aligned}$$

The initial condition $y''(0) = 2$ provides

$$\begin{aligned} 2 &= 16C_1 + (6C_3 - 9C_2) + (6C_3 + 4C_2) \\ 2 &= 16C_1 - 5C_2 + 12C_3. \end{aligned}$$

The augmented matrix reduces

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -4 & 2 & 3 & -1 \\ 16 & -5 & 12 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 2/15 \\ 0 & 1 & 0 & -2/15 \\ 0 & 0 & 1 & -1/15 \end{pmatrix}.$$

Thus, $C_1 = 2/15$, $C_2 = -2/15$, $C_3 = -1/15$, and the solution is

$$y(t) = \frac{2}{15}e^{-4t} + e^{2t}\left(-\frac{2}{15}\cos 3t - \frac{1}{15}\sin 3t\right).$$

43. If $y^{(4)} + 8y'' + 16y = 0$, then the characteristic equation is

$$\lambda^4 + 8\lambda^2 + 16 = (\lambda^2 + 4)^2 = 0.$$

Therefore, the roots are $\pm 2i$, each of which has algebraic multiplicity 2. Therefore, the general solution is

$$y(t) = C_1 \cos 2t + C_2 t \cos 2t + C_3 \sin 2t + C_4 t \sin 2t.$$

The initial condition $y(0) = 0$ provides $C_1 = 0$. The derivative of the general solution is

$$\begin{aligned} y'(t) &= -2C_1 \sin 2t + C_2(\cos 2t - 2t \sin 2t) \\ &\quad + 2C_3 \cos 2t + C_4(\sin 2t + 2t \cos 2t). \end{aligned}$$

The initial condition $y'(0) = -1$ generates $C_2 + 2C_3 = -1$. The second derivative of the general solution is

$$\begin{aligned} y''(t) &= -4C_1 \cos 2t + C_2(-4 \sin 2t - 4t \cos 2t) \\ &\quad - 4C_3 \sin 2t + C_4(4 \cos 2t - 4t \sin 2t). \end{aligned}$$

The initial condition $y''(0) = 2$ generates $-4C_1 + 4C_4 = 2$. The third derivative of the general solution is:

$$\begin{aligned} y'''(t) &= 8C_1 \sin 2t + C_2(-12 \cos 2t + 8t \sin 2t) \\ &\quad - 8C_3 \cos 2t + C_4(-12 \sin 2t - 8t \cos 2t) \end{aligned}$$

The initial condition $y'''(0) = 0$ generates $-12C_2 - 8C_3 = 0$. The augmented matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 \\ -4 & 0 & 0 & 4 & 2 \\ 0 & -12 & -8 & 0 & 0 \end{pmatrix}$$

reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -3/4 \\ 0 & 0 & 0 & 1 & 1/2 \end{pmatrix}.$$

Thus, $C_1 = 0$, $C_2 = 1/2$, $C_3 = -3/4$, and $C_4 = 1/2$. Therefore, the solution of the initial value problem is

$$y(t) = \frac{1}{2}t \cos 2t - \frac{3}{4} \sin 2t + \frac{1}{2}t \sin 2t.$$

44. Recall that for $\mathbf{a} = (a_1, a_2, \dots, a_q)^T \in \mathbf{R}^q$, we define

$$y_{\mathbf{a}}(t) = (a_1 + a_2 t + \dots + a_q t^{q-1})e^{\lambda t}.$$

Now, let $\mathbf{b} = (b_1, b_2, \dots, b_q) \in \mathbf{R}$ and $\alpha, \beta \in \mathbf{R}$. Then,

$$\alpha \mathbf{a} + \beta \mathbf{b} = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \dots, \alpha a_q + \beta b_q)$$

and

$$\begin{aligned} y_{\alpha \mathbf{a} + \beta \mathbf{b}}(t) &= [(\alpha a_1 + \beta b_1) + (\alpha a_2 + \beta b_2)t + \dots \\ &\quad + (\alpha a_q + \beta b_q)t^{q-1}]e^{\lambda t} \\ &= (\alpha a_1 + \alpha a_2 t + \dots + \alpha a_q t^{q-1})e^{\lambda t} \\ &\quad + (\beta b_1 + \beta b_2 t + \dots + \beta b_q t^{q-1})e^{\lambda t} \\ &= \alpha [(a_1 + a_2 t + \dots + a_q t^{q-1})e^{\lambda t}] \\ &\quad + \beta [(b_1 + b_2 t + \dots + b_q t^{q-1})e^{\lambda t}] \\ &= \alpha y_{\mathbf{a}}(t) + \beta y_{\mathbf{b}}(t). \end{aligned}$$

Thus, $y_{\alpha \mathbf{a} + \beta \mathbf{b}} = \alpha y_{\mathbf{a}} + \beta y_{\mathbf{b}}$.

45. First, we must show that the set V is closed under addition. Let $\mathbf{a}$ and $\mathbf{b}$ be elements of $V \subset \mathbf{R}^q$. Then, $y_{\mathbf{a}}$ and $y_{\mathbf{b}}$ are solutions of

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0. \quad (*)$$

However, $y_{\mathbf{a}} + y_{\mathbf{b}}$, being a linear combination of solutions of $(*)$, is also a solution of $(*)$. However, by 8.31,

$$y_{\mathbf{a}} + y_{\mathbf{b}} = y_{\mathbf{a} + \mathbf{b}}.$$

Recall that the set $V \subset \mathbf{R}^q$ is defined

$$V = \{\mathbf{a} : y_{\mathbf{a}} \text{ is a solution of } (*)\}.$$

Therefore, $y_{\mathbf{a} + \mathbf{b}}$ is a solution of $(*)$ and $\mathbf{a} + \mathbf{b} \in V$. Therefore, V is closed under addition. Next, we

must show that V is closed under scalar multiplication. Let $\mathbf{a} \in V$ and let $\alpha \in \mathbf{R}$ be a scalar. Then, by definition of V , $y_{\mathbf{a}}$ is a solution of $(*)$. However, $\alpha y_{\mathbf{a}}$, being a linear combination of solutions of $(*)$, is also a solution of $(*)$. By 8.31,

$$\alpha y_{\mathbf{a}} = y_{\alpha \mathbf{a}}.$$

Hence, $y_{\alpha \mathbf{a}}$ is a solution of $(*)$ and $\alpha \mathbf{a} \in V$. Therefore, V is closed under scalar multiplication and is a subspace of $\mathbf{R}^q$.

46. Recall that

$$y_j(t) = P_j(t)e^{\lambda t}$$

are independent solutions of

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0, \quad (**)$$

for $i = 1, 2, \dots, q$. Recall that $\mathbf{a} \in V \subset \mathbf{R}^q$ iff $y_{\mathbf{a}}$ is a solution of $(**)$. For each $j = 1, 2, \dots, q$, let $\mathbf{a}_j$ be the vector of coefficients of the polynomial $P_j(t)$. Thus, $y_{\mathbf{a}_j} = y_j$ for $j = 1, 2, \dots, q$. Let

$$C_1 \mathbf{a}_1 + C_2 \mathbf{a}_2 + \dots + C_q \mathbf{a}_q = \mathbf{0}.$$

Then,

$$y_{C_1 \mathbf{a}_1 + C_2 \mathbf{a}_2 + \dots + C_q \mathbf{a}_q} = y_{\mathbf{0}}.$$

Then

$$C_1 y_{\mathbf{a}_1} + C_2 y_{\mathbf{a}_2} + \dots + C_q y_{\mathbf{a}_q} = y_{\mathbf{0}}.$$

But, $y_j = y_{\mathbf{a}_j}$, so

$$C_1 y_j + C_2 y_j + \dots + C_q y_j = y_{\mathbf{0}}.$$

Note that $y_{\mathbf{0}}$ is the zero polynomial. However, the y_j 's are given as q independent solutions. Thus, $C_1 = C_2 = \dots = C_q = 0$ and $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_q$ are independent.

Section 9.9. Inhomogeneous Systems

1. If

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} e^t \\ e^t \end{pmatrix},$$

then the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2),$$

generating eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. The associated eigenvectors are

$$A - I = \begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \text{and}$$

$$A - 2I = \begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Thus, the homogeneous solution is $\mathbf{y}_h = C_1\mathbf{y}_1 + C_2\mathbf{y}_2$, where

$$\mathbf{y}_1(t) = e^t \mathbf{v}_1 = \begin{pmatrix} 3e^t \\ -2e^t \end{pmatrix} \quad \text{and}$$

$$\mathbf{y}_2(t) = e^{2t} \mathbf{v}_2 = \begin{pmatrix} 2e^{2t} \\ -e^{2t} \end{pmatrix}.$$

The fundamental matrix is

$$Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{pmatrix} 3e^t & 2e^{2t} \\ -2e^t & -e^{2t} \end{pmatrix}.$$

The inverse¹ of $Y(t)$ is calculated

$$\begin{aligned} Y^{-1}(t) &= \frac{1}{e^{3t}} \begin{pmatrix} -e^{2t} & -2e^{2t} \\ 2e^t & 3e^t \end{pmatrix} \\ &= \begin{pmatrix} -e^{-t} & -2e^{-t} \\ 2e^{-2t} & 3e^{-2t} \end{pmatrix}. \end{aligned}$$

Hence,

$$Y^{-1}(t)\mathbf{f}(t) = \begin{pmatrix} -e^{-t} & -2e^{-t} \\ 2e^{-2t} & 3e^{-2t} \end{pmatrix} \begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} -3 \\ 5e^{-t} \end{pmatrix},$$

and

$$\int Y^{-1}(t)\mathbf{f}(t) dt = \int \begin{pmatrix} -3 \\ 5e^{-t} \end{pmatrix} dt = \begin{pmatrix} -3t \\ -5e^{-t} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \mathbf{y}_p &= Y(t) \int Y^{-1}\mathbf{f}(t) dt \\ &= \begin{pmatrix} 3e^t & 2e^{2t} \\ -2e^t & -e^{2t} \end{pmatrix} \begin{pmatrix} -3t \\ -5e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} -9te^t - 10e^t \\ 6te^t + 5e^t \end{pmatrix}. \end{aligned}$$

Finally, the general solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^t \begin{pmatrix} 3 \\ -2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &\quad + \begin{pmatrix} -9te^t - 10e^t \\ 6te^t + 5e^t \end{pmatrix}. \end{aligned}$$

2. The matrix

$$A = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, with associated eigenvectors $\mathbf{v}_1 = (2, 1)^T$ and $\mathbf{v}_2 = (1, 1)^T$. Thus, the homogeneous solution is $\mathbf{y}_h = C_1\mathbf{y}_1 + C_2\mathbf{y}_2$, where

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The fundamental matrix is

$$Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{pmatrix} 2e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}.$$

The inverse is calculated with

$$Y^{-1}(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 2e^t \end{pmatrix}.$$

¹ Perhaps the easiest way to invert a 2×2 matrix is to use the following fact:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence,

$$\begin{aligned} Y^{-1}(t)\mathbf{f}(t) &= \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 2e^t \end{pmatrix} \begin{pmatrix} e^{-t} \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} e^{-2t} - 1 \\ -1 + 2e^{2t} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \int Y^{-1}\mathbf{f}(t) dt &= \int \begin{pmatrix} e^{-2t} - 1 \\ 2e^{2t} - 1 \end{pmatrix} dt \\ &= \begin{pmatrix} -\frac{1}{2}e^{-2t} - t \\ e^{2t} - t \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{y}_p &= Y(t) \int Y^{-1}\mathbf{f}(t) dt \\ &= \begin{pmatrix} 2e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}e^{-2t} - t \\ e^{2t} - t \end{pmatrix} \\ &= \begin{pmatrix} -e^{-t} - 2te^t + e^t - te^{-t} \\ -\frac{1}{2}e^{-t} - te^t + e^t - te^{-t} \end{pmatrix}. \end{aligned}$$

Finally, the general solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} -e^{-t} - 2te^t + e^t - te^{-t} \\ -\frac{1}{2}e^{-t} - te^t + e^t - te^{-t} \end{pmatrix}. \end{aligned}$$

3. If

$$A = \begin{pmatrix} -3 & 6 \\ -2 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

then the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - \lambda = \lambda(\lambda - 1),$$

generating eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$. The associated eigenvectors are

$$\begin{aligned} A - 0I &= \begin{pmatrix} -3 & 6 \\ -2 & 4 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{and} \\ A - I &= \begin{pmatrix} -4 & 6 \\ -2 & 3 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \end{aligned}$$

Thus, the homogeneous solution is $\mathbf{y}_h = C_1 \mathbf{y}_1 + C_2 \mathbf{y}_2$, where

$$\begin{aligned} \mathbf{y}_1(t) &= e^{0t} \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \\ \mathbf{y}_2(t) &= e^t \mathbf{v}_2 = \begin{pmatrix} 3e^t \\ 2e^t \end{pmatrix}. \end{aligned}$$

The fundamental matrix is

$$Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{pmatrix} 2 & 3e^t \\ 1 & 2e^t \end{pmatrix}.$$

The inverse of $Y(t)$ is calculated

$$Y^{-1}(t) = \frac{1}{e^t} \begin{pmatrix} 2e^t & -3e^t \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -e^{-t} & 2e^{-t} \end{pmatrix}.$$

Hence,

$$\begin{aligned} Y^{-1}(t)\mathbf{f}(t) &= \begin{pmatrix} 2 & -3 \\ -e^{-t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 5e^{-t} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \int Y^{-1}(t)\mathbf{f}(t) dt &= \int \begin{pmatrix} -6 \\ 5e^{-t} \end{pmatrix} dt \\ &= \begin{pmatrix} -6t \\ -5e^{-t} \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{y}_p &= Y(t) \int Y^{-1}\mathbf{f}(t) dt \\ &= \begin{pmatrix} 2 & 3e^t \\ 1 & 2e^t \end{pmatrix} \begin{pmatrix} -6t \\ -5e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} -12t - 15 \\ -6t - 10 \end{pmatrix}. \end{aligned}$$

Finally, the general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 3e^t \\ 2e^t \end{pmatrix} + \begin{pmatrix} -12t - 15 \\ -6t - 10 \end{pmatrix}.$$

4. The matrix

$$A = \begin{pmatrix} -3 & 10 \\ -3 & 8 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$, with associated eigenvectors $\mathbf{v}_1 = (2, 1)^T$ and $\mathbf{v}_2 = (5, 3)^T$. Thus,

the homogeneous solution is $\mathbf{y}_h = C_1 \mathbf{y}_1 + C_2 \mathbf{y}_2$, where

$$\mathbf{y}_1(t) = e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^{3t} \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

The fundamental matrix is

$$Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{pmatrix} 2e^{2t} & 5e^{3t} \\ e^{2t} & 3e^{3t} \end{pmatrix}.$$

The inverse is calculated with

$$\begin{aligned} Y^{-1}(t) &= \frac{1}{e^{5t}} \begin{pmatrix} 3e^{3t} & -5e^{3t} \\ -e^{2t} & 2e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 3e^{-2t} & -5e^{-2t} \\ -e^{-3t} & 2e^{-3t} \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} Y^{-1}(t)\mathbf{f}(t) &= \begin{pmatrix} 3e^{-2t} & -5e^{-2t} \\ -e^{-3t} & 2e^{-3t} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -11e^{-2t} \\ 5e^{-3t} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \int Y^{-1}(t)\mathbf{f}(t) dt &= \int \begin{pmatrix} -11e^{-2t} \\ 5e^{-3t} \end{pmatrix} dt \\ &= \begin{pmatrix} 11e^{-2t}/2 \\ -5e^{-3t}/3 \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{y}_p &= Y(t) \int Y^{-1}(t)\mathbf{f}(t) dt \\ &= \begin{pmatrix} 2e^{2t} & 5e^{3t} \\ e^{2t} & 3e^{3t} \end{pmatrix} \begin{pmatrix} 11e^{-2t}/2 \\ -5e^{-3t}/3 \end{pmatrix} \\ &= \begin{pmatrix} 8/3 \\ 1/2 \end{pmatrix}. \end{aligned}$$

Finally, the general solution is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 8/3 \\ 1/2 \end{pmatrix}.$$

5. A has eigenvalues $2 \pm i$, and associated with the eigenvalue $2 + i$ has eigenvector $\mathbf{w} = (-1 - i, 1)^T$.

Hence the homogeneous equation has complex solution

$$\begin{aligned} \mathbf{z}(t) &= e^{(2+i)t} \begin{pmatrix} -1-i \\ 1 \end{pmatrix} \\ &= e^{2t} [\cos t + i \sin t] \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= e^{2t} \left\{ \begin{pmatrix} \sin t - \cos t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t - \sin t \\ \sin t \end{pmatrix} \right\} \end{aligned}$$

Thus the homogeneous equation has the real solutions

$$\begin{aligned} \mathbf{y}_1(t) &= \operatorname{Re} \mathbf{z}(t) = e^{2t} \begin{pmatrix} \sin t - \cos t \\ \cos t \end{pmatrix} \quad \text{and} \\ \mathbf{y}_2(t) &= \operatorname{Im} \mathbf{z}(t) = e^{2t} \begin{pmatrix} -\cos t - \sin t \\ \sin t \end{pmatrix}. \end{aligned}$$

The fundamental matrix is

$$Y(t) = e^{2t} \begin{pmatrix} \sin t - \cos t & -\cos t - \sin t \\ \cos t & \sin t \end{pmatrix}.$$

Its inverse is

$$Y^{-1}(t) = e^{-2t} \begin{pmatrix} \sin t & \cos t + \sin t \\ -\cos t & \sin t - \cos t \end{pmatrix}.$$

Hence

$$\begin{aligned} Y^{-1}(t)\mathbf{f}(t) &= e^{-2t} \begin{pmatrix} \sin t & \cos t + \sin t \\ -\cos t & \sin t - \cos t \end{pmatrix} \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} \sin t + \cos t \\ \sin t - \cos t \end{pmatrix}, \end{aligned}$$

and

$$\int Y^{-1}(t)\mathbf{f}(t) dt = \begin{pmatrix} \sin t - \cos t \\ -\cos t - \sin t \end{pmatrix}.$$

Then the particular solution is

$$\begin{aligned} \mathbf{y}_p(t) &= Y(t) \int Y^{-1}\mathbf{f}(t) dt \\ &= e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \end{aligned}$$

The general solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{2t} \begin{pmatrix} \sin t - \cos t \\ \cos t \end{pmatrix} \\ &\quad + C_2 e^{2t} \begin{pmatrix} -\cos t - \sin t \\ \sin t \end{pmatrix} + e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \end{aligned}$$

6. The matrix

$$A = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}$$

has eigenvalues $3 \pm i$. Associated with $\lambda = 3 + i$ is $\mathbf{w} = (-1 - i, 1)^T$, so the homogeneous equation has complex solution

$$\begin{aligned} \mathbf{z}(t) &= e^{(3+i)t} \begin{pmatrix} -1-i \\ 1 \end{pmatrix} \\ &= e^{3t}(\cos t + i \sin t) \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= e^{3t} \left(\cos t \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) \\ &\quad + i e^{3t} \left(\cos t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right). \end{aligned}$$

The real and imaginary parts of $\mathbf{z}(t)$ help form the fundamental matrix

$$Y(t) = e^{3t} \begin{pmatrix} -\cos t + \sin t & -\cos t - \sin t \\ \cos t & \sin t \end{pmatrix}.$$

Its inverse is

$$Y^{-1}(t) = e^{-3t} \begin{pmatrix} \sin t & \cos t + \sin t \\ -\cos t & -\cos t + \sin t \end{pmatrix}.$$

Hence,

$$\begin{aligned} Y^{-1}(t)\mathbf{f}(t) &= e^{-3t} \begin{pmatrix} \sin t & \cos t + \sin t \\ -\cos t & -\cos t + \sin t \end{pmatrix} \begin{pmatrix} t \\ e^{3t} \end{pmatrix} \\ &= e^{-3t} \begin{pmatrix} t \sin t + e^{3t}(\cos t + \sin t) \\ -t \cos t + e^{3t}(-\cos t + \sin t) \end{pmatrix} \\ &= \begin{pmatrix} t e^{-3t} \sin t + \cos t + \sin t \\ -t e^{-3t} \cos t - \cos t + \sin t \end{pmatrix}. \end{aligned}$$

Needless to say, $\int Y^{-1}(t)\mathbf{f}(t) dt$ is a tough antiderivative to find. We will use a CAS (computer algebra system) to calculate

$$\begin{aligned} \mathbf{y}_p(t) &= Y(t) \int Y^{-1}(t)\mathbf{f}(t) dt \\ &= \begin{pmatrix} -t/5 - 1/50 + 2e^{3t} \\ -t/10 - 3/50 - e^{3t} \end{pmatrix}. \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1 e^{3t} \begin{pmatrix} -\cos t + \sin t \\ \cos t \end{pmatrix} \\ &\quad + C_2 e^{3t} \begin{pmatrix} -\cos t - \sin t \\ \sin t \end{pmatrix} \\ &\quad + \begin{pmatrix} -t/5 - 1/50 + 2e^{3t} \\ -t/10 - 3/50 - e^{3t} \end{pmatrix}. \end{aligned}$$

7. A has eigenvalues 0, 2, and 1, with corresponding eigenvalues $(-1, 2, 0)^T$, $(1, 0, 1)^T$, and $(0, 3, 1)^T$. Thus

$$V = \begin{pmatrix} -1 & e^{2t} & 0 \\ 2 & 0 & 3e^t \\ 0 & e^{2t} & e^t \end{pmatrix}.$$

If we form the augmented matrix $[V, I]$ and use row operations to reduce to row echelon form $[I, V^{-1}]$, we discover that

$$V^{-1} = \begin{pmatrix} -3 & -1 & 3 \\ -2e^{-2t} & -e^{-2t} & 3e^{-2t} \\ 2e^{-t} & e^{-t} & -2e^{-t} \end{pmatrix}.$$

Then

$$V^{-1}\mathbf{f} = \begin{pmatrix} \sin t \\ 0 \\ 0 \end{pmatrix}.$$

Hence $\int V^{-1}\mathbf{f} dt = (-\cos t, 0, 0)^T$, and the particular solution is

$$\begin{aligned} \mathbf{y}_p(t) &= V \int V^{-1}\mathbf{f} dt \\ &= \begin{pmatrix} -1 & e^{2t} & 0 \\ 2 & 0 & 3e^t \\ 0 & e^{2t} & e^t \end{pmatrix} \begin{pmatrix} -\cos t \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t \\ -2 \cos t \\ 0 \end{pmatrix}. \end{aligned}$$

The general solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 3e^t \\ e^t \end{pmatrix} \\ &\quad + \begin{pmatrix} \cos t \\ -2 \cos t \\ 0 \end{pmatrix}. \end{aligned}$$

8. The matrix

$$A = \begin{pmatrix} 1 & -18 & 8 \\ 0 & 14 & -6 \\ 0 & 35 & -15 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3 = 1$, with associated eigenvectors $\mathbf{v}_1 = (2, -3, -7)^T$, $\mathbf{v}_2 = (-2, 2, 5)^T$, and $\mathbf{v}_3 = (1, 0, 0)^T$. Thus, the homogeneous solution is $\mathbf{y}_h = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + C_3\mathbf{y}_3$, where

$$\mathbf{y}_1(t) = \begin{pmatrix} 2 \\ -3 \\ -7 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-t} \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}, \quad \text{and} \\ \mathbf{y}_3(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The fundamental matrix is

$$Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)] \\ = \begin{pmatrix} 2 & -2e^{-t} & e^t \\ -3 & 2e^{-t} & 0 \\ -7 & 5e^{-t} & 0 \end{pmatrix},$$

with inverse

$$Y^{-1}(t) = \begin{pmatrix} 0 & -5 & 2 \\ 0 & -7e^t & 3e^t \\ e^{-t} & -4e^{-t} & 2e^{-t} \end{pmatrix}.$$

Hence,

$$Y^{-1}(t)\mathbf{f}(t) = \begin{pmatrix} 0 & -5 & 2 \\ 0 & -7e^t & 3e^t \\ e^{-t} & -4e^{-t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ = \begin{pmatrix} -5 \\ -7e^t \\ -4e^{-t} \end{pmatrix},$$

and

$$\int Y^{-1}(t)\mathbf{f}(t) dt = \int \begin{pmatrix} -5 \\ -7e^t \\ -4e^{-t} \end{pmatrix} dt \\ = \begin{pmatrix} -5t \\ -7e^t \\ 4e^{-t} \end{pmatrix}.$$

Thus,

$$\mathbf{y}_p(t) = Y(t) \int Y^{-1}(t)\mathbf{f}(t) dt \\ = \begin{pmatrix} 2 & -2e^{-t} & e^t \\ -3 & 2e^{-t} & 0 \\ -7 & 5e^{-t} & 0 \end{pmatrix} \begin{pmatrix} -5t \\ -7e^t \\ 4e^{-t} \end{pmatrix} \\ = \begin{pmatrix} -10t + 18 \\ 15t - 14 \\ 35t - 35 \end{pmatrix}.$$

Finally, the general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 2 \\ -3 \\ -7 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} + C_3 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ + \begin{pmatrix} -10t + 18 \\ 15t - 14 \\ 35t - 35 \end{pmatrix}.$$

9. A has eigenvalues 0, -2, and -1, with corresponding eigenvectors $(-1, 4, 1)^T$, $(-3, 2, 0)^T$, and $(0, 3, 1)^T$, so

$$V = \begin{pmatrix} -1 & -3e^{-2t} & 0 \\ 4 & 2e^{-2t} & 3e^{-t} \\ 1 & 0 & e^{-t} \end{pmatrix}.$$

If we form the augmented matrix $[V, I]$ and use row operations to reduce to row echelon form $[I, V^{-1}]$, we discover that

$$V^{-1} = \begin{pmatrix} 2 & 3 & -9 \\ -e^{2t} & -e^{2t} & 3e^{2t} \\ -2e^t & -3e^t & 10e^t \end{pmatrix}.$$

Then $V^{-1}\mathbf{f} = (6, -2e^{2t}, -6e^t)^T$, and $\int V^{-1}\mathbf{f} dt = (6t, -e^{2t}, -6e^t)^T$. Hence the particular solution is

$$\mathbf{y}_p(t) = V \int V^{-1}\mathbf{f} dt \\ = \begin{pmatrix} -1 & -3e^{-2t} & 0 \\ 4 & 2e^{-2t} & 3e^{-t} \\ 1 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 6t \\ -e^{2t} \\ -6e^t \end{pmatrix} \\ = \begin{pmatrix} 3 - 6t \\ 24t - 20 \\ 6t - 6 \end{pmatrix}.$$

The general solution is

$$\mathbf{y}(t) = C_1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -3e^{-2t} \\ 2e^{-2t} \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 3e^{-t} \\ e^{-t} \end{pmatrix} + \begin{pmatrix} 3-6t \\ 24t-20 \\ 6t-6 \end{pmatrix}.$$

10. The matrix

$$A = \begin{pmatrix} 11 & -7 & -4 \\ -6 & 4 & 2 \\ 42 & -27 & -15 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 0$, with associated eigenvectors $\mathbf{v}_1 = (1, -2, 6)^T$, $\mathbf{v}_2 = (1, 0, 3)^T$, and $\mathbf{v}_3 = (1, 1, 1)^T$. Thus, the homogeneous solution is $\mathbf{y}_h = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + C_3\mathbf{y}_3$, where

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \text{and} \\ \mathbf{y}_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The fundamental matrix is

$$Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)] \\ = \begin{pmatrix} e^t & e^{-t} & 1 \\ -2e^t & 0 & 1 \\ 6e^t & 3e^{-t} & 1 \end{pmatrix},$$

with inverse

$$Y^{-1}(t) = \begin{pmatrix} 3e^{-t} & -2e^{-t} & -e^{-t} \\ -8e^t & 5e^t & 3e^t \\ 6 & -3 & -2 \end{pmatrix}.$$

Hence,

$$Y^{-1}(t)\mathbf{f}(t) = \begin{pmatrix} 3e^{-t} & 2e^{-t} & -e^{-t} \\ -8e^t & 5e^t & 3e^t \\ 6 & -3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ = \begin{pmatrix} 3e^{-t} \\ -8e^t \\ 6 \end{pmatrix},$$

and

$$\int Y^{-1}(t)\mathbf{f}(t) dt = \int \begin{pmatrix} 3e^{-t} \\ -8e^t \\ 6 \end{pmatrix} dt \\ = \begin{pmatrix} -3e^{-t} \\ -8e^t \\ 6t \end{pmatrix}.$$

Thus,

$$\mathbf{y}_p(t) = Y(t) \int Y^{-1}(t)\mathbf{f}(t) dt \\ = \begin{pmatrix} e^t & e^{-t} & 1 \\ -2e^t & 0 & 1 \\ 6e^t & 3e^{-t} & 1 \end{pmatrix} \begin{pmatrix} -3e^{-t} \\ -8e^t \\ 6t \end{pmatrix} \\ = \begin{pmatrix} 6t-11 \\ 6t+6 \\ 6t-42 \end{pmatrix}.$$

Finally, the general solution is

$$\mathbf{y}(t) = C_1 e^t \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ + \begin{pmatrix} 6t-11 \\ 6t+6 \\ 6t-42 \end{pmatrix}.$$

11. The independent solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ in Example 9.16 provide a fundamental matrix

$$Y = [\mathbf{y}_1, \mathbf{y}_2] \\ = e^{-2t/5} \begin{pmatrix} -5c(t) & -5s(t) \\ 3c(t)-4s(t) & 4c(t)+3s(t) \end{pmatrix},$$

where $c(t) = \cos(4t/5)$ and $s(t) = \sin(4t/5)$. It's actually possible to invert this by hand if you use the fact that $(\alpha A)^{-1} = (1/\alpha)A^{-1}$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence,

$$Y^{-1} = e^{2t/5} \frac{1}{-20} \begin{pmatrix} 4c(t)+3s(t) & 5s(t) \\ -3c(t)+4s(t) & -5c(t) \end{pmatrix}.$$

Then,

$$Y^{-1}\mathbf{f} = e^{2t/5} \begin{pmatrix} -c(t) + (1/2)s(t) \\ -(1/2)c(t) - s(t) \end{pmatrix}.$$

At this point, we're facing a tricky integration by parts, so we will use a computer to integrate for us.

$$\mathbf{v} = \int Y^{-1}\mathbf{f} = e^{2t/5} \begin{pmatrix} -c(t) - (3/4)s(t) \\ (3/4)c(t) - s(t) \end{pmatrix}.$$

Finally, we will again let the computer do the multiplication in the final step,

$$\mathbf{y}_p = Y\mathbf{v} = \begin{pmatrix} 5 \\ 0 \end{pmatrix},$$

which is the same particular solution found in Example 9.16.

12. The natural guess is $\mathbf{y}_p = \mathbf{a}$, where $\mathbf{a} = (a_1, a_2)^T$. thus, $\mathbf{y}'_p = \mathbf{0}$, so consequently, $A\mathbf{y}_p + \mathbf{f} = \mathbf{0}$. Thus,

$$\mathbf{y}_p = A^{-1}(-\mathbf{f}) = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1/2 \end{pmatrix}.$$

Thus, $\mathbf{y}_p = (2, -1/2)^T$.

13. The natural guess is $\mathbf{y}_p = \mathbf{a}$, where $\mathbf{a} = (a_1, a_2)^T$. thus, $\mathbf{y}'_p = \mathbf{0}$, so consequently, $A\mathbf{y}_p + \mathbf{f} = \mathbf{0}$. Thus,

$$\mathbf{y}_p = A^{-1}(-\mathbf{f}) = \begin{pmatrix} -6 & 8 \\ -4 & 6 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -5/2 \end{pmatrix}.$$

Thus, $\mathbf{y}_p = (-3, -5/2)^T$.

14. If $\mathbf{y}_p = \mathbf{a}e^t$, where $\mathbf{a} = (a_1, a_2)^T$, then

$$\mathbf{y}'_p = \mathbf{a}e^t = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t.$$

Furthermore,

$$\begin{aligned} A\mathbf{y}_p + \mathbf{f} &= \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t \\ &= \begin{pmatrix} -a_1 + 2 \\ a_1 - 2a_2 - 1 \end{pmatrix} e^t. \end{aligned}$$

Comparing, $a_1 = -a_1 + 2$ and $a_2 = a_1 - 2a_2 - 1$, and we easily find that $a_1 = 1$ and $a_2 = 0$. Thus,

$$\mathbf{y}_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t = \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

15. If $\mathbf{y}_p = \mathbf{a} \cos t + \mathbf{b} \sin t$, where $\mathbf{a} = (a_1, a_2)^T$ and $\mathbf{b} = (b_1, b_2)^T$, then

$$\mathbf{y}'_p = -\mathbf{a} \sin t + \mathbf{b} \cos t = \begin{pmatrix} -a_1 \sin t + b_1 \cos t \\ -a_2 \sin t + b_2 \cos t \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} A\mathbf{y}_p + \mathbf{f} &= A(\mathbf{a} \cos t + \mathbf{b} \sin t) + \mathbf{f} \\ &= \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos t \\ &\quad + \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ \sin t \end{pmatrix} \\ &= \begin{pmatrix} -2a_1 \cos t - 2b_1 \sin t \\ (a_1 - a_2) \cos t + (b_1 - b_2 + 1) \sin t \end{pmatrix}. \end{aligned}$$

Comparing coefficients of sine and cosine in the vector components, we have the following system of equations.

$$\begin{aligned} 2a_1 + b_1 &= 0 \\ a_1 - 2b_1 &= 0 \\ a_1 - a_2 - b_2 &= 0 \\ a_2 + b_1 - b_2 &= -1 \end{aligned}$$

This system is easily solved and $a_1 = 0$, $a_2 = -1/2$, $b_1 = 0$, and $b_2 = 1/2$. Thus,

$$\begin{aligned} \mathbf{y}_p &= \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \sin t \\ &= \begin{pmatrix} 0 \\ (-1/2) \cos t + (1/2) \sin t \end{pmatrix}. \end{aligned}$$

16. If $\mathbf{y}_p = \mathbf{a}t + \mathbf{b}$, then

$$\mathbf{y}'_p = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

and

$$\begin{aligned} A\mathbf{y}_p + \mathbf{f} &= A(\mathbf{a}t + \mathbf{b}) + \mathbf{f} \\ &= \begin{pmatrix} -3 & 6 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \end{aligned}$$

or,

$$A\mathbf{y}_p + \mathbf{f} = \begin{pmatrix} (-3a_1 + 6a_2)t + (-3b_1 + 6b_2 + 3) \\ (-2a_1 + 4a_2)t + (-2b_1 + 4b_2 + 4) \end{pmatrix}.$$

Comparing coefficients of the polynomial entries (e.g., $0 = -3a_1 + 6a_2$ and $a_1 = -3b_1 + 6b_2 + 3$), we get the following system.

$$\begin{aligned} a_1 - 2a_2 &= 0 \\ a_1 + 3b_1 - 6b_2 &= 3 \\ a_2 + 2b_1 - 4b_2 &= 4 \end{aligned}$$

Solving, $a_1 = -12$, $a_2 = -6$, $b_1 = 5 + 2b_2$, with b_2 free. Letting $b_2 = 0$, we get $a_1 = -12$, $a_2 = -6$, $b_1 = 5$, and $b_2 = 0$, providing the particular solution

$$\mathbf{y}_p = \begin{pmatrix} -12 \\ -6 \end{pmatrix} t + \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

17. If $\mathbf{y}_p = (\mathbf{a}t + \mathbf{b})e^{-t}$, then

$$\begin{aligned} \mathbf{y}'_p &= \mathbf{a}e^{-t} - (\mathbf{a}t + \mathbf{b})e^{-t} = [-\mathbf{a}t + (\mathbf{a} - \mathbf{b})]e^{-t} \\ &= \begin{pmatrix} -a_1t + (a_1 - b_1) \\ -a_2t + (a_2 - b_2) \end{pmatrix} e^{-t}. \end{aligned}$$

Next,

$$\begin{aligned} A\mathbf{y}_p + \mathbf{f} &= A(\mathbf{a}t + \mathbf{b})e^{-t} + \mathbf{f} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1t + b_1 \\ a_2t + b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}, \\ &= \begin{pmatrix} (a_1 + 2a_2)t + (b_1 + 2b_2 + 1) \\ (2a_1 + a_2)t + (2b_1 + b_2) \end{pmatrix} e^{-t}. \end{aligned}$$

Thus,

$$\begin{aligned} &\begin{pmatrix} -a_1t + (a_1 - b_1) \\ -a_2t + (a_2 - b_2) \end{pmatrix} \\ &= \begin{pmatrix} (a_1 + 2a_2)t + (b_1 + 2b_2 + 1) \\ (2a_1 + a_2)t + (2b_1 + b_2) \end{pmatrix}. \end{aligned}$$

Comparing coefficients of the polynomial entries in these vectors (e.g., $-a_1 = (a_1 + 2a_2)$) leads to the system

$$\begin{aligned} a_1 + a_2 &= 0 \\ a_1 - 2b_1 - 2b_2 &= 1 \\ a_2 - 2b_1 - 2b_2 &= 0. \end{aligned}$$

The solution of this system is $a_1 = 1/2$, $a_2 = -1/2$, $b_1 = -1/4 - b_2$, where b_2 is free. Choosing $b_2 = -1/8$ provides the solution $a_1 = 1/2$, $a_2 = -1/2$, $b_1 = -1/8$, and $b_2 = -1/8$, so

$$\mathbf{y}_p = (\mathbf{a}t + \mathbf{b})e^{-t} = \begin{pmatrix} (1/2)t - 1/8 \\ (-1/2)t - 1/8 \end{pmatrix} e^{-t},$$

which is identical to that found in Example 9.15.

18. The current coming into the node at “a” must equal the current coming out of the same node. Hence,

$$i = i_1 + i_2. \quad (9.16)$$

Traversing (clockwise) the leftmost loop containing emf, resistor, and inductor, Kirchhoff’s voltage law provides

$$-30 + 10i + 0.02i'_1 = 0. \quad (9.17)$$

Traversing (clockwise) the outer loop containing emf, both resistors, and the far inductor,

$$-30 + 10i + 20i_2 + 0.04i'_2 = 0. \quad (9.18)$$

If we now substitute $i = i_1 + i_2$ into equations (9.17) and (9.18), then a little algebra provides us with the following system (in the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$).

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}' = \begin{bmatrix} -500 & -500 \\ -250 & -750 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1500 \\ 750 \end{bmatrix} \quad (9.19)$$

A computer provides the following eigenvalue-eigenvector pairs for the coefficient matrix in equation (9.19):

$$-1000 \longrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad -250 \longrightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Hence, the homogeneous solution is

$$\mathbf{x} = C_1 e^{-1000t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-250t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \quad (9.20)$$

Now for a particular solution, let’s try the form $\mathbf{x}_p = (a_1, a_2)^T$. Substituting this informed guess in equation (9.19), we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -500 & -500 \\ -250 & -750 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 1500 \\ 750 \end{bmatrix}.$$

It is easy to show that $a_1 = 3$ and $a_2 = 0$, revealing the particular solution $\mathbf{x}_p = (3, 0)^T$. Therefore, the general solution is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, or

$$\mathbf{x} = C_1 e^{-1000t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-250t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \quad (9.21)$$

Initially the currents i_1 and i_2 both equal zero. That is,

$$\mathbf{x}(0) = \begin{bmatrix} i_1(0) \\ i_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Substituting this initial condition in the general solution (9.21),

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

from which it is easily seen that $C_1 = 1$ and $C_2 = -1$. Thus, the solution satisfying the initial condition is

$$\mathbf{x} = e^{-1000t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-250t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Because, $\mathbf{x} = (i_1, i_2)^T$, we have solutions for the current through the inductors.

$$\begin{aligned} i_1(t) &= e^{-1000t} + 2e^{-250t} + 3 \\ i_2(t) &= e^{-1000t} - e^{-250t} \end{aligned}$$

19. (a) The current entering node “a” must equal the current coming out of node “a”.

$$i_1 = i_2 + i_3 \quad (9.22)$$

Traversing (clockwise) the leftmost loop containing emf, resistor, inductor, and resistor, KVL provides

$$-E + 8i_1 + 2i'_1 + 4i_3 = 0. \quad (9.23)$$

Traversing (clockwise) the rightmost loop containing inductor and resistor, KVL provides

$$i'_2 - 4i_3 = 0. \quad (9.24)$$

Substitute (9.22) in both (9.23) and (9.24), then solve the resulting equations for i'_1 and i'_2 .

$$i'_1 = -6i_1 + 2i_2 + \sin t \quad (9.25)$$

$$i'_2 = 4i_1 - 4i_2 \quad (9.26)$$

Note that we've also substituted $E = 2 \sin t$. Place the system in matrix form

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{f}, \quad (9.27)$$

where

$$\mathbf{A} = \begin{bmatrix} -6 & 2 \\ 4 & -4 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} \sin t \\ 0 \end{bmatrix},$$

and $\mathbf{y} = (i_1, i_2)^T$. It is not difficult to find the eigenvalue-eigenvector pairs

$$-2 \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad -8 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

which provide the homogeneous solution

$$\mathbf{y}_h = C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (9.28)$$

Because of the form of $\mathbf{f} = (\sin t, 0)^T$, we will try

$$\mathbf{y}_p = \sin t \begin{bmatrix} a \\ b \end{bmatrix} + \cos t \begin{bmatrix} c \\ d \end{bmatrix} \quad (9.29)$$

as a particular solution. Substituting in the left hand side of equation (9.27),

$$\mathbf{y}'_p = \cos t \begin{bmatrix} a \\ b \end{bmatrix} - \sin t \begin{bmatrix} c \\ d \end{bmatrix}. \quad (9.30)$$

Substituting in the right hand side of (9.27),

$$\begin{aligned} \mathbf{A}\mathbf{y}_p + \mathbf{f} &= \begin{bmatrix} -6 & 2 \\ 4 & -4 \end{bmatrix} \left[\sin t \begin{bmatrix} a \\ b \end{bmatrix} + \cos t \begin{bmatrix} c \\ d \end{bmatrix} \right] \\ &\quad + \begin{bmatrix} \sin t \\ 0 \end{bmatrix} \\ &= \cos t \begin{bmatrix} -6c + 2d \\ 4c - 4d \end{bmatrix} \\ &\quad + \sin t \begin{bmatrix} -6a + 2b + 1 \\ 4a - 4b \end{bmatrix} \end{aligned} \quad (9.31)$$

Comparing coefficients of $\cos t$ and $\sin t$ in equations (9.30) and (9.31) leads to the system of equations

$$\begin{aligned} a + 6c - 2d &= 0 \\ b - 4c + 4d &= 0 \\ 6a - 2b - c &= 1 \\ -4a + 4b - d &= 0. \end{aligned} \quad (9.32)$$

A computer or calculator provides a solution, which when substituted in (9.29), provides the particular solution $\mathbf{y}_p$. Finally, the general solution is $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$.

$$\begin{aligned} \mathbf{y} &= C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &+ \sin t \begin{bmatrix} 14/65 \\ 12/65 \end{bmatrix} + \cos t \begin{bmatrix} -5/65 \\ -8/65 \end{bmatrix}. \end{aligned} \quad (9.33)$$

The initial current through each inductor is zero, so the initial condition for our system is $\mathbf{y}(0) = (0, 0)^T$. In (9.33), this provides

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -5/65 \\ -8/65 \end{bmatrix}. \quad (9.34)$$

It is not difficult to solve this equation for $C_1 = 1/15$ and $C_2 = 2/195$, providing the final solution

$$\begin{aligned} \mathbf{y} &= \frac{1}{15} e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{195} e^{-8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &+ \sin t \begin{bmatrix} 14/65 \\ 12/65 \end{bmatrix} + \cos t \begin{bmatrix} -5/65 \\ -8/65 \end{bmatrix}. \end{aligned} \quad (9.35)$$

Recalling that $\mathbf{y} = (i_1, i_2)^T$ and equating components of (9.35), we can find the current in each inductor as a function of time.

$$\begin{aligned} i_1 &= \frac{1}{15} e^{-2t} + \frac{2}{195} e^{-8t} + \frac{14}{65} \sin t \\ &\quad - \frac{1}{13} \cos t \\ i_2 &= \frac{2}{15} e^{-2t} - \frac{2}{195} e^{-8t} + \frac{12}{65} \sin t \\ &\quad - \frac{8}{65} \cos t. \end{aligned} \quad (9.36)$$

(b) Note that the homogeneous solution

$$\mathbf{y}_h = C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (9.37)$$

can be written in the form

$$\mathbf{y}_h = Y\mathbf{c}, \quad (9.38)$$

where

$$Y(t) = \begin{bmatrix} e^{-2t} & e^{-8t} \\ 2e^{-2t} & -e^{-8t} \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

We will seek a particular solution by replacing $\mathbf{c}$ in (9.38) with $\mathbf{v}(t)$,

$$\mathbf{y}_p(t) = Y(t)\mathbf{v}(t), \quad (9.39)$$

where $\mathbf{v}(t)$ is a vector of functions to be determined. Substituting equation (9.38) in

$$\begin{aligned} \mathbf{y} &= A\mathbf{y} + \mathbf{f} \\ (Y\mathbf{v})' &= A(Y\mathbf{v}) + \mathbf{f} \\ Y\mathbf{v}' + Y'\mathbf{v} &= AY\mathbf{v} + \mathbf{f} \end{aligned}$$

Using the fact that $Y' = AY$,

$$\begin{aligned} Y\mathbf{v}' &= \mathbf{f} \\ \mathbf{v}' &= Y^{-1}\mathbf{f}. \end{aligned}$$

To determine $\mathbf{v}$, we need to perform the calculation

$$\mathbf{v} = \int Y^{-1} \vec{f}. \quad (9.40)$$

To that end, we must first calculate Y^{-1} . We invoke the shortcut for finding the inverse of a 2×2 matrix.

$$\begin{aligned} Y^{-1} &= \frac{1}{\det(Y)} \begin{bmatrix} -e^{-8t} & -e^{-8t} \\ -2e^{-2t} & e^{-2t} \end{bmatrix} \\ &= -\frac{1}{3} e^{10t} \begin{bmatrix} -e^{-8t} & -e^{-8t} \\ -2e^{-2t} & e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}/3 & e^{2t}/3 \\ 2e^{8t}/3 & -e^{8t}/3 \end{bmatrix} \end{aligned}$$

Next,

$$\begin{aligned} Y^{-1}\mathbf{f} &= \begin{bmatrix} e^{2t}/3 & e^{2t}/3 \\ 2e^{8t}/3 & -e^{8t}/3 \end{bmatrix} \begin{bmatrix} \sin t \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (e^{2t} \sin t)/3 \\ (2e^{8t} \sin t)/3 \end{bmatrix}. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \mathbf{v} &= \int Y^{-1}\mathbf{f} \\ &= \begin{bmatrix} (-e^{2t} \cos t)/15 + (2e^{2t} \sin t)/15 \\ (-2e^{8t} \cos t)/195 + (16e^{8t} \sin t)/195 \end{bmatrix} \end{aligned}$$

Using (9.39),

$$\begin{aligned} \mathbf{y}_p &= Y\mathbf{v} \\ &= \begin{bmatrix} (-\cos t)/13 + (14 \sin t)/65 \\ (-8 \cos t)/65 + (12 \sin t)/65 \end{bmatrix}. \end{aligned}$$

Finally, the general solution is $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$.

$$\begin{aligned} \mathbf{y} &= C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &\quad + \begin{bmatrix} (-\cos t)/13 + (14 \sin t)/65 \\ (-8 \cos t)/65 + (12 \sin t)/65 \end{bmatrix}. \end{aligned} \quad (9.41)$$

The initial current through each inductor is zero, so the initial condition for our system is $\mathbf{y}(0) = (0, 0)^T$. In (9.41), this provides

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1/13 \\ -8/65 \end{bmatrix}. \quad (9.42)$$

It is not difficult to solve this equation for $C_1 = 1/15$ and $C_2 = 2/195$, providing the final solution

$$\begin{aligned} \mathbf{y} &= \frac{1}{15} e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{195} e^{-8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &\quad + \begin{bmatrix} (-\cos t)/13 + (14 \sin t)/65 \\ (-8 \cos t)/65 + (12 \sin t)/65 \end{bmatrix}. \end{aligned} \quad (9.43)$$

Recalling that $\mathbf{y} = (i_1, i_2)^T$ and equating components of (9.43), we can find the current in

each inductor as a function of time.

$$\begin{aligned} i_1 &= \frac{1}{15} e^{-2t} + \frac{2}{195} e^{-8t} - \frac{1}{13} \cos t \\ &\quad + \frac{14}{65} \sin t \\ i_2 &= \frac{2}{15} e^{-2t} - \frac{2}{195} e^{-8t} - \frac{8}{65} \cos t \\ &\quad + \frac{12}{65} \sin t. \end{aligned} \quad (9.44)$$

20. A has eigenvalues -1 and 3 , with corresponding eigenvectors $(0, 1)^T$ and $(-1, 1)^T$. Thus

$$Y(t) = \begin{pmatrix} 0 & -e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix}.$$

$$Y(0) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad Y(0)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

so

$$e^{tA} = Y(t)Y(0)^{-1} = \begin{pmatrix} e^{3t} & 0 \\ e^{-t} - e^{3t} & e^{-t} \end{pmatrix}.$$

21. A has eigenvalues 1 and 3 , with corresponding eigenvectors $(2, 1)^T$ and $(1, 1)^T$. Thus

$$Y(t) = \begin{pmatrix} 2e^t & e^{3t} \\ e^t & e^{-t} \end{pmatrix}.$$

$$Y(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad Y(0)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

so

$$e^{tA} = Y(t)Y(0)^{-1} = \begin{pmatrix} 2e^t - e^{3t} & 2e^{3t} - 2e^t \\ e^t - e^{3t} & 2e^{3t} - e^t \end{pmatrix}.$$

22. A has eigenvalues $\pm i$. An eigenvector corresponding to i is $(2 - i, 5)^T$. Hence a complex solution is

$$\begin{aligned} \mathbf{z}(t) &= e^{it}\mathbf{w} \\ &= [\cos t + i \sin t] \left[\begin{pmatrix} 2 \\ 5 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} 2 \cos t + \sin t \\ 5 \cos t \end{pmatrix} + i \begin{pmatrix} 2 \sin t - \cos t \\ 5 \sin t \end{pmatrix}. \end{aligned}$$

The real and imaginary parts of $\mathbf{z}$ are solutions to the homogeneous equation, and so

$$V(t) = \begin{pmatrix} 2 \cos t + \sin t & 2 \sin t - \cos t \\ 5 \cos t & 5 \sin t \end{pmatrix}.$$

Thus

$$V(0) = \begin{pmatrix} 2 & -1 \\ 5 & 0 \end{pmatrix} \quad \text{and}$$

$$V(0)^{-1} = \begin{pmatrix} 0 & 1/5 \\ -1 & 2/5 \end{pmatrix},$$

and

$$e^{tA} = V(t)V(0)^{-1}$$

$$= \begin{pmatrix} \cos t - 2 \sin t & \sin t \\ -5 \sin t & \cos t + 2 \sin t \end{pmatrix}.$$

23. A has eigenvalues $1 \pm i$. An eigenvector corresponding to i is $(1 + i, 1)^T$. Hence a complex solution is

$$\mathbf{z}(t) = e^{(1+i)t} \mathbf{w}$$

$$= e^t [\cos t + i \sin t] \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$= e^t \left[\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix} \right].$$

The real and imaginary parts of $\mathbf{z}$ are solutions to the homogeneous equation, and so

$$V(t) = \begin{pmatrix} \cos t - \sin t & \cos t + \sin t \\ \cos t & \sin t \end{pmatrix}.$$

Thus

$$V(0) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and}$$

$$V(0)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

and

$$e^{tA} = V(t)V(0)^{-1}$$

$$= \begin{pmatrix} \cos t + \sin t & -2 \sin t \\ \sin t & \cos t - \sin t \end{pmatrix}.$$

24. A has eigenvalue $\lambda = -3$, which has algebraic multiplicity 2 and geometric multiplicity 1. Thus

$$e^{tA} = e^{\lambda t} e^{t(A-\lambda I)}$$

$$= e^{-3t} [I + t(A + 3I)]$$

$$= e^{-3t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}.$$

25. A has eigenvalue $\lambda = 2$, which has algebraic multiplicity 2 and geometric multiplicity 1. Thus

$$e^{tA} = e^{\lambda t} e^{t(A-\lambda I)}$$

$$= e^{2t} [I + t(A - 2I)]$$

$$= e^{2t} \begin{pmatrix} 1-2t & t \\ -4t & 1+2t \end{pmatrix}.$$

26. A has eigenvalues 2 and -1 with eigenvectors $(-1, 1)^T$ and $(-1, 2)^T$. Thus

$$Y(t) = \begin{pmatrix} -e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix},$$

$$Y(0) = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \quad \text{and}$$

$$Y(0)^{-1} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix},$$

Thus

$$e^{tA} = Y(t)Y(0)^{-1}$$

$$= \begin{pmatrix} 2e^{2t} - e^{-t} & e^{2t} - e^{-t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{pmatrix}.$$

The solution to the initial value problem is

$$\mathbf{y}(t) = e^{tA} \mathbf{y}_0$$

$$= \begin{pmatrix} 2e^{2t} - e^{-t} & e^{2t} - e^{-t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{2t} - 2e^{-t} \\ 4e^{-t} - 3e^{2t} \end{pmatrix}.$$

27. A has eigenvalues -4 and -1 with eigenvectors $(-1, 1)^T$ and $(-1, 2)^T$. Thus

$$Y(t) = \begin{pmatrix} -e^{-4t} & -e^{-t} \\ e^{-4t} & 2e^{-t} \end{pmatrix},$$

$$Y(0) = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}.$$

Thus

$$e^{tA} = Y(t)Y(0)^{-1}$$

$$= \begin{pmatrix} 2e^{-4t} - e^{-t} & e^{-4t} - e^{-t} \\ 2e^{-t} - 2e^{-4t} & 2e^{-t} - e^{-4t} \end{pmatrix}.$$

The solution to the initial value problem is

$$\begin{aligned}\mathbf{y}(t) &= e^{tA}\mathbf{y}_0 \\ &= \begin{pmatrix} 2e^{-4t} - e^{-t} & e^{-4t} - e^{-t} \\ 2e^{-t} - 2e^{-4t} & 2e^{-t} - e^{-4t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-4t} - e^{-t} \\ 2e^{-t} - 2e^{-4t} \end{pmatrix}.\end{aligned}$$

28. A has eigenvalues $\pm 2i$. Associated with the eigenvalue $2i$ there is the eigenvector $(1 + i, 2)^T$. The associated complex solution is

$$\begin{aligned}\mathbf{z}(t) &= e^{2it} \begin{pmatrix} 1 + i \\ 2 \end{pmatrix} \\ &= \left[\cos 2t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \sin 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &\quad + i \left[\cos 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right].\end{aligned}$$

The real and imaginary parts of $\mathbf{z}$ are a fundamental set of solutions, so we can take

$$Y(t) = \begin{pmatrix} \cos 2t - \sin 2t & \cos 2t + \sin 2t \\ 2 \cos 2t & 2 \sin 2t \end{pmatrix}.$$

Then

$$Y(0) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad Y(0)^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1 & -1/2 \end{pmatrix},$$

so

$$\begin{aligned}e^{tA} &= Y(t)Y(0)^{-1} \\ &= \begin{pmatrix} \cos 2t + \sin 2t & -\sin 2t \\ 2 \sin 2t & \cos 2t - \sin 2t \end{pmatrix}.\end{aligned}$$

The solution to the initial value problem is

$$\begin{aligned}\mathbf{y}(t) &= e^{tA}\mathbf{y}_0 \\ &= \begin{pmatrix} \cos 2t + \sin 2t & -\sin 2t \\ 2 \sin 2t & \cos 2t - \sin 2t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}.\end{aligned}$$

29. A has one eigenvalue, -3 , with multiplicity 2. Hence

$$\begin{aligned}e^{tA} &= e^{-3t} e^{t(A+3I)} \\ &= e^{-3t} [I + t(A + 3I)] \\ &= e^{-3t} \begin{pmatrix} 1 + 2t & t \\ -4t & 1 - 2t \end{pmatrix}.\end{aligned}$$

The solution to the initial value problem is

$$\begin{aligned}\mathbf{y}(t) &= e^{tA}\mathbf{y}_0 \\ &= e^{-3t} \begin{pmatrix} 1 + 2t & t \\ -4t & 1 - 2t \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= e^{-3t} \begin{pmatrix} 2 + 3t \\ -1 - 6t \end{pmatrix}.\end{aligned}$$

30. Let $x_1(t)$ and $x_2(t)$ represent the amount of salt (in pounds) in the upper and lower tanks, respectively. The rate at which salt flows into the upper tank is found by multiplying the solution rate by the concentration of the salt solution entering the upper tank.

$$\text{Rate In} = 5 \text{ gal/min} \times 0.2 \text{ lb/gal} = 1 \text{ lb/min}$$

Similarly, salt leaves the upper tank at the following rate.

$$\text{Rate Out} = 5 \text{ gal/min} \times \frac{x_1}{200} \text{ lb/gal} = \frac{x_1}{40} \text{ lb/min}$$

Thus, the rate at which the salt content is changing in the upper tank is $dx_1/dt = \text{Rate In} - \text{Rate Out}$, or

$$\frac{dx_1}{dt} = 1 - \frac{x_1}{40}. \quad (9.45)$$

A similar argument shows that the salt content in the lower tank is changing according to the equation

$$\frac{dx_2}{dt} = \frac{x_1}{40} - \frac{x_2}{40}. \quad (9.46)$$

In matrix form, the system defined by equations (9.45) and (9.46) is

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \quad (9.47)$$

where $\mathbf{x} = (x_1, x_2)^T$ and

$$A = \begin{bmatrix} -1/40 & 0 \\ 1/40 & -1/40 \end{bmatrix} \quad \text{and} \quad \vec{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

First, find the homogeneous solution $\mathbf{x}_h$, satisfying $\mathbf{x}'_h = A\mathbf{x}_h$. Note that matrix A is lower triangular, so

its eigenvalues are found on its main diagonal. Note the repeated eigenvalue $-1/40$. Thus,

$$\begin{aligned} e^{tA} &= e^{-t/40} (I + t(A + (1/40)I)) \\ &= e^{-t/40} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 1/40 & 0 \end{bmatrix} \right) \\ &= e^{-t/40} \begin{bmatrix} 1 & 0 \\ t/40 & 1 \end{bmatrix}. \end{aligned} \quad (9.48)$$

Thus, the homogeneous solution is

$$\mathbf{x}_h = e^{tA} \mathbf{c}, \quad (9.49)$$

where $\mathbf{c} = (C_1, C_2)^T$ is constant. To find a particular solution $\vec{x}_p$, set

$$\mathbf{x}_p = e^{tA} \mathbf{v}, \quad (9.50)$$

where $\mathbf{v}$ is a vector of functions to be determined. Substitute equation (9.50) into equation (9.47).

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} + \mathbf{f} \\ e^{At} \mathbf{v}' + A e^{tA} &= A e^{tA} \mathbf{v} + \mathbf{f} \\ e^{tA} \mathbf{v}' &= \mathbf{f} \\ \mathbf{v}' &= e^{-tA} \mathbf{f} \end{aligned}$$

Thus,

$$\mathbf{v} = \int_0^t e^{-sA} \mathbf{f}(s) ds, \quad (9.51)$$

where we've chosen a particular integral whose value is zero at $t = 0$. We first need to calculate e^{-tA} . Because of the form of e^{tA} in equation (9.48), and the fact that the inverse of (αA) is $(1/\alpha)A^{-1}$, where α is any scalar and A is any matrix, it is a simple matter to calculate

$$e^{-tA} = e^{t/40} \begin{bmatrix} 1 & 0 \\ -t/40 & 1 \end{bmatrix}. \quad (9.52)$$

Next,

$$e^{tA} \mathbf{f} = e^{t/40} \begin{bmatrix} 1 & 0 \\ -t/40 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{t/40} \\ -te^{t/40}/40 \end{bmatrix}. \quad (9.53)$$

We now use equation (9.51) to compute $\vec{v}$.

$$\begin{aligned} \mathbf{v} &= \int_0^t e^{-sA} \mathbf{f}(s) ds \\ &= \int_0^t \begin{bmatrix} e^{s/40} \\ -se^{s/40}/40 \end{bmatrix} ds \\ &= \begin{bmatrix} 40e^{t/40} - 40 \\ -te^{t/40} + 40e^{t/40} - 40 \end{bmatrix}. \end{aligned}$$

Finally, equation (9.50) will find the required particular solution.

$$\begin{aligned} \mathbf{x}_p &= e^{tA} \mathbf{v} \\ &= e^{-t/40} \begin{bmatrix} 1 & 0 \\ t/40 & 1 \end{bmatrix} \begin{bmatrix} 40e^{t/40} - 40 \\ -te^{t/40} + 40e^{t/40} - 40 \end{bmatrix} \\ &= \begin{bmatrix} 40 - 40e^{-t/40} \\ -te^{-t/40} - 40e^{-t/40} + 40 \end{bmatrix}. \end{aligned}$$

Thus, the general solution is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, or equivalently,

$$\mathbf{x} = e^{tA} \mathbf{c} + e^{tA} \int_0^t e^{-sA} \mathbf{f}(s) ds. \quad (9.54)$$

Therefore, the initial condition $\mathbf{x}(0) = (0, 0)^T$ (both tanks contain pure water to start — no salt) provides

$$\begin{aligned} \mathbf{x}(0) &= e^{0A} \mathbf{c} + e^{0A} \int_0^0 e^{-sA} \mathbf{f}(s) ds \\ \mathbf{0} &= \mathbf{c}. \end{aligned}$$

Subbing this back in equation (9.54) provides the final solution,

$$\begin{aligned} \mathbf{x} &= e^{tA} \int_0^t e^{-sA} \mathbf{f}(s) ds \\ &= \begin{bmatrix} 40 - 40e^{-t/40} \\ -te^{-t/40} - 40e^{-t/40} + 40 \end{bmatrix}. \end{aligned} \quad (9.55)$$

Thus,

$$\begin{aligned} x_1(t) &= 40 - 40e^{-t/40} \\ x_2(t) &= -te^{-t/40} - 40e^{-t/40} + 40. \end{aligned} \quad (9.56)$$

31. The matrix A has eigenvalues -1 and 5 , with associated eigenvectors $(0, 1)^T$ and $(1, 1)^T$. Thus

$$Y(t) = \begin{pmatrix} 0 & e^{5t} \\ e^{-t} & e^{5t} \end{pmatrix},$$

$$Y(0) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad Y(0)^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence

$$e^{tA} = Y(t)Y(0)^{-1} = \begin{pmatrix} e^{5t} & 0 \\ e^{5t} - e^{-t} & e^{-t} \end{pmatrix}.$$

The solution to the initial value problem is

$$\mathbf{y}(t) = e^{(t-t_0)A} \mathbf{y}_0 = \begin{pmatrix} -e^{5t-5} \\ 4e^{1-t} - e^{5t-5} \end{pmatrix}.$$

32. The matrix A has eigenvalues -1 and -2 with associated eigenvectors $(-1, 1)^T$ and $(-1, 2)^T$. Hence

$$Y(t) = \begin{pmatrix} -e^{-t} & -e^{-2t} \\ e^{-t} & 2e^{-2t} \end{pmatrix},$$

$$Y(0) = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$$

and

$$Y(0)^{-1} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} e^{tA} &= Y(t)Y(0)^{-1} \\ &= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{pmatrix}. \end{aligned}$$

The solution to the initial value problem is

$$\mathbf{y}(t) = e^{(t-t_0)A} \mathbf{y}_0 = \begin{pmatrix} 2e^{-t-1} - e^{-2t-2} \\ 2e^{-2t-2} - 2e^{-t-1} \end{pmatrix}.$$

33. A has eigenvalues $\pm 3i$. The eigenvalue $3i$ has associated eigenvector $(2 + i, 5)^T$. The associated solution is

$$\begin{aligned} \mathbf{z}(t) &= \left[\cos 3t \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \sin 3t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &\quad + i \left[\cos 3t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right]. \end{aligned}$$

Thus

$$Y(t) = \begin{pmatrix} 2 \cos 3t - \sin 3t & \cos 3t + 2 \sin 3t \\ 5 \cos 3t & 5 \sin 3t \end{pmatrix},$$

$$Y(0) = \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix} \quad \text{and} \quad Y(0)^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 5 & -2 \end{pmatrix}.$$

Hence

$$\begin{aligned} e^{tA} &= Y(t)Y(0)^{-1} \\ &= \begin{pmatrix} \cos 3t + 2 \sin 3t & -\sin 3t \\ 5 \sin 3t & \cos 3t - 2 \sin 3t \end{pmatrix}. \end{aligned}$$

The solution to the initial value problem is

$$\begin{aligned} \mathbf{y}(t) &= e^{(t-t_0)A} \mathbf{y}_0 \\ &= \begin{pmatrix} -\cos 3(t-1) - 2 \sin 3(t-1) \\ -5 \sin 3(t-1) \end{pmatrix}. \end{aligned}$$

34. A has the single eigenvalue -4 . Hence

$$\begin{aligned} e^{tA} &= e^{-4t} e^{t(A+4I)} \\ &= e^{-4t} [I + t(A+4I)] \\ &= e^{-4t} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}. \end{aligned}$$

The solution to the initial value problem is

$$\mathbf{y}(t) = e^{(t-t_0)A} \mathbf{y}_0 = e^{8-4t} \begin{pmatrix} -2 \\ 2t-2 \end{pmatrix}.$$

35. The matrix A has eigenvalues -2 , 3 , and 1 , with associated eigenvectors $(0, 1, 0)^T$, $(-2, 0, 1)^T$, and $(-1, 1, 1)^T$. Hence

$$Y(t) = \begin{pmatrix} 0 & -2e^{3t} & -e^t \\ e^{-2t} & 0 & e^t \\ 0 & e^{3t} & e^t \end{pmatrix},$$

$$Y(0) = \begin{pmatrix} 0 & -2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and}$$

$$Y(0)^{-1} = \begin{pmatrix} -1 & 1 & -2 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

Thus

$$\begin{aligned} e^{tA} &= Y(t)Y(0)^{-1} \\ &= \begin{pmatrix} 2e^{3t} - e^t & 0 & 2e^{3t} - 2e^t \\ e^t - e^{-2t} & e^{-2t} & 2e^t - 2e^{-2t} \\ e^t - e^{3t} & 0 & 2e^t - e^{3t} \end{pmatrix}. \end{aligned}$$

36. A has eigenvalues $\pm i$ and 1. Associated to the eigenvalue i is the eigenvector $\mathbf{w} = (-1 - i, 0, 2)^T$. This leads to the complex solution

$$\mathbf{z}(t) = e^{it}\mathbf{w} = \left[\cos t \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] + i \left[\cos t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right].$$

The real and imaginary parts of $\mathbf{z}(t)$ provide two linearly independent solutions. We get a third from the eigenvalue 1 and its eigenvector $(0, 1, 2)^T$. It is $\mathbf{y}_3(t) = e^t(0, 1, 2)^T$. Hence we have the fundamental matrix

$$Y(t) = \begin{pmatrix} \sin t - \cos t & -\cos t - \sin t & 0 \\ 0 & 0 & e^t \\ 2 \cos t & 2 \sin t & 2e^t \end{pmatrix}.$$

We have

$$Y(0) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix} \quad \text{and} \quad Y(0)^{-1} = \begin{pmatrix} 0 & -1 & 1/2 \\ -1 & 1 & -1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Finally

$$e^{tA} = Y(t)Y(0)^{-1} = \begin{pmatrix} \cos t + \sin t & -2 \sin t & \sin t \\ 0 & e^t & 0 \\ -2 \sin t & 2e^t - 2 \cos t + 2 \sin t & \cos t - \sin t \end{pmatrix}.$$

37. A has eigenvalues $-2, -1, 1$, and 2 with associated eigenvectors $(0, 1, 0, 1)^T$, $(-1, 1, 1, 0)^T$, $(0, 1, 0, 0)^T$, and $(1, 2, 0, 1)^T$. Thus we have the fundamental matrix

$$Y(t) = \begin{pmatrix} 0 & -e^{-t} & 0 & e^{2t} \\ e^{-2t} & e^{-t} & e^t & 2e^{2t} \\ 0 & e^{-t} & 0 & 0 \\ e^{-2t} & 0 & 0 & e^{2t} \end{pmatrix}.$$

We have

$$Y(0) = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y(0)^{-1} = \begin{pmatrix} -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Finally,

$$e^{tA} = Y(t)Y(0)^{-1} = \begin{pmatrix} e^{2t} & 0 & e^{2t} - e^{-t} & 0 \\ 2e^{2t} - e^t - e^{-2t} & e^t & 2e^{2t} - 2e^t + e^{-t} - e^{-2t} & e^{-2t} - e^t \\ 0 & 0 & e^{-t} & 0 \\ e^{2t} - e^{-2t} & 0 & e^{2t} - e^{-2t} & e^{-2t} \end{pmatrix}.$$

38. A has eigenvalues $-1, -2$, and -3 . -1 has algebraic multiplicity 2 and geometric multiplicity 1. The vector $\mathbf{v}_1 = (-2, 1, 0, 1)^T$ is an eigenvector and $\mathbf{v}_2 = (0, 0, 1, 1)^T$ is a generalized eigenvector with $(A + I)\mathbf{v}_2 = -\mathbf{v}_1$. Hence $\mathbf{v}_1$ leads to the solution $\mathbf{y}_1(t) = e^{-t}(-2, 1, 0, 1)^T$, and $\mathbf{v}_2$ leads to the solution

$$\mathbf{y}_2(t) = e^{-t}[\mathbf{v}_2 + t(A + I)\mathbf{v}_2] = e^{-t}[\mathbf{v}_2 - t\mathbf{v}_1] = e^{-t} \begin{pmatrix} 2t \\ -t \\ 1 \\ 1 - t \end{pmatrix}.$$

The eigenvalue -2 has eigenvector $(-1, 0, 0, 1)^T$, and leads to the solution $\mathbf{y}_3(t) = e^{-2t}(-1, 0, 0, 1)^T$. The eigenvalue -3 has eigenvector $(0, 0, 1, 0)^T$ and leads to the solution $\mathbf{y}_4(t) = e^{-3t}(0, 0, 1, 0)^T$. Thus we have the fundamental matrix

$$Y(t) = \begin{pmatrix} -2e^{-t} & 2te^{-t} & -e^{-2t} & 0 \\ e^{-t} & -te^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 & e^{-3t} \\ e^{-t} & e^{-t}(1-t) & e^{-2t} & 0 \end{pmatrix}.$$

Then

$$Y(0) = \begin{pmatrix} -2 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Y(0)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & -2 & 0 & 0 \\ -1 & -1 & 1 & -1 \end{pmatrix}.$$

Finally

$$e^{tA} = Y(t)Y(0)^{-1} = \begin{pmatrix} e^{-2t} + 2te^{-t} & e^{-t}(2t-2) + 2e^{-2t} & 0 & 2te^{-t} \\ -te^{-t} & e^{-t}(1-t) & 0 & -te^{-t} \\ e^{-t} - e^{-3t} & e^{-t} - e^{-3t} & e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t}(1-t) - e^{-2t} & e^{-t}(2-t) - 2e^{-3t} & 0 & e^{-t}(1-t) \end{pmatrix}.$$

39. We write

$$\mathbf{y}(t) = e^{tA} \left[\mathbf{y}_0 + \int_0^t e^{-sA} \mathbf{f}(s) ds \right].$$

We can now prove the result by direct substitution.

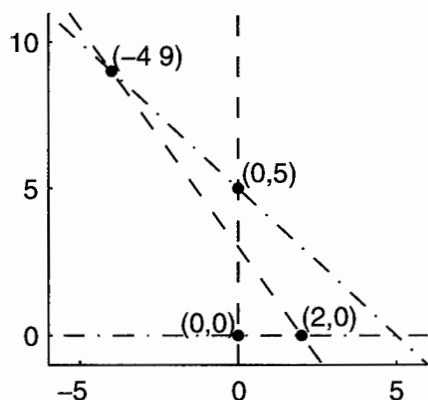
$$\mathbf{y}'(t) = A\mathbf{y}(t) + e^{tA} [e^{-tA} \mathbf{f}(t)] = A\mathbf{y}(t) + \mathbf{f}(t).$$

$$\mathbf{y}(0) = e^{0A} \mathbf{y}_0 = \mathbf{y}_0.$$

Chapter 10. Linear Systems with Constant Coefficients

Section 10.1. The Linearization of a Nonlinear System

1. The x -nullcline consists of the two lines defined by $x = 0$ and $3x + 2y = 6$. It is shown dashed in the accompanying figure. The y -nullcline consists of the two lines defined by $y = 0$ and $x + y = 5$. It is shown dot-dashed in the accompanying figure. The equilibrium points are found by looking for the intersections. They are $(0, 0)$, $(0, 5)$, $(2, 0)$, and $(-4, 9)$.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 6 - 6x - 2y & -2x \\ -y & 5 - x - 2y \end{pmatrix}.$$

At $(0, 0)$,

$$J(0, 0) = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

has eigenvalues 6 and 5 so $(0, 0)$ is a nodal source.

At $(0, 5)$,

$$J(0, 5) = \begin{pmatrix} -4 & 0 \\ -5 & -5 \end{pmatrix}$$

has eigenvalues -4 and -5 so $(0, 5)$ is a nodal sink. At $(2, 0)$,

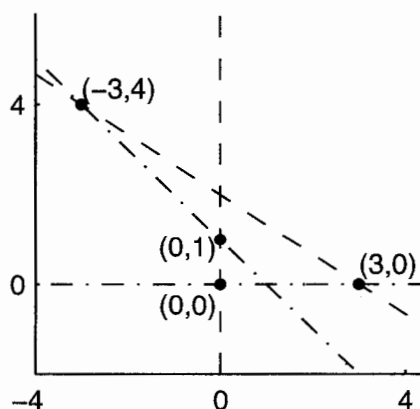
$$J(2, 0) = \begin{pmatrix} -6 & -4 \\ 0 & 3 \end{pmatrix}$$

has eigenvalues -6 and 3 so $(2, 0)$ is a saddle. At $(-4, 9)$,

$$J(-4, 9) = \begin{pmatrix} 12 & 8 \\ -9 & -9 \end{pmatrix}$$

has determinant $D = -36$ so $(-4, 9)$ is a saddle.

2. The x -nullcline consists of the two lines defined by $x = 0$ and $2x + 3y = 6$. It is shown dashed in the accompanying figure. The y -nullcline consists of the two lines defined by $y = 0$ and $x + y = 1$. It is shown dot-dashed in the accompanying figure. The equilibrium points are found by looking for the intersections. They are $(0, 0)$, $(0, 1)$, $(3, 0)$, and $(-3, 4)$.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 6 - 4x - 3y & -3x \\ -y & 1 - x - 2y \end{pmatrix}.$$

At $(0, 0)$,

$$J(0, 0) = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$$

has eigenvalues 6 and 1, so $(0, 0)$ is a nodal source.

At $(0, 1)$

$$J(0, 1) = \begin{pmatrix} 3 & 0 \\ -1 & -1 \end{pmatrix}$$

has eigenvalues 3 and -1 , so $(0, 1)$ is a saddle. At $(3, 0)$

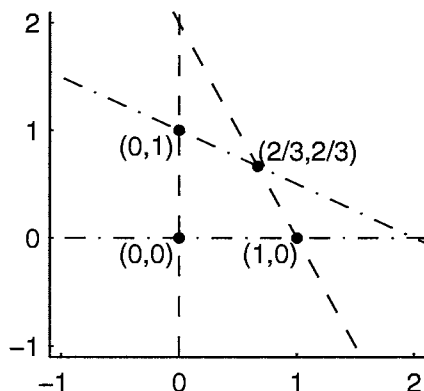
$$J(3, 0) = \begin{pmatrix} -6 & -9 \\ 0 & -2 \end{pmatrix}$$

has eigenvalues -6 and -2 , so $(3, 0)$ is a nodal sink. At $(-3, 4)$

$$J(-3, 4) = \begin{pmatrix} 6 & 9 \\ -4 & -4 \end{pmatrix}$$

has determinant $D = 12$, trace $T = 2$, and discriminant $T^2 - 4D = -44 < 0$, so $(-3, 4)$ is a spiral source.

3. The x -nullcline consists of the two lines defined by $x = 0$ and $2x + y = 2$. It is shown dashed in the accompanying figure. The y -nullcline consists of the two lines defined by $y = 0$ and $x + 2y = 2$. It is shown dot-dashed in the accompanying figure. The equilibrium points are found by looking for the intersections. They are $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(2/3, 2/3)$.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 2 - 4x - y & -x \\ -y & 2 - x - 4y \end{pmatrix}.$$

At $(0, 0)$,

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

has eigenvalues 2 and 2, so $(0, 0)$ is a degenerate source. The type of equilibrium point for the nonlinear system is unknown. At $(0, 1)$

$$J(0, 1) = \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}$$

has eigenvalues 1 and -2 , so $(0, 1)$ is a saddle. At $(1, 0)$

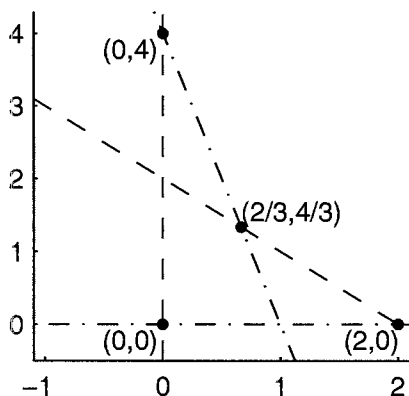
$$J(1, 0) = \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}$$

has eigenvalues -2 and 1, so $(1, 0)$ is a saddle. At $(2/3, 2/3)$

$$J(2/3, 2/3) = \begin{pmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{pmatrix}$$

has determinant $D = 4/3$, trace $T = -8/3$, and discriminant $T^2 - 4D = 16/9 > 0$, so $(2/3, 2/3)$ is a nodal sink.

4. The x -nullcline consists of the two lines defined by $x = 0$ and $x + y = 2$. It is shown dashed in the accompanying figure. The y -nullcline consists of the two lines defined by $y = 0$ and $4x + y = 4$. It is shown dot-dashed in the accompanying figure. The equilibrium points are found by looking for the intersections. They are $(0, 0)$, $(0, 4)$, $(2, 0)$, and $(2/3, 4/3)$.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 2 - 2x - y & -x \\ -4y & 4 - 4x - 2y \end{pmatrix}.$$

At $(0, 0)$,

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

has eigenvalues 2 and 4, so $(0, 0)$ is a nodal source.

At $(0, 4)$,

$$J(0, 4) = \begin{pmatrix} -2 & 0 \\ -16 & -4 \end{pmatrix}$$

has eigenvalues -2 and -4 , so $(0, 4)$ is a nodal sink.

At $(2, 0)$

$$J(2, 0) = \begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix}$$

has eigenvalues -2 and -4 , so $(2, 0)$ is a nodal sink.

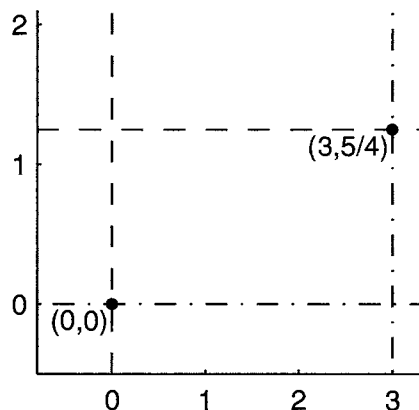
At $(2/3, 4/3)$

$$J(2/3, 4/3) = \begin{pmatrix} -2/3 & -2/3 \\ -16/3 & -4/3 \end{pmatrix}$$

has determinant $D = -24/9$, so $(2/3, 4/3)$ is a saddle.

5. The x -nullcline consists of the two lines defined by $x = 0$ and $y = 5/4$. It is shown dashed in the accompanying figure. The y -nullcline consists of the

two lines defined by $y = 0$ and $x = 3$. It is shown dot-dashed in the accompanying figure. The equilibrium points are found by looking for the intersections. They are $(0, 0)$, and $(3, 5/4)$.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 4y - 5 & 4x \\ -y & 3 - x \end{pmatrix}.$$

At $(0, 0)$

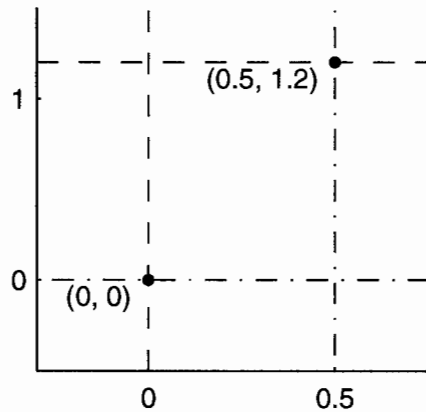
$$J(0, 0) = \begin{pmatrix} -5 & 0 \\ 0 & 3 \end{pmatrix}$$

has eigenvalues -5 and 3 , so $(0, 0)$ is a saddle. At $(3, 5/4)$

$$J(3, 5/4) = \begin{pmatrix} 0 & 12 \\ -5/4 & 0 \end{pmatrix}$$

has determinant $D = 15$ and trace $T = 0$, so $(3, 5/4)$ is a linear center. Since a center is not generic, we can not classify the equilibrium point.

6. The x -nullcline consists of the two lines defined by $x = 0$ and $y = 1.2$. It is shown dashed in the accompanying figure. The y -nullcline consists of the two lines defined by $y = 0$ and $x = 0.5$. It is shown dot-dashed in the accompanying figure. The equilibrium points are found by looking for the intersections. They are $(0, 0)$, and $(0.5, 1.2)$.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 1.2 - y & -x \\ y & -0.5 + x \end{pmatrix}.$$

At $(0, 0)$

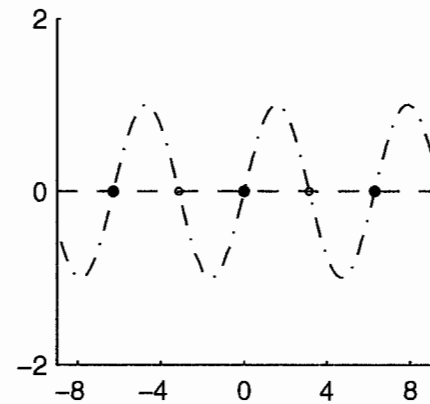
$$J(0, 0) = \begin{pmatrix} 1.2 & 0 \\ 0 & -0.5 \end{pmatrix}$$

has eigenvalues 1.2 and -0.5 , so $(0, 0)$ is a saddle. At $(0.5, 1.2)$

$$J(0.5, 1.2) = \begin{pmatrix} 0 & -0.5 \\ 1.2 & 0 \end{pmatrix}$$

has determinant $D = 0.6$, and trace $T = 0$, so $(0.5, 1.2)$ is a linear center. Since a center is not generic, we can not classify the equilibrium point.

7. The x -nullcline consists of the line $y = 0$. It is shown dashed in the accompanying figure. The y -nullcline consists of the curve $y = -\sin x$. It is shown dot-dashed in the accompanying figure. The equilibrium points are where $0 = -\sin x$. Hence $x = k\pi$, where k is any integer. The equilibrium points are $(k\pi, 0)$, where k is any integer.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -\cos x & -1 \end{pmatrix}.$$

If k is even,

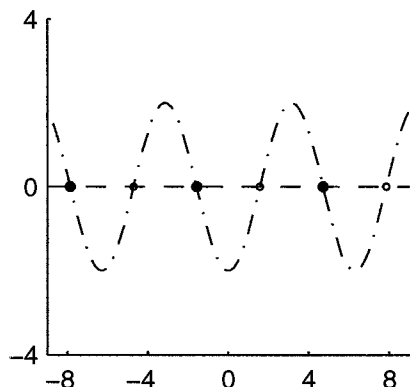
$$J(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

has determinant $D = 1$, trace $T = -1$, and discriminant $T^2 - 4D = -3$, so $(k\pi, 0)$ is a spiral sink. These are shown by solid dots in the figure. If k is odd,

$$J(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

has determinant $D = -1$, so $(k\pi, 0)$ is a saddle. These are shown by open dots in the figure.

8. The x -nullcline consists of the line $y = 0$. It is shown dashed in the accompanying figure. The y -nullcline consists of the curve $y = -2\cos x$. It is shown dot-dashed in the accompanying figure. The equilibrium points are where $0 = -2\cos x$. Hence $x = \pi/2 + k\pi$, where k is any integer. The equilibrium points are $(\pi/2 + k\pi, 0)$, where k is any integer.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ \sin x & -0.5 \end{pmatrix}.$$

If k is even,

$$J(\pi/2 + k\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -0.5 \end{pmatrix}$$

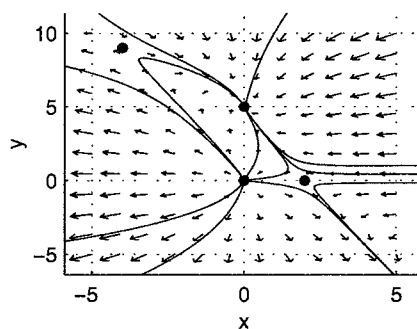
has determinant $D = -1$, so $(\pi/2 + k\pi, 0)$ is a saddle. These are shown by open dots in the figure. If k is odd,

$$J(\pi/2 + k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -0.5 \end{pmatrix}$$

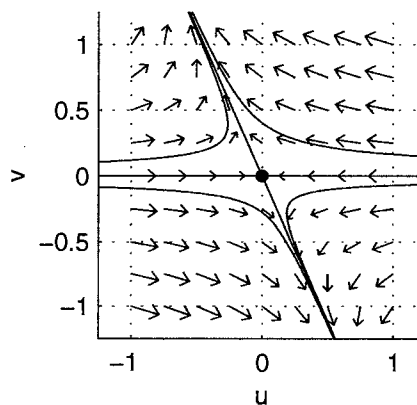
has determinant $D = 1$, trace $T = -0.5$, and discriminant $T^2 - 4D = -3.75$, so $(\pi/2 + k\pi, 0)$ is a spiral sink. These are shown by solid dots in the figure.

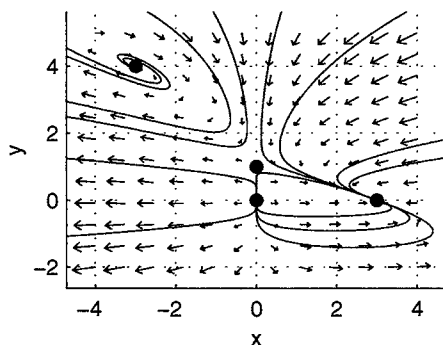
9. Near the point $(2, 0)$, the nonlinear system

10. Near the point $(3, 0)$, the nonlinear system

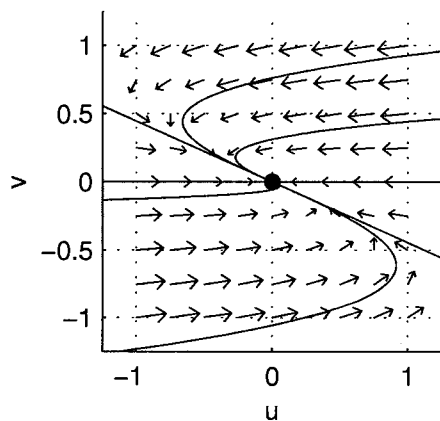


has linearization:

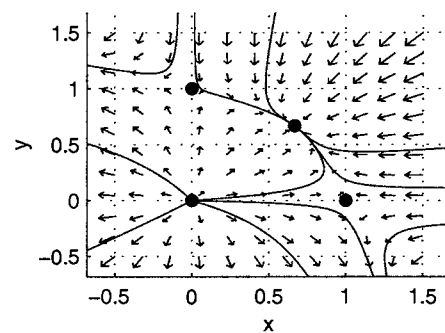




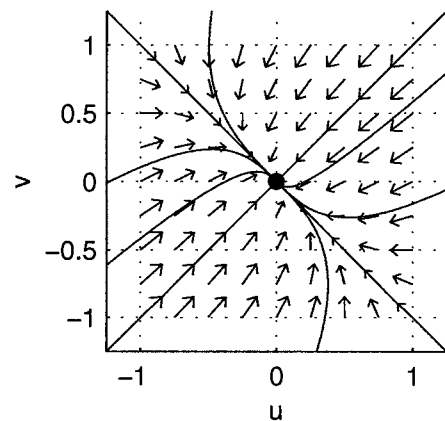
has linearization:



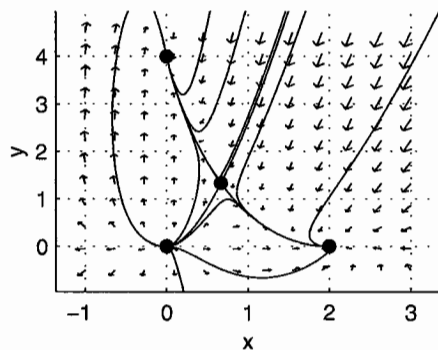
11. Near the point $(2/3, 2/3)$, the nonlinear system



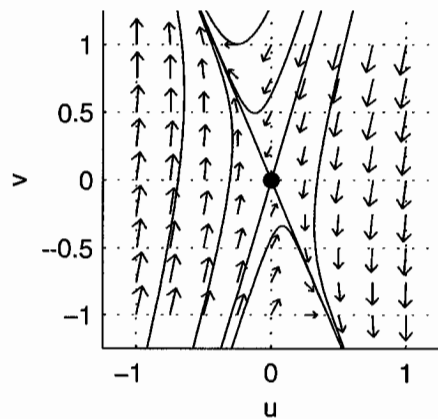
has linearization:



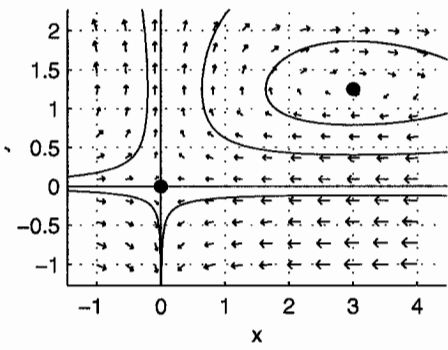
12. Near the point $(2/3, 4/3)$, the nonlinear system



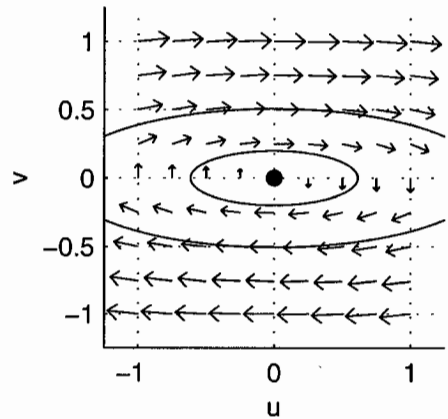
has linearization:



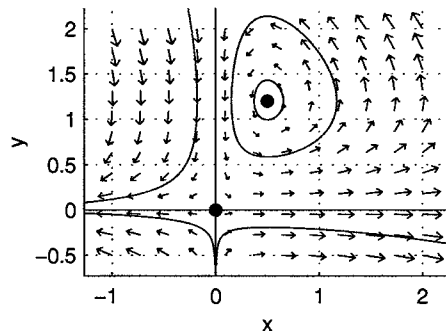
13. Near the point $(3, 5/4)$, the nonlinear system



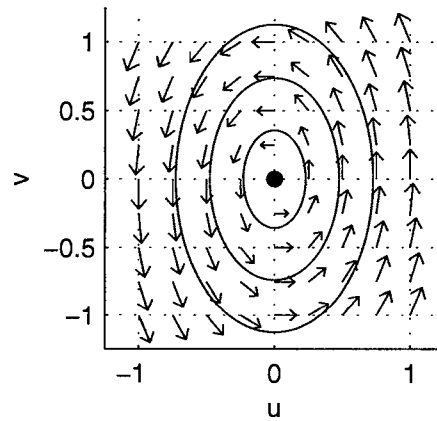
has linearization:



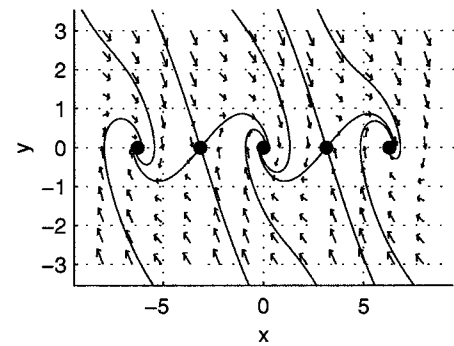
14. Near the point $(1/2, 6/5)$, the nonlinear system



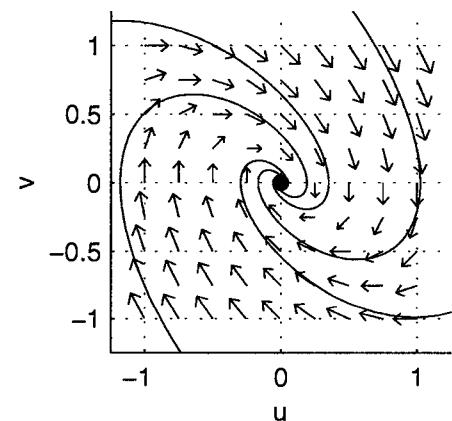
has linearization:



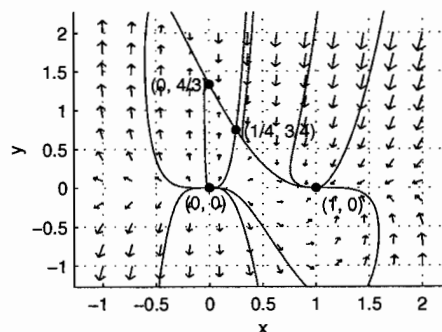
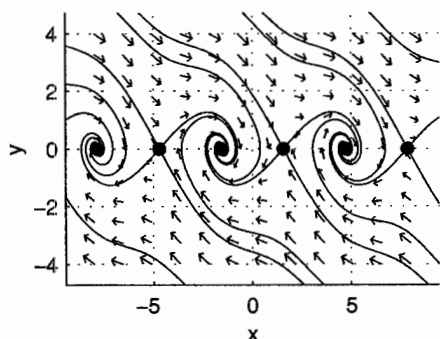
15. Near the point $(2\pi, 0)$, the nonlinear system



has linearization:

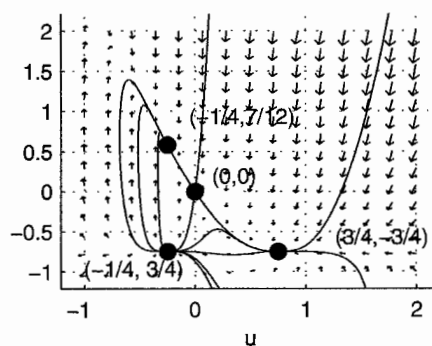
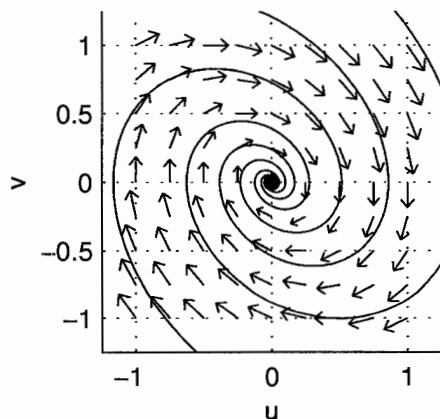


16. Near the point $(-\pi/2, 0)$, the nonlinear system



(b) Phase portrait and equilibrium points.

has linearization:



We have

$$\begin{aligned}
 u' &= (x - 1/4)' \\
 &= x' \\
 &= (1 - x - y)x \\
 &= [1 - (1/4 + u) - (3/4 + v)](1/4 + u) \\
 &= -(u + v)(1/4 + u) \\
 &= -u/4 - v/4 - u^2 - uv.
 \end{aligned}$$

17. (a) Phase portrait and equilibrium points.

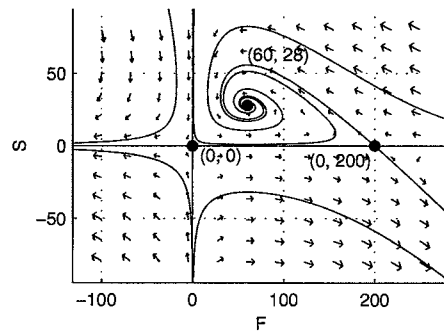
and

$$\begin{aligned}
 v' &= (y - 3/4)' \\
 &= y' \\
 &= (4 - 7x - 3y)y \\
 &= [4 - 7(1/4 + u) \\
 &\quad - 3(3/4 + v)](3/4 + v) \\
 &= -(7u + 3v)(3/4 + v) \\
 &= -21u/4 - 9v/4 - 7uv - 3v^2
 \end{aligned}$$

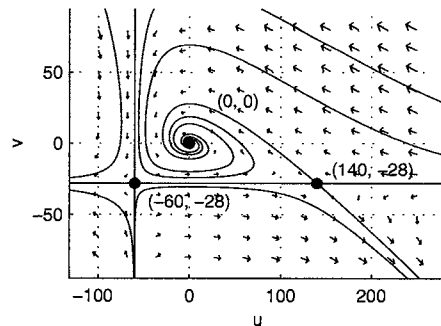
The effect is a translation of the figure, but the essential features are the same.

- (c) If we throw away the terms having degree 2 or more we get the linearization.

18. (a) Phase portrait and equilibrium points.



- (b) Phase portrait and equilibrium points.



We have

$$\begin{aligned}
 u' &= F' \\
 &= [0.4 - 0.002(60 + u) \\
 &\quad - 0.01(28 + v)](60 + u) \\
 &= -(0.002u + 0.01v)(60 + u) \\
 &= -0.12u - 0.6v - 0.002u^2 - 0.01uv
 \end{aligned}$$

and

$$\begin{aligned}
 v' &= S' \\
 &= [-0.3 + 0.005(60 + u)](28 + v) \\
 &= 0.005u(28 + v) \\
 &= 0.14u + 0.005uv
 \end{aligned}$$

The effect is a translation of the figure, but the essential features are the same.

- (c) If we throw away the terms of degree 2 or more we get the linearization.

19. (a) Differentiating both sides of $r^2 = x^2 + y^2$ we get

$$2r \frac{dr}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.$$

Dividing by 2 we get the first result. Next differentiating both sides of $\tan \theta = y/x$ we get

$$\sec^2 \theta \frac{d\theta}{dt} = \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) / x^2.$$

Since $\sec \theta = r/x$, multiplying by x^2/r^2 yields the second result.

- (b) First we have

$$\begin{aligned}
 r \frac{dr}{dt} &= x[y + \alpha x(x^2 + y^2)] \\
 &\quad + y[-x + \alpha y(x^2 + y^2)] \\
 &= \alpha r^4.
 \end{aligned}$$

Dividing by r we get the result. Next we have

$$\begin{aligned}
 r^2 \frac{d\theta}{dt} &= x[-x + \alpha y(x^2 + y^2)] \\
 &\quad - y[y + \alpha x(x^2 + y^2)] \\
 &= -x^2 - y^2.
 \end{aligned}$$

Dividing by r^2 we get the result.

- (c) Integrating we get $\theta(t) = -t + C$. Hence θ is decreasing at a uniform rate. Hence the solution curve is spiraling clockwise. Since $r' = \alpha r^3$, we see that if $\alpha < 0$, r is decreasing, and the curve is spiraling into the origin. If $\alpha > 0$, r is increasing and the curve is spiraling out of the origin.

20. (a) The Jacobian is

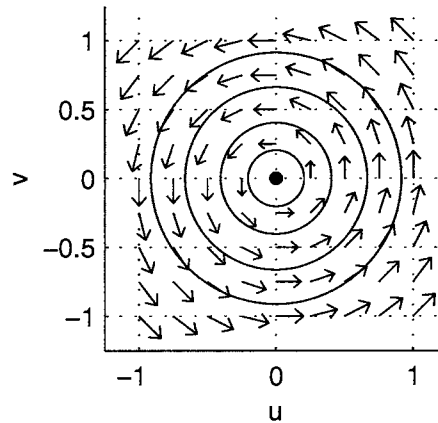
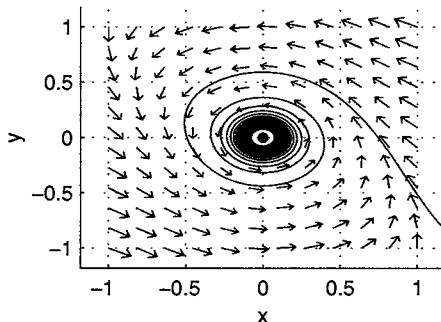
$$J = \begin{pmatrix} -3x^2 & -1 \\ 1 & 0 \end{pmatrix}.$$

At the origin, we have

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The determinant is 1 and the trace is 0, so this is a linear center.

- (b) The first figure is the original nonlinear system and the second is its linearization near $(0, 0)$.



- (c)

$$r \frac{dr}{dt} = x(-y - x^3) + yx = -x^4$$

- (d)

$$r^2 \frac{d\theta}{dt} = xx - y(-y - x^3) = r^2 + x^3y.$$

Dividing by r^2 we get the result. Using polar coordinates,

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta,$$

then we can write

$$\frac{x^3y}{r^2} = \frac{(r^3 \cos^3 \theta)(r \sin \theta)}{r^2} = r^2 \cos^3 \theta \sin \theta.$$

Note that $|\cos \theta| \leq 1$ and $|\sin \theta| \leq 1$ for all θ , so

$$|r^2 \cos^3 \theta \sin \theta| \leq |r^2|.$$

Hence, as $r \rightarrow 0$, $x^3y/r^2 \rightarrow 0$.

- (e) Since $dr/dt = -x^4/r \leq 0$, r is nonincreasing, and it is decreasing except where $x = 0$. Since $|x| \leq r$ and $|y| \leq r$, we have $d\theta/dt = 1 + x^3y/r^2 > 1 - r^4/r^2 = 1 - r^2$. Thus if $r < 1$, θ is strictly increasing. Thus at least when $r < 1$ the solution curve is spiraling counterclockwise and approaching the origin.

21. We have the model

$$\begin{aligned}x_1' &= (a_1 - b_1x_1 + c_1x_2)x_1 \\x_2' &= (a_2 - b_2x_2 + c_2x_1)x_2\end{aligned}$$

where $a_1 < 0$, $b_1 = 0$, $c_1 > 0$, $a_2 > 0$, $b_2 > 0$, and $c_2 > 0$.

22. We have the model

$$\begin{aligned}x_1' &= (a_1 - b_1x_1 + c_1x_2)x_1 \\x_2' &= (a_2 - b_2x_2 + c_2x_1)x_2\end{aligned}$$

where $a_1 < 0$, $b_1 = 0$, $c_1 > 0$, $a_2 > 0$, $b_2 > 0$, and $c_2 < 0$.

23. We have the model

$$\begin{aligned}x_1' &= (a_1 - b_1x_1 + c_1x_2)x_1 \\x_2' &= (a_2 - b_2x_2 + c_2x_1)x_2\end{aligned}$$

where $a_1 < 0$, $b_1 = 0$, $c_1 > 0$, $a_2 < 0$, $b_2 = 0$, and $c_2 > 0$.

24. We have the model

$$\begin{aligned}x_1' &= (a_1 - b_1x_1 + c_1x_2)x_1 \\x_2' &= (a_2 - b_2x_2 + c_2x_1)x_2\end{aligned}$$

where $a_1 > 0$, $b_1 > 0$, $c_1 > 0$, $a_2 > 0$, $b_2 > 0$, and $c_2 < 0$.

25. We have the model

$$\begin{aligned}x_1' &= (a_1 - b_1x_1 + c_1x_2)x_1 \\x_2' &= (a_2 - b_2x_2 + c_2x_1)x_2\end{aligned}$$

where $a_1 > 0$, $b_1 > 0$, $c_1 > 0$, $a_2 > 0$, $b_2 > 0$, and $c_2 > 0$.

26.

$$\begin{aligned}x_1' &= (a_1 - b_1x_1 - c_1x_2)x_1 \\x_2' &= (a_2 - b_2x_2 - c_2x_3 + d_2x_1)x_2 \\x_3' &= (-a_3 + d_3x_2)x_3\end{aligned}$$

where all of the constants are positive.

27. The model is

$$\begin{aligned}x_1' &= (a_1 + b_{11}x_1 + b_{12}x_2 + b_{13}x_3)x_1 \\x_2' &= (a_2 + b_{21}x_1 + b_{22}x_2 + b_{23}x_3)x_2 \\x_3' &= (a_3 + b_{31}x_1 + b_{32}x_2 + b_{33}x_3)x_3.\end{aligned}$$

The constant a_j is the reproductive rate of the population x_j in the absence of the others. The constant b_{jj} is negative if there is a logistic limit, and zero if not. For $i \neq j$ the constant b_{ij} measures the effect of x_j on the reproduction of x_i . If $b_{ij} > 0$ the effect is to increase the reproductive rate, and if $b_{ij} < 0$ the reproductive rate is diminished.

————— × —————

Section 10.2. Long-Term Behavior of Solutions

1. The equilibrium points are $(0, 0)$, $(1, 1)$, and $(-1, 1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 1 - y & -x \\ 2x & -1 \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$(0, 0)$ is a saddle point. Since

$$J(1, 1) = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$$

has determinant $D = 2$, trace $T = -1$, and discriminant $T^2 - 4D < 0$, $(1, 1)$ is a spiral sink. $J(-1, 1)$ has the same trace and determinant so it is also a spiral sink.

2. The equilibrium points are $(0, 0)$ and $(3, 2)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 2 - y & -x \\ y & -3 + x \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix},$$

the origin is a saddle. Since

$$J(3, 2) = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}$$

has determinant $D = 6 > 0$ and trace $T = 0$, $(3, 2)$ is a linear center. It is not possible to classify this equilibrium point for the nonlinear system.

3. The equilibrium points are $(0, 0)$ and $(-5, 1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 1-y & 9(1-y) - (x+9y) \\ -1 & -5 \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$$

has determinant $D = 4$, trace $T = -4$, and discriminant $T^2 - 4D = 0$, the origin is a nongeneric sink. Since

$$J(-5, 1) = \begin{pmatrix} 0 & -4 \\ -1 & -5 \end{pmatrix}$$

has determinant $D = -4 < 0$, $(-5, 1)$ is a saddle.

4. The equilibrium points are $(0, 0)$, $(1, -1)$, and $(-1, 1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 1 & 1 \\ -2xy & 1 - x^2 \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$(0, 0)$ is a nongeneric source and the equilibrium point cannot be classified. Since

$$J(1, -1) = J(-1, 1) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

has determinant $D = -2$, both $(1, -1)$ and $(-1, 1)$ are saddles.

5. The equilibrium points are $(0, 0, 0)$ and $(2/3, 4/9, 2/9)$. The Jacobian is

$$J(x, y, z) = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ 2x & 0 & -2 \end{pmatrix}.$$

Since

$$J(0, 0, 0) = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{pmatrix},$$

the origin is a sink (the eigenvalues are -1 , -1 , and -2). Since

$$J(2/3, 4/9, 2/9) = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ 4/3 & 0 & -2 \end{pmatrix}$$

the characteristic polynomial is $p(\lambda) = (\lambda + 1)(\lambda^2 + 3\lambda - 2)$, and the eigenvalues are -1 , $(-3 \pm \sqrt{17})/2$. Since $(-3 + \sqrt{17})/2 = 0.5616 > 0$, $(2/3, 4/9, 2/9)$ is unstable.

6. The equilibrium points are $(0, 0, 0)$ and $(2, 2, 2)$. The Jacobian is

$$J(x, y, z) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2y & -2 \end{pmatrix}.$$

Since

$$J(0, 0, 0) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

we see that the origin is a sink (the eigenvalues are -1 , -1 , and -2). Since

$$J(2, 2, 2) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 4 & -2 \end{pmatrix},$$

the characteristic polynomial is $p(\lambda) = -(\lambda + 1)(\lambda^2 + 3\lambda - 2)$, and the eigenvalues are -1 , $(-3 \pm \sqrt{17})/2$. Since $(-3 + \sqrt{17})/2 = 0.5616 > 0$, $(2, 2, 2)$ is unstable.

7. The equilibrium points are $(0, 0, 0)$ and $(1, 1, 1)$. The Jacobian is

$$J(x, y, z) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2x & 2y & -2 \end{pmatrix}.$$

Since

$$J(0, 0, 0) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -2 \end{pmatrix},$$

the characteristic polynomial is $p(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda - 1)$. Since 1 is an eigenvalue, $(0, 0, 0)$ is unstable. Since

$$J(1, 1, 1) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -2 \end{pmatrix},$$

the characteristic polynomial is $p(\lambda) = (\lambda^3 + 2\lambda^2 + 3\lambda + 2)$. The roots are $\lambda = -1$ and $\lambda = (-1 \pm \sqrt{7}i)/2$. Since all have negative real part, $(1, 1, 1)$ is asymptotically stable.

8. The equilibrium points are $(0, 0, 0)$ and $(-2, 2, 2)$. The Jacobian is

$$J(x, y, z) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 2y & -2 \end{pmatrix}.$$

Since

$$J(0, 0, 0) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix},$$

the characteristic polynomial is $p(\lambda) = (\lambda + 2)(\lambda^2 - 1)$. Since $\lambda = 1$ is a root, $(0, 0, 0)$ is unstable. Since,

$$J(-2, 2, 2) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 4 & -2 \end{pmatrix},$$

the characteristic polynomial is $p(\lambda) = (\lambda - 1)(\lambda^2 + 3\lambda - 2)$. since $\lambda = 1$ is a root, $(-2, 2, 2)$ is unstable.

9. The Jacobian at the origin is

$$J = \begin{pmatrix} -1 & 4 & 4 \\ 0 & -3 & -2 \\ 0 & 4 & 3 \end{pmatrix}.$$

The characteristic polynomial is $p(\lambda) = (\lambda + 1)(\lambda^2 - 1)$. Since $\lambda = 1$ is a root, the origin is unstable.

10. The Jacobian at the origin is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ -2 & -2 & 0 \end{pmatrix}.$$

The characteristic polynomial is $p(\lambda) = \lambda(\lambda + 1)^2$. There are two negative eigenvalues, but since 0 is also an eigenvalue, it is not possible to classify the equilibrium point using the Jacobian.

11. The Jacobian at the origin is

$$J = \begin{pmatrix} -5 & 6 & 0 \\ -3 & 1 & 1 \\ -3 & 2 & -1 \end{pmatrix}.$$

Using a computer we find that the eigenvalues are -2.3971 and $-1.3015 \pm 2.6583i$. Since all have negative real part, the origin is asymptotically stable.

12. The Jacobian at the origin is

$$J = \begin{pmatrix} 1 & 6 & 6 \\ 0 & 1 & 0 \\ -3 & -15 & -5 \end{pmatrix}.$$

The characteristic equation $p(\lambda) = (\lambda - 1)(\lambda^2 + 4\lambda + 13)$ shows that $\lambda = 1$ is an eigenvalue. Hence the origin is unstable.

13. The Jacobian at the origin is

$$J = \begin{pmatrix} 6 & -4 & 0 \\ 12 & -8 & 0 \\ 9 & -5 & -3 \end{pmatrix}.$$

The characteristic polynomial is $p(\lambda) = \lambda(\lambda + 2)(\lambda + 3)$. Although there are two negative roots, 0 is also an eigenvalue, so it is not possible to classify the equilibrium point using the Jacobian.

14. The Jacobian at the origin is

$$J = \begin{pmatrix} -5 & -1 & 1 \\ 2 & -2 & -1 \\ 28 & 18 & 1 \end{pmatrix}.$$

Using a computer we find that the eigenvalues are 2.5311 , -5.5311 , and -3 . Since one of these is positive, the origin is unstable.

15. The Jacobian at the origin is

$$J = \begin{pmatrix} 1 & 3 & -3 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ -9 & -9 & 11 & -2 \end{pmatrix}.$$

The characteristic polynomial is $p(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda^2 + \lambda + 7)$. Since all roots have negative real part, the origin is asymptotically stable.

16. The Jacobian at the origin is

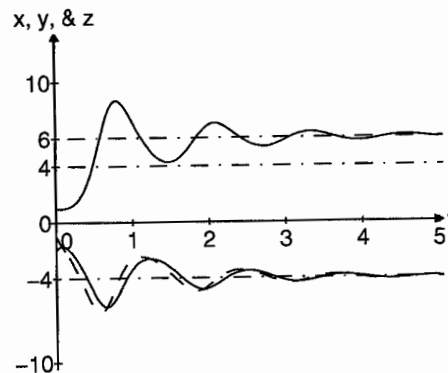
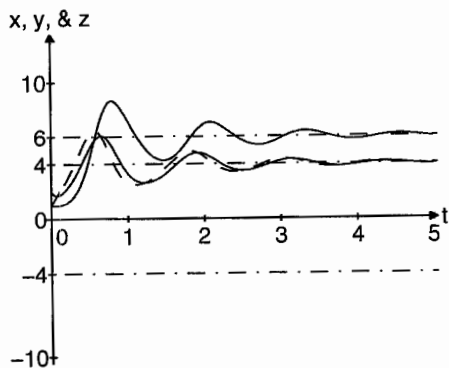
$$J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & -2 \\ 6 & 9 & -2 & -18 \\ 1 & 0 & 0 & -2 \end{pmatrix}.$$

The characteristic polynomial is $p(\lambda) = (\lambda+1)^2(\lambda+2)^2$. Since all of the roots are negative, the origin is asymptotically stable.

17. The equilibrium points are
- $\mathbf{0}$
- ,
- $\mathbf{c}^+ = (4, 4, 6)^T$
- , and
- $\mathbf{c}^- = (-4, -4, 6)^T$
- . Since
- $r > 1$
- we know that
- $\mathbf{0}$
- is unstable.

$$J(\mathbf{c}^\pm) = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & \mp 4 \\ \pm 4 & \pm 4 & -8/3 \end{pmatrix}.$$

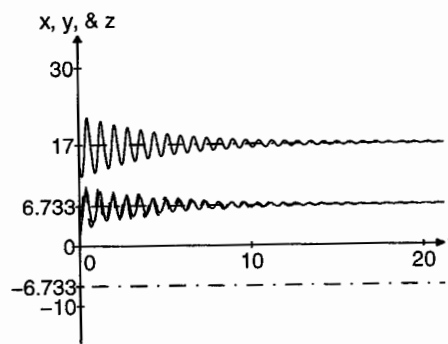
Both matrices have eigenvalues -12.1055 , and $-0.7806 \pm 5.0818i$. Hence $\mathbf{c}^+$ and $\mathbf{c}^-$ are asymptotically stable. In the figures below we computed and plotted the solutions starting at $(2, 1, 1)$ and $(-2, -1, 1)$. Notice how they converge to $\mathbf{c}^+$ and $\mathbf{c}^-$, respectively.

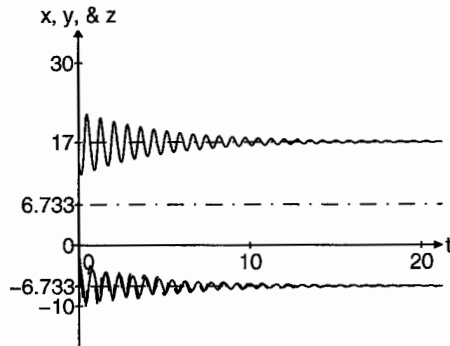


18. The equilibrium points are
- $\mathbf{0}$
- ,
- $\mathbf{c}^+ = (6.733, 6.733, 17)^T$
- , and the third equilibrium point is
- $\mathbf{c}^- = (-6.733, -6.733, 17)^T$
- . Since
- $r > 1$
- we know that
- $\mathbf{0}$
- is unstable.

$$J(\mathbf{c}^\pm) = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & \mp 6.733 \\ \pm 6.733 & \pm 6.733 & -8/3 \end{pmatrix}.$$

Both matrices have eigenvalues -13.2100 , and $-0.2283 \pm 8.215i$. Hence $\mathbf{c}^+$ and $\mathbf{c}^-$ are asymptotically stable. In the figures below we computed and plotted the solutions starting at $(1, 5, 14)^T$ and $(-1, -5, 14)^T$. Notice how they converge to $\mathbf{c}^+$ and $\mathbf{c}^-$, respectively.

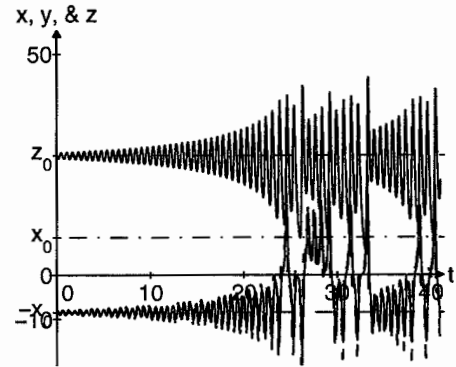
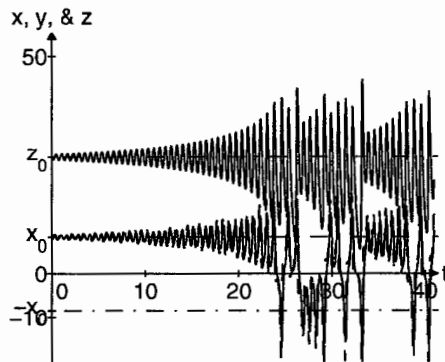




19. The equilibrium points are $\mathbf{0}$, $\mathbf{c}^+ = (x_0, x_0, z_0)^T$, and $\mathbf{c}^- = (x_0, x_0, z_0)^T$, where $x_0 = \sqrt{b(r-1)} \approx 8.4853$, and $z_0 = r - 1 = 27$. Since $r > 1$ we know that $\mathbf{0}$ is unstable.

$$J(\mathbf{c}^\pm) = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & \mp 8.4853 \\ \pm 8.4853 & \pm 8.4853 & -8/3 \end{pmatrix}.$$

Both matrices have eigenvalues -13.8546 , and $0.094 \pm 10.1945i$. Since the last two have positive real part, $\mathbf{c}^+$ and $\mathbf{c}^-$ are unstable. In the figures below we computed and plotted the solutions starting at $(8, 8, 27)^T$ and $(-8, -8, 27)^T$. Notice that these starting positions are very close to $\mathbf{c}^+$ and $\mathbf{c}^-$. The solutions move away and then oscillate around $\mathbf{c}^+$ and $\mathbf{c}^-$ in turn.



20. The three equilibrium solutions corresponding to the equilibrium points $\mathbf{0}$, $\mathbf{c}^+$, and $\mathbf{c}^-$ do not tend to the attractor.
21. (a) The system displays all of the interaction among the three species.
- (b) If all populations are positive, then we must have the three equations

$$\begin{aligned} 2x_2 &= 1 \\ -2x_1 + 4x_3 &= 0.5 \\ x_2 + 2x_3 &= 2 \end{aligned}$$

The unique solution is $\mathbf{x} = (5/4, 1/2, 3/4)^T$. The Jacobian J is:

$$\begin{pmatrix} -1 + 2x_2 & 2x_1 & 0 \\ -2x_2 & -1/2 - 2x_1 + 4x_3 & 4x_2 \\ 0 & -x_3 & 2 - x_2 - 4x_3 \end{pmatrix}$$

At the equilibrium point this is

$$J = \begin{pmatrix} 0 & 5/2 & 0 \\ -1 & 0 & 2 \\ 0 & -3/4 & -3/2 \end{pmatrix}.$$

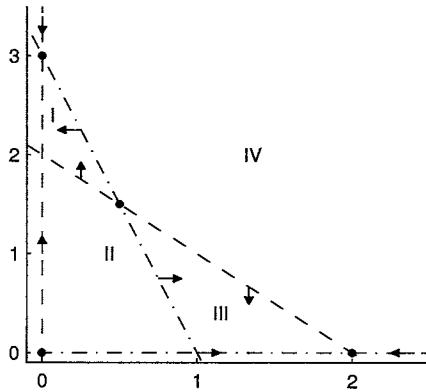
The eigenvalues are -1.0610 and $-0.2195 \pm 1.8671i$. Since all have negative real part, the equilibrium point is asymptotically stable.

Section 10.3. Invariant Sets and the Use of Nullclines

1. The x -axis is defined by $y = 0$. Notice that if $x(t)$ solves the logistic equation $x' = (2 - x)x$ and $y(t) = 0$ then x and y are a solution to the system in the exercise. Every point in the x -axis is contained in such a solution curve, which stays in the x -axis. Hence the x -axis is invariant. Similarly the functions $x(t) = 0$ and $y(t)$ defined by $y' = (3 - y)y$ are solutions to the system in the exercise, and their solution curves exhaust the y -axis, making it invariant. A solution curve starting in one of the four quadrants must stay in that quadrant, because to get out it has to cross one of the axes. It cannot do so because the unique solution curve through any point in the axis must be entirely contained in the axis.
2. The x -axis is defined by $y = 0$. Notice that if $x(t)$ solves the logistic equation $x' = 4x(1 - x)$ and $y(t) = 0$ then x and y are a solution to the system in the exercise. Every point in the x -axis is contained in such a solution curve, which stays in the x -axis. Hence the x -axis is invariant. Similarly the functions $x(t) = 0$ and $y(t)$ defined by $y' = (3 - y)y$ are solutions to the system in the exercise, and their solution curves exhaust the y -axis, making it invariant. A solution curve starting in one of the four quadrants must stay in that quadrant, because to get out it has to cross one of the axes. It cannot do so because the unique solution curve through any point in the axis must be entirely contained in the axis.
3. The x -axis is defined by $y = 0$. Notice that if $x(t)$ solves the logistic equation $x' = (1 - x)x$ and $y(t) = 0$ then x and y are a solution to the system in the exercise. Every point in the x -axis is contained in such a solution curve, which stays in the x -axis. Hence the x -axis is invariant. Similarly the functions $x(t) = 0$ and $y(t)$ defined by $y' = (6 - 3y)y$ are solutions to the system in the exercise, and their solution curves exhaust the y -axis, making it invariant. A solution curve starting in one of the four quadrants must stay in that quadrant, because to get out it has to cross one of the axes. It cannot do so because the unique solution curve through any point in the axis must be entirely contained in the axis.
4. The x -axis is defined by $y = 0$. Notice that if $x(t)$ solves the logistic equation $x' = 4x(3 - x)$ and $y(t) = 0$ then x and y are a solution to the system in the exercise. Every point in the x -axis is contained in such a solution curve, which stays in the x -axis. Hence the x -axis is invariant. Similarly the functions $x(t) = 0$ and $y(t)$ defined by $y' = (1 - y)y$ are solutions to the system in the exercise, and their solution curves exhaust the y -axis, making it invariant. A solution curve starting in one of the four quadrants must stay in that quadrant, because to get out it has to cross one of the axes. It cannot do so because the unique solution curve through any point in the axis must be entirely contained in the axis.
5. The axes are invariant, so no solution curves can cross them. Along the line $x = 3$ we have $x' = (2 - x - y)x = -3(1 + y) < 0$, so x is decreasing. Thus the solution curve is moving into S . Along the line $y = 3$ we have $y' = (3 - 3x - y)y = -9x < 0$, so y is decreasing. Again the solution curves are moving into S . Consequently all solution curves cross the boundary of S going into S . Hence S is invariant.
6. The axes are invariant, so no solution curves can cross them. Along the line $x = 3$ we have $x' = 4x(1 - x) - xy = -24 - 3y < 0$, so x is decreasing. Thus the solution curve is moving into S . Along the line $y = 4$ we have $y' = y(3 - y) - xy = -4 - 4x < 0$, so y is decreasing. Again the solution curves are moving into S . Consequently all solution curves cross the boundary of S going into S . Hence S is invariant.
7. The axes are invariant, so no solution curves can cross them. Along the line $x = 4$ we have $x' = (1 - x - y)x = -4(3 + y) < 0$, so x is decreasing. Thus the solution curve is moving into S . Along the line $y = 3$ we have $y' = (6 - 2x - 3y)y = -3(3 + 2x) < 0$, so y is decreasing. Again the solution curves are moving into S . Consequently no solution curves cross the boundary of S going out of S . Hence S is invariant.
8. The axes are invariant, so no solution curves can cross them. Along the line $x = 4$ we have $x' = 4x(3 - x) - 3xy = -16 - 12y < 0$, so x is decreasing. Thus the solution curve is moving into S . Along the line $y = 5$ we have $y' = y(1 - y) - xy =$

$-20 - 5x < 0$, so y is decreasing. Again the solution curves are moving into S . Consequently all solution curves cross the boundary of S going into S . Hence S is invariant.

9. The x -nullcline is the union of the two lines $x = 0$ and $x + y = 2$, shown dashed.



The y -nullcline is the union of the two lines $y = 0$ and $3x + y = 3$, shown dot-dashed. These intersect in the four equilibrium points $(0, 0)$, $(2, 0)$, $(0, 3)$, and $(1/2, 3/2)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 2 - 2x - y & -x \\ -3y & 3 - 3x - 2y \end{pmatrix}.$$

At $(0, 0)$,

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

has eigenvalues 2 and 3, so the origin is a nodal source. At $(2, 0)$,

$$J(2, 0) = \begin{pmatrix} -2 & -2 \\ 0 & -3 \end{pmatrix},$$

has eigenvalues -2 and -3 , so $(2, 0)$ is a nodal sink. At $(0, 3)$,

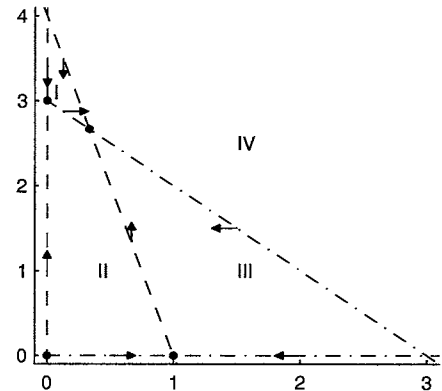
$$J(0, 3) = \begin{pmatrix} -1 & 0 \\ -9 & -3 \end{pmatrix},$$

has eigenvalues -1 and -3 , so $(0, 3)$ is also a nodal sink. Finally,

$$J(1/2, 3/2) = \begin{pmatrix} -1/2 & -1/2 \\ -9/2 & -3/2 \end{pmatrix}$$

has determinant $D = -3/2 < 0$, so $(1/2, 3/2)$ is a saddle. The flow of the solutions along the nullclines is shown by the arrows. This information shows that regions I and III are invariant. The solution curves flow from region II into regions I and III, except the one stable solution curve for the saddle at $(1/2, 3/2)$. The solution curves in region IV can flow directly to the sinks, or into regions I and III, with the exception of the one stable solution curve for the saddle at $(1/2, 3/2)$. Finally solution curves in the invariant region I flow to the sink at $(0, 3)$ and the solution curves in the invariant region III flow to the sink at $(2, 0)$.

10. The x -nullcline is the union of the two lines $x = 0$ and $4x + y = 4$, shown dashed.



The y -nullcline is the union of the two lines $y = 0$ and $x + y = 3$, shown dot-dashed. These intersect in the four equilibrium points $(0, 0)$, $(1, 0)$, $(0, 3)$, and $(1/3, 8/3)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 4 - 8x - y & -x \\ -y & 3 - x - 2y \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$$

has eigenvalues 4 and 3, the origin is a nodal source. Since

$$J(1, 0) = \begin{pmatrix} -4 & -1 \\ 0 & 2 \end{pmatrix}$$

has eigenvalues -4 and 2 , $(1, 0)$ is a saddle. Since

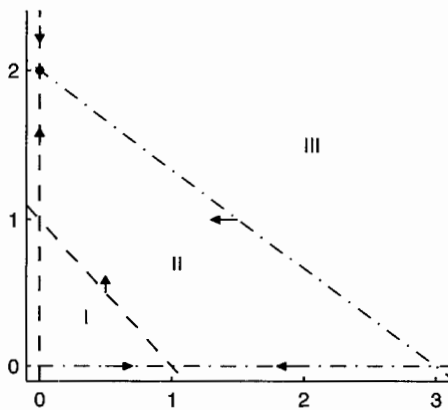
$$J(0, 3) = \begin{pmatrix} 1 & 0 \\ -3 & -3 \end{pmatrix}$$

has eigenvalues 1 and -3 , $(0, 3)$ is also a saddle. Finally,

$$J(1/3, 8/3) = \begin{pmatrix} -4/3 & -1/3 \\ -8/3 & -8/3 \end{pmatrix}$$

has determinant $D = 8/3$, trace $T = -4$, and discriminant $T^2 - 4D > 0$ so $(1/3, 8/3)$ is a nodal sink. The flow of the solutions along the nullclines is shown by the arrows. This information shows that regions I and III are invariant. The solution curves flow from region II into regions I and III. The solution curves in region IV also flow into regions I and III. Finally solution curves in the invariant regions I and III flow to the sink at $(1/3, 8/3)$. Thus all solutions in the positive quadrant flow to the sink at $(1/3, 8/3)$.

11. The x -nullcline is the union of the two lines $x = 0$ and $x + y = 1$, shown dashed.



The y -nullcline is the union of the two lines $y = 0$ and $2x + 3y = 6$, shown dot-dashed. These intersect in the four equilibrium points $(0, 0)$, $(1, 0)$,

$(0, 2)$, and $(-3, 4)$. Since we are only interested in the positive quadrant, the last of these is not shown. The Jacobian is

$$J(x, y) = \begin{pmatrix} 1 - 2x - y & -x \\ -2y & 6 - 2x - 6y \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

has eigenvalues 1 and 6 , the origin is a nodal source. Since

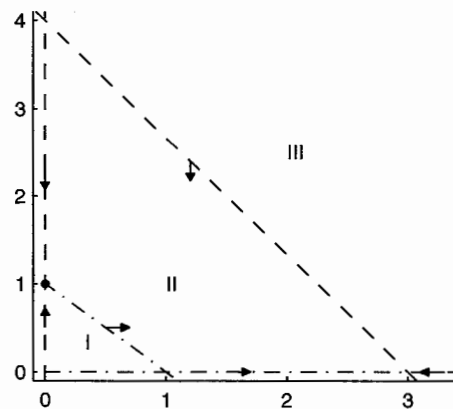
$$J(1, 0) = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$$

has eigenvalues -1 and 4 , $(1, 0)$ is a saddle. Since

$$J(0, 2) = \begin{pmatrix} -1 & 0 \\ -4 & -2 \end{pmatrix}$$

has eigenvalues -1 and -2 , $(0, 2)$ is a nodal sink. The flow of the solutions along the nullclines is shown by the arrows. This information shows that the solution curves flow from regions I and III into region II. Region II is invariant, and all solution curves flow to the sink at $(0, 2)$. Thus all solution curves in the positive quadrant flow to the sink at $(0, 2)$.

12. The x -nullcline is the union of the two lines $x = 0$ and $4x + 3y = 12$, shown dashed.



The y -nullcline is the union of the two lines $y = 0$ and $x + y = 1$, shown dot-dashed. These intersect in the four equilibrium points $(0, 0)$, $(3, 0)$, $(0, 1)$, and $(9, -8)$. Since we are only interested in the positive quadrant, the last of these is not shown. The Jacobian is

$$J(x, y) = \begin{pmatrix} 12 - 8x - 3y & -3x \\ -y & 1 - x - 2y \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 12 & 0 \\ 0 & 1 \end{pmatrix}$$

has eigenvalues 12 and 1, the origin is a nodal source.

Since

$$J(3, 0) = \begin{pmatrix} -12 & -9 \\ 0 & -2 \end{pmatrix}$$

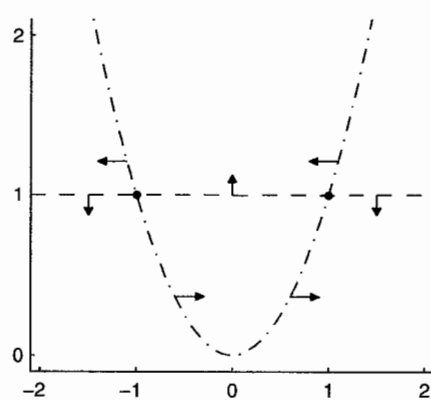
has eigenvalues -12 and -2 , $(3, 0)$ is a nodal sink.

Since

$$J(0, 1) = \begin{pmatrix} 9 & 0 \\ -1 & -1 \end{pmatrix}$$

has eigenvalues 9 and -1 , $(0, 1)$ is a saddle. The flow of the solutions along the nullclines is shown by the arrows. This information shows that the solution curves flow from regions I and III into region II. Region II is invariant, and all solution curves flow to the sink at $(3, 0)$. Thus all solution curves in the positive quadrant flow to the sink at $(3, 0)$.

13. The x -nullcline is the line $y = 1$ shown dashed. The y -nullcline is the parabola $y = x^2$ shown dot-dashed.



These intersect in the equilibrium points $(\pm 1, 1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 0 & -1 \\ -2x & 1 \end{pmatrix}.$$

Since

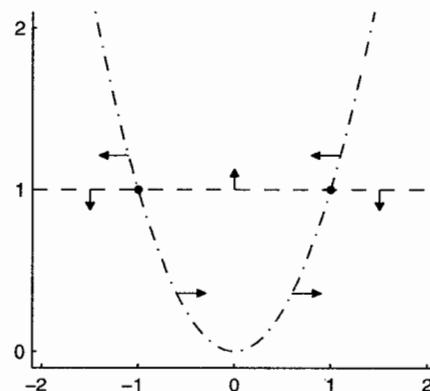
$$J(1, 1) = \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$$

has determinant $D = -2$, so $(1, 1)$ is a saddle. Since

$$J(-1, 1) = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$$

has determinant $D = 2$, trace $T = 1$, and discriminant $T^2 - 4D < 0$, $(-1, 1)$ is a spiral source. The flow of the solutions along the nullclines is shown by the arrows.

14. The x -nullcline is the line $y = x$ shown dashed. The y -nullcline is the parabola $x = y^2$ shown dot-dashed.



These intersect in two equilibrium points $(0, 0)$ and $(1, 1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} -1 & 1 \\ 1 & -2y \end{pmatrix}.$$

Since

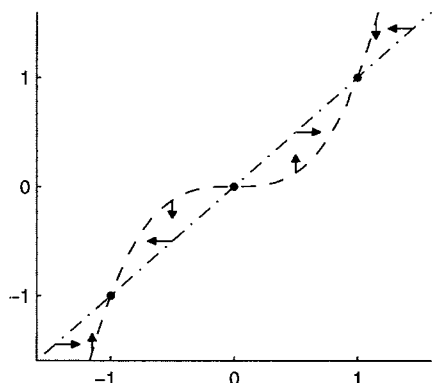
$$J(0, 0) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

has determinant $D = -1$, the origin is a saddle. Since

$$J(1, 1) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

has determinant $D = 1$, trace $T = -3$ and discriminant $T^2 - 4D > 0$, $(1, 1)$ is a nodal sink. The flow of the solutions along the nullclines is shown by the arrows.

15. The x -nullcline is the cubic curve $y = x^3$ shown dashed. The y -nullcline is the line $x = y$ shown dot-dashed.



These intersect in three equilibrium points $(0, 0)$, $(1, 1)$ and $(-1, -1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} -3x^2 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

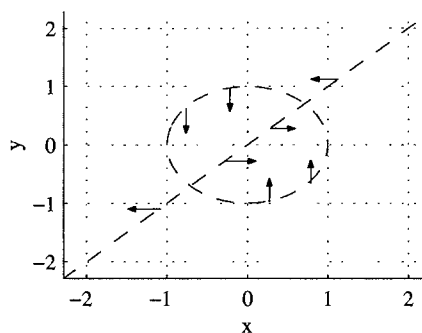
has determinant $D = -1$, the origin is a saddle. At the other two equilibrium points

$$J = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix},$$

which has determinant $D = 2$, trace $T = -4$, and discriminant $T^2 - 4D > 0$. Hence $(1, 1)$ and $(-1, -1)$ are nodal sinks. The flow of the solutions

along the nullclines is shown by the arrows. The nullclines split all of $\mathbb{R}^2$ into six regions. Three of them are invariant. The flow of all solutions can be followed as they converge to one of the two sinks, except for the two stable solutions for the saddle point.

16. The x -nullcline is the circle $x^2 + y^2 = 1$ shown dashed. The y -nullcline is the line $y = x$ shown dot-dashed.



These intersect in two equilibrium points $(\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} -2x & -2y \\ 1 & -1 \end{pmatrix}.$$

Since

$$J(\sqrt{2}/2, \sqrt{2}/2) = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ 1 & -1 \end{pmatrix}$$

has trace $T = -1 - \sqrt{2}$, determinant $D = 2\sqrt{2}$, and discriminant $T^2 - 4D = 3 - 6\sqrt{2} < 0$, $(\sqrt{2}/2, \sqrt{2}/2)$ is a spiral sink. Since

$$J(-\sqrt{2}/2, -\sqrt{2}/2) = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 1 & -1 \end{pmatrix}$$

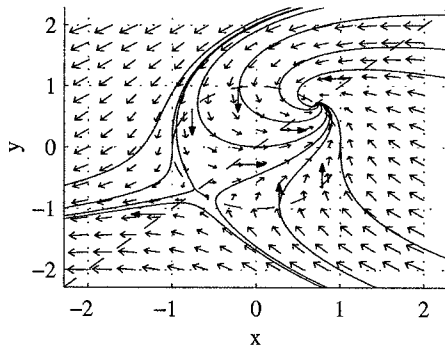
has determinant $D = -2\sqrt{2} < 0$, the equilibrium point at $(-\sqrt{2}/2, -\sqrt{2}/2)$ is a saddle. Testing at $(x, y) = (0, 0)$,

$$1 - x^2 - y^2 = 1 - 0^2 - 0^2 > 0.$$

This indicates that x' is positive within the boundary of the circle and negative without. Testing at $(x, y) = (1, 0)$,

$$x - y = 1 - 0 > 0.$$

This indicates that y' is positive below the line and negative above. This behavior is indicated with arrows placed on the nullclines in the figure above. The preceding analysis is verified in the phase portrait that follows.



17. (a) If

$$\begin{aligned} u' &= u(1 - u + av) \\ v' &= rv(1 - v + bu), \end{aligned}$$

the u -nullclines are

$$u = 0 \quad \text{and} \quad 1 - u + av = 0,$$

and the v -nullclines are

$$v = 0 \quad \text{and} \quad 1 - v + bu = 0.$$

The u -nullcline $u = 0$ will intersect the v -nullclines $v = 0$ and $1 - v + bu = 0$ at $(0, 0)$ and $(0, 1)$, respectively. The u -nullcline $1 - u + av = 0$ intersects the v -nullcline $v = 0$ at $(1, 0)$. What remains is to find where the u -nullcline $1 - u + av = 0$ meets the v -nullcline $1 - v + bu = 0$. Solving the first for u ,

$$u = 1 + av,$$

and substituting into the v -nullcline,

$$\begin{aligned} 1 - v + b(1 + av) &= 0 \\ v &= \frac{1 + b}{1 - ab} \end{aligned}$$

Substituting this result in $u = 1 + av$,

$$\begin{aligned} u &= 1 + \frac{a(1 + b)}{1 - ab} \\ u &= \frac{1 + a}{1 - ab}. \end{aligned}$$

(b) In the case that $ab > 1$, we have that $1 - ab < 0$. This implies that the equilibrium point

$$\tilde{u} = \frac{1 + a}{1 - ab} \quad \text{and} \quad \tilde{v} = \frac{1 + b}{1 - ab}$$

lies in the third quadrant and will be ignored. The Jacobian is

$$J(u, v) = \begin{pmatrix} 1 - 2u + av & au \\ brv & r - 2rv + bru \end{pmatrix}.$$

Thus,

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$$

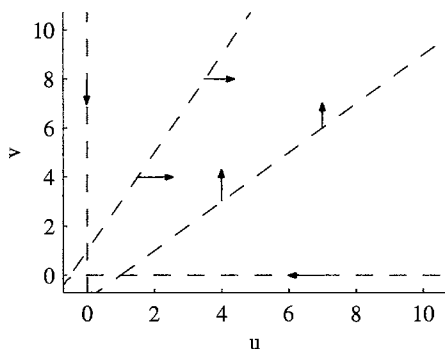
and the origin is a source ($\lambda_1 = 1$, $\lambda_2 = r$). At $(1, 0)$,

$$J(1, 0) = \begin{pmatrix} -1 & a \\ 0 & r(1 + b) \end{pmatrix}$$

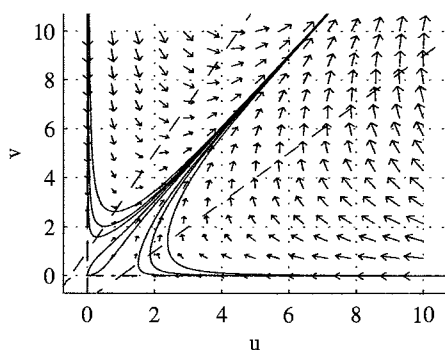
and $(1, 0)$ is a saddle ($\lambda_1 = -1$, $\lambda_2 = r(1 + b)$). Finally, at $(0, 1)$,

$$J(0, 1) = \begin{pmatrix} 1 + a & 0 \\ br & -r \end{pmatrix}$$

and $(0, 1)$ is also a saddle ($\lambda_1 = 1 + a$, $\lambda_2 = -r$). Because we restrict the analysis to the first quadrant, we can assume u and v are greater than zero. Thus, $u' = u(1 - u + av)$ is positive precisely when $1 - u + av$ is greater than zero, which occurs above the nullcline $1 - u + av = 0$. Thus, motion along the v -nullcline, $1 - v + bu = 0$, is strictly to the right. A similar analysis shows that motion across the u -nullcline, $1 - u + av = 0$ is strictly upward (at least in the first quadrant).



There is now enough information to draw the phase portrait.



Note that both populations grow without bound if we start with any two initial nonzero populations.

- (c) In the case that $ab < 1$, the Jacobian analysis at the points $(0, 0)$, $(1, 0)$, and $(0, 1)$ remains the same. However, this time the equilibrium point

$$\tilde{u} = \frac{1+a}{1-ab} \quad \text{and} \quad \tilde{v} = \frac{1+b}{1-ab}$$

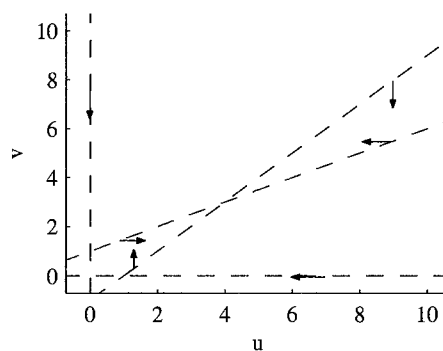
lies in the first quadrant. Noting that $1 - \tilde{u} + a\tilde{v} = 0$ and $1 - \tilde{v} + b\tilde{u} = 0$,

$$J(\tilde{u}, \tilde{v}) = \begin{pmatrix} 1 - 2\tilde{u} + a\tilde{v} & a\tilde{u} \\ br\tilde{v} & r(1 - 2\tilde{v} + b\tilde{u}) \end{pmatrix} = \begin{pmatrix} -\tilde{u} & a\tilde{u} \\ br\tilde{v} & -r\tilde{v} \end{pmatrix}.$$

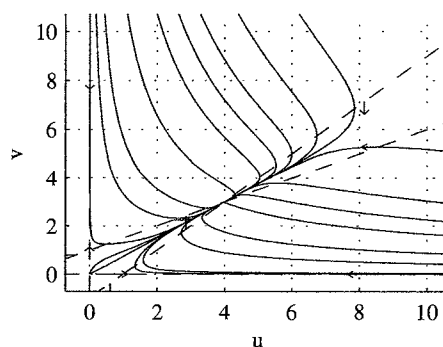
The trace $T = -\tilde{u} - r\tilde{v} < 0$. The determinant $D = r\tilde{u}\tilde{v}(1 - ab) > 0$, because of the stipulation that $ab < 1$. Further,

$$\begin{aligned} T^2 - 4D &= (-\tilde{u} - r\tilde{v})^2 - 4r\tilde{u}\tilde{v}(1 - ab) \\ &= (\tilde{u} - r\tilde{v})^2 + 4abr\tilde{u}\tilde{v} \\ &> 0. \end{aligned}$$

Hence, the equilibrium point is a nodal sink. As in part (b), $u' > 0$ when $1 - u + av > 0$, which occurs above and to the left of the u -nullcline $1 - u + av = 0$. In a similar manner, $v' = 0$ when $1 - v + bu > 0$, which occurs below the v -nullcline $1 - v + bu = 0$.



This information makes the phase portrait an easy task.



Note that any two positive initial conditions lead to a solution that flows into the sink at $(\tilde{u}, \tilde{v})$.

18. (a) If

$$\begin{aligned}\frac{dN}{dt} &= N \left(\alpha_1 \left(\frac{C}{1+C} \right) - 1 \right) \\ \frac{dC}{dt} &= - \left(\frac{C}{1+C} \right) N - C + \alpha_2\end{aligned}$$

then the N -nullclines are

$$N = 0 \quad \text{or} \quad C = \frac{1}{\alpha_1 - 1}.$$

The N -nullcline $N = 0$ intersects the C -nullcline,

$$- \left(\frac{C}{1+C} \right) N - C + \alpha_2 = 0$$

at $C = \alpha_2$, leading to the equilibrium point $(N_2, C_2) = (0, \alpha_2)$. The N -nullcline $C = 1/(\alpha_1 - 1)$ intercepts the C -nullcline at

$$\begin{aligned}\frac{NC}{1+C} &= \alpha_2 - C \\ N &= \frac{(\alpha_2 - C)(1+C)}{C} \\ N &= \frac{\left(\alpha_2 - \frac{1}{\alpha_1 - 1} \right) \left(1 + \frac{1}{\alpha_1 - 1} \right)}{\frac{1}{\alpha_1 - 1}} \\ N &= \alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1} \right).\end{aligned}$$

Thus,

$$(N_1, C_1) = \left(\alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1} \right), \frac{1}{\alpha_1 - 1} \right)$$

is a second equilibrium point.

(b) If $\alpha_1 > 1$ and $\alpha_2 > 1/(\alpha_1 - 1)$, then

$$(N_1, C_1) = \left(\alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1} \right), \frac{1}{\alpha_1 - 1} \right)$$

lies in the first quadrant. The Jacobian is

$$J(N, C) = \begin{pmatrix} \frac{\alpha_1 C}{1+C} - 1 & \frac{\alpha_1 N}{(1+C)^2} \\ -\frac{C}{1+C} & \frac{-N}{(1+C)^2} - 1 \end{pmatrix}.$$

If we evaluate the Jacobian at (N_1, C_1) , and use the fact that $C_1/(1+C_1) = 1/\alpha_1$, then

$$J(N_1, C_1) = \begin{pmatrix} 0 & \alpha_1 A \\ -\frac{1}{\alpha_1} & -A - 1 \end{pmatrix},$$

where $A = N_1/(1+C_1)^2$. Note that the trace $T = -A - 1$, a negative number. Furthermore, the determinant $D = A$, a positive number, so the equilibrium point (N_1, C_1) is stable.

(c) At $(N_2, C_2) = (0, \alpha_2)$,

$$\begin{aligned}J(N_2, C_2) &= \begin{pmatrix} \frac{\alpha_1 \alpha_2}{1+\alpha_2} - 1 & 0 \\ -\frac{\alpha_2}{1+\alpha_2} & -1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 B - 1 & 0 \\ -B & -1 \end{pmatrix},\end{aligned}$$

where $B = \alpha_2/(1+\alpha_2)$. In this case, the determinant is

$$\begin{aligned}D &= 1 - \alpha_1 B \\ &= 1 - \frac{\alpha_1 \alpha_2}{1+\alpha_2} \\ &= \frac{1 + \alpha_2 - \alpha_1 \alpha_2}{1 + \alpha_2}.\end{aligned}$$

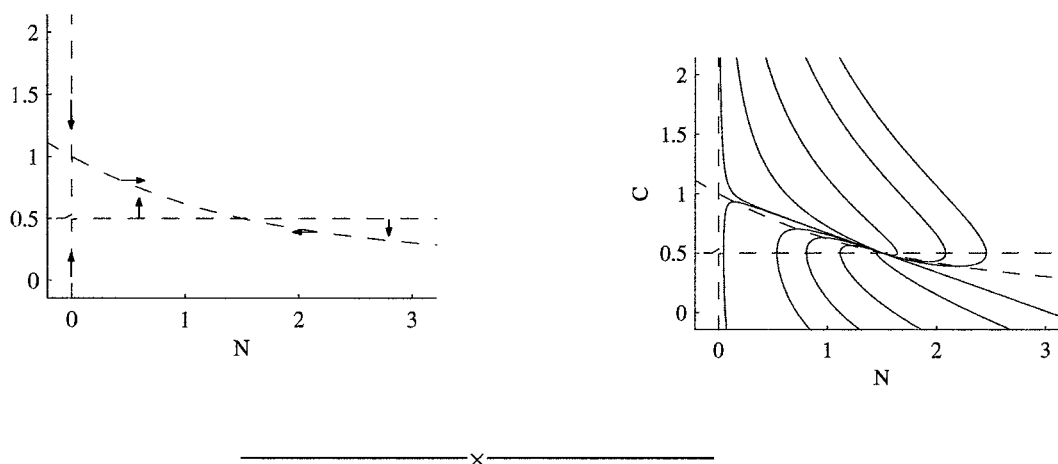
However, with the assumption that $\alpha_2 > 1/(\alpha_1 - 1)$ and $\alpha_1 > 1$,

$$\begin{aligned}\alpha_2(\alpha_1 - 1) &> 1 \\ 0 &> 1 + \alpha_2 - \alpha_1 \alpha_2.\end{aligned}$$

Therefore, $D < 0$ and the equilibrium point at (N_2, C_2) is a saddle.

(d) With $\alpha_1 = 3$ and $\alpha_2 = 1$, we sketch the nullclines and indicate the flow of solutions across the nullclines.

A phase portrait is easily obtained.

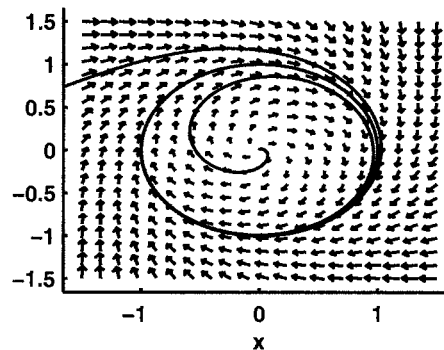


Section 10.4. Long-Term Behavior of Solutions to Planar Systems

1. For $r^2 = x^2 + y^2$ we have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[y + x(1 - r)] + y[-x + y(1 - r)] \\ &= r^2(1 - r). \end{aligned}$$

Thus $r' = r(1 - r)$. For this differential equation $r = 1$ is a stable equilibrium point. Hence $r = 1$ is an attracting limit cycle.

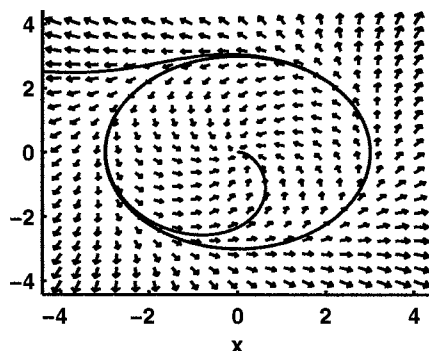


2. For $r^2 = x^2 + y^2$ we have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[-y + x(r - 3)] + y[x + y(r - 3)] \\ &= r^2(r - 3). \end{aligned}$$

Thus $r' = r(r - 3)$. For this differential equation

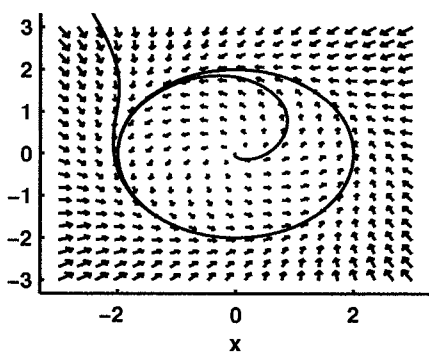
$r = 3$ is an unstable equilibrium point. Hence $r = 3$ is a repelling limit cycle.



3. For $r^2 = x^2 + y^2$ we have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[-3y + x(4 - r^2)] + y[3x + y(4 - r^2)] \\ &= r^2(4 - r^2). \end{aligned}$$

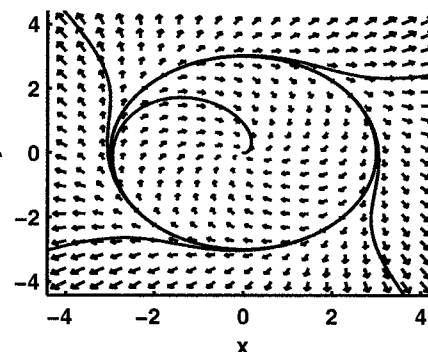
Thus $r' = r(4 - r^2)$. For this differential equation $r = 2$ is a stable equilibrium point. Hence $r = 2$ is an attracting limit cycle.



4. For $r^2 = x^2 + y^2$ we have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[5y + x(r^2 - 9)] + y[-5x + y(r^2 - 9)] \\ &= r^2(r^2 - 9). \end{aligned}$$

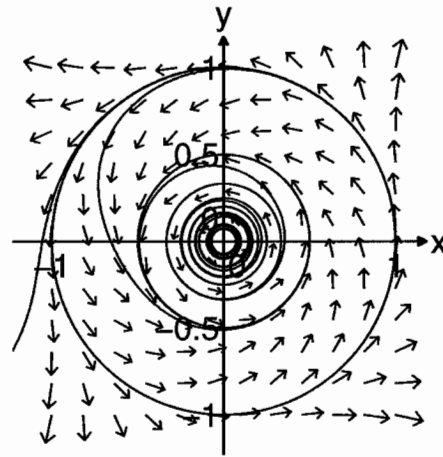
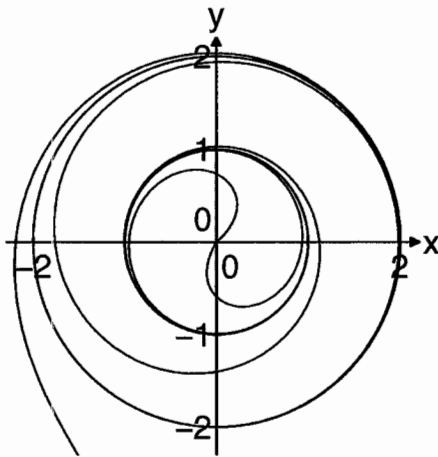
Thus $r' = r(r^2 - 9)$. For this differential equation $r = 3$ is an unstable equilibrium point. Hence $r = 3$ is a repelling limit cycle.



5. For $r^2 = x^2 + y^2$ we have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[-y + x(r - 3 + 2/r)] \\ &\quad + y[x + y(r - 3 + 2/r)] \\ &= r^3 - 3r^2 + 2r. \end{aligned}$$

Thus $r' = (r - 2)(r - 1)$, so $r = 1$ is an attracting limit cycle, while $r = 2$ is a repelling limit cycle.



6. For $r^2 = x^2 + y^2$ we have

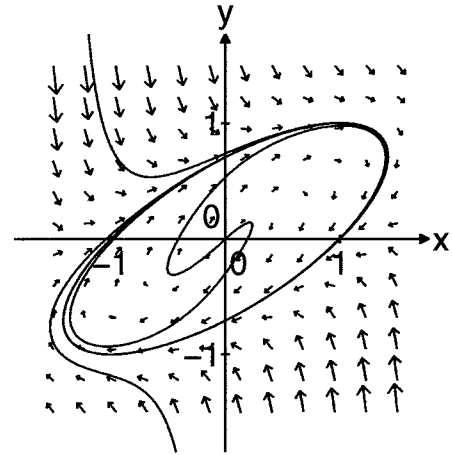
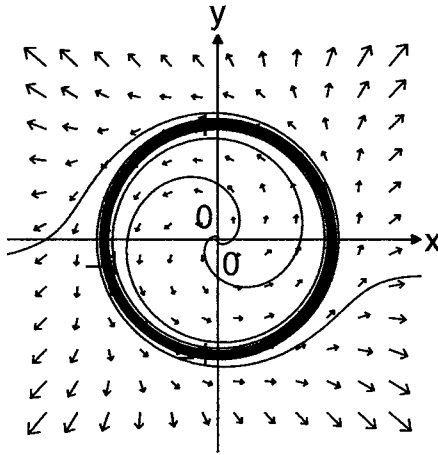
$$\begin{aligned} rr' &= xx' + yy' \\ &= x[-y + xr^2 \sin(\pi/r)] \\ &\quad + y[x + yr^2 \sin(\pi/r)] \\ &= r^4 \sin(\pi/r). \end{aligned}$$

Thus $r' = r^3 \sin(\pi/r)$. This equation has equilibrium points when $\pi/r = n\pi$, or when $r = 1/n$. The equilibrium point is stable if n is even and unstable if n is odd. Consequently we have equilibrium points at $r = 1/n$, which are attracting if n is even and repelling if n is odd.

7. For $r^2 = x^2 + y^2$ we have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[y + x(r^2 - 1)^2] \\ &\quad + y[-x + y(r^2 - 1)^2] \\ &= r^2(r^2 - 1)^2. \end{aligned}$$

Thus $r' = r(r^2 - 1)^2$, so $r = 1$ is an unstable limit cycle. Orbits with $r < 1$ spiral out to the circle $r = 1$, and those with $r > 1$ spiral out and away from the circle $r = 1$.



8. If $F(t) = F(x(t), y(t))$, then

$$\begin{aligned} F' &= 2(x-y)x' + 2(2y-x)y' \\ &= 2(x-y)[-x+2y] \\ &\quad + 2(2y-x)[-x+y] \\ &\quad + (x-2y)(x^2-2xy+2y^2-1) \\ &= -2(2y-x)^2[F(x,y)-1]. \end{aligned}$$

Thus if $F(x, y) < 1$, $F' \geq 0$, so F is increasing. Similarly if $F(x, y) > 1$, F is decreasing. Thus as t increases the solution moves so that $F(t) \rightarrow 1$. Hence the solution approaches the set

$$F(x, y) = x^2 - 2xy + 2y^2 = 1.$$

Since $F(x, y) = (x-y)^2 + y^2$, this set is an ellipse.

9. We verify, using the Jacobian, that the origin is a spiral source. For $r^2 = x^2 + y^2$ we have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[x+y-x(x^2+3y^2)] \\ &\quad + y[-x+2y-2y^3] \\ &= (x^2+2y^2)(1-r^2). \end{aligned}$$

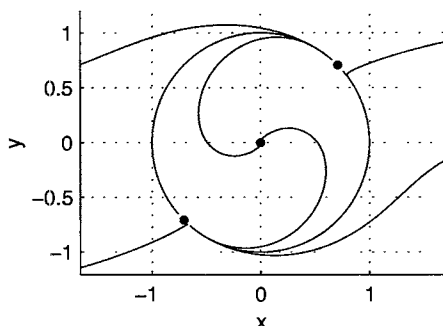
Thus $r' = (x^2+2y^2)(1-r^2)/r$. This means that r is increasing for $r < 1$ and decreasing for $r > 1$. Consequently all solution curves starting away from the origin tend to the unit circle, which is invariant. Notice that $r' = 0$ if and only if $r = 1$. Thus all equilibrium points, except for the origin, are on the unit circle.

We also have

$$\begin{aligned} r^2\theta' &= xy' - yx' \\ &= -(x^2 - xy - y^2) + xyr^2 \end{aligned}$$

On the unit circle $r = 1$ we have $\theta' = -(x-y)^2$. Hence $\theta' = 0$ only if $x = y$. Thus the equilibrium points on the unit circle are at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. The Jacobian has determinant 0 at these points. Experimentally we see that there are solutions starting close to these points that move

away. These equilibrium points are unstable. Nevertheless, every solution curve not through the origin is attracted to them!



10. If R is closed and positively invariant then it contains the limit sets of any solution curve that starts in R . By Theorem 4.1, these limit sets are not empty. According to Bendixson's theorem such a limit set must be an equilibrium point, a closed solution curve or a planar graph. Since R contains no equilibrium points, it cannot be the first of these. In addition the vertices of any planar graph must be equilibrium points, so this is impossible as well. Hence the limit sets must be closed solution curves, and so there must be a closed solution curve in R .

11. (a) For $r^2 = x^2 + y^2$,

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[y] + y[-x + y(1 - 3x^2 - 2y^2)] \\ &= y^2(1 - 3x^2 - 2y^2). \end{aligned}$$

- (b) We have

$$2r^2 = 2x^2 + 2y^2 \leq 3x^2 + 2y^2,$$

and

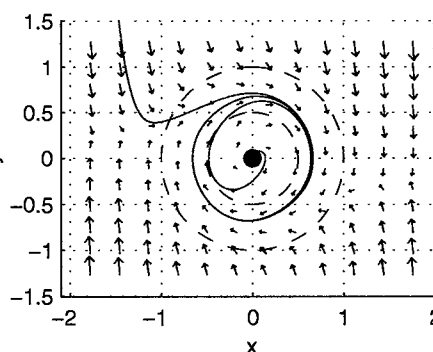
$$3r^2 = 3x^2 + 3y^2 \geq 3x^2 + 2y^2.$$

On the inner boundary of R we have $r = 1/2$ so $(1/2)r' = y^2(1 - 3x^2 - 2y^2) \geq y^2(1 - 3r^2) \geq$

$y^2/4 \geq 0$. Thus $r \geq 0$ and r is not decreasing and so the solution curve stays in R . On the outer boundary of R we have $r = 1$ and $1r' = y^2(1 - 3x^2 - 2y^2) \leq y^2(1 - 2r^2) \leq -y^2 \leq 0$. Thus, $r' \leq 0$ and r is not increasing there and the solution curve stays in R .

- (c) The only equilibrium points of the system is $(0, 0)$, and therefore there are none in R . By the Poincaré-Bendixson theorem (Theorem 4.10) there must be closed solution curve in R .

- (d) A limit cycle.



12. For $r^2 = x^2 + y^2$,

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[x + y - x(x^2 + 3y^2)] + y[-x + y - 2y^3] \\ &= x^2 + y^2 - [x^4 + 3x^2y^2 + 2y^4] \\ &= x^2 + y^2 - (x^2 + y^2)(x^2 + 2y^2) \\ &= r^2[1 - (x^2 + 2y^2)] \end{aligned}$$

Thus $r' = r[1 - (x^2 + 2y^2)]$. Since $r^2 \leq x^2 + 2y^2 \leq 2r^2$, we have $r[1 - 2r^2] \leq r' \leq r[1 - r^2]$. We get an invariant domain by taking $R = \{(x, y) | 1/2 \leq r \leq 2\}$. Since we are given that the origin is the only equilibrium point, R contains none. Thus the Poincaré-Bendixson theorem applies and R contains a closed solution curve. If we start with a solution curve on the boundary of R where $r' \neq 0$ we see that this is not a closed curve and so the limit set for this solution curve must be a limit cycle.

13. For
- $r^2 = x^2 + y^2$
- ,

$$\begin{aligned}
 rr' &= xx' + yy' \\
 &= x[x - y - x(3x^2 + y^2)] \\
 &\quad + y[x + y - y(2x^2 + y^2)] \\
 &= r^2 - [3x^4 + 3x^2y^2 + y^4].
 \end{aligned}$$

Notice that $r^4 \leq 3x^4 + 3x^2y^2 + y^4 \leq 3r^4$. Thus we have $r^2 - 3r^4 \leq rr' \leq r^2 - r^4$, or $r(1 - 3r^2) \leq r' \leq r(1 - r^2)$. From this we conclude that the annulus R defined by $1/3 \leq r \leq 1$ is invariant, since we are given that it contains no equilibrium points, it must contain a closed solution curve, which must be a limit cycle.

14. Use the chain rule to differentiate
- $E(x, y)$
- with respect to
- t
- .

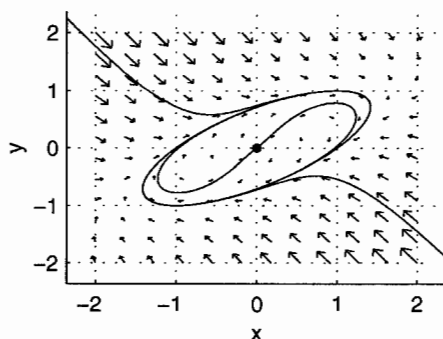
$$\begin{aligned}
 \frac{dE}{dt} &= E_x x' + E_y y' \\
 &= E_x [E_y - E_x E] + E_y [-E_x - E_y E] \\
 &= -(E_x^2 + E_y^2)E.
 \end{aligned}$$

If we divide this equation by E we get

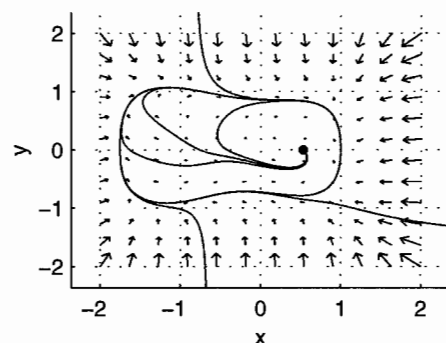
$$\frac{d}{dt} \ln |E| = \frac{1}{E} \frac{dE}{dt} = -(E_x^2 + E_y^2) \leq 0.$$

Thus $\ln |E|$ is decreasing, which means that $|E(t)|$ is decreasing. Thus the zero set of E is attracting.

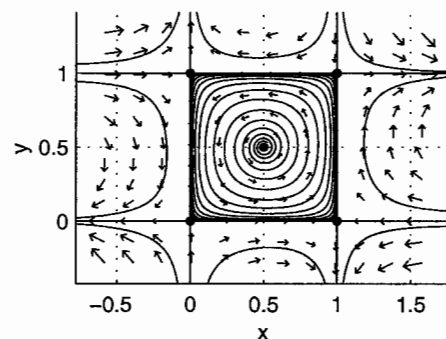
15. The zero set
- $E(x, y) = 0$
- contains the limit set.



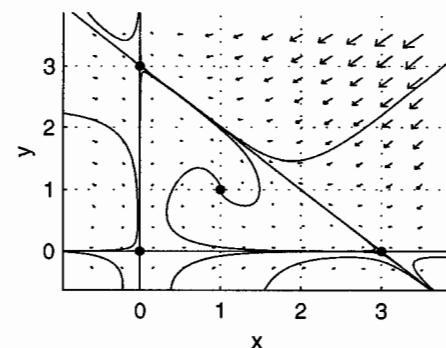
16. The zero set
- $E(x, y) = 0$
- contains the limit set.

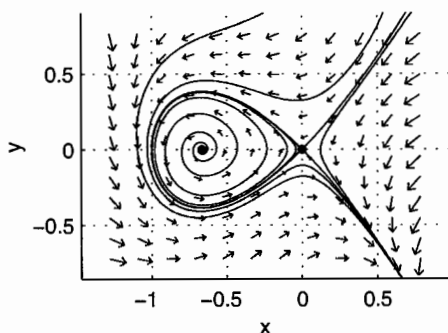
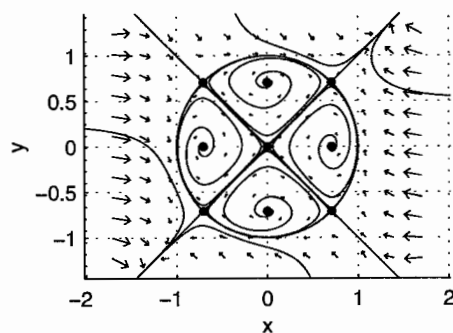
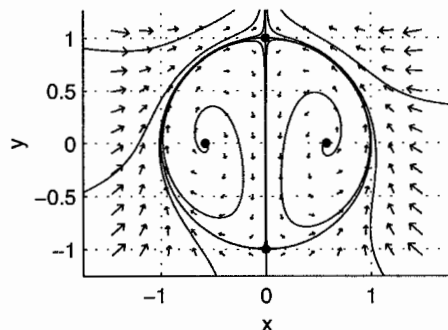


17. The zero set
- $E(x, y) = 0$
- contains the limit set.



18. The zero set
- $E(x, y) = 0$
- contains the limit set.



19. The zero set $E(x, y) = 0$ contains the limit set.

22. The zero set $E(x, y) = 0$ contains the limit set.

20. The zero set $E(x, y) = 0$ contains the limit set.

23. (a) The first equation means that $y = 0$ at any equilibrium point, and the second equation becomes $x = 0$. Thus the origin is the only equilibrium point. The Jacobian is

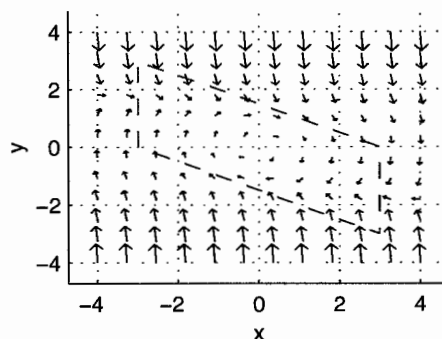
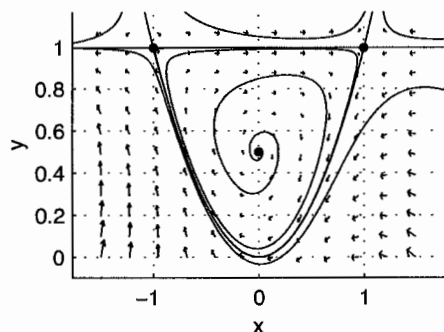
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 1 - 3y^2 \end{pmatrix},$$

and

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Since $J(0, 0)$ has trace $T = 1$ and determinant $D = 1$, we conclude that the origin is a spiral source.

(b) Does the vector field curve inward?

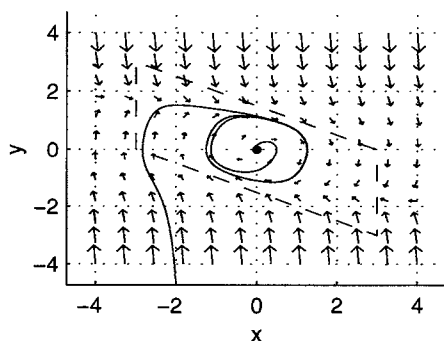
21. The zero set $E(x, y) = 0$ contains the limit set.


- (c) The sides of the parallelogram are straight line segments defined by

$$\begin{aligned}x &= 3, & \text{for } -3 \leq y \leq 0 \\x &= -3, & \text{for } 0 \leq y \leq 3 \\x + 2y &= 3, & \text{for } 0 \leq y \leq 3 \\x + 2y &= -3, & \text{for } 0 \leq y \leq 3\end{aligned}$$

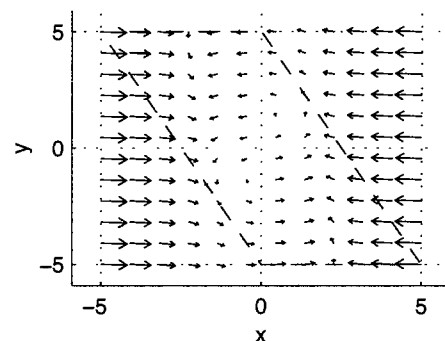
On the line $x = 3$ we have $x' = y \leq 0$, so x is decreasing. On the line $x = -3$ we have $x' = y \geq 0$, so x is increasing. On the line $x + 2y = 3$ we have $(x + 2y)' = x' + 2y' = 3y - 2x - 2y^3$. We substitute $x = 3 - 2y$ to get $(x + 2y)' = 7y - 7 - 2y^3$. We need this quantity to be negative for $0 \leq y \leq 3$. Using calculus we find that its maximum is at $y = \sqrt{7/3}$, and its value there is -2.4358 . Similarly, on the line $x + 2y = -3$, we find that $(x + 2y)' \geq 2.4358$. Thus the parallelogram is invariant.

- (d) The region is invariant, so it contains the limit set of any solution curve. The only equilibrium point is at the origin, and it is a spiral source, so it cannot be the limit set. According to the Bendixson alternatives, the only possible limit set which does not contain an equilibrium point is a closed solution curve. Starting with a solution curve on the boundary of the parallelogram which is moving into the parallelogram, we see that the limit set must be a limit cycle.



24. First we notice that the origin is the only equilibrium point, and this is a nodal source. We can use the

parallelogram with vertices $(-5, 5)$, $(0, 5)$, $(5, -5)$, and $(0, -5)$.



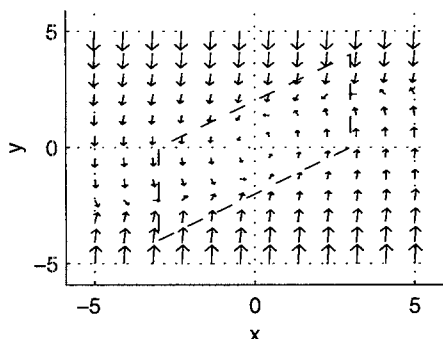
The sides are defined by the equations

$$\begin{aligned}y &= 5, & \text{for } -5 \leq x \leq 0 \\y &= -5, & \text{for } 0 \leq x \leq 5 \\2x + y &= 5, & \text{for } 0 \leq x \leq 5 \\2x + y &= -5, & \text{for } 0 \leq x \leq 5\end{aligned}$$

On the line $y = 5$ we have $y' = x \leq 0$, so y is decreasing. On the line $y = -5$ we have $y' = x \geq 0$, so y is increasing. On the line $2x + y = 5$ we have $(2x + y)' = 2x' + y' = 7x - 2y - 2x^3$. We substitute $y = 5 - 2x$ to get $(x + 2y)' = 11x - 10 - 2x^3$. We need this quantity to be negative for $0 \leq x \leq 5$. Using calculus we find that its maximum is at $x = \sqrt{11/6}$, and its value there is -0.0706 . Similarly, on the line $2x + y = -5$, we find that $(2x + y)' \geq 0.0706$. Thus the parallelogram is invariant. Since this is so, it contains the limit set of any solution curve. The only equilibrium point is at the origin, and it is a nodal source, so it cannot be the limit set. According to the Bendixson alternatives, the only possible limit set which does not contain an equilibrium point is a closed solution curve. Starting with a solution curve on the boundary of the parallelogram which is moving into the parallelogram, we see that the limit set must be a limit cycle.

25. First we notice that the origin is the only equilibrium point, and this is a spiral source. We can use the

parallelogram with vertices $(3, 4)$, $(3, 0)$, $(-3, -4)$, and $(-3, 0)$.



The sides are defined by the equations

$$\begin{aligned} x &= 3, & \text{for } 0 \leq y \leq 4 \\ x &= -3, & \text{for } -4 \leq y \leq 0 \\ 3y - 2x &= 3, & \text{for } 0 \leq y \leq 4 \\ 3y - 2x &= -3, & \text{for } -4 \leq y \leq 0 \end{aligned}$$

On the line $x = 3$ we have $x' = -y \leq 0$, so x is decreasing. On the line $x = -3$ we have $x' = -y \geq 0$, so x is increasing. On the line $3y - 2x = 3$, we have

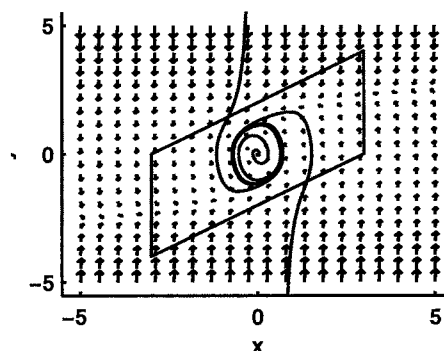
$$\begin{aligned} (3y - 2x)' &= 3y' - 2x' \\ &= 3[y + 3x - y^3] + 2y \\ &= 5y + 3x - 3y^3. \end{aligned}$$

On the line we have $3y - 2x = 3$, or $x = 3(y - 1)/2$. making this substitution we have

$$\begin{aligned} (3y - 2x)' &= 5y + 3[3(y - 1)/2] - 3y^3 \\ &= 19y/2 - 9/2 - 3y^3. \end{aligned}$$

Using calculus we find that the maximum of this quantity is at $y = \sqrt{19/6}$ and its value there is -4.5 . Thus along this line $(3y - 2x)' < 0$. Similarly, on the line $3y - 2x = -3$ we find that $(3y - 2x)' > 4.5 > 0$. Thus the parallelogram is invariant. Since this is so, it contains the limit set of

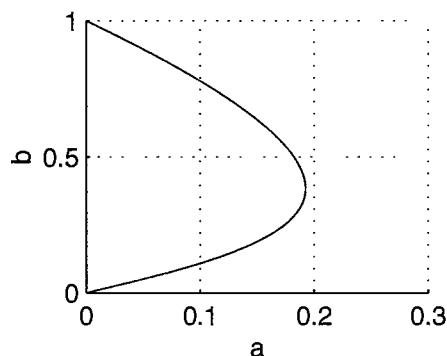
any solution curve. the only equilibrium point is at the origin, and it is a spiral source, so it cannot be the limit set. According to the Bendixson alternatives, the only possible limit set which does not contain an equilibrium point is a closed solution curve. Starting with a solution curve on the boundary of the parallelogram which is moving into the parallelogram, we see that the limit set must be a limit cycle.



26. (a) At an equilibrium point, $a = x - x^2y$ and $b = x^2y$. Hence $x_0 = a + b$, and $y_0 = b/(a + b)^2$ is the only equilibrium point. The Jacobian at this point is

$$J = \begin{pmatrix} -1 + 2b/(a + b) & (a + b)^2 \\ -2b/(a + b) & -(a + b)^2 \end{pmatrix}.$$

- (b) The Jacobian has determinant $D = (a + b)^2 > 0$. If the trace $T = (b - a - (a + b)^3)/(a + b) > 0$ then (x_0, y_0) will be a source, and the Poincaré-Bendixson theorem (Theorem 4.10) implies that there is a limit cycle.



The points (a, b) to the left of the curve in the above figure satisfy $b - a > (a + b)^3$.

27. Any equilibrium point contained in limiting planar graph is vertex of the graph and must be the meeting place of two edges. Each of these edges must be a solution curve. Furthermore, since the edges are contained in the limit of a solution curve, one of the edges must be a stable solution curve for (x_0, y_0) , and the other must be an unstable solution curve for (x_0, y_0) . The only type of equilibrium point with a nonsingular determinant is a saddle. Clearly one of the edges is a stable separatrix and the other is an unstable separatrix.

Section 10.5. Conserved Quantities

1. We must solve $dy/dv = -v/(2y)$. Separating the variables we get $2y dy = -v dv$. Integrating we get $y^2 = -v^2/2 + C$. Hence the quantity $E(y, v) = y^2 + v^2/2$ is conserved.

2. If

$$\begin{aligned} y' &= v \\ v' &= 3 - y, \end{aligned}$$

then

$$\frac{dv}{dy} = \frac{dv/dt}{dy/dt} = \frac{3-y}{v}.$$

Separate the variables and integrate.

$$\begin{aligned} \int v dv &= \int (3-y) dy \\ \frac{1}{2}v^2 &= 3y - \frac{1}{2}y^2 + C \end{aligned}$$

Set $E(y, v) = (1/2)v^2 - 3y + (1/2)y^2$. Now,

$$\begin{aligned} \frac{dE}{dt} &= \frac{\partial E}{\partial y} \frac{dy}{dt} + \frac{\partial E}{\partial v} \frac{dv}{dt} \\ &= (-3+y)(v) + (v)(3-y) \\ &= 0. \end{aligned}$$

Thus, $E = E(y, v)$ is constant on solution trajectories.

3. We must solve $dy/dv = v/(y^2 - y)$. Separating variables we get $(y^2 - y) dy = v dv$. Integrating we get $y^3/3 - y^2/2 = v^2/2 + C$. Hence the quantity $E(y, v) = y^3/3 - y^2/2 - v^2/2$ is conserved.

4. If

$$\begin{aligned} y' &= v \\ v' &= 2y - y^2, \end{aligned}$$

then

$$\frac{dv}{dy} = \frac{dv/dt}{dy/dt} = \frac{2y - y^2}{v}.$$

Separate the variables and integrate.

$$\begin{aligned} \int v dv &= \int (2y - y^2) dy \\ \frac{1}{2}v^2 &= y^2 - \frac{1}{3}y^3 + C \end{aligned}$$

Set $E(y, v) = (1/2)v^2 - y^2 + (1/3)y^3$. Now,

$$\begin{aligned} \frac{dE}{dt} &= \frac{\partial E}{\partial y} \frac{dy}{dt} + \frac{\partial E}{\partial v} \frac{dv}{dt} \\ &= (-2y + y^2)(v) + (v)(2y - y^2) \\ &= 0. \end{aligned}$$

Thus, $E = E(y, v)$ is constant on solution trajectories.

5. We must solve $dy/dv = v/(y - y^3)$. Separating variables we get $(y - y^3) dy = v dv$. Integrating we get $y^2/2 - y^4/4 = v^2/2 + C$. Hence the quantity $E(y, v) = y^2/2 - y^4/4 - v^2/2$ is conserved.

6. If

$$\begin{aligned}y' &= v \\v' &= -2y + y^3,\end{aligned}$$

then

$$\frac{dv}{dy} = \frac{dv/dt}{dy/dt} = \frac{-2y + y^3}{v}.$$

Separate the variables and integrate.

$$\begin{aligned}\int v dv &= \int (-2y + y^3) dy \\ \frac{1}{2}v^2 &= -y^2 + \frac{1}{4}y^4 + C\end{aligned}$$

Set $E(y, v) = (1/2)v^2 + y^2 - (1/4)y^4$. Now,

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial E}{\partial y} \frac{dy}{dt} + \frac{\partial E}{\partial v} \frac{dv}{dt} \\ &= (2y - y^3)(v) + (v)(-2y + y^3) \\ &= 0.\end{aligned}$$

Thus, $E = E(y, v)$ is constant on solution trajectories.

7. We must solve $dx/dy = e^y/e^x$. Separating variables we get $e^x dx = e^y dy$. Integrating we get $e^x = e^y + C$. Hence the quantity $E(x, y) = e^x - e^y$ is conserved.

8. If

$$\begin{aligned}x' &= x(1 - 2y) \\ y' &= y(-1 + 2x),\end{aligned}$$

then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(-1 + 2x)}{x(1 - 2y)}.$$

Separate the variables and integrate.

$$\begin{aligned}\frac{1 - 2y}{y} dy &= \frac{-1 + 2x}{x} dx \\ \int \left(\frac{1}{y} - 2 \right) dy &= \int \left(-\frac{1}{x} + 2 \right) dx \\ \ln |y| - 2y &= -\ln |x| + 2x + C\end{aligned}$$

Set $E(x, y) = \ln |x| + \ln |y| - 2y - 2x$. Now,

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} \\ &= \left(\frac{1}{x} - 2 \right) x(1 - 2y) \\ &\quad + \left(\frac{1}{y} - 2 \right) y(-1 + 2x) \\ &= (1 - 2x)(1 - 2y) - (1 - 2y)(1 - 2x) \\ &= 0.\end{aligned}$$

Thus, $E = E(x, y)$ is constant on solution trajectories.

9. We must solve $dx/dy = y/(-\sin x)$. Separating variables we get $-\sin x dx = y dy$. Integrating we get $\cos x = y^2/2 + C$. Hence the quantity $E(x, y) = \cos x - y^2/2$ is conserved.

10. If

$$\begin{aligned}x' &= 1/(1 + y^2) \\ y' &= e^x,\end{aligned}$$

then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^x}{1/(1 + y^2)} = e^x(1 + y^2).$$

Separate the variables and integrate.

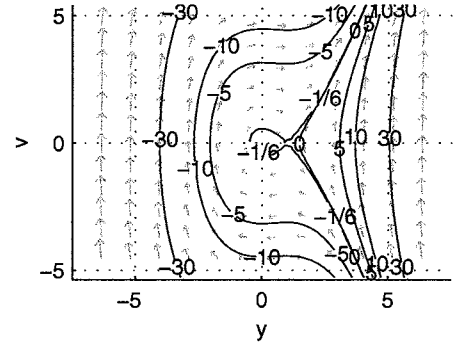
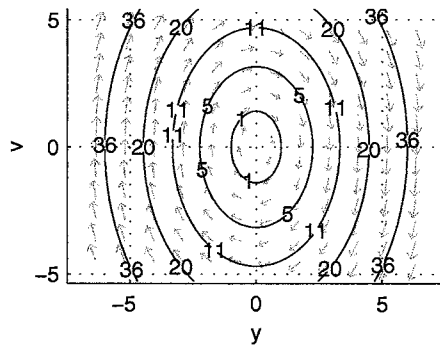
$$\begin{aligned}\int \frac{dy}{1 + y^2} &= \int e^x dx \\ \tan^{-1} y &= e^x + C\end{aligned}$$

Set $E(x, y) = \tan^{-1} y - e^x$. Now,

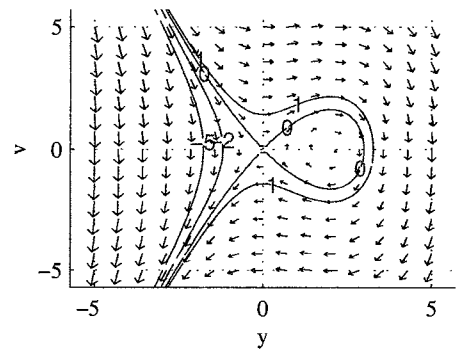
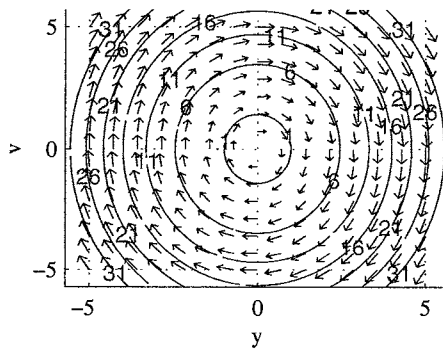
$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} \\ &= (-e^x) \left(\frac{1}{1 + y^2} \right) + \left(\frac{1}{1 + y^2} \right) (e^x) \\ &= 0.\end{aligned}$$

Thus, $E = E(x, y)$ is constant on solution trajectories.

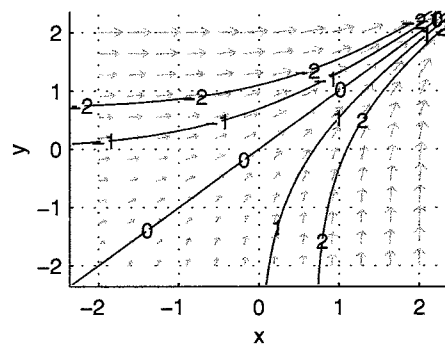
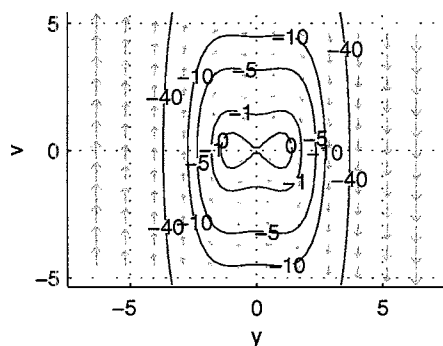
11. The conserved quantity is $E(y, v) = y^2 + v^2/2$.



12. The conserved quantity is $E(y, v) = (1/2)v^2 - 3y + (1/2)y^2$. 14. The conserved quantity is $E(y, v) = (1/2)v^2 - y^2 + (1/3)y^3$.

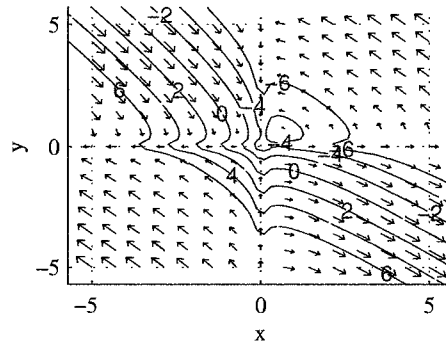
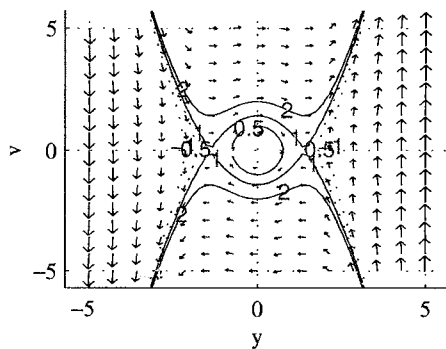


13. The conserved quantity is $E(y, v) = y^3/3 - y^2/2 - v^2/2$. 15. The conserved quantity is $E(y, v) = y^2/2 - y^4/4 - v^2/2$.



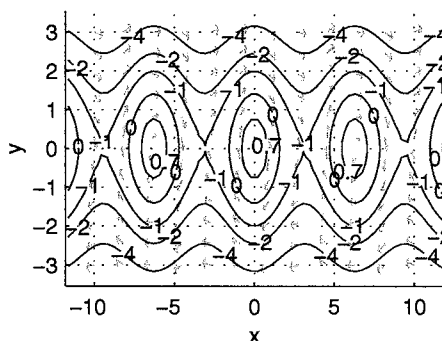
16. The conserved quantity is $E(y, v) = (1/2)v^2 + y^2 - (1/4)y^4$.

18. The conserved quantity is $E(x, y) = \ln |x| + \ln |y| - 2y - 2x$.

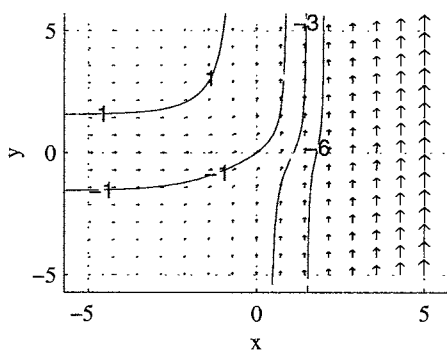


17. The conserved quantity is $E(x, y) = e^x - e^y$.

19. The conserved quantity is $E(x, y) = \cos x - y^2/2$.



20. The conserved quantity is $E(x, y) = \tan^{-1} y - e^x$.



21. The conserved quantity is $E(y, v) = y^2 + v^2/2$.

$$\begin{aligned} E' &= 2yy' + vv' \\ &= 2yv + v(-v - 2y) \\ &= -v^2 \end{aligned}$$

22. Let

$$\begin{aligned} y' &= v \\ v' &= 3 - v - y, \end{aligned}$$

and $E(y, v) = (1/2)v^2 - 3y + (1/2)y^2$. Then

$$\begin{aligned} \frac{dE}{dt} &= \frac{\partial E}{\partial y} \frac{dy}{dt} + \frac{\partial E}{\partial v} \frac{dv}{dt} \\ &= (-3 + y)(v) + (v)(3 - v - y) \\ &= -v^2. \end{aligned}$$

23. The conserved quantity is $E(y, v) = y^4/4 - y^2 - v^2/2$.

$$\begin{aligned} E' &= (y^3 - 2y)y' - vv' \\ &= (y^3 - 2y)v - v(-v - 2y + y^3) \\ &= v^2 \end{aligned}$$

24. Let

$$\begin{aligned} x' &= y \\ y' &= -y - \sin x, \end{aligned}$$

and $E(x, y) = \cos x - y^2/2$. Then,

$$\begin{aligned} \frac{dE}{dt} &= \frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} \\ &= (-\sin x)(y) + (-y)(-y - \sin x) \\ &= y^2. \end{aligned}$$

25. (a) If

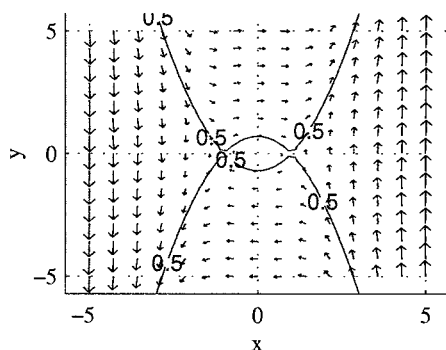
$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + x^3, \end{aligned}$$

and $H(x, y) = x^2 - (1/2)x^4 + y^2$, then

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} \\ &= (2x - 2x^3)(y) + (2y)(-x + x^3) \\ &= 0. \end{aligned}$$

and H is conserved along solution curves.

- (b) Using a computer, the level curve $H(x, y) = 1/2$ is easily sketched.



What's difficult is to determine initial conditions on the level curve. However, if $x = 0$, then

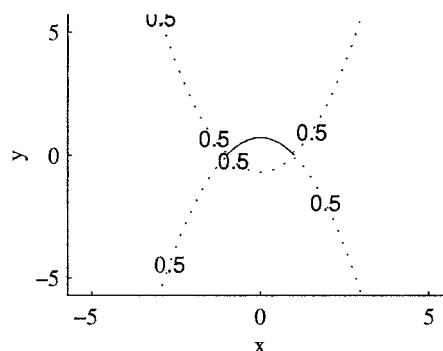
$$H(x, y) = \frac{1}{2}$$

$$x^2 - \frac{1}{2}x^4 + y^2 = \frac{1}{2}$$

becomes

$$y^2 = \frac{1}{2}$$

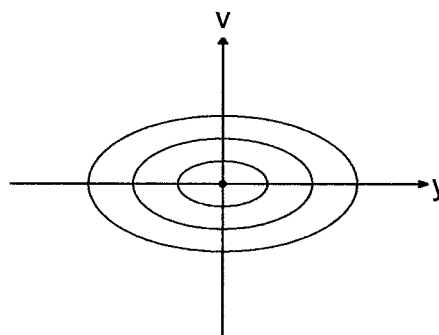
and $y = \pm\sqrt{2}/2$. We can now use our numerical solver to sketch a solution with initial condition $(0, \sqrt{2}/2)$. Note how the solution is invariant on the level curve $H(x, y) = 1/2$, both positively and negatively.



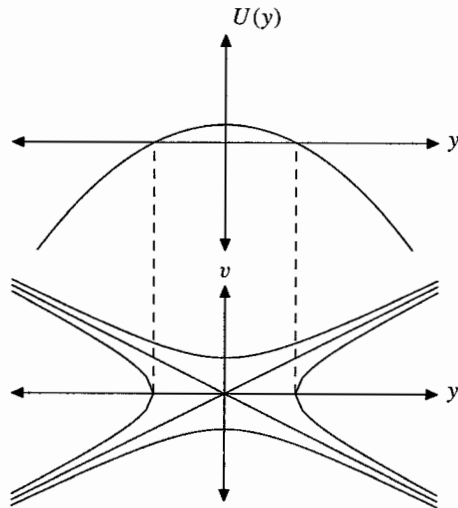
A similar strategy can be employed to start solution trajectories on the remaining pieces of the level curve $H(x, y) = 1/2$.

Section 10.6. Nonlinear Mechanics

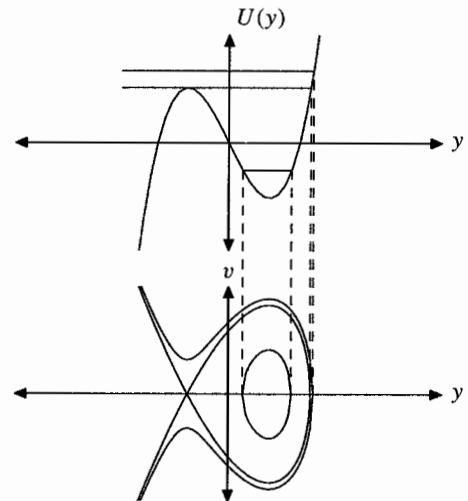
1. The potential function has a minimum at $y = 0$. Hence, the phase portrait will have an equilibrium point at $(0, 0)$ that will be a center.



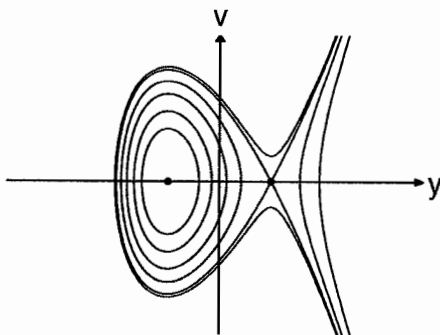
2. The potential function has a maximum at $y = 0$. Hence, the phase portrait will have an equilibrium point at $(0, 0)$ that will be a saddle.



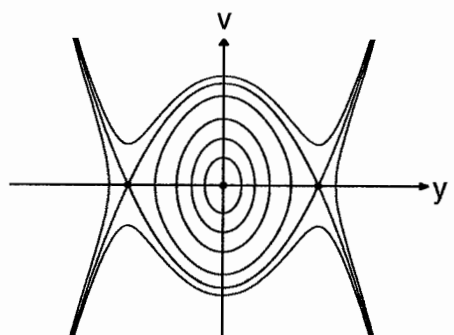
4. The potential has a minimum to the right of $y = 0$. Hence the phase portrait has a center at this value of y . The potential has maxima to the left of $y = 0$, so the phase portrait will exhibit a saddle at this value of y .



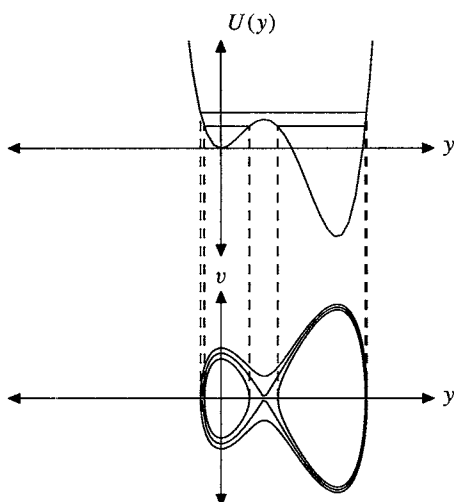
3. The potential curve has a minimum somewhere to the left of the v -axis, say at y_1 . Consequently, the phase portrait will show an equilibrium point at $(y_1, 0)$ that is a center. Returning to the potential function, you can see a maximum to the right of the v -axis, say at y_2 . Thus, the phase portrait will show a saddle at $(y_2, 0)$.



5. The potential curve has maxima both to the right and left of the v -axis, say at y_1 , and by symmetry, at about $-y_1$. Therefore, the phase portrait will show saddle equilibrium points at $(y_1, 0)$ and $(-y_1, 0)$. Returning to the potential function, you can see a minimum at the origin. Hence, there will be a center in the phase portrait at the origin as well.



6. The potential has minima at $y = 0$ and at a value to the right of $y = 0$. Hence, the phase portrait will exhibit equilibrium points for these values of y that are centers. The potential also has a maximum to the right of $y = 0$, so the phase portrait will exhibit an equilibrium point at this value of y that is a saddle.



7. The potential function has derivative $U'(y) = -f(y)$. Thus,

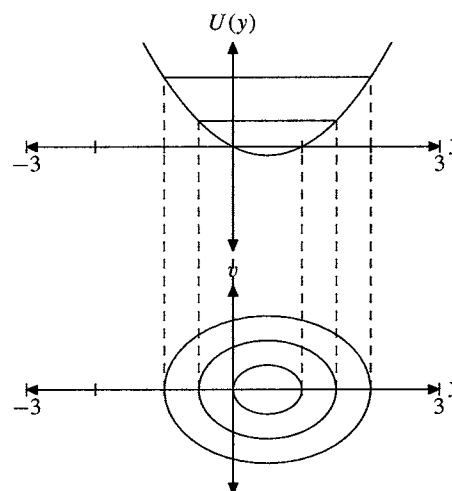
$$U'(y) = -(1 - 2y)$$

$$U(y) = -y + y^2.$$

The extrema of $U(y)$ occur when $U'(y) = 0$. Hence,

$$\begin{aligned} U'(y) &= 0 \\ -(1 - 2y) &= 0 \\ y &= \frac{1}{2}. \end{aligned}$$

As indicated in the plot that follows, the potential function has a minimum at $y = 1/2$, which leads to a center in the phase plane.



8. The potential function has derivative $U'(y) = -f(y)$. Thus,

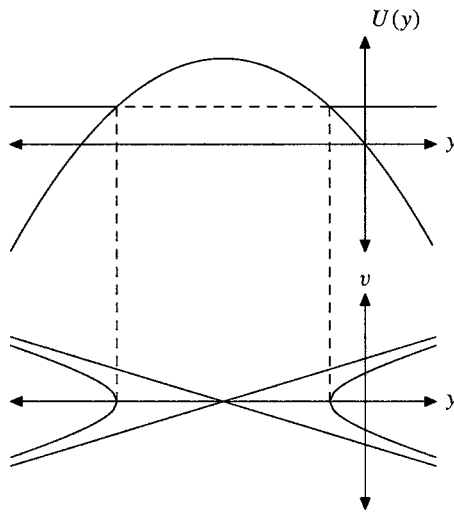
$$U'(y) = -(y + 4)$$

$$U(y) = -\frac{1}{2}y^2 - 4y.$$

The extrema of $U(y)$ occur when $U'(y) = 0$. Hence,

$$\begin{aligned} U'(y) &= 0 \\ -(y + 4) &= 0 \\ y &= -4. \end{aligned}$$

As indicated in the plot that follows, the potential function has a maximum at $y = -4$, which leads to a saddle in the phase plane.



9. The potential function has derivative $U'(y) = -f(y)$. Thus,

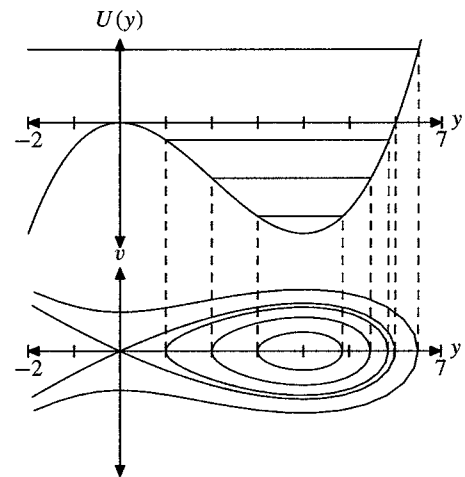
$$U'(y) = -(4y - y^2)$$

$$U(y) = -2y^2 + \frac{1}{3}y^3.$$

The extrema of $U(y)$ occur when $U'(y) = 0$. Hence,

$$\begin{aligned} U'(y) &= 0 \\ -(4y - y^2) &= 0 \\ y(y - 4) &= 0 \\ y &= 0, 4. \end{aligned}$$

As indicated in the plot that follows, the potential function has a minimum at $y = 4$, which leads to a center in the phase plane. The potential also has a maximum at $y = 0$, which leads to a saddle in the phase plane.



10. The potential function has derivative $U'(y) = -f(y)$. Thus,

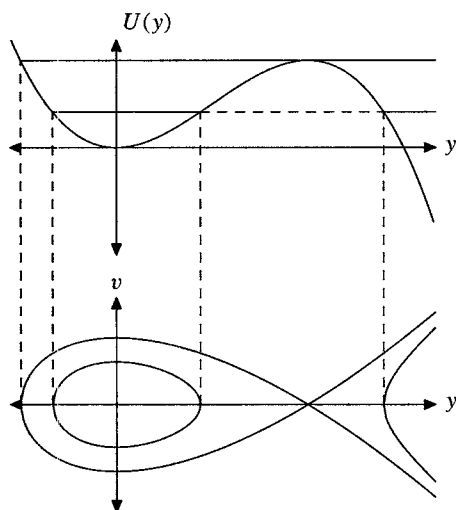
$$U'(y) = -(y^2 - 9y)$$

$$U(y) = -\frac{1}{3}y^3 + \frac{9}{2}y^2.$$

The extrema of $U(y)$ occur when $U'(y) = 0$. Hence,

$$\begin{aligned} U'(y) &= 0 \\ -(y^2 - 9y) &= 0 \\ -y(y - 9) &= 0 \\ y &= 0, 9. \end{aligned}$$

As indicated in the plot that follows, the potential function has a minimum at $y = 0$, leading to a center in the phase plane. The potential also has a maximum at $y = 9$, leading to a saddle in the phase plane.



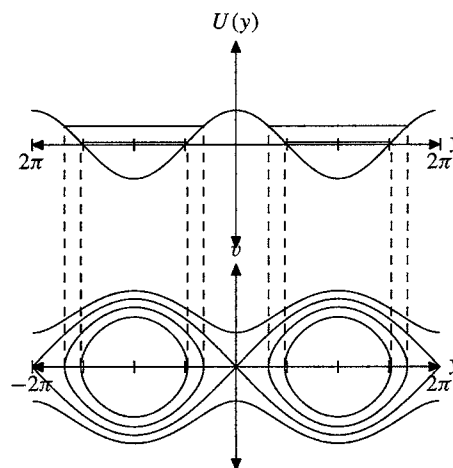
11. The potential function has derivative $U'(y) = -f(y)$. Thus,

$$\begin{aligned} U'(y) &= -\sin y \\ U(y) &= \cos y. \end{aligned}$$

The extrema of $U(y)$ occur when $U'(y) = 0$. Hence,

$$\begin{aligned} U'(y) &= 0 \\ -\sin y &= 0 \\ y &= -\pi, 0, \pi. \end{aligned}$$

As indicated in the plot that follows, the potential function has minima at $y = -\pi$ and $y = \pi$, which leads to centers in the phase plane. The potential also has a maximum at $y = 0$, which leads to a saddle point in the phase plane. Of course, this discussion can be extended beyond the interval $[-2\pi, 2\pi]$. In that case, the minima would occur at $y = k\pi$, where k is an odd integer, and the maxima would occur at $y = k\pi$, where k is an even integer. These would lead to centers at $(k\pi, 0)$, k odd, and saddles at $(k\pi, 0)$, k even.



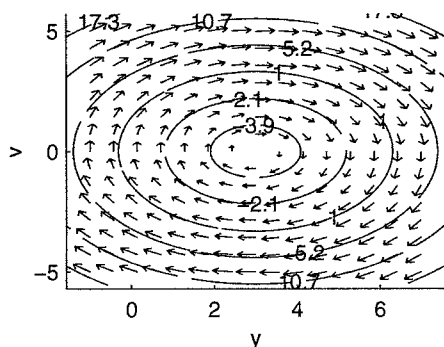
12. The potential function has derivative $U'(y) = -f(y)$. Thus,

$$\begin{aligned} U'(y) &= -(-\cos y) \\ U(y) &= \sin y. \end{aligned}$$

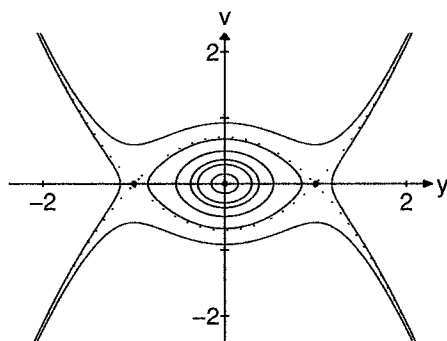
The extrema of $U(y)$ occur when $U'(y) = 0$. Hence,

$$\begin{aligned} U'(y) &= 0 \\ \cos y &= 0 \\ y &= \frac{\pi}{2} + k\pi, \quad k \text{ an integer.} \end{aligned}$$

In the plot that follows, we restrain our discussion to the domain $[-3\pi/2, 5\pi/2]$. The potential function has maxima at $-3\pi/2$, $\pi/2$, and $5\pi/2$, leading to saddle points in the phase plane. The potential has minima at $-\pi/2$ and $3\pi/2$, leading to centers in the phase plane.



13. The phase portrait



indicates the presence of saddles at $(-1, 0)$ and $(1, 0)$. In addition, there is a center at $(0, 0)$. Because the potential

$$\begin{aligned} U(y) &= - \int f(y) dy \\ &= - \int (y^3 - y) dy \\ &= -\frac{1}{4}y^4 + \frac{1}{2}y^2, \end{aligned}$$

the energy is given by

$$\begin{aligned} \frac{1}{2}v^2 + U(y) &= E \\ \frac{1}{2}v^2 - \frac{1}{4}y^4 + \frac{1}{2}y^2 &= E. \end{aligned}$$

The presence of a saddle at $y = -1$ tells us that the potential is maximized at this value. But,

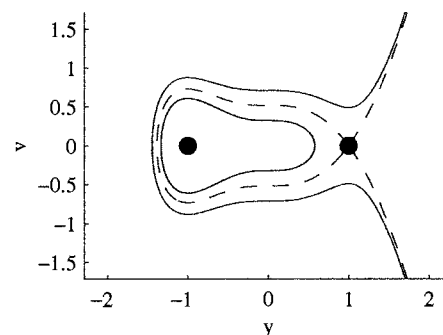
$$U(-1) = -\frac{1}{4}(-1)^4 + \frac{1}{2}(-1)^2 = \frac{1}{4}.$$

Thus, the equation of the separatrix is

$$\frac{1}{2}v^2 - \frac{1}{4}y^4 + \frac{1}{2}y^2 = \frac{1}{4}.$$

You will need an implicit function plotter to plot the separatrix as we have done on our figure (dotted lines).

14. The phase portrait



indicates the presence of a saddle at $(1, 0)$. In addition, there is a center at $(-1, 0)$. Because the potential

$$\begin{aligned} U(y) &= - \int f(y) dy \\ &= - \int y^2(y^2 - 1) dy \\ &= -\frac{1}{5}y^5 + \frac{1}{3}y^3, \end{aligned}$$

the energy is given by

$$\begin{aligned}\frac{1}{2}v^2 + U(y) &= E \\ \frac{1}{2}v^2 - \frac{1}{5}y^5 + \frac{1}{3}y^3 &= E.\end{aligned}$$

The presence of a saddle at $y = 1$ tells us that the potential is maximized at this value. But,

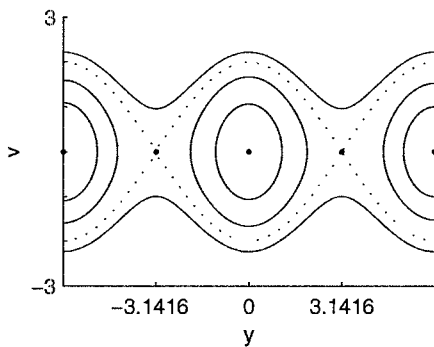
$$U(1) = -\frac{1}{5}(1)^5 + \frac{1}{3}(1)^3 = \frac{2}{15}.$$

Thus, the equation of the separatrix is

$$\frac{1}{2}v^2 - \frac{1}{5}y^5 + \frac{1}{3}y^3 = \frac{2}{15}.$$

You will need an implicit function plotter to plot the separatrix as we have done in our figure (dashed lines).

15. The phase portrait



indicates the presence of saddles at $(-\pi, 0)$ and $(\pi, 0)$. In addition, there are centers at $(-2\pi, 0)$, $(0, 0)$, and $(2\pi, 0)$. Because the potential

$$U(y) = -\int f(y) dy = \int \sin y dy = -\cos y,$$

the energy is given by

$$\begin{aligned}\frac{1}{2}v^2 + U(y) &= E \\ \frac{1}{2}v^2 - \cos y &= E.\end{aligned}$$

The presence of a saddle at $y = -\pi$ tells us that the potential is maximized at this value. But,

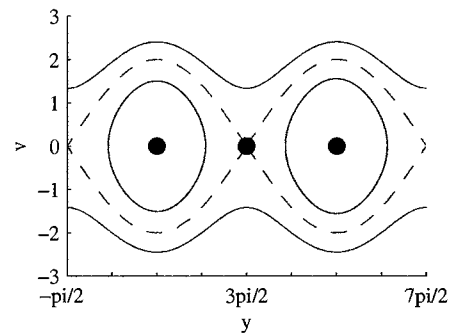
$$U(-\pi) = -\cos \pi = 1.$$

Thus, the equation of the separatrix is

$$\frac{1}{2}v^2 - \cos y = 1.$$

You can solve this equation for v , but we have chosen to use our implicit function plotter to plot the separatrix in our phase portrait (dotted lines). Finally, this discussion has focused on the interval $[-2\pi, 2\pi]$, but can easily be extended to a larger interval.

16. The phase portrait



indicates the presence of saddles at $(-\pi/2, 0)$, $(3\pi/2, 0)$, and $(7\pi/2, 0)$. In addition, there are centers at $(\pi/2, 0)$ and $(5\pi/2, 0)$. We've restricted y so that $-\pi/2 \leq y \leq 7\pi/2$, but you can easily extend this domain. Because the potential is

$$U(y) = -\int f(y) dy = -\int \cos y dy = -\sin y,$$

the energy is given by

$$\begin{aligned}\frac{1}{2}v^2 + U(y) &= E \\ \frac{1}{2}v^2 - \sin y &= E.\end{aligned}$$

The presence of a saddle at $-\pi/2$ tells us that the potential is maximized at this value. But,

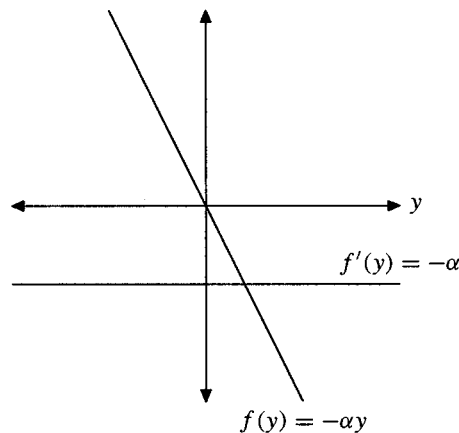
$$U(-\pi/2) = -\sin(-\pi/2) = 1.$$

Thus, the equation of the separatrix is

$$\frac{1}{2}v^2 - \sin y = 1.$$

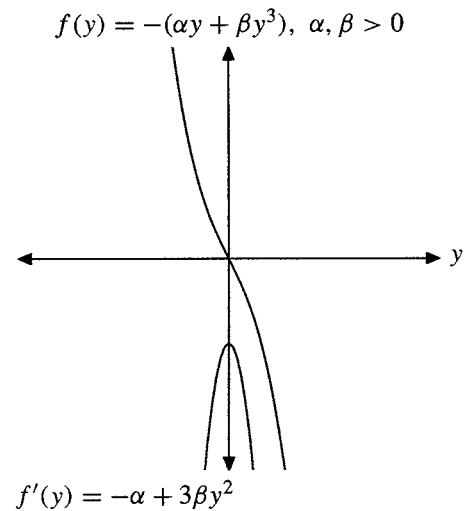
You can solve this equation for v , but we have chosen to use our implicit function plotter to plot the separatrix in our phase portrait (dashed lines).

17. The linear spring defined by $f(y) = -\alpha y$, $\alpha > 0$, is shown in the following graph.

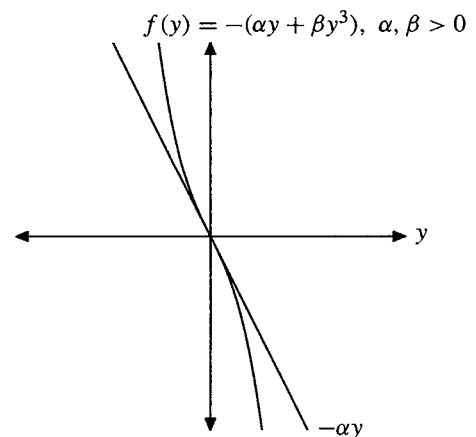


- Note that the force is negative when the displacement is positive (and positive when the displacement is negative).
- The magnitude of the force increases with the displacement. The further the spring is displaced from equilibrium, the more force is required to displace the spring an additional amount.
- The derivative of the force, $f'(y)$, is constant. Thus, the *stiffness* of the spring remains constant, regardless of the displacement.

If we add a term to the forcing function, as in $f(y) = -(\alpha y + \beta y^3)$, where $\beta > 0$, as y increases, it is intuitive that the function will decrease quicker. This is easily seen in the graph of f .



Note that the derivative, the stiffness, is no longer constant. Indeed, as the spring is displaced in the positive direction, the *magnitude* of the stiffness increases (the derivative is becoming more negative, but it is increasing in magnitude). This means that we have a *hard* spring, as the restoring force increases at a greater rate with displacement than that of the linear spring. This is easily seen when you graph the hard spring and linear spring in the same plot.



Similar comments are in order for the soft spring, which occurs when $f(y) = -(\alpha y + \beta y^3)$, $\alpha > 0$, $\beta < 0$.

18. (a) If $\beta > 0$, then

$$y'' + \alpha y + \beta y^3 = 0$$

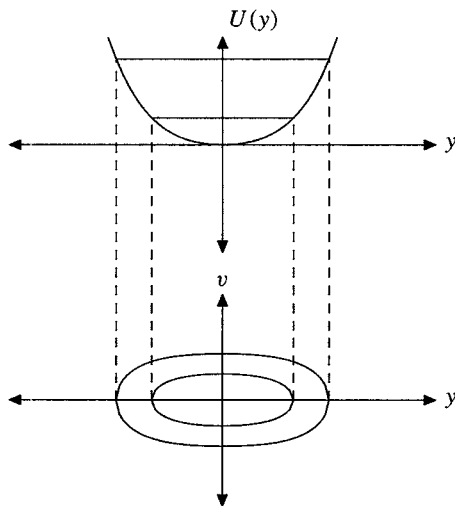
models a spring-mass system ($m = 1$) with a *hard* spring as its restoring force. If we let $v = y'$, then the second order equation can be written as a system of first order equations.

$$\begin{aligned} y' &= v \\ v' &= -(\alpha y + \beta y^3). \end{aligned}$$

Because the potential

$$\begin{aligned} U(y) &= -\int f(y) dy \\ &= \int (\alpha y + \beta y^3) dy \\ &= \frac{1}{2}\alpha y^2 + \frac{1}{4}\beta y^4, \end{aligned}$$

and $\alpha, \beta > 0$, the potential function must have the shape shown in the following diagram.



Consequently, all trajectories in the phase plane are closed orbits circling an isolated equilibrium point at the origin.

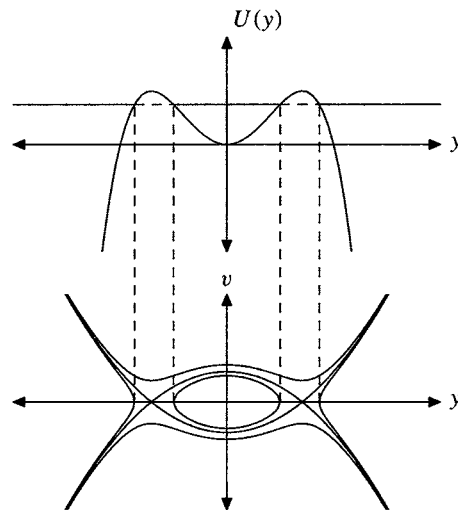
- (b) If $\beta < 0$, then the system

$$\begin{aligned} y' &= v \\ v' &= -(\alpha y + \beta y^3) \end{aligned}$$

models a spring-mass system with a *soft* spring as its restoring force. Because the potential

$$\begin{aligned} U(y) &= -\int f(y) dy \\ &= \int (\alpha y + \beta y^3) dy \\ &= \frac{1}{2}\alpha y^2 + \frac{1}{4}\beta y^4, \end{aligned}$$

and $\alpha > 0$ but $\beta < 0$, the potential function must have the shape shown in the following diagram.



The separatrices occur where the potential function achieves a maximal energy. Differentiating,

$$U'(y) = \alpha y + \beta y^3 = y(\alpha + \beta y^2).$$

The maxima of U occur where

$$\begin{aligned} \alpha + \beta y^2 &= 0 \\ y &= \pm \sqrt{-\frac{\alpha}{\beta}}. \end{aligned}$$

Thus, if we set $E = U(\sqrt{-\alpha/\beta})$, then the separatrices have equations $v = \pm \sqrt{2(E - U(y))}$.

19. (a) If absolute value is not used, then the damping term becomes $-b\omega^2$. Note that in this case the damping is always negative, and not opposite the motion. However, if the damping term $-b\omega|\omega|$ is used, then

$$-b\omega|\omega| = \begin{cases} -b\omega^2, & \text{if } \omega \geq 0, \\ b\omega^2, & \text{if } \omega < 0. \end{cases}$$

Therefore, if the damping is define as $-b\omega|\omega|$, then the damping is always opposite the motion.

To find the equilibrium points, set both right-hand sides of the system equal to zero.

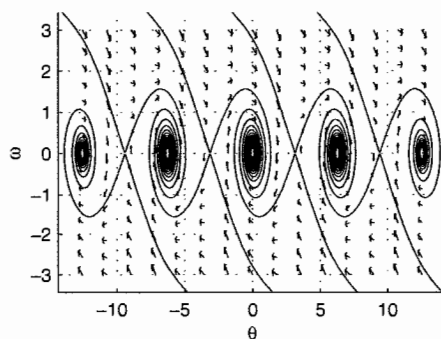
$$\begin{aligned} \omega &= 0 \\ -a \sin \theta - b\omega|\omega| &= 0 \end{aligned}$$

Thus, $\omega = 0$ and the second equation becomes

$$-a \sin \theta = 0,$$

which leads to equilibrium points at $(k\pi, 0)$, where k is an integer.

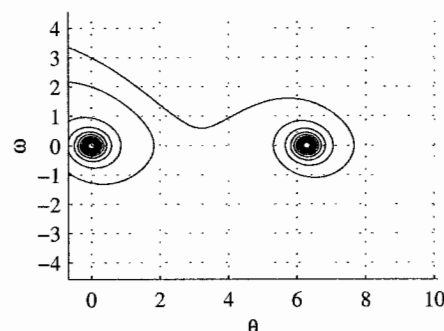
- (b) In the image that follows, only the separatrices have been drawn. This is enough to indicate that spiral sinks occur at $(k\pi, 0)$, where k is an even integer, and saddles occur at $(k\pi, 0)$, where k is an odd integer.



20. (a) With $a = 1$ and $b = 0.2$,

$$\begin{aligned} \theta' &= \omega \\ \omega' &= -a \sin \theta - b\omega|\omega| \end{aligned}$$

is entered into our solver and solution trajectories started, one at $(0, 2)$, a second at $(0, 3)$. Note that the second solution “goes over the top” and settles at the equilibrium point $(2\pi, 0)$.



Further refinement and experiment reveals that the initial condition $(\theta, \bar{\omega}) = (0, 2.7878)$ makes the point that “separates” solutions that tend to $(0, 0)$ from those that “settle” to $(2\pi, 0)$.

- (b) If $\omega = \sqrt{a}y$, then

$$\begin{aligned} \frac{d\omega}{dt} &= \sqrt{a} \frac{dy}{dt} \\ &= \sqrt{a} \frac{dy}{d\theta} \frac{d\theta}{dt} \\ &= \sqrt{a} \frac{dy}{d\theta} \omega \\ &= \sqrt{a} \frac{dy}{d\theta} (\sqrt{a}y) \\ &= ay \frac{dy}{d\theta}. \end{aligned}$$

Note the use of the chain rule and the fact (first equation) that $\omega = d\theta/dt$. Now

$$\begin{aligned} \frac{d\omega}{dt} &= -a \sin \theta - b\omega|\omega| \\ ay \frac{dy}{d\theta} &= -a \sin \theta - b(\sqrt{a}y)|\sqrt{a}y|, \end{aligned}$$

and if $y > 0$, then

$$ay \frac{dy}{d\theta} = -a \sin \theta - aby^2.$$

Dividing by a and rearranging,

$$y \frac{dy}{d\theta} + by^2 = -\sin \theta.$$

Multiplying by 2 and using $(d/d\theta)y^2 = 2y dy/d\theta$,

$$\begin{aligned} 2y \frac{dy}{d\theta} + 2by^2 &= -2\sin \theta \\ \frac{d}{d\theta}y^2 + 2by^2 &= -2\sin \theta. \end{aligned}$$

This is linear in y^2 . Using an integrating factor,

$$\begin{aligned} e^{2b\theta} \frac{d}{d\theta}y^2 + 2be^{2b\theta}y^2 &= -2e^{2b\theta}\sin \theta \\ \frac{d}{d\theta}(e^{2b\theta}y^2) &= -2e^{2b\theta}\sin \theta. \end{aligned}$$

Thus,

$$e^{2b\theta}y^2 = -2 \int e^{2b\theta}\sin \theta d\theta + C_1.$$

Integrating by parts,

$$e^{2b\theta}y^2 = \frac{2e^{2b\theta}}{1+4b^2}\cos \theta - \frac{4be^{2b\theta}}{1+4b^2}\sin \theta + C_1$$

and

$$y^2 = \frac{2}{1+4b^2}\cos \theta - \frac{4b}{1+4b^2}\sin \theta + C_1e^{-2b\theta}.$$

- (c) The stable solution does not enter the saddle point at $(n\pi, 0)$, n odd, but it gets arbitrarily close. If $\omega = 0$, then $\sqrt{a}y = \omega$ implies that $y = 0$. Substitute $(\theta, y) = (n\pi, 0)$ to find the integration constant, C_1 .

$$\begin{aligned} 0 &= \frac{2}{1+4b^2}\cos n\pi - \frac{4b}{1+4b^2}\sin n\pi \\ &\quad + C_1e^{-2bn\pi} \\ 0 &= -\frac{2}{1+4b^2} + C_1e^{-2bn\pi} \\ C_1 &= \frac{2e^{2bn\pi}}{1+4b^2} \end{aligned}$$

Thus,

$$\begin{aligned} y^2 &= \frac{2}{1+4b^2}\cos \theta - \frac{4b}{1+4b^2}\sin \theta \\ &\quad + \frac{2e^{2bn\pi}}{1+4b^2}e^{-2b\theta}. \end{aligned}$$

The stable solution crosses the vertical axis at $\theta = 0$. Thus,

$$y^2 = \frac{2}{1+4b^2} + \frac{2e^{2bn\pi}}{1+4b^2}.$$

But $\sqrt{a}y = \omega$, so

$$\begin{aligned} \frac{\omega^2}{a} &= \frac{2}{1+4b^2}(1 + e^{2bn\pi}) \\ \omega^2 &= \frac{2a}{1+4b^2}(1 + e^{2bn\pi}). \end{aligned}$$

Using $a = 1$, $b = 0.2$, and $n = 1$,

$$\begin{aligned} \omega^2 &\approx 7.78204418 \\ \omega &\approx 2.789631549. \end{aligned}$$

Note how well this agrees with the approximation found in part (a).

21. If the quantity H is conserved along solutions of the system

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy, \end{aligned}$$

then

$$\begin{aligned} \frac{dH}{dt} &= 0 \\ \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}y' &= 0. \end{aligned}$$

Then,

$$\begin{aligned} 0 &= (2Ax + By)(ax + by) \\ &\quad + (Bx + 2Cy)(cx + dy) \\ 0 &= (2aA + cB)x^2 + (2bA + (a+d)B \\ &\quad + 2cC)xy + (bB + 2dC)y^2. \end{aligned}$$

If this is true for nontrivial solutions $(x(t), y(t))$, then we get the following system of equations by equating coefficients.

$$\begin{aligned} 2aA + cB &= 0 \\ 2bA + (a+d)B + 2cC &= 0 \\ bB + 2dC &= 0 \end{aligned}$$

If this system has a nontrivial solution for A , B , and C , then the determinant of the coefficient matrix is zero. Thus,

$$\begin{vmatrix} 2a & c & 0 \\ 2b & a+d & 2c \\ 0 & b & 2d \end{vmatrix} = 0$$

$$2a \begin{vmatrix} a+d & 2c \\ b & 2d \end{vmatrix} - c \begin{vmatrix} 2b & 2c \\ 0 & 2d \end{vmatrix} = 0$$

$$2a(2ad + 2d^2 - 2bc) - c(4bd) = 0$$

$$a^2d + ad^2 - abc - bcd = 0$$

$$(ad - bc)(a + d) = 0.$$

Thus, the determinant of the matrix of the system either has determinant or trace equal to zero.

22. If

$$x' = f(x, y) = y$$

$$y' = g(x, y) = -x + x^3,$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y) = 0, \text{ and}$$

$$-\frac{\partial g}{\partial y} = -\frac{\partial}{\partial y}(-x + x^3) = 0.$$

Therefore, the system is Hamiltonian. Next,

$$\frac{\partial H}{\partial y} = f(x, y) = y, \text{ so}$$

$$H(x, y) = \frac{1}{2}y^2 + \phi(x).$$

But,

$$\frac{\partial H}{\partial x} = -g(x) = -(-x + x^3), \text{ so}$$

$$\phi'(x) = x - x^3.$$

Thus, $\phi(x) = (1/2)x^2 - (1/4)x^4$ and

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4.$$

23. If

$$x' = f(x, y) = -2x - 3y^2,$$

$$y' = g(x, y) = -3x^2 + 2y,$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(-2x - 3y^2) = -2, \text{ and}$$

$$-\frac{\partial g}{\partial y} = -\frac{\partial}{\partial y}(-3x^2 + 2y) = -2.$$

Therefore, the system is Hamiltonian. Next,

$$\frac{\partial H}{\partial y} = f(x, y) = -2x - 3y^2, \text{ so}$$

$$H(x, y) = -2xy - y^3 + \phi(x).$$

But,

$$\frac{\partial H}{\partial x} = -g(x, y) = -(-3x^2 + 2y), \text{ so}$$

$$-2y + \phi'(x) = 3x^2 - 2y$$

$$\phi'(x) = 3x^2.$$

Thus, $\phi(x) = x^3$ and

$$H(x, y) = -2xy - y^3 + x^3.$$

24. If

$$x' = f(x, y) = 5x - 3y^2,$$

$$y' = g(x, y) = 3x^2 - 4y,$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(5x - 3y^2) = 5, \text{ and}$$

$$-\frac{\partial g}{\partial y} = -\frac{\partial}{\partial y}(3x^2 - 4y) = 4.$$

Therefore, $\partial f/\partial x \neq \partial g/\partial y$ and the system is *not* Hamiltonian.

25. If

$$x' = f(x, y) = 3y^2,$$

$$y' = g(x, y) = -3x^2$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3y^2) = 0, \text{ and}$$

$$-\frac{\partial g}{\partial y} = -\frac{\partial}{\partial y}(-3x^2) = 0.$$

Therefore, the system is Hamiltonian. Next,

$$\frac{\partial H}{\partial y} = f(x, y) = 3y^2, \text{ so}$$

$$H(x, y) = y^3 + \phi(x).$$

But,

$$\frac{\partial H}{\partial x} = -g(x, y) = -(-3x^2), \text{ so}$$

$$\phi'(x) = 3x^2.$$

Thus, $\phi(x) = x^3$ and

$$H(x, y) = y^3 + x^3.$$

26. If

$$x' = f(x, y) = x - 4y,$$

$$y' = g(x, y) = 4x + 2y,$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x - 4y) = 1, \text{ and}$$

$$-\frac{\partial g}{\partial y} = -\frac{\partial}{\partial y}(4x + 2y) = -2.$$

Therefore, $\partial f/\partial x \neq -\partial g/\partial y$ and the system is *not* Hamiltonian.

27. If

$$x' = f(x, y) = -x + 2y,$$

$$y' = g(x, y) = -2x + y$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(-x + 2y) = -1, \text{ and}$$

$$-\frac{\partial g}{\partial y} = -\frac{\partial}{\partial y}(-2x + y) = -1.$$

Therefore, the system is Hamiltonian. Next,

$$\frac{\partial H}{\partial y} = f(x, y) = -x + 2y, \text{ so}$$

$$H(x, y) = -xy + y^2 + \phi(x).$$

But,

$$\frac{\partial H}{\partial x} = -g(x, y) = -(-2x + y), \text{ so}$$

$$-y + \phi'(x) = 2x - y$$

$$\phi'(x) = 2x.$$

Thus, $\phi(x) = x^2$ and

$$H(x, y) = -xy + y^2 + x^2.$$

28. If

$$x' = f(x, y) = -2y \sin(x^2 + y^2),$$

$$y' = g(x, y) = 2x \sin(x^2 + y^2),$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(-2y \sin(x^2 + y^2))$$

$$= -4xy \cos(x^2 + y^2)$$

and

$$-\frac{\partial g}{\partial y} = -\frac{\partial}{\partial y}(2x \sin(x^2 + y^2))$$

$$= -4xy \cos(x^2 + y^2).$$

Therefore, the system is Hamiltonian. Next,

$$\frac{\partial H}{\partial y} = f(x, y) = -2y \sin(x^2 + y^2), \text{ so}$$

$$H(x, y) = \cos(x^2 + y^2) + \phi(x).$$

But,

$$\frac{\partial H}{\partial x} = -g(x, y) = -2x \sin(x^2 + y^2),$$

so

$$-2x \sin(x^2 + y^2) + \phi'(x) = -2x \sin(x^2 + y^2)$$

$$\phi'(x) = 0.$$

Thus, $\phi(x) = C$, where C is an arbitrary constant. Picking $C = 0$,

$$H(x, y) = \cos(x^2 + y^2).$$

29. If

$$x' = f(x, y) = \cos x,$$

$$y' = g(x, y) = -y \sin x + 2x$$

then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\cos x) = -\sin x, \text{ and}$$

$$-\frac{\partial g}{\partial y} = -\frac{\partial}{\partial y}(-y \sin x + 2x) = \sin x.$$

Therefore, the system is not Hamiltonian.

30. (a) If we let $y = \dot{x}$, then $\ddot{x} = -2x + x^2$ becomes

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -2x + x^2.\end{aligned}$$

Checking,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(y) = 0, \text{ and} \\ -\frac{\partial g}{\partial y} &= -\frac{\partial}{\partial y}(-2x + x^2) = 0,\end{aligned}$$

so the system is Hamiltonian. Next,

$$\begin{aligned}\frac{\partial H}{\partial y} &= f(x, y) = y, \text{ so} \\ H(x, y) &= \frac{1}{2}y^2 + \phi(x).\end{aligned}$$

But,

$$\begin{aligned}\frac{\partial H}{\partial x} &= -g(x, y) = -(-2x + x^2), \text{ so} \\ \phi'(x) &= 2x - x^2.\end{aligned}$$

Thus, $\phi(x) = x^2 - (1/3)x^3$ and

$$H(x, y) = \frac{1}{2}y^2 + x^2 - \frac{1}{3}x^3.$$

- (b) The system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -2x + x^2\end{aligned}$$

has equilibrium points at $(0, 0)$ and $(2, 0)$. We know that the system is Hamiltonian,

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x},\end{aligned}$$

with Hamiltonian $H(x, y) = (1/2)y^2 + x^2 - (1/3)x^3$. Thus, the Jacobian is

$$J(x, y) = \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial y \partial x} \end{pmatrix},$$

or, equivalently,

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -(2-2x) & 0 \end{pmatrix}.$$

At $(2, 0)$,

$$J(2, 0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

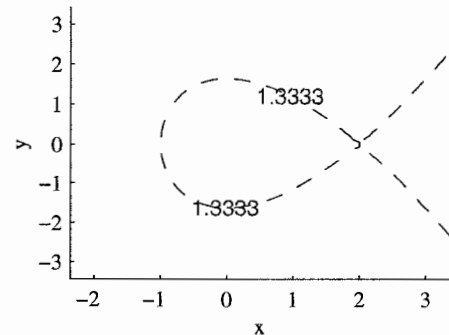
Note that the determinant is $D = -2 < 0$, so $(2, 0)$ is a saddle. We know H is conserved on solutions of the system, so we expect that the “energy” along the separatrices will be identical to the energy at the saddle point. However,

$$H(2, 0) = \frac{1}{2}(0)^2 + (2)^2 - \frac{1}{3}(2)^3 = \frac{4}{3}.$$

Thus, the level curve $H(x, y) = 4/3$, or

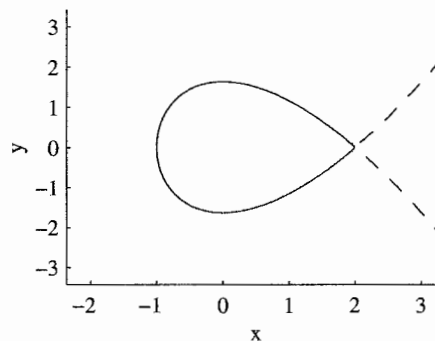
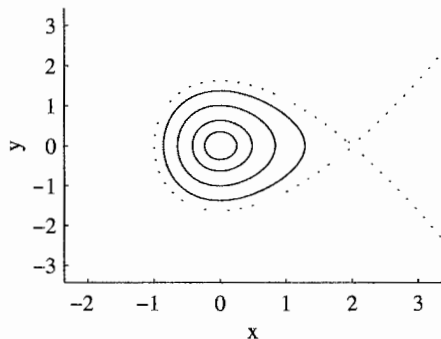
$$\frac{1}{2}y^2 + x^2 - \frac{1}{3}x^3 = \frac{4}{3}$$

should contain the stable and unstable orbits of the saddle point at $(2, 0)$. Indeed, the image produced by our solver verifies our solution.



The loop-like region bound by the separatrices is where oscillations occur, which is clearly

shown in the next figure.



In a similar manner, two more distinct solutions can be found in the tails of the “fish” that is the level curve $H(x, y) = 4/3$.

31. We compute that

- (c) If $H(x, y) = (1/2)y^2 + x^2 - (1/3)x^3$. Then $H(-1, 0) = 4/3$. Thus, if we use our numerical solver and start a solution at $(-1, 0)$, then the solution will remain on our level curve in both forward and backward time, as shown in the next figure.

$$\begin{aligned}\dot{E} &= \frac{\partial E}{\partial y_1} y'_1 + \frac{\partial E}{\partial y_2} y'_2 + \frac{\partial E}{\partial v_1} v'_1 + \frac{\partial E}{\partial v_2} v'_2 \\ &= -f_1 v_1 - f_2 v_2 + v_1 f_1 + v_2 f_2 \\ &= 0.\end{aligned}$$

Hence E is a conserved quantity.

—————x—————

Section 10.7. The Method of Lyapunov

1. If $V(x, y) = x^2 + 2y^2$, then $V(0, 0) = 0$, but $V(x, y) > 0$ for all $(x, y) \in \mathbf{R}^2$ that are different from $(0, 0)$. Thus, V is positive definite.
2. If $V(x, y) = -x^2 - 2y^2$, then $V(0, 0) = 0$. However, $V(x, y) = -(x^2 + 2y^2) < 0$ for all $(x, y) \in \mathbf{R}^2$ that are different from $(0, 0)$. Therefore, V is negative definite.
3. If $V(x, y) = x + y^2$, then $V(0, 0) = 0$. However, V is both positive and negative in any neighborhood containing $(0, 0)$. Thus, V is neither positive definite or negative definite, nor is it positive semidefinite or negative semidefinite.
4. If $V(x, y) = (x + y)^2$, then $V(0, 0) = 0$. Furthermore, $V(x, y) \geq 0$ for all $(x, y) \in \mathbf{R}^2$. Note that any open neighborhood of $(0, 0)$ must contain a portion of the line $x + y = 0$, where V is identically zero. Thus, V is positive semidefinite.
5. If $V(x, y) = -2y^2$, then $V(0, 0) = 0$. Furthermore, $V(x, y) \leq 0$ for all $(x, y) \in \mathbf{R}^2$. Note that any open neighborhood of $(0, 0)$ must contain a portion of the y -axis, where V is identically zero. Thus, V is negative semidefinite.
6. If $V(x, y) = x^2$, then $V(0, 0) = 0$. Furthermore, $V(x, y) \geq 0$ for all $(x, y) \in \mathbf{R}^2$. Note that any

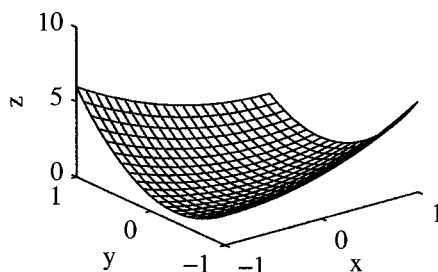
open neighborhood of $(0, 0)$ must contain a portion of the y -axis, where V is identically zero. Thus, V is positive semidefinite.

7. If $V(x, y) = x^2 + y$, then $V(0, 0) = 0$. However, V is both positive and negative in any neighborhood containing $(0, 0)$. Thus, V is neither positive definite or negative definite, nor is it positive semidefinite or negative semidefinite.
8. If $V(x, y) = 2xy - x^2 - y^2$, then $V(0, 0) = 0$. Furthermore,

$$V(x, y) = -(x^2 - 2xy + y^2) = -(x - y)^2,$$

so $V(x, y) \leq 0$ for all $(x, y) \in \mathbf{R}^2$. Note that any open neighborhood of $(0, 0)$ must contain a portion of the line $x - y = 0$, where V is identically zero. Thus, V is negative semidefinite.

9. Note that the surface determined by $V(x, y) = x^2 - 2xy + 3y^2$ appears to lie completely above the xy -plane.



Indeed, $V(0, 0) = 0$ and

$$\begin{aligned} V(x, y) &= x^2 - 2xy + 3y^2 \\ &= (x^2 - 2xy + y^2) + 2y^2 \\ &= (x - y)^2 + 2y^2. \end{aligned}$$

In order that $V(x, y) = 0$, both $x - y$ and y must simultaneously equal zero. This can only happen at $(0, 0)$. Thus, $V(x, y) > 0$ for all $(x, y) \in \mathbf{R}^2$ different from $(0, 0)$ and V is positive definite.

10. If $V(x, y) = x^2 - 6xy + 9y^2$, then $V(0, 0) = 0$. Furthermore, $V(x, y) = (x - 3y)^2 \geq 0$ for all $(x, y) \in \mathbf{R}^2$. Note that any neighborhood of $(0, 0)$ must contain a portion of the line $x - 3y = 0$, where V is identically zero. Thus, V is positive semidefinite.

11. A standard test from multivariable calculus requires two conditions in order that $V(x, y) = ax^2 + 2bxy + cy^2$ have a minimum at $(0, 0)$.

- The second derivative, $V_{xx}(0, 0) = 2a > 0$, which means that a must be greater than zero.
- The second derivative test, $V_{xx}(0, 0)V_{yy}(0, 0) - V_{xy}(0, 0)^2 = (2a)(2c) - (2b)^2 > 0$. This requires that $ac - b^2 > 0$.

You can also use linear algebra. Note that

$$V(x, y) = ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus, $V(x, y) = \mathbf{x}^T A \mathbf{x}$, where

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Therefore, $V(x, y) > 0$ for all $(x, y) \neq (0, 0)$ is equivalent to saying that $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Note that if $\mathbf{x} = (1, 0)^T$, then

$$\mathbf{x}^T A \mathbf{x} > 0 \quad \Rightarrow \quad a > 0.$$

Further, if λ is an eigenvalue with associated eigenvector $\mathbf{x}$, then

$$0 < \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T \lambda \mathbf{x} = \lambda \|\mathbf{x}\|^2.$$

Because $\|\mathbf{x}\| \neq 0$, this implies that all eigenvalues of the matrix A are positive. Because the determinant of matrix A equals the product of its eigenvalues, and all the eigenvalues of matrix A are positive, this in turn implies that the determinant of matrix A is positive. Thus,

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0 \quad \text{or} \quad ac - b^2 > 0.$$

12. (a) Since S is negative definite with a maximum at (x_0, y_0) , $V(x, y) = -S(x, y)$ is positive definite with a minimum at (x_0, y_0) . In addition

$$\begin{aligned}\dot{V} &= -\dot{S} \\ &= -\left[\frac{\partial S}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial S}{\partial y} \cdot \frac{dy}{dt}\right] \\ &= -|\nabla S|^2 \\ &\leq 0,\end{aligned}$$

so V is negative semidefinite.

- (b) We want to use part (a), so we look for a function S with

$$\frac{\partial S}{\partial x} = 2y - 2x, \quad \text{and} \quad \frac{\partial S}{\partial y} = 2x - 2y - 4y^3.$$

Integrating the first equation we get

$$S(x, y) = \int 2y - 2x \, dx = 2yx - x^2 + \phi(y).$$

With this equation for S we have

$$2x - 2y - 4y^3 = \frac{\partial S}{\partial y} = 2x + \phi'(y).$$

Thus $\phi'(y) = -2y - 4y^3$. A solution to this equation is $\phi(y) = -y^2 - y^4$. Hence we take $S(x, y) = 2yx - x^2 - y^2 - y^4$. Since this can be written as $S(x, y) = -(x - y)^2 - y^4$, we see that S is negative definite. Using part (a) we see that $V(x, y) = -S(x, y) = (x - y)^2 + y^4$ is a Lyapunov function.

13. If

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - y^3\end{aligned}$$

and $V(x, y) = x^2 + y^2$, then

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} \\ &= 2x(y) + 2y(-x - y^3) \\ &= -2y^4.\end{aligned}$$

14. If

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - y^3\end{aligned}$$

and $V(x, y) = x^3 + y^3$, then

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} \\ &= (3x^2)(y) + (3y^2)(-x - y^3) \\ &= 3x^2y - 3xy^2 - 3y^5\end{aligned}$$

15. If

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - y^3\end{aligned}$$

and $V(x, y) = x^2 + xy - y^2$, then

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} \\ &= (2x + y)(y) + (x - 2y)(-x - y^3) \\ &= 2y^4 - xy^3 + y^2 + 4xy - x^2\end{aligned}$$

16. If

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - y^3\end{aligned}$$

and $V(x, y) = 2x^2 + 3xy - y^2$, then

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} \\ &= (4x + 3y)(y) + (3x - 2y)(-x - y^3) \\ &= 3y^2 + 6xy - 3x^2 - 3xy^3 + 2y^4.\end{aligned}$$

17. (a) If

$$\begin{aligned}\frac{dx}{dt} &= -y - x^3 \\ \frac{dy}{dt} &= x - y^3,\end{aligned}$$

then the Jacobian is

$$J(x, y) = \begin{pmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{pmatrix}.$$

Evaluating at the origin,

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

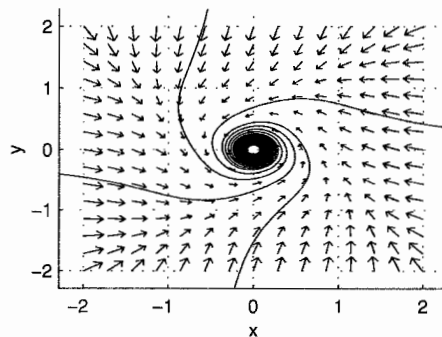
the trace is $T = 0$ and the determinant is $D = 1$. Therefore, the linearization predicts that the equilibrium point at the origin is a center.

- (b) If $V(x, y) = x^2 + y^2$, then $V(0, 0) = 0$, but $V(x, y) > 0$ for all $(x, y) \in \mathbf{R}^2$ different from $(0, 0)$. Hence, V is positive definite on $\mathbf{R}^2$. Further,

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} \\ &= 2x(-y - x^3) + 2y(x - y^3) \\ &= -2(x^4 + y^4). \end{aligned}$$

Note that $\dot{V}(0, 0) = 0$, but $\dot{V}(x, y) < 0$ for all $(x, y) \in \mathbf{R}^2$ different from $(0, 0)$. Hence, $\dot{V}$ is negative definite, and the equilibrium point at the origin is asymptotically stable.

- (c) Note that the equilibrium point is a spiral sink. Note further that our solver stopped short of the sink at the origin, but the theory of part (b) assures us that solutions should continue to spiral into the origin, provided we let enough time pass in our solver.



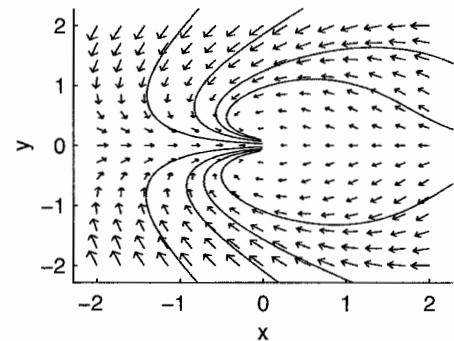
18. If

$$\begin{aligned} x' &= -x - 2y^2 \\ y' &= 2xy - y^3 \end{aligned}$$

and $V(x, y) = x^2 + y^2$, then V is positive definite with minimum at $(0, 0)$. Further,

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} \cdot x' + \frac{\partial V}{\partial y} \cdot y' \\ &= 2x(-x - 2y^2) + 2y(2xy - y^3) \\ &= -2(x^2 + y^4), \end{aligned}$$

so $\dot{V}$ is negative on $\mathbf{R}^2$. Thus the origin is asymptotically stable, as shown in the following phase portrait.



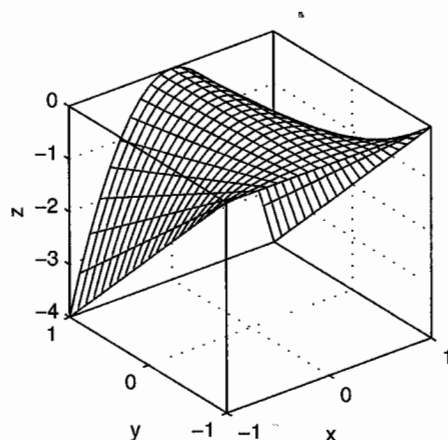
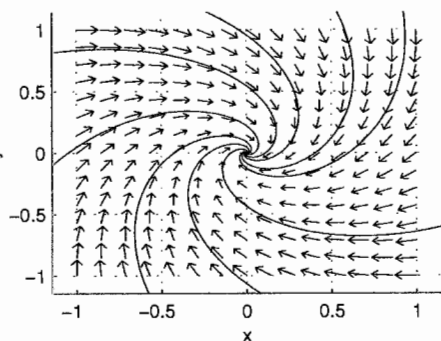
19. If

$$\begin{aligned} x' &= -x + y \\ y' &= -x - y, \end{aligned}$$

and $V(x, y) = x^2 + y^2$, then V is positive definite with minimum at $(0, 0)$. Further,

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} \cdot x' + \frac{\partial V}{\partial y} \cdot y' \\ &= 2x(-x + y) + 2y(-x - y) \\ &= -2(x^2 + y^2), \end{aligned}$$

so $\dot{V}$ is negative definite on $\mathbf{R}^2$. Thus, the origin is asymptotically stable as shown in the following phase portrait.



20. If

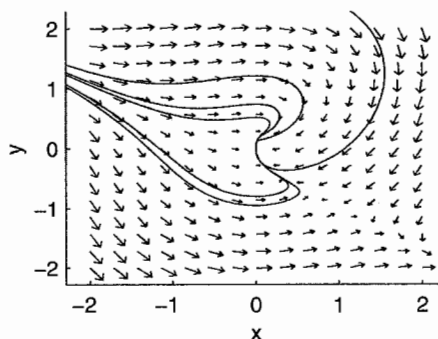
$$\begin{aligned}x' &= -x + y^2 \\y' &= -xy - x^2\end{aligned}$$

and V is the positive definite function defined by $V(x, y) = x^2 + y^2$, then

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x} \cdot x' + \frac{\partial V}{\partial y} \cdot y' \\&= 2x(-x + y^2) + 2y(-xy - x^2) \\&= -2x^2(1 + y)\end{aligned}$$

A plot of the surface represented by V reveals that it appears to lie below the xy -plane on the domain $D = \{(x, y) : -1 < x, y < 1\}$, with the exception that $\dot{V} = 0$ on $\{(x, y) : x = 0\}$. Thus, $\dot{V}$ is negative semidefinite on D .

Theorem 7.10 tell us that any forward limit set lies in $\{(x, y) : \dot{V} = 0\}$, which is the y -axis. However, any limit set is the union of solution curves and the only solution curve of the system that lies in the y -axis is the origin. Therefore, the origin is asymptotically stable. We used a stiff numerical solver to produce the following phase portrait.



21. If

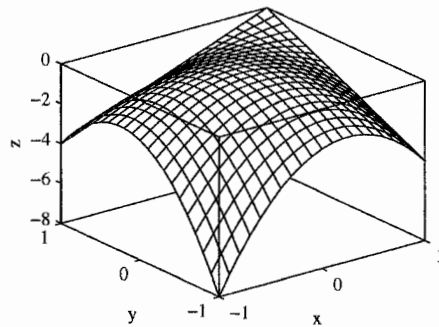
$$\begin{aligned}x' &= -x + xy \\y' &= -y + xy,\end{aligned}$$

and V is the positive definite function defined by

$V(x, y) = x^2 + y^2$, then

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x} \cdot x' + \frac{\partial V}{\partial y} \cdot y' \\ &= 2x(-x + xy) + 2y(-y + xy) \\ &= -2(x^2 - x^2y + y^2 - xy^2).\end{aligned}$$

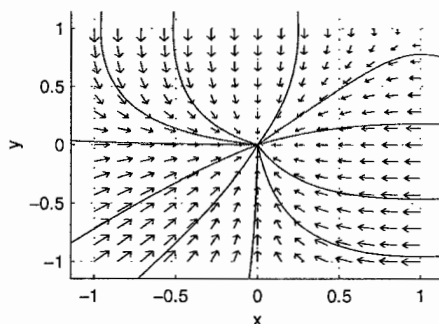
A plot of the surface represented by $\dot{V}$ is revealing in that it appears to lie below the xy -plane on the domain $D = \{(x, y) : x, y < 1\}$.



Indeed, we can write

$$\dot{V}(x, y) = -2(x^2(1 - y) + y^2(1 - x)),$$

so, if we restrict x and y so that both are less than 1, then $\dot{V}$ is negative definite on this restricted domain. Thus, the equilibrium point at the origin is asymptotically stable, as shown in the following phase portrait.



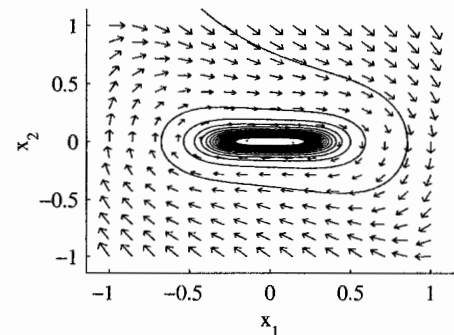
22. If $V(x_1, x_2) = (1/2)x_1^4 + x_2^2$, then $V(0, 0) = 0$ and $V(x_1, x_2) > 0$ for all $(x_1, x_2) \in \mathbf{R}^2$ that are different from $(0, 0)$. Hence, V is positive definite. If

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2^3 - x_1^3,\end{aligned}$$

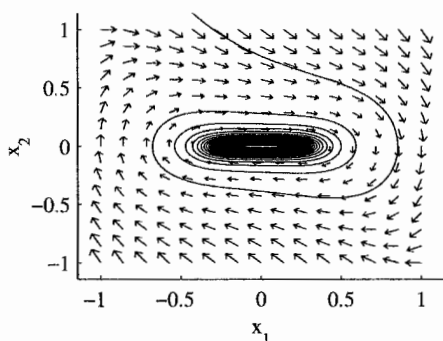
then

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial V}{\partial x_2} \cdot \dot{x}_2 \\ &= (2x_1^3)(x_2) + (2x_2)(-x_2^3 - x_1^3) \\ &= 2x_1^3x_2 - 2x_2^4 - 2x_1^3x_2 \\ &= -2x_2^4,\end{aligned}$$

so $\dot{V}(x_1, x_2) \leq 0$ on any neighborhood containing the origin. That is, V is negative semidefinite. According to Theorem 7.10, the limit set of every solution curve is contained in $\dot{V} = 0$, which is the x_1 -axis, defined by $x_2 = 0$. However, a limit set must be a union of solution curves, and the only solution curve contained in the x_1 -axis is the equilibrium point at the origin. Hence every solution is attracted to the origin. Note that in our first attempt, our trajectory pulls up short of the origin, seemingly contradicting the analysis above.



An obvious difficulty is the fact that the approach to equilibrium is extremely slow. However, we lengthened the interval of computation in our second attempt and all seems well.



23. (a) If

$$x'' = -x' + \frac{1}{3}(x')^3 - x,$$

and we let $x_1 = x$ and $x_2 = x'$, then

$$x_1' = x_2$$

$$x_2' = -x_2 + \frac{1}{3}x_2^3 - x_1.$$

The function $V(x_1, x_2) = x_1^2 + x_2^2$ is positive definite with minimum at $(0, 0)$. Further,

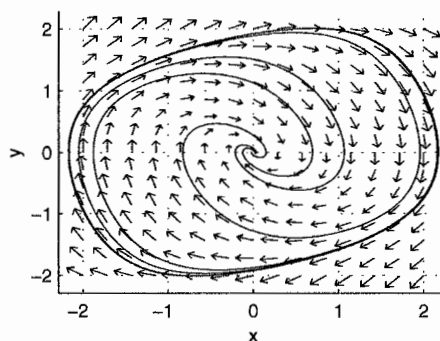
$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1} \cdot x_1' + \frac{\partial V}{\partial x_2} \cdot x_2' \\ &= 2x_1(x_2) + 2x_2(-x_2 + \frac{1}{3}x_2^3 - x_1) \\ &= -2x_2^2 + \frac{2}{3}x_2^4 \\ &= -\frac{2}{3}x_2^2(3 - x_2^2).\end{aligned}$$

We can make $\dot{V}(x_1, x_2) \leq 0$ by choosing

$$\begin{aligned}3 - x_2^2 &> 0 \\ x_2^2 &< 3 \\ |x_2| &< \sqrt{3}.\end{aligned}$$

Thus, provided we stay in the domain $\{(x_1, x_2) : |x_2| < \sqrt{3}\}$, then V is negative semi-definite and the equilibrium point at the origin is stable.

(b) In part (a), we saw that $V(x_1, x_2) = x_1^2 + x_2^2$ is positive definite on $\{(x_1, x_2) : |x_2| < \sqrt{3}\}$. According to Theorem 7.10, the limit set of every solution curve is contained in the set $\{(x_1, x_2) : \dot{V}(x_1, x_2) = 0\}$. Since $\dot{V}(x_1, x_2) = 0$ only when $x_2 = 0$, this set is the x_1 axis. However, a limit set must be a union of solution curves, and the only solution curve contained entirely in the x_1 axis is the equilibrium point at the origin. Thus, the origin is asymptotically stable, as shown in the following phase portrait.



24. If we let $x_1 = x$ and $x_2 = \dot{x}$, then the second order equation $\ddot{x} + \dot{x} + x^3 = 0$ can be written as a system of first order equations.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - x_1^3$$

As a first attempt, try $V(x_1, x_2) = x_1^2 + x_2^2$ and calculate

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial V}{\partial x_2} \cdot \dot{x}_2 \\ &= 2x_1x_2 + 2x_2(-x_2 - x_1^3) \\ &= 2x_1x_2 - 2x_2^2 - 2x_1^3x_2.\end{aligned}$$

It's a shame that the first and last terms don't cancel, leaving us with just $-2x_2^2$. With this in mind, we try $V(x_1, x_2) = (1/2)x_1^4 + x_2^2$. Thus,

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial V}{\partial x_2} \cdot \dot{x}_2 \\ &= 2x_1^3x_2 + 2x_2(-x_2 - x_1^3) \\ &= -2x_2^2.\end{aligned}$$

Thus, $\dot{V}$ is negative semidefinite. We can now apply Theorem 7.10 to argue that the origin is asymptotically stable.

25. If

$$\ddot{x} + (\dot{x})^3 + x^3 = 0,$$

and we let $x_1 = x$ and $x_2 = \dot{x}$, then

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2^3 - x_1^3.\end{aligned}$$

From here, we use $V(x_1, x_2) = (1/2)x_1^4 + x_2^2$ and argue as we did in Exercise 22 that the equilibrium point at the origin is asymptotically stable.

26. (a) Note that $1 - \cos \theta > 0$ on $(-\pi, \pi)$. Thus, if

$$V(\theta, \omega) = \frac{1}{2}mL^2\omega^2 + mgL(1 - \cos \theta),$$

then $V(0, 0) = 0$ and $V(\theta, \omega) > 0$ for all (θ, ω) in $\{(\theta, \omega) : \theta^2 + \omega^2 < \pi^2\}$ that are different from $(0, 0)$. Hence, V is positive definite.

(b) If

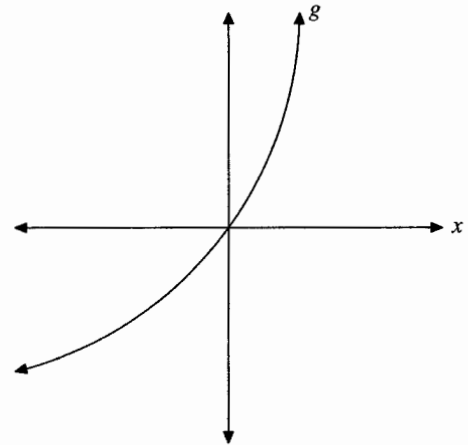
$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{\mu}{m}\omega - \frac{g}{L}\sin \theta,\end{aligned}$$

then

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial \theta} \cdot \dot{\theta} + \frac{\partial V}{\partial \omega} \cdot \dot{\omega} \\ &= (mgL \sin \theta)(\omega) \\ &\quad + (mL^2\omega) \left(-\frac{\mu}{m}\omega - \frac{g}{L}\sin \theta \right) \\ &= mgL\omega \sin \theta - \mu L^2\omega^2 - mgL\omega \sin \theta \\ &= -\mu L^2\omega^2,\end{aligned}$$

and V is negative semidefinite.

27. (a) If $xg(x) > 0$ for all x , this says that x and $g(x)$ must agree in sign. That is, when $x > 0$, so must $g(x) > 0$. Similarly, if $x < 0$, then $g(x) < 0$. One such example is shown in the following graph.



Another example is $g(x) = \sin x$, provided we restrict the domain to $|x| < \pi/2$.

Because the graph of g changes sign at $x = 0$ (negative to the left of the origin, positive to the right), the continuity of g requires that $g(0) = 0$. If we define

$$G(x) = \int_0^x g(u) du,$$

then $G(0) = 0$ and $G'(x) = g(x)$, so $G'(x) < 0$ if $-k < x < 0$ and $G'(x) > 0$ if $0 < x < k$. Thus, G is decreasing to the left of zero and increasing to the right, making $G(0) = 0$ an absolute minimum (at least on $(-k, k)$).

(b) If we let $y = x'$, then because $x'' = -f(x)x' - g(x)$, we can write

$$\begin{aligned}x' &= y \\ y' &= -f(x)y - g(x).\end{aligned}$$

If

$$V(x, y) = \frac{1}{2}y^2 + G(x),$$

then because we've shown that $G(0) = 0$ and G has an absolute minimum value of zero at $x = 0$ on $(-k, k)$, we can see that $V(0, 0) = 0$ and $V(x, y) > 0$ for all $(x, y) \neq (0, 0)$, provided that we restrict the neighborhood so that $-k < x < k$. Thus, V is positive definite with minimum at $(0, 0)$, provided $-k < x < k$.

(c) Furthermore,

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x} \cdot x' + \frac{\partial V}{\partial y} \cdot y' \\ &= G'(x)y + y(-f(x)y - g(x)) \\ &= g(x)y + y(-f(x)y - g(x)) \\ &= -f(x)y^2.\end{aligned}$$

Note that $\dot{V}(0, 0) = 0$. Further, because it is given that $f(x) > 0$ for all x , $\dot{V}$ is negative semi-definite on $\{(x, y) : -k < x < k\}$. Thus, by Theorem 7.10, the forward limit set of all trajectories must lie in the set $\dot{V}(x, y) = 0$, which because $f(x) > 0$ for all x , must consist of that part of the x -axis between $-k$ and k . However, a limit set must be a union of solution curves, and the only solution curve contains in the x -axis is the equilibrium point at the origin. Therefore, the equilibrium point at the origin is asymptotically stable.

28. (a)

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y' + \frac{\partial V}{\partial z}z' \\ &= 2rx[-ax + ay] + 2ay[rx - y - xz] \\ &\quad + 2a(z - 2r)[-bz + xy] \\ &= -2a[rx^2 + y^2 + b(z - r)^2 - br^2].\end{aligned}$$

(b) Assuming that m is the smallest of the three numbers 1, $1/a$ and $b/2a$, and the inequality

$(z - r)^2 \geq (z - 2r)^2/2 - r^2$, we have

$$\begin{aligned}rx^2 + y^2 + b(z - r)^2 &\geq mrx^2 + may^2 + 2ma(z - r)^2 \\ &\geq m[rx^2 + ay^2 \\ &\quad + 2a((z - 2r)^2/2 - r^2)] \\ &\geq mV - 2amr^2 \\ &\geq mV - br^2\end{aligned}$$

(c) Using (a) and (b) we have

$$\begin{aligned}\dot{V} &= -2a[rx^2 + y^2 + b(z - r)^2 - br^2] \\ &\leq -2a[mV - br^2 - br^2] \\ &\leq -2maV + 4abr^2.\end{aligned}$$

(d) If (x, y, z) is not in the ellipsoid, then

$$\begin{aligned}\dot{V} &\leq -2maV + 4abr^2 \\ &\leq -2ma \cdot 3br^2/m + 4abr^2 \\ &\leq -2abr^2 < 0\end{aligned}$$

(e) On the boundary of R , where $V(x, y, z) = 3br^2/m$, we have $\dot{V} \leq -2abr^2 < 0$, so the solution curve is moving into R . Hence R is positively invariant.

Along any portion of a solution curve outside of R V is strictly decreasing. When it decreases to the point where $V(x, y, z) \leq 3br^2/m$, then $(x, y, z) \in R$. Consequently every solution curve ends up in R .

—————x—————

Section 10.8. Predator-Prey Systems

1. The system

$$\begin{aligned}x' &= x(a - ex - by) = ax - ex^2 - bxy \\ y' &= y(-c + dx) = -cy + dxy,\end{aligned}$$

has an equilibrium point which is the intersection of

the nullclines

$$\begin{aligned}a - ex - by &= 0 \\ -c + dx &= 0.\end{aligned}$$

This equilibrium point has coordinates,

$$(x_0, y_0) = \left(\frac{c}{d}, \frac{ad - ec}{bd} \right).$$

The equilibrium point (x_0, y_0) lies in the first quadrant if $y_0 > 0$, which is true only if $ad - ec > 0$. The Jacobian of the system is

$$J(x, y) = \begin{pmatrix} a - 2ex - by & -bx \\ dy & -c + dx \end{pmatrix}.$$

If we evaluate the Jacobian at the equilibrium point (x_0, y_0) , we have

$$\begin{aligned} J(x_0, y_0) &= \begin{pmatrix} a - 2ex_0 - by_0 & -bx_0 \\ dy_0 & -c + dx_0 \end{pmatrix} \\ &= \begin{pmatrix} (a - ex_0 - by_0) - ex_0 & -bx_0 \\ dy_0 & -c + dx_0 \end{pmatrix} \\ &= \begin{pmatrix} -ex_0 & -bx_0 \\ dy_0 & 0 \end{pmatrix}. \end{aligned}$$

This last result holds true because the equilibrium point is the intersection of the nullclines $a - ex - by = 0$ and $-c + dx = 0$. Consequently, $a - ex_0 - by_0 = 0$ and $-c + dx_0 = 0$. Now, the trace and determinant of $J(x_0, y_0)$ are easily determined. The trace is $T = -ex_0$, and because $x_0 > 0$, the trace is negative. The determinant is $D = bdx_0y_0$, and since both x_0 and y_0 lie in the positive quadrant, the determinant is also positive. Therefore, we have determined that the equilibrium point is some kind of sink.

To determine the kind of sink (nodal or spiral), we must examine $T^2 - 4D$.

2. (a) First write the equation in the form

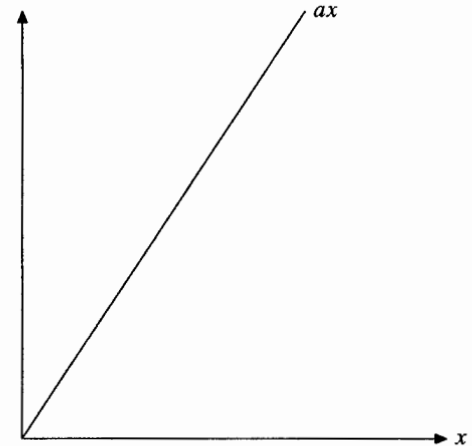
$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K} \right) - axy.$$

For sake of argument, suppose that the unit of time is hours. Then the units on dx/dt is prey per hour (prey/hr). Therefore, both terms on the right hand side of the equation must have the same units; i.e., prey per hour (prey/hr). Because the term axy has units prey per hour, then

$$\begin{aligned} ax \text{ prey}/(\text{predator hr}) \times y \text{ predators} \\ = axy \text{ prey/hr,} \end{aligned}$$

shows clearly that ax has units of prey per predator per hour. Therefore, ax is the *predation rate*.

- (b) Here is a sketch of the predation rate versus the prey x .

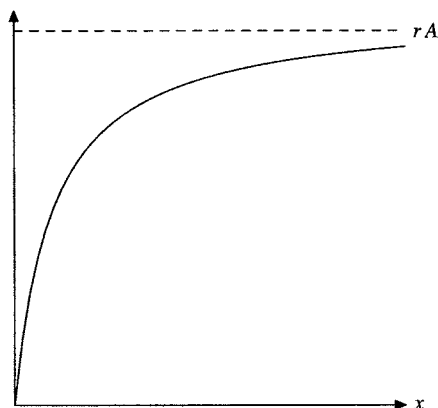


This graph indicates that the predation rate grows without bound as the prey increases. This is not a satisfactory model, because it does not take into consideration the fact that the predators will soon become satiated with the abundance of prey and the predation rate will slow.

- (c) First rewrite equation (8.9) as

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K} \right) - \frac{rAx}{x + B}y.$$

Thus, the predation rate is $rAx/(x + B)$. The sketch follows.



For small prey populations, the predation rate is low. This could be because it is hard to find the prey to the prey taken per predator per unit time is extremely low. However, as the prey increases, the number of encounters between predators and prey increase and the prey taken per predator per unit time increases. However, when the prey become abundant, the predators become satiated and the prey taken per predator per unit time begins to slow and level off.

3. In our derivation of the predator-prey equations, we assumed that the increase in predators would be proportional to the number of contacts between predators and prey, and therefore to the product xy . This lead us to the equation

$$y' = -sx + bxy,$$

where x is the prey and y the predators. Again, the term bxy assumes that the probability that a predator will attack is the same for every encounter. This may be reasonable for small prey populations, but as the prey increases, many encounters will occur when the predator has recently eaten and is not hungry. There are a number of ways to account for this, but assuming again that the predation is proportional to $xy/(x + D)$ for some constant D is a good way to satisfy this difficulty. Note that for large values of x , this quantity becomes approximately proportional to y , and not the larger quantity xy . Thus, we replace

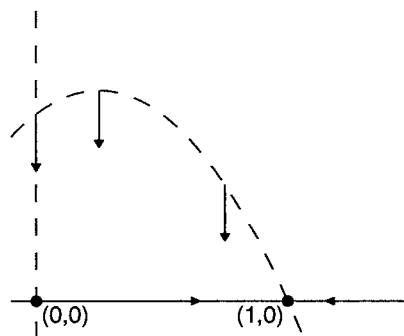
the term bxy with $cxy/(x + D)$ and write

$$\begin{aligned} y' &= -sy + \frac{cxy}{x + D} \\ y' &= -sy \left(1 - \frac{Cx}{x + D} \right). \end{aligned}$$

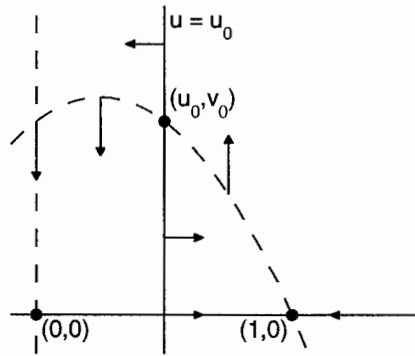
4. The u -nullcline has two pieces. The first is the v -axis defined by $u = 0$, and the second is the curve defined by $1 - u - v/(u + b) = 0$. If we solve the latter equation for v , we get $v = (u + b)(1 - u)$. This is the equation of a parabola with intercepts at $-b$ and 1 , and positive in between. This is true independent of the value of u_0 . The u -nullclines are plotted with dashed lines in the following figures.

The v -nullcline consists of the u -axis, defined by $v = 0$, and the line $u = u_0$. The v -nullclines are plotted with solid lines in the following figures.

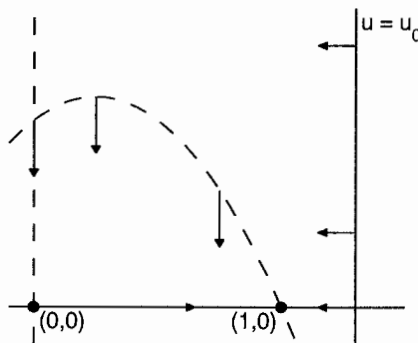
In the case when $u_0 < 0$, this portion of the v -nullcline does not affect the behavior for positive populations. When we compute the direction of the field along the other nullclines we get the result below.



In the case when $0 < u_0 < 1$ the line $u = u_0$ is important. The equilibrium point (u_0, v_0) is in the positive quadrant, and we get the information as plotted below.



In the case when $1 < u_0$ the line $u = u_0$ is again important. However, $v_0 < 0$ so the equilibrium point (u_0, v_0) is not in the positive quadrant. We get the information as plotted below.



5. When $u_0 < 0$, the parabolic portion of the u -nullcline divides the positive quadrant into two pieces. The direction of the nullclines show that the lower region is invariant, and that all solution curves in this region approach the equilibrium point at $(1, 0)$. All solution curves in the upper region move to the right and down, and are drawn into the lower region and then to $(1, 0)$, or are drawn directly to $(1, 0)$. Thus all solution curves end up at $(1, 0)$.

When $u_0 > 1$, the parabolic portion of the u -nullcline and the line $u = u_0$ divide the positive quadrant into three pieces. Again the region below the parabola is invariant, and all solutions are drawn to the equilibrium point at $(1, 0)$. All solution curves starting to the left of the line $u = u_0$ move to the left and ultimately into the region between the line $u = u_0$ and the parabola. Once there they move to the left and down, either into the region below the parabola and then on to $u = u_0$, or they move directly to $u = u_0$. Again all solution curves approach the equilibrium point at $(1, 0)$.

6. (a) If it takes N prey to support one predator, then one prey can support $1/N$ predator, and x prey can support x/N predators.
(b) We should set $K = x/N$. Then (8.16) becomes

$$y' = sy \left(1 - \frac{Ny}{x} \right).$$

Chapter 11. Series Solutions to Differential Equations

Section 11.1. Review of Power Series

1. If $\sum_{n=0}^{\infty} x^n/(n+1)$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{n+2}} \\ &= \lim_{n \rightarrow \infty} \frac{1+2/n}{1+1/n} = 1.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

2. If $\sum_{n=0}^{\infty} x^n/\sqrt{n+3}$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n+3}}}{\frac{1}{\sqrt{n+4}}} \\ &= \lim_{n \rightarrow \infty} \sqrt{\frac{1+4/n}{1+3/n}} = 1.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

3. If $\sum_{n=0}^{\infty} nx^n$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1+1/n} = 1.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

4. If $\sum_{n=0}^{\infty} \sqrt{n}x^{n+1}$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+1}} \\ &= \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1+1/n}} = 1.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

5. If $\sum_{n=0}^{\infty} x^{n+1}/n!$, then

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n \rightarrow \infty} (n+1) = \infty.$$

Thus, the radius of convergence is $R = \infty$; i.e., the series converges for all x .

6. If $\sum_{n=0}^{\infty} (x-2)^n/(n^2+1)$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2+1}}{\frac{1}{(n+1)^2+1}} \\ &= \lim_{n \rightarrow \infty} \frac{n^2+2n+2}{n^2+1} \\ &= \lim_{n \rightarrow \infty} \frac{1+2/n+2/n^2}{1+1/n^2} = 1.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

7. If $\sum_{n=0}^{\infty} n!(x-1)^n$, then

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Therefore, the series diverges for all $x \neq 0$.

8. If $\sum_{n=0}^{\infty} [2^n/(n+1)]x^n$, then

$$\frac{|a_n|}{|a_{n+1}|} = \frac{\frac{2^n}{n+1}}{\frac{2^{n+1}}{n+2}} = \frac{n+2}{n+1} \cdot \frac{2^n}{2^{n+1}} = \frac{1+2/n}{1+1/n} \cdot \frac{1}{2}.$$

Thus, $\lim_{n \rightarrow \infty} |a_n|/|a_{n+1}| = 1/2$. Therefore, the radius of convergence is $R = 1/2$.

9. If
- $\sum_{n=2}^{\infty} x^n / \ln n$
- , then

$$\frac{|a_n|}{|a_{n+1}|} = \frac{\frac{1}{\ln n}}{\frac{1}{\ln(n+1)}} = \frac{\ln(n+1)}{\ln n}.$$

As $n \rightarrow \infty$, both numerator and denominator go to infinity, so we can apply L'Hôpital's rule.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} &= \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n} = 1. \end{aligned}$$

Thus, the radius of convergence is $R = 1$.

10. If
- $\sum_{n=0}^{\infty} [3^n/n!](x+2)^n$
- , then

$$\frac{|a_n|}{|a_{n+1}|} = \frac{3^n}{n!} \cdot \frac{(n+1)!}{3^{n+1}} = \frac{n+1}{3}.$$

Thus, $\lim_{n \rightarrow \infty} |a_n|/|a_{n+1}| = \infty$ and the radius of convergence is $R = \infty$; i.e., the series converges for all x .

11. If
- $\sum_{n=1}^{\infty} nx^n/(1 \cdot 3 \cdot 5 \cdots (2n-1))$
- , note that

$$\begin{aligned} a_n &= \frac{n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\ &= \frac{n(2 \cdot 4 \cdot 6 \cdots (2n-2))}{(2n-1)!} \\ &= \frac{n \cdot 2^{n-1} (1 \cdot 2 \cdot 3 \cdots (n-1))}{(2n-1)!} \\ &= \frac{2^{n-1} n!}{(2n-1)!}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{|a_n|}{|a_{n+1}|} &= \frac{2^{n-1} n!}{(2n-1)!} \cdot \frac{(2n+1)!}{2^n (n+1)!} = \frac{2n(2n+1)}{2(n+1)} \\ &= \frac{2n+1}{1+1/n}. \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} |a_n|/|a_{n+1}| = \infty$ and the radius of convergence is $R = \infty$; i.e., the series converges for all x .

12. If
- $\sum_{n=0}^{\infty} (n!x^n)/(2n)!$
- , then

$$\begin{aligned} \frac{|a_n|}{|a_{n+1}|} &= \frac{n!}{(2n)!} \cdot \frac{(2n+2)!}{(n+1)!} = \frac{(2n+2)(2n+1)}{n+1} \\ &= 2(2n+1). \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} |a_n|/|a_{n+1}| = \infty$ and the radius of convergence is $R = \infty$; i.e., the series converges for all x .

13. We know that

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots + \frac{(-1)^n t^{2n}}{(2n)!} + \cdots$$

and the radius of convergence is $R = \infty$; i.e., the series converges for all x . Thus, letting $t = 2x$,

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \cdots + \frac{(-1)^n (2x)^{2n}}{(2n)!} + \cdots$$

and the series converges for all x .

14. We know that

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots + \frac{(-1)^n t^{2n+1}}{(2n+1)!} + \cdots$$

and the radius of convergence is $R = \infty$; i.e., the series converges for all t . Thus, letting $t = x^2$,

$$\begin{aligned} \sin x^2 &= x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots \\ &\quad + \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} + \cdots \\ &= x^2 - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{4n+2}}{(2n+1)!} + \cdots \end{aligned}$$

and the series converges for all x .

15. Derivatives reveal the following details.

$$\begin{aligned} f(x) &= \sin x \Rightarrow f(\pi) = 0 \\ f'(x) &= \cos x \Rightarrow f'(\pi) = -1 \\ f''(x) &= -\sin x \Rightarrow f''(\pi) = 0 \\ f'''(x) &= -\cos x \Rightarrow f'''(\pi) = 1, \end{aligned}$$

after which this same pattern repeats itself in blocks of four. Therefore,

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!} (x - \pi)^n \\ &= f(\pi) + f'(\pi)(x - \pi) + \frac{f''(\pi)}{2!} (x - \pi)^2 \\ &\quad + \frac{f'''(\pi)}{3!} (x - \pi)^3 + \cdots \\ &= -1(x - \pi) + \frac{1}{3!}(x - \pi)^3 - \frac{1}{5!}(x - \pi)^5 \\ &\quad + \frac{1}{7!}(x - \pi)^7 - \cdots \\ &\quad + \frac{(-1)^{n+1}}{(2n+1)!} (x - \pi)^{2n+1} + \cdots.\end{aligned}$$

Alternatively,

$$\begin{aligned}\sin x &= \sin(x - \pi + \pi) \\ &= \sin(x - \pi) \cos \pi + \sin \pi \cos(x - \pi) \\ &= -\sin(x - \pi).\end{aligned}$$

Now let $t = x - \pi$ in the series.

$$-\sin t = -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots + \frac{(-1)^{n+1}t^{2n+1}}{(2n+1)!} + \cdots.$$

Thus,

$$\begin{aligned}\sin x &= -\sin(x - \pi) \\ &= -(x - \pi) + \frac{1}{3!}(x - \pi)^3 - \frac{1}{5!}(x - \pi)^5 + \cdots \\ &\quad + \frac{(-1)^{n+1}}{(2n+1)!} (x - \pi)^{2n+1} + \cdots.\end{aligned}$$

The radius of convergence is $R = \infty$.

16. Derivatives reveal the following details.

$$\begin{aligned}f(x) &= \ln x \Rightarrow f(1) = 0 \\ f'(x) &= \frac{1}{x} \Rightarrow f'(1) = 1 \\ f''(x) &= -\frac{1}{x^2} \Rightarrow f''(1) = -1 \\ f'''(x) &= \frac{2}{x^3} \Rightarrow f'''(1) = 2 \\ f^{(4)}(x) &= -\frac{2 \cdot 3}{x^4} \Rightarrow f^{(4)}(1) = -2 \cdot 3 \\ f^{(5)}(x) &= \frac{2 \cdot 3 \cdot 4}{x^5} \Rightarrow f^{(5)}(1) = 2 \cdot 3 \cdot 4\end{aligned}$$

We conjecture that for $n \geq 1$,

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n},$$

so

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!.$$

Hence,

$$\begin{aligned}\ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n \\ &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!} (x-1)^2 \\ &\quad + \frac{f'''(1)}{3!} (x-1)^3 + \cdots + \frac{f^{(n)}(1)}{n!} (x-1)^n \\ &\quad + \cdots \\ &= 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots \\ &\quad + \frac{(-1)^{n-1}}{n} (x-1)^n + \cdots,\end{aligned}$$

for $n \geq 1$. That is,

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n.$$

If we shift the index,

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}.$$

Alternatively, we can expand $1/(1+t)$ in a geometric series.

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots$$

Integrate.

$$\begin{aligned}\ln(1+t) &= C + t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \cdots \\ &\quad + \frac{(-1)^n}{n+1}t^{n+1} + \cdots\end{aligned}$$

At $t = 0$,

$$\begin{aligned}\ln(1+0) &= C + 0 \\ C &= 0.\end{aligned}$$

Let $t = x - 1$.

$$\begin{aligned}\ln x &= \ln(1 + (x - 1)) \\ &= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 \\ &\quad - \frac{1}{4}(x - 1)^4 + \cdots \\ &\quad + \frac{(-1)^n}{n+1}(x - 1)^{n+1} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x - 1)^{n+1}.\end{aligned}$$

The ratio test shows that the radius of convergence is $R = 1$.

17. Derivatives reveal the following details.

$$\begin{aligned}f(x) &= \frac{1}{x} \Rightarrow f(3) = \frac{1}{3} \\ f'(x) &= -\frac{1}{x^2} \Rightarrow f'(3) = -\frac{1!}{3^2} \\ f''(x) &= \frac{2}{x^3} \Rightarrow f''(3) = \frac{2!}{3^3} \\ f'''(x) &= -\frac{3 \cdot 2}{x^4} \Rightarrow f'''(3) = -\frac{3!}{3^4} \\ f^{(4)}(x) &= \frac{4 \cdot 3 \cdot 2}{x^5} \Rightarrow f^{(4)}(3) = \frac{4!}{3^5}.\end{aligned}$$

We conjecture that

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \Rightarrow f^{(n)}(3) = \frac{(-1)^n n!}{3^{n+1}},$$

for $n \geq 0$. Thus,

$$\begin{aligned}\frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x - 3)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x - 3)^n \\ &= \frac{1}{3} - \frac{1}{3^2}(x - 3) + \frac{1}{3^3}(x - 3)^2 - \frac{1}{3^4}(x - 3)^3 \\ &\quad + \cdots + \frac{(-1)^n}{3^{n+1}}(x - 3)^n + \cdots.\end{aligned}$$

The ratio test shows that the radius of convergence is $R = 1$.

18. Taking derivatives is prohibitively difficult. However,

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x),$$

provided both numerator and denominator are positive.

Expand $1/(1+t)$ in a geometric series.

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots$$

Integrate.

$$\ln(1+t) = C + t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \cdots + \frac{(-1)^n}{n+1}t^{n+1} + \cdots$$

At $t = 0$,

$$\begin{aligned}\ln(1+0) &= C + 0 \\ C &= 0.\end{aligned}$$

The mathematics here is valid provided $|t| < 1$. Thus, if $t = x$, then

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^n}{n+1}x^{n+1} + \cdots,$$

and with $t = -x$,

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^{n+1}}{n+1} - \cdots$$

Thus,

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots \\ &\quad + \frac{2}{2n+1}x^{2n+1} + \cdots,\end{aligned}$$

valid for $|x| < 1$. The ratio test shows that the radius of convergence is $R = 1$.

19. We have

$$\begin{aligned}f(x) &= (1+x) \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^n + x \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1} \\ &= \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} x^n \\ &= 1 + 2 \sum_{n=1}^{\infty} x^n.\end{aligned}$$

20.

$$\begin{aligned}f(x) &= (1+x^2) \sum_{n=1}^{\infty} \frac{x^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n} + x^2 \sum_{n=1}^{\infty} \frac{x^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n} + \sum_{n=1}^{\infty} \frac{x^{n+2}}{n} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n} + \sum_{n=3}^{\infty} \frac{x^n}{n-2} \\ &= x + \frac{x^2}{2} + \sum_{n=3}^{\infty} \left(\frac{1}{n} + \frac{1}{n-2}\right) x^n\end{aligned}$$

21.

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} x^n - \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \\ &= \sum_{n=0}^{\infty} 0 \frac{x^n}{n} - \sum_{n=0}^{\infty} \frac{x^n}{n+1} \\ &= \sum_{n=0}^{\infty} \left(1 - \frac{1}{n+1}\right) x^n \\ &= \sum_{n=0}^{\infty} \frac{n}{n+1} x^n\end{aligned}$$

22.

$$\begin{aligned}f(x) &= (1+x^2) \sin x \\ &= (1+x^2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+3} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n+1} \\ &= x + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{(2n+1)!} - \frac{1}{(2n-1)!} \right] x^{2n+1}\end{aligned}$$

23.

$$\begin{aligned}f(x) &= e^x - e^{-x} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\ &= \sum_{n=0}^{\infty} [1 - (-1)^n] \frac{x^n}{n!} \\ &= 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\end{aligned}$$

24.

$$\begin{aligned}
 f(x) &= e^x + \sum_{n=2}^{\infty} \frac{x^{n-2}}{n(n+1)} \\
 &= \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{(n+2)(n+3)} \\
 &= \sum_{n=0}^{\infty} \left[\frac{1}{n!} + \frac{1}{(n+2)(n+3)} \right] x^n
 \end{aligned}$$

25. The derivatives reveal the following details.

$$\begin{aligned}
 f(x) &= \frac{1}{3+x} \Rightarrow f(0) = \frac{1}{3} \\
 f'(x) &= \frac{-1}{(3+x)^2} \Rightarrow f'(0) = -\frac{1}{3^2} \\
 f''(x) &= \frac{2}{(3+x)^3} \Rightarrow f''(0) = \frac{2!}{3^3} \\
 f'''(x) &= \frac{-3 \cdot 2}{(3+x)^4} \Rightarrow f'''(0) = \frac{-3!}{3^4}.
 \end{aligned}$$

We conjecture that

$$f^{(n)}(x) = \frac{(-1)^n n!}{(3+x)^{n+1}} \Rightarrow f^{(n)}(0) = \frac{(-1)^n n!}{3^{n+1}}.$$

Thus,

$$\begin{aligned}
 \frac{1}{3+x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
 &= \frac{1}{3} - \frac{1}{3^2} x + \frac{1}{3^3} x^2 - \frac{1}{3^4} x^3 + \cdots \\
 &\quad + \frac{(-1)^n}{3^{n+1}} x^n + \cdots.
 \end{aligned}$$

Alternatively, we can expand in a geometric series.

$$\begin{aligned}
 \frac{1}{3+x} &= \frac{1}{3} \cdot \frac{1}{1 - (-x/3)} \\
 &= \frac{1}{3} \left[1 - \frac{x}{3} + \frac{x^2}{3^2} - \frac{x^3}{3^3} + \cdots \right. \\
 &\quad \left. + \frac{(-1)^n x^n}{3^n} + \cdots \right] \\
 &= \frac{1}{3} - \frac{x}{3^2} + \frac{x^2}{3^3} - \frac{x^3}{3^4} + \cdots \\
 &\quad + \frac{(-1)^n x^n}{3^{n+1}} + \cdots
 \end{aligned}$$

The argument is valid provided $|x/3| < 1$, or, equivalently, $|x| < 3$. The radius of convergence is $R = 3$.

26. The derivatives reveal the following details.

$$\begin{aligned}
 f(x) &= \frac{1}{x} \Rightarrow f(2) = \frac{1}{2} \\
 f'(x) &= -\frac{1}{x^2} \Rightarrow f'(2) = -\frac{1}{2^2} \\
 f''(x) &= \frac{2}{x^3} \Rightarrow f''(2) = \frac{2!}{2^3} \\
 f'''(x) &= \frac{-3 \cdot 2}{x^4} \Rightarrow f'''(2) = \frac{-3!}{2^4}.
 \end{aligned}$$

We conjecture that

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \Rightarrow f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}.$$

Thus,

$$\begin{aligned}
 \frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n \\
 &= \frac{1}{2} - \frac{1}{2^2} (x-2) + \frac{1}{2^3} (x-2)^2 - \frac{1}{2^4} (x-2)^3 \\
 &\quad + \cdots + \frac{(-1)^n}{2^{n+1}} (x-2)^n + \cdots.
 \end{aligned}$$

Alternatively, we can expand in a geometric series.

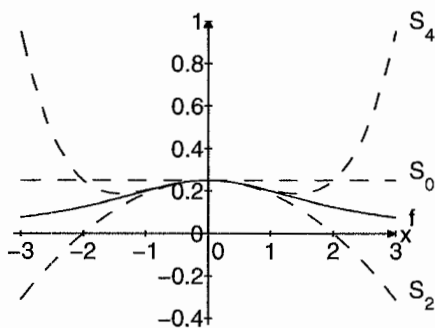
$$\begin{aligned}
 \frac{1}{x} &= \frac{1}{2 + (x-2)} \\
 &= \frac{1}{2} \cdot \frac{1}{1 + (x-2)/2} \\
 &= \frac{1}{2} \left\{ 1 - \frac{x-2}{2} + \frac{(x-2)^2}{2^2} - \cdots + \right. \\
 &\quad \left. \frac{(-1)^n (x-2)^n}{2^n} + \cdots \right\} \\
 &= \frac{1}{2} - \frac{1}{2^2} (x-2) + \frac{1}{2^3} (x-2)^2 - \frac{1}{2^4} (x-2)^3 \\
 &\quad + \cdots + \frac{(-1)^n}{2^{n+1}} (x-2)^n + \cdots
 \end{aligned}$$

This argument is valid provided $|(x-2)/2| < 1$, or equivalently, $|x-2| < 2$. Thus, the radius of convergence is $R = 2$.

27. We expand in a geometric series.

$$\begin{aligned}\frac{1}{4+x^2} &= \frac{1}{4} \cdot \frac{1}{1+x^2/4} \\ &= \frac{1}{4} \left\{ 1 - \frac{x^2}{4} + \frac{x^4}{4^2} - \frac{x^6}{4^3} + \cdots \right. \\ &\quad \left. + \frac{(-1)^n x^{2n}}{4^n} + \cdots \right\} \\ &= \frac{1}{4} - \frac{x^2}{4^2} + \frac{x^4}{4^3} - \frac{x^6}{4^4} + \cdots \\ &\quad + \frac{(-1)^n x^{2n}}{4^{n+1}} + \cdots,\end{aligned}$$

provided $|x| < 2$.



28. We know that

$$\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \cdots + \frac{(-1)^{n-1}}{n}t^n + \cdots,$$

for $n \geq 1$. Thus,

$$\begin{aligned}\ln(1-x^2) &= \ln(1+(-x^2)) \\ &= -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \cdots \\ &\quad - \frac{1}{n}x^{2n} - \cdots,\end{aligned}$$

provided $|x| \leq 1$.

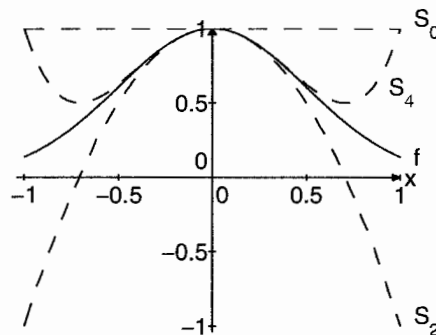
29. We know that

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots.$$

Thus,

$$\begin{aligned}e^{-2x^2} &= 1 - 2x^2 + \frac{2^2x^4}{2!} - \frac{2^3x^6}{3!} + \cdots \\ &\quad + \frac{(-1)^n 2^n x^{2n}}{n!} + \cdots,\end{aligned}$$

valid for all x .



30. Using the series expansion for e^x ,

$$\begin{aligned}\sinh 3x &= \frac{e^{3x} - e^{-3x}}{2} \\ &= \frac{1}{2} \left\{ \left(1 + 3x + \frac{3^2x^2}{2!} + \cdots + \frac{3^n x^n}{n!} + \cdots \right) \right. \\ &\quad \left. - \left(1 - 3x + \frac{3^2x^2}{2!} - \cdots + \frac{(-1)^n 3^n x^n}{n!} + \cdots \right) \right\} \\ &= \frac{1}{2} \left\{ 2(3x) + 2 \left(\frac{3^3 x^3}{3!} \right) + \cdots \right. \\ &\quad \left. + 2 \left(\frac{3^{2n+1} x^{2n+1}}{(2n+1)!} \right) + \cdots \right\} \\ &= 3x + \frac{3^3 x^3}{3!} + \cdots + \frac{3^{2n+1} x^{2n+1}}{(2n+1)!} + \cdots,\end{aligned}$$

valid for all x .

31. Derivatives reveal these details.

$$\begin{aligned}f(x) &= \frac{1}{x^2} \Rightarrow f(2) = \frac{1}{2^2} \\f'(x) &= \frac{-2}{x^3} \Rightarrow f'(2) = \frac{-2}{2^3} \\f''(x) &= \frac{3 \cdot 2}{x^4} \Rightarrow f''(2) = \frac{3!}{2^4} \\f'''(x) &= \frac{-4 \cdot 3 \cdot 2}{x^5} \Rightarrow f'''(2) = \frac{-4!}{2^5}\end{aligned}$$

We conjecture that

$$f^{(n)}(x) = \frac{(-1)^n(n+1)!}{x^{n+2}},$$

so

$$f^{(n)}(2) = \frac{(-1)^n(n+1)!}{2^{n+2}}.$$

Thus,

$$\begin{aligned}\frac{1}{x^2} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n \\&= \frac{1}{2^2} - \frac{2}{2^3}(x-2) + \frac{3}{2^4}(x-2)^2 \\&\quad - \cdots + \frac{(-1)^n(n+1)}{2^{n+2}}(x-2)^n + \cdots.\end{aligned}$$

Note that

$$\frac{|a_n|}{|a_{n+1}|} = \frac{2(n+1)}{n+2} \rightarrow 2$$

as $n \rightarrow \infty$. Thus, the radius of convergence is $R = 2$.

32. Derivatives reveal these details.

$$\begin{aligned}f(x) &= \frac{1}{x^4} \Rightarrow f(1) = 1 \\f'(x) &= \frac{-4}{x^5} \Rightarrow f'(1) = \frac{-4!}{3 \cdot 2} \\f''(x) &= \frac{5 \cdot 4}{x^6} \Rightarrow f''(1) = 5 \cdot 4 = \frac{5!}{3!} \\f'''(x) &= \frac{-6 \cdot 5 \cdot 4}{x^7} \Rightarrow f'''(1) = -6 \cdot 5 \cdot 4 = \frac{6!}{3!}\end{aligned}$$

We conjecture that

$$f^{(n)}(x) = \frac{(-1)^n(n+3)!}{6x^{n+4}},$$

so

$$f^{(n)}(1) = \frac{(-1)^n(n+3)!}{3!}.$$

Thus,

$$\begin{aligned}\frac{1}{x^4} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n \\&= 1 - \frac{4!}{3!}(x-1) + \frac{5!}{2!3!}(x-1)^2 \\&\quad - \cdots + \frac{(-1)^n(n+3)!}{n!3!}(x-1)^n + \cdots.\end{aligned}$$

Readers might recognize the binomial coefficients.

$$\begin{aligned}\frac{1}{x^4} &= \binom{3}{3} - \binom{4}{3}(x-1) + \binom{5}{3}(x-1)^2 \\&\quad - \binom{6}{3}(x-1)^3 + \cdots \\&\quad + (-1)^n \binom{n+3}{3}(x-1)^n + \cdots\end{aligned}$$

33. Start with the geometric series,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

valid for $|x| < 1$. Differentiate.

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Shift the index of the sum on the right.

$$\begin{aligned}\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} (n+1)x^n \\ \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n \\ \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^n + \frac{1}{1-x}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} nx^n &= \frac{1}{(1-x)^2} - \frac{1}{1-x} \\ \sum_{n=0}^{\infty} nx^n &= \frac{1 - (1-x)}{(1-x)^2} \\ \sum_{n=0}^{\infty} nx^n &= \frac{x}{(1-x)^2},\end{aligned}$$

provided $|x| < 1$.

34. Integrate.

$$\begin{aligned}\sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} \\ \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} &= -\ln|1-x| + C\end{aligned}$$

Let $x = 0$ in the above.

$$\begin{aligned}0 &= -\ln|1-0| + C \\ C &= 0\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|.$$

Multiply both sides by $1/x$.

$$\begin{aligned}\frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} &= -\frac{1}{x} \ln|1-x| \\ \sum_{n=0}^{\infty} \frac{x^n}{n+1} &= -\frac{1}{x} \ln|1-x|.\end{aligned}$$

This argument is valid provided $|x| < 1$.

35. If $f(x) = \tan x$, then

$$\begin{aligned}f'(x) &= \sec^2 x, \\ f''(x) &= 2 \sec^2 x \tan x, \quad \text{and} \\ f'''(x) &= 2 \sec^2 x (1 + 3 \tan^2 x).\end{aligned}$$

Thus,

$$\begin{aligned}f(x) &\approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &\approx x + \frac{x^3}{3}.\end{aligned}$$

36. We know that

$$\begin{aligned}e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots\end{aligned}$$

The product is $f(x) = e^{x^2} \cos x$, where the first four terms are as follows.

The first term is 1. The second term is

$$\frac{-x^2}{2!} + x^2 = \frac{1}{2}x^2.$$

The third term is

$$\frac{x^4}{2!} - \frac{x^4}{2!} + \frac{x^4}{4!} = \frac{x^4}{24}.$$

The fourth term is

$$\frac{x^6}{3!} - \frac{x^6}{2!2!} + \frac{x^6}{4!} - \frac{x^6}{6!} = -\frac{31}{720}x^6.$$

Thus,

$$e^{x^2} \cos x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{31}{720}x^6 + \cdots$$

37. We're given $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and we wish to find

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n x^n,$$

or equivalently

$$\begin{aligned}1 &= f(x) \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n \\ &= \sum_{n=0}^{\infty} c_n x^n,\end{aligned}$$

where

$$c_n = \sum_{k=0}^n a_{n-k} b_k.$$

Thus,

$$1 = a_0 b_0$$

$$b_0 = \frac{1}{a_0},$$

provided $a \neq 0$. The remaining coefficients on the right must equal zero. Thus,

$$0 = a_1 b_0 + a_0 b_1$$

$$b_1 = -\frac{a_1 b_0}{a_0}.$$

And, in general,

$$0 = a_n b_0 + a_{n-1} b_1 + \cdots + a_1 b_{n-1} + a_0 b_n$$

$$b_n = -\frac{a_1 b_{n-1} + \cdots + a_n b_0}{a_0},$$

for $n \geq 1$.

Section 11.2. Series Solutions near Ordinary Points

1. The equation $y' = 3y$ is separable with solution $y(x) = Ce^{3x}$. With $y(0) = a_0$, this becomes $y(x) = a_0 e^{3x}$. We look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating, then shifting the index,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Substituting,

$$\begin{aligned} 0 &= y' - 3y \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+1) a_{n+1} - 3a_n] x^n. \end{aligned}$$

Setting the coefficients equal to zero, we have the recurrence formula,

$$\begin{aligned} (n+1) a_{n+1} - 3a_n &= 0 \\ a_{n+1} &= \frac{3a_n}{n+1}, \quad n \geq 0. \end{aligned}$$

Thus, with $y(0) = a_0$,

$$a_1 = 3a_0,$$

$$a_2 = \frac{3}{2} a_1 = \frac{3 \cdot 3}{2} a_0,$$

$$a_3 = \frac{3}{3} a_2 = \frac{3 \cdot 3 \cdot 3}{2 \cdot 3} a_0,$$

$$a_4 = \frac{3}{4} a_3 = \frac{3 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 4} a_0.$$

We conjecture that the general coefficient is

$$a_n = \frac{3^n}{n!} a_0.$$

Thus,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n \\ &= a_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = a_0 e^{3x}. \end{aligned}$$

This agrees nicely with the separable solution above.

2. The equation $y' + 2y = 0$ is separable with solution $y(x) = Ce^{-2x}$. With $y(0) = a_0$, this becomes $y(x) = a_0 e^{-2x}$. We look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating, then shifting the index,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Substituting,

$$\begin{aligned} 0 &= y' + 2y \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+1) a_{n+1} + 2a_n] x^n. \end{aligned}$$

Setting the coefficients equal to zero, we have the recurrence formula

$$\begin{aligned} (n+1) a_{n+1} + 2a_n &= 0 \\ a_{n+1} &= \frac{-2a_n}{n+1}, \quad n \geq 0. \end{aligned}$$

Thus, with $y(0) = a_0$,

$$\begin{aligned} a_1 &= -2a_0, \\ a_2 &= \frac{-2a_1}{2} = \frac{2 \cdot 2}{2} a_0, \\ a_3 &= \frac{-2a_2}{3} = \frac{-2 \cdot 2 \cdot 2}{3 \cdot 2} a_0, \\ a_4 &= \frac{-2a_3}{4} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{4 \cdot 3 \cdot 2} a_0. \end{aligned}$$

We conjecture that the general coefficient is

$$a_n = \frac{(-1)^n 2^n}{n!} a_0.$$

Thus,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} a_0 x^n \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} \\ &= a_0 e^{-2x}. \end{aligned}$$

This agrees nicely with the separable solution above.

3. The equation $y' = -x^3 y$ is separable.

$$\begin{aligned} \frac{dy}{y} &= -x^3 dx \\ \ln y &= -\frac{1}{4} x^4 + C \\ y &= C e^{-x^4/4}. \end{aligned}$$

With $y(0) = a_0$, this gives $y = a_0 e^{-x^4/4}$. We look for a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned} 0 &= y' + x^3 y \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} + x^3 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+3}. \end{aligned}$$

To add these two series, we shift the index in the first one so that powers of x are the same.

$$\begin{aligned} \sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n'=-3}^{\infty} (n'+4) a_{n'+4} x^{n'+3} \\ &= \sum_{n=-3}^{\infty} (n+4) a_{n+4} x^{n+3} \end{aligned}$$

Thus,

$$\begin{aligned}
 0 &= \sum_{n=-3}^{\infty} (n+4)a_{n+4}x^{n+3} + \sum_{n=0}^{\infty} a_n x^{n+3} \\
 &= a_1 + 2a_2x + 3a_3x^2 + \sum_{n=0}^{\infty} (n+4)a_{n+4}x^{n+3} \\
 &\quad + \sum_{n=0}^{\infty} a_n x^{n+3} \\
 &= a_1 + 2a_2x + 3a_3x^2 \\
 &\quad + \sum_{n=0}^{\infty} [(n+4)a_{n+4} + a_n] x^{n+3}.
 \end{aligned}$$

Setting coefficients equal to zero, $a_1 = a_2 = a_3 = 0$ and the recurrence formula

$$\begin{aligned}
 (n+4)a_{n+4} + a_n &= 0 \\
 a_{n+4} &= \frac{-a_n}{n+4}, \quad n \geq 0.
 \end{aligned}$$

Thus, with $y(0) = a_0$,

$$\begin{aligned}
 a_4 &= -\frac{1}{4}a_0 \\
 a_5 &= -\frac{1}{5}a_1 = 0 \\
 a_6 &= -\frac{1}{6}a_2 = 0 \\
 a_7 &= -\frac{1}{7}a_3 = 0 \\
 a_8 &= -\frac{1}{8}a_4 = \frac{1}{4 \cdot 8}a_0.
 \end{aligned}$$

You can see that every fourth term is meaningful, with terms in between equaling zero. Thus,

$$\begin{aligned}
 a_{12} &= -\frac{1}{12}a_8 = -\frac{1}{4 \cdot 8 \cdot 12}a_0 \\
 a_{16} &= -\frac{1}{16}a_{12} = \frac{1}{4 \cdot 8 \cdot 12 \cdot 16}a_0.
 \end{aligned}$$

Note that

$$a_{16} = a_{4(4)} = \frac{1}{4^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4} a_0.$$

We conjecture that

$$a_{4n} = \frac{(-1)^n}{4^n n!} a_0.$$

Because intervening terms are zero,

$$\begin{aligned}
 y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
 &= \sum_{n=0}^{\infty} a_{4n} x^{4n} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} a_0 x^{4n} \\
 &= a_0 \sum_{n=0}^{\infty} \frac{(-x^4/4)^n}{n!} \\
 &= a_0 e^{-x^4/4}.
 \end{aligned}$$

This agrees nicely with the separable solution found above.

4. The equation $y' = x^2 y$ is separable.

$$\begin{aligned}
 \frac{dy}{y} &= x^2 dx \\
 \ln y &= \frac{1}{3}x^3 + C \\
 y &= C e^{x^3/3}
 \end{aligned}$$

With $y(0) = a_0$, this becomes $y(x) = a_0 e^{x^3/3}$. We seek a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned}
 0 &= y' - x^2 y \\
 &= \sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n \\
 &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+2}.
 \end{aligned}$$

To add these two series, we shift the index of the first one so that the powers of x are the same.

$$\begin{aligned}\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n'=-2}^{\infty} (n'+3) a_{n'+3} x^{n'+2} \\ &= \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2}\end{aligned}$$

Thus,

$$\begin{aligned}0 &= \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= a_1 + 2a_2 x + \sum_{n=0}^{\infty} (n+3) a_{n+3} x^{n+2} \\ &\quad - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= a_1 + 2a_2 x + \sum_{n=0}^{\infty} [(n+3) a_{n+3} - a_n] x^{n+2}.\end{aligned}$$

Setting coefficients equal to zero, $a_1 = a_2 = 0$ and the recurrence formula,

$$\begin{aligned}(n+3) a_{n+3} - a_n &= 0 \\ a_{n+3} &= \frac{a_n}{n+3}, \quad n \geq 0.\end{aligned}$$

Thus, with $y(0) = a_0$,

$$\begin{aligned}a_3 &= \frac{1}{3} a_0 \\ a_4 &= \frac{1}{4} a_1 = 0 \\ a_5 &= \frac{1}{5} a_2 = 0 \\ a_6 &= \frac{1}{6} a_3 = \frac{1}{3 \cdot 6} a_0.\end{aligned}$$

You can see that every third term is meaningful with terms inbetween equalling zero. Thus,

$$\begin{aligned}a_6 &= \frac{1}{9} a_3 = \frac{1}{3 \cdot 6 \cdot 9} a_0, \\ a_{12} &= \frac{1}{12} a_9 = \frac{1}{3 \cdot 6 \cdot 9 \cdot 12} a_0.\end{aligned}$$

Note that

$$a_{12} = a_{3(4)} = \frac{1}{3^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4} a_0.$$

We conjecture that

$$a_{3n} = \frac{1}{3^n \cdot n!} a_0.$$

Because intervening terms are zero,

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} a_{3n} x^{3n} \\ &= \sum_{n=0}^{\infty} \frac{1}{3^n \cdot n!} a_0 x^{3n} \\ &= a_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} \\ &= a_0 e^{x^3/3}.\end{aligned}$$

This agrees nicely with the separable solution found above.

5. The equation $(1-x)y' + y = 0$ is separable with solution

$$\begin{aligned}(1-x) \frac{dy}{dx} &= -y \\ \frac{dy}{y} &= \frac{dx}{x-1} \\ \ln y &= \ln(x-1) + C \\ y &= C(x-1).\end{aligned}$$

With $y(0) = a_0$, $C = -a_0$ and $y = a_0(1-x)$. We seek a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and derivative

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned} 0 &= (1-x)y' + y \\ &= (1-x) \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Shifting the index on the first term and noting that the second term

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

we write

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+1) a_{n+1} + (1-n) a_n] x^n. \end{aligned}$$

Setting the coefficients equal to zero,

$$\begin{aligned} (n+1) a_{n+1} + (1-n) a_n &= 0 \\ a_{n+1} &= \frac{n-1}{n+1} a_n, \quad n \geq 0. \end{aligned}$$

Thus, a_0 is free and

$$\begin{aligned} a_1 &= (-1) a_0 \\ a_2 &= 0 a_1 = 0 \\ a_3 &= \frac{1}{3} a_2 = 0. \end{aligned}$$

Indeed, $a_n = 0, n \geq 2$. Thus,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x = a_0(1-x),$$

comparing nicely with the separable solution above.

6. The equation $(1+x)y' - y = 0$ is separable and

$$\begin{aligned} (1+x) \frac{dy}{dx} &= y \\ \frac{dy}{y} &= \frac{dx}{x+1} \\ \ln y &= \ln(x+1) + C \\ y &= C(x+1), \end{aligned}$$

with $y(0) = a_0$, $C = a_0$ and $y(x) = a_0(1+x)$. We seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

with derivative

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned} 0 &= (1+x)y' - y \\ &= (1+x) \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Shifting the index of the first term and noting that the second term

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n,$$

we write

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+1) a_{n+1} + (n-1) a_n] x^n. \end{aligned}$$

Setting coefficients equal to zero,

$$\begin{aligned} (n+1) a_{n+1} + (n-1) a_n &= 0 \\ a_{n+1} &= \frac{1-n}{n+1} a_n, \quad n \geq 0. \end{aligned}$$

Thus,

$$\begin{aligned} a_1 &= a_0 \\ a_2 &= 0 a_1 = 0 \\ a_3 &= -\frac{1}{3} a_2 = 0 \end{aligned}$$

and note that $a_n = 0$ for $n \geq 2$. Thus,

$$\begin{aligned} y(x) &= a_0 + a_1x \\ &= a_0 + a_0x \\ &= a_0(1 + x), \end{aligned}$$

which agrees with the separable solution found above.

7. The equation is separable.

$$\begin{aligned} (x-4) \frac{dy}{dx} &= -y \\ \frac{dy}{y} &= \frac{-dx}{x-4} \\ \ln y &= -\ln(x-4) + C \\ \ln y &= \ln \frac{D}{x-4} \\ y &= \frac{D}{x-4} \end{aligned}$$

With $y(0) = a_0$,

$$\begin{aligned} a_0 &= \frac{D}{0-4} \\ D &= -4a_0, \end{aligned}$$

and

$$y(x) = \frac{-4a_0}{x-4}.$$

We look for a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned} 0 &= (x-4)y' + y \\ &= (x-4) \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Noting that the first term is

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

and shifting the index of the second term, we write

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 4(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [n a_n - 4(n+1) a_{n+1} + a_n] x^n. \end{aligned}$$

Setting the coefficients equal to zero,

$$\begin{aligned} n a_n - 4(n+1) a_{n+1} + a_n &= 0 \\ a_{n+1} &= \frac{1}{4} a_n, \quad n \geq 0. \end{aligned}$$

Thus,

$$\begin{aligned} a_1 &= \frac{1}{4} a_0 \\ a_2 &= \frac{1}{4} a_1 = \frac{1}{4^2} a_0 \\ a_3 &= \frac{1}{4} a_2 = \frac{1}{4^3} a_0. \end{aligned}$$

We conjecture that

$$a_n = \frac{1}{4^n} a_0.$$

Therefore,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{4^n}.$$

However, this can be written

$$\begin{aligned} y(x) &= a_0 \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n \\ &= a_0 \left(\frac{1}{1-x/4}\right) \\ &= \frac{-4a_0}{x-4}, \end{aligned}$$

which agrees nicely with our separable solution.

8. The equation is separable.

$$\begin{aligned}(2-x)\frac{dy}{dx} &= y \\ \frac{dy}{y} &= \frac{dx}{2-x} \\ \ln y &= -\ln(2-x) + C \\ \ln y &= \ln \frac{D}{2-x} \\ y &= \frac{D}{2-x}\end{aligned}$$

where we've used $C = \ln D$. With $y(0) = a_0$,

$$a_0 = \frac{D}{2-0} = \frac{D}{2}.$$

Thus, $D = 2a_0$ and

$$y(x) = \frac{2a_0}{2-x}.$$

We look for a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned}0 &= (2-x)y' - y \\ &= (2-x) \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} 2n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n.\end{aligned}$$

Noting that the second term

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

and shifting the index of the first term,

$$\begin{aligned}0 &= \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [2(n+1)a_{n+1} - (n+1)a_n] x^n.\end{aligned}$$

Setting coefficients equal to zero,

$$\begin{aligned}0 &= 2(n+1)a_{n+1} - (n+1)a_n \\ a_{n+1} &= \frac{1}{2} a_n, \quad n \geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}a_1 &= \frac{1}{2} a_0, \\ a_2 &= \frac{1}{2} a_1 = \frac{1}{2^2} a_0, \\ a_3 &= \frac{1}{2} a_2 = \frac{1}{2^3} a_0.\end{aligned}$$

We conjecture that

$$a_n = \frac{1}{2^n} a_0.$$

Thus,

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n} a_0 x^n \\ &= a_0 \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n.\end{aligned}$$

However, this series is geometric and can be written

$$\begin{aligned}y(x) &= a_0 \frac{1}{1-x/2} \\ &= \frac{2a_0}{2-x},\end{aligned}$$

which agrees nicely with the separable solution above.

9. The equation is separable.

$$\begin{aligned}(2-x)\frac{dy}{dx} &= -2y \\ \frac{dy}{y} &= \frac{-2dx}{2-x} \\ \ln y &= 2\ln(2-x) + C \\ \ln y &= \ln D(2-x)^2 \\ y &= D(2-x)^2\end{aligned}$$

With $y(0) = a_0$,

$$a_0 = D(2-0)^2 = 4D.$$

Thus, $D = a_0/4$ and

$$y(x) = \frac{1}{4}a_0(2-x)^2.$$

We look for a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned}0 &= (2-x)y' + 2y \\ &= (2-x) \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} 2n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n.\end{aligned}$$

Noting that the second term

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

and shifting the index of the first term,

$$\begin{aligned}0 &= \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=0}^{\infty} [2(n+1)a_{n+1} - n a_n + 2a_n] x^n.\end{aligned}$$

Setting coefficients equal to zero,

$$\begin{aligned}0 &= 2(n+1)a_{n+1} + (2-n)a_n \\ a_{n+1} &= \frac{n-2}{2(n+1)} a_n, \quad n \geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}a_1 &= -a_0 \\ a_2 &= \frac{-1}{4} a_1 = \frac{1}{4} a_0, \\ a_3 &= 0,\end{aligned}$$

and $a_n = 0$ for all $n \geq 3$. Thus,

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 \left[1 - x + \frac{1}{4} x^2 \right].\end{aligned}$$

However, this can be written

$$\begin{aligned}y(x) &= \frac{1}{4}a_0 [4 - 4x + x^2] \\ &= \frac{1}{4}a_0 (2-x)^2,\end{aligned}$$

which agrees nicely with the separable solution above.

10. The equation is separable.

$$\begin{aligned}(x-3)\frac{dy}{dx} &= -2y \\ \frac{dy}{y} &= \frac{-2dx}{x-3} \\ \ln y &= -2\ln(x-3) + C \\ \ln y &= \ln \frac{D}{(x-3)^2} \\ y &= \frac{D}{(x-3)^2}\end{aligned}$$

where we've used $C = \ln D$. With $y(0) = a_0$,

$$\begin{aligned}a_0 &= \frac{D}{(0-3)^2} \\ D &= 9a_0,\end{aligned}$$

and

$$y(x) = \frac{9a_0}{(x-3)^2}.$$

We look for a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned} (x-3) \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 3 n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^n &= 0. \end{aligned}$$

Note that the first term

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n.$$

Shifting the index of the second term,

$$\sum_{n=1}^{\infty} 3 n a_n x^{n-1} = \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n.$$

Thus,

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n \\ &\quad + \sum_{n=0}^{\infty} 2 a_n x^n \\ 0 &= \sum_{n=0}^{\infty} [n a_n - 3(n+1) a_{n+1} + 2 a_n] x^n \\ 0 &= \sum_{n=0}^{\infty} [(n+2) a_n - 3(n+1) a_{n+1}] x^n. \end{aligned}$$

Setting the coefficients equal to zero,

$$\begin{aligned} 0 &= (n+2) a_n - 3(n+1) a_{n+1} \\ a_{n+1} &= \frac{n+2}{3(n+1)} a_n, \quad n \geq 0. \end{aligned}$$

Thus,

$$\begin{aligned} a_1 &= \frac{2}{3} a_0 \\ a_2 &= \frac{3}{3 \cdot 2} a_1 = \frac{3}{3^2} a_0 \\ a_3 &= \frac{4}{3 \cdot 3} a_2 = \frac{4}{3^3} a_0 \\ a_4 &= \frac{5}{3 \cdot 4} a_3 = \frac{5}{3^4} a_0. \end{aligned}$$

We conjecture that

$$a_n = \frac{n+1}{3^n} a_0, \quad n \geq 1.$$

Thus,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \frac{n+1}{3^n} a_0 x^n \\ &= a_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n. \end{aligned}$$

Now our separable solution can be written.

$$\begin{aligned} y(x) &= \frac{9a_0}{(x-3)^2} \\ &= \frac{d}{dx} \left(\frac{-9a_0}{x-3} \right) \\ &= \frac{d}{dx} \left(\frac{3a_0}{1-x/3} \right) \\ &= 3a_0 \frac{d}{dx} \sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n \\ &= 3a_0 \sum_{n=1}^{\infty} \frac{n x^{n-1}}{3^n} \\ &= a_0 \sum_{n=1}^{\infty} \frac{n x^{n-1}}{3^{n-1}} \end{aligned}$$

If we shift the index,

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n,$$

which agrees nicely with our series solution.

11. The equation $y'' = y'$ has characteristic equation $\lambda^2 - \lambda = 0$ and roots $\lambda = 0, 1$. Thus, we have independent solutions $y_1(x) = 1$ and $y_2(x) = e^x$ and general solution $y(x) = a_0 + a_1 e^x$ where $y(0) = a_0$ and $y'(0) = a_1$. We seek a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\begin{aligned} y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ &= \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^{n-1}. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= y'' - y' \\ &= \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} [n(n+1) a_{n+1} - n a_n] x^{n-1}. \end{aligned}$$

Setting coefficients equal to zero,

$$n[(n+1)a_{n+1} - a_n] = 0, \quad n \geq 1.$$

Thus,

$$a_{n+1} = \frac{a_n}{n+1}, \quad n \geq 1.$$

Thus, a_0 and a_1 are arbitrary, and

$$\begin{aligned} a_2 &= \frac{1}{2} a_1 \\ a_3 &= \frac{1}{3} a_2 = \frac{1}{2 \cdot 3} a_1 \\ a_4 &= \frac{1}{4} a_3 = \frac{1}{2 \cdot 3 \cdot 4} a_1. \end{aligned}$$

Note that for $n \geq 2$,

$$a_n = \frac{1}{n!} a_1.$$

Thus, the general solution can be written

$$\begin{aligned} y(x) &= a_0 + \sum_{n=1}^{\infty} \frac{a_1}{n!} x^n \\ &= a_0 + a_1 \sum_{n=1}^{\infty} \frac{x^n}{n!} \\ &= a_0 + a_1 e^x, \end{aligned}$$

which agrees nicely with the separable solution above.

12. The equation $y'' = 9y$ has characteristic equation $\lambda^2 - 9 = 0$ and roots $\lambda = \pm 3$. Thus, there are independent solutions $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ and general solution $y(x) = C_1 e^{3x} + C_2 e^{-3x}$. With $y(0) = a_0$ and $y'(0) = a_1$,

$$y(x) = \frac{3a_0 + a_1}{6} e^{3x} + \frac{3a_0 - a_1}{6} e^{-3x}.$$

We seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$0 = y'' - 9y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 9 \sum_{n=0}^{\infty} a_n x^n.$$

Shifting the index on the first term,

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 9a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 9a_n]x^n.$$

Setting the coefficients equal to zero, we see that a_0 and a_1 are arbitrary and

$$(n+2)(n+1)a_{n+2} - 9a_n = 0 \quad \text{and} \quad a_{n+2} = \frac{9a_n}{(n+2)(n+1)}, \quad n \geq 0.$$

Thus,

$$a_2 = \frac{9a_0}{2 \cdot 1} = \frac{3^2}{2!} a_0, \quad a_4 = \frac{9a_2}{4 \cdot 3} = \frac{9 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} a_0 = \frac{3^4}{4!} a_0, \quad \text{and} \quad a_6 = \frac{9a_4}{6 \cdot 5} = \frac{9 \cdot 9 \cdot 9}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} a_0 = \frac{3^6}{6!} a_0$$

We conjecture that

$$a_{2n} = \frac{9^n}{(2n)!} a_0 = \frac{3^{2n}}{(2n)!} a_0.$$

Similarly,

$$a_3 = \frac{9a_1}{3 \cdot 2} = \frac{3^2}{3!} a_1, \quad a_5 = \frac{9a_3}{5 \cdot 4} = \frac{9 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2} a_1 = \frac{3^4}{5!} a_1, \quad \text{and} \quad a_7 = \frac{9a_5}{7 \cdot 6} = \frac{9 \cdot 9 \cdot 9}{7 \cdot 6 \cdot 4 \cdot 3 \cdot 2} a_1 = \frac{3^6}{7!} a_1$$

We conjecture that

$$a_{2n+1} = \frac{9^n}{(2n+1)!} a_1 = \frac{3^{2n}}{(2n+1)!} a_1.$$

Thus,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \left[1 + \frac{3^2}{2!} x^2 + \frac{3^4}{4!} x^4 + \frac{3^6}{6!} x^6 + \cdots \right] + a_1 \left[x + \frac{3^2}{3!} x^3 + \frac{3^4}{5!} x^5 + \frac{3^6}{7!} x^7 + \cdots \right]$$

Now,

$$\begin{aligned} & \frac{3a_0 + a_1}{6} e^{3x} + \frac{3a_0 - a_1}{6} e^{-3x} \\ &= \frac{3a_0 + a_1}{6} \left[1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \cdots \right] + \frac{3a_0 - a_1}{6} \left[1 - 3x + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} - \cdots \right] \\ &= \frac{3a_0}{6} \left[2 + 2\frac{(3x)^2}{2!} + 2\frac{(3x)^4}{4!} + \cdots \right] + \frac{a_1}{6} \left[2(3x) + 2\frac{(3x)^3}{3!} + 2\frac{(3x)^5}{5!} + \cdots \right] \\ &= a_0 \left[1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots \right] + a_1 \left[x + \frac{3^2 x^3}{3!} + \frac{3^4 x^5}{5!} + \cdots \right], \end{aligned}$$

which is identical to the solution found above by power series technique.

13. The characteristic equation of $y'' + y = 0$ is $\lambda^2 + 1 = 0$ with zeros $\lambda = \pm i$. This yields independent solutions $y_1(x) = \cos x$ and $y_2(x) = \sin x$ and general solution $y(x) = C_1 \cos x + C_2 \sin x$. With $y(0) = a_0$, $a_0 = y(0) = C_1$. Differentiate.

$$y'(x) = -C_1 \sin x + C_2 \cos x$$

With $y'(0) = a_1$, $a_1 = y'(0) = C_2$. Thus,

$$y(x) = a_0 \cos x + a_1 \sin x.$$

We look for solutions having form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, with second derivative

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting,

$$0 = y'' + y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n.$$

Shift the index of the first term.

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n]x^n$$

Set the coefficients equal to zero.

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad \text{and} \quad a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0$$

Thus,

$$a_2 = \frac{-a_0}{2 \cdot 1}, \quad a_4 = \frac{-a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \quad \text{and} \quad a_6 = \frac{-a_4}{6 \cdot 5} = \frac{-a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

We conjecture that

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}.$$

Similarly,

$$a_3 = \frac{-a_1}{3 \cdot 2}, \quad a_5 = \frac{-a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \quad \text{and} \quad a_7 = \frac{-a_5}{7 \cdot 6} = \frac{-a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}.$$

We conjecture that

$$a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}.$$

Thus,

$$\begin{aligned} y(x) &= a_0 \left[1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots \right] \\ &\quad + a_1 \left[x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots \right] \\ &= a_0 \cos x + a_1 \sin x, \end{aligned}$$

which agrees nicely with the result above.

14. The equation $y'' + 4y = 0$ has characteristic equation $\lambda^2 + 4 = 0$ and roots $\lambda = \pm 2i$. This leads to the independent solutions $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$ and the general solution $y(x) = C_1 \cos 2x + C_2 \sin 2x$. With $y(0) = a_0$,

$$a_0 = y(0) = C_1.$$

Differentiate.

$$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x$$

With $y'(0) = a_1$,

$$a_1 = y'(0) = 2C_2.$$

Thus, $C_2 = a_1/2$ and

$$y(x) = a_0 \cos 2x + \frac{1}{2} a_1 \sin 2x.$$

We seek a solution having form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$0 = y'' + 4y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 4 \sum_{n=0}^{\infty} a_n x^n.$$

Shifting the second term,

$$0 = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} 4a_{n-2} x^{n-2} = \sum_{n=2}^{\infty} [n(n-1)a_n + 4a_{n-2}] x^{n-2}.$$

Setting coefficients equal to zero,

$$n(n-1)a_n + 4a_{n-2} = 0 \quad \text{and} \quad a_n = \frac{-4a_{n-2}}{n(n-1)}, \quad n \geq 2.$$

Thus, a_0 and a_1 are arbitrary and

$$a_2 = \frac{-4a_0}{2 \cdot 1}, \quad a_4 = \frac{-4a_2}{4 \cdot 3} = \frac{4 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} a_0, \quad \text{and} \quad a_6 = \frac{-4a_4}{6 \cdot 5} = \frac{-4 \cdot 4 \cdot 4}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} a_0.$$

We conjecture that

$$a_{2n} = \frac{(-1)^n 4^n}{(2n)!} a_0 = \frac{(-1)^n 2^{2n}}{(2n)!} a_0.$$

Also,

$$a_3 = \frac{-4a_1}{3 \cdot 2}, \quad a_5 = \frac{-4a_3}{5 \cdot 4} = \frac{4 \cdot 4}{5 \cdot 4 \cdot 3 \cdot 2} a_1, \quad \text{and} \quad a_7 = \frac{-4a_5}{7 \cdot 6} = \frac{-4 \cdot 4 \cdot 4}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} a_1.$$

We conjecture that

$$a_{2n+1} = \frac{(-1)^n 4^n}{(2n+1)!} a_1 = \frac{(-1)^n 2^{2n}}{(2n+1)!} a_1.$$

Thus,

$$\begin{aligned}
 y(x) &= a_0 \left[1 - \frac{4}{2!}x^2 + \frac{4^2}{4!}x^4 - \frac{4^3}{6!}x^6 + \cdots + \frac{(-1)^n 2^{2n}}{(2n)!}x^{2n} + \cdots \right] \\
 &\quad + a_1 \left[x - \frac{4}{3!}x^3 + \frac{4^2}{5!}x^5 - \frac{4^3}{7!}x^7 + \cdots + \frac{(-1)^n 2^{2n}}{(2n+1)!}x^{2n+1} + \cdots \right] \\
 &= a_0 \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots + \frac{(-1)^n (2x)^{2n}}{(2n)!} + \cdots \right] \\
 &\quad + \frac{a_1}{2} \left[2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots + \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} + \cdots \right] \\
 &= a_0 \cos 2x + \frac{a_1}{2} \sin 2x,
 \end{aligned}$$

which agrees nicely with the solution found above.

15. In the equation $y'' + x^2 y = 0$, the coefficient of y is a polynomial, clearly analytic at $x = 0$. Thus, $x = 0$ is an ordinary point. According to Theorem 2.29, the solutions will have an infinite radius of convergence. We look for a solution having the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$0 = y'' + x^2 y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2}.$$

Shifting the index of the first term,

$$0 = \sum_{n=-2}^{\infty} (n+4)(n+3) a_{n+4} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 2a_2 + 6a_3 x + \sum_{n=0}^{\infty} [(n+4)(n+3) a_{n+4} + a_n] x^{n+2}.$$

Setting the coefficients equal to zero, $a_2 = a_3 = 0$ and

$$(n+4)(n+3) a_{n+4} + a_n = 0 \quad \text{or} \quad a_{n+4} = \frac{-a_n}{(n+4)(n+3)}.$$

Thus,

$$a_4 = \frac{-a_0}{4 \cdot 3}, \quad a_8 = \frac{-a_4}{8 \cdot 7} = \frac{a_0}{4 \cdot 8 \cdot 3 \cdot 7}, \quad \text{and} \quad a_{12} = \frac{-a_8}{12 \cdot 11} = \frac{-a_0}{4 \cdot 8 \cdot 12 \cdot 3 \cdot 7 \cdot 11}.$$

Note that

$$a_{16} = \frac{-a_{12}}{16 \cdot 15} = \frac{a_0}{4 \cdot 8 \cdot 12 \cdot 16 \cdot 3 \cdot 7 \cdot 11 \cdot 15} = \frac{a_0}{4^4 (1 \cdot 2 \cdot 3 \cdot 4) \cdot [3 \cdot 7 \cdot 11 \cdot 15]}.$$

We conjecture that

$$a_{4n} = \frac{(-1)^n a_0}{4^n \cdot n! [3 \cdot 7 \cdot \cdots \cdot (4n-1)]}, \quad n \geq 1.$$

Similarly,

$$a_5 = \frac{-a_1}{5 \cdot 4}, \quad a_9 = \frac{-a_5}{9 \cdot 8} = \frac{a_1}{5 \cdot 9 \cdot 4 \cdot 8}, \quad \text{and} \quad a_{13} = \frac{-a_9}{13 \cdot 12} = \frac{-a_1}{5 \cdot 9 \cdot 13 \cdot 4 \cdot 8 \cdot 12}.$$

Note that

$$a_{17} = \frac{-a_{13}}{17 \cdot 16} = \frac{a_1}{5 \cdot 9 \cdot 13 \cdot 17 \cdot 4 \cdot 8 \cdot 12 \cdot 16} = \frac{a_1}{4^4(1 \cdot 2 \cdot 3 \cdot 4)[5 \cdot 9 \cdot 13 \cdot 17]}.$$

We conjecture

$$a_{4n+1} = \frac{(-1)^n a_1}{4^n n! [5 \cdot 9 \cdot 13 \cdot \dots \cdot (4n+1)]}.$$

Finally, since $a_2 = a_3 = 0$, it is easy to see that $a_6 = a_{10} = a_{14} = \dots = 0$, and $a_7 = a_{11} = \dots = 0$. Thus,

$$y(x) = a_0 \left[1 - \frac{1}{4 \cdot 3} x^4 + \frac{1}{4 \cdot 8 \cdot 3 \cdot 7} x^8 - \dots \right] + a_1 \left[x - \frac{1}{5 \cdot 4} x^5 + \frac{1}{5 \cdot 9 \cdot 4 \cdot 8} x^9 - \dots \right].$$

Choose $y_1(x)$ with $1 = y_1(0) = a_0$ and $0 = y_1'(0) = a_1$.

$$y_1(x) = 1 - \frac{1}{4 \cdot 3} x^4 + \frac{1}{4 \cdot 8 \cdot 3 \cdot 7} x^8 - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{4^n n! [3 \cdot 7 \cdot \dots \cdot (4n-1)]}.$$

Choose $y_2(x)$ with $0 = y_2(0) = a_0$ and $1 = y_2'(0) = a_1$.

$$y_2(x) = x - \frac{1}{5 \cdot 4} x^5 + \frac{1}{5 \cdot 9 \cdot 4 \cdot 8} x^9 - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4^n n! [5 \cdot 9 \cdot \dots \cdot (4n+1)]}.$$

Note that the solutions are chosen so that

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so the solutions are independent.

16. In the equation $y'' + xy = 0$, the coefficient of y is a polynomial, clearly analytic at $x = 0$. Thus, $x = 0$ is an ordinary point. According to Theorem 2.29, all solutions will have radius of convergence $R = \infty$. We look for a solution having the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$0 = y'' + xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1}.$$

Shifting the index of the first term,

$$0 = \sum_{n=-1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2) a_{n+3} + a_n] x^{n+1}.$$

Setting coefficients equal to zero, $a_2 = 0$ and

$$(n+3)(n+2)a_{n+3} + a_n = 0 \quad \text{or} \quad a_{n+3} = -\frac{a_n}{(n+2)(n+3)}, \quad n \geq 0.$$

Thus,

$$a_3 = -\frac{a_0}{2 \cdot 3}, \quad a_6 = -\frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6} = \frac{4}{6!} a_0, \quad \text{and} \quad a_9 = -\frac{a_6}{8 \cdot 9} = -\frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} = -\frac{4 \cdot 7}{9!} a_0.$$

We conjecture that

$$a_{3n} = (-1)^n \frac{4 \cdot 7 \cdots (3n-2)}{(3n)!} a_0, \quad n \geq 1.$$

Similarly,

$$a_4 = -\frac{a_1}{3 \cdot 4}, \quad a_7 = -\frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7} = \frac{2 \cdot 5}{7!} a_1, \quad \text{and} \quad a_{10} = -\frac{a_7}{9 \cdot 10} = -\frac{2 \cdot 5 \cdot 8}{10!} a_1.$$

We conjecture that

$$a_{3n+1} = (-1)^n \frac{2 \cdot 5 \cdots (3n-1)}{(3n+1)!} a_1.$$

Finally, because $a_2 = 0$, it is easy to see that $a_5 = a_8 = a_{11} = \cdots = 0$. Thus

$$y(x) = a_0 \left[1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \cdots \right] + a_1 \left[x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \cdots \right].$$

Choose $y_1(x)$ with $1 = y_1(0) = a_0$ and $0 = y_1'(0) = a_1$.

$$y_1(x) = 1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \cdots = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{4 \cdot 7 \cdots (3n-2)}{(3n)!} x^{3n}.$$

Choose $y_2(x)$ with $0 = y_2(0) = a_0$ and $1 = y_2'(0) = a_1$.

$$y_2(x) = x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \cdots = x + \sum_{n=1}^{\infty} (-1)^n \frac{2 \cdot 5 \cdots (3n-1)}{(3n+1)!} x^{3n+1}.$$

Note that the solutions are chosen so that

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so the solutions are independent.

17. The coefficients of y' and y are $p(x) = 2x$ and $q(x) = -1$, both polynomials are analytic at $x = 0$. Thus, $x = 0$ is an ordinary point. According to Theorem 2.29, the solutions will have an infinite radius of convergence. We seek a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$\begin{aligned} 0 &= y'' + 2xy' - y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Shifting the index of the first term and noting that the second term

$$\sum_{n=1}^{\infty} 2na_n x^n = \sum_{n=0}^{\infty} 2na_n x^n,$$

we write

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (2n-1)a_n] x^n.$$

Setting coefficients equal to zero,

$$(n+2)(n+1)a_{n+2} + (2n-1)a_n = 0 \quad \text{or} \quad a_{n+2} = \frac{1-2n}{(n+2)(n+1)} a_n, \quad n \geq 0.$$

Thus,

$$a_2 = \frac{1}{2 \cdot 1} a_0, \quad a_4 = \frac{-3}{4 \cdot 3} a_2 = \frac{-3}{4 \cdot 3 \cdot 2 \cdot 1} a_0, \quad \text{and} \quad a_6 = \frac{-7}{6 \cdot 5} a_4 = \frac{7 \cdot 3}{6!} a_0.$$

Note that

$$a_8 = \frac{-11}{8 \cdot 7} a_6 = \frac{-11 \cdot 7 \cdot 3}{8!} a_0.$$

We conjecture that

$$a_{2n} = \frac{(-1)^{n+1}(3 \cdot 7 \cdot 11 \cdots (4n-5))}{(2n)!} a_0, \quad n > 1.$$

Similarly,

$$a_3 = \frac{-1}{3 \cdot 2} a_1, \quad a_5 = \frac{-5}{5 \cdot 4} a_3 = \frac{5}{5 \cdot 4 \cdot 3 \cdot 2} a_1, \quad \text{and} \quad a_7 = \frac{-9}{7 \cdot 6} a_5 = \frac{-9 \cdot 5}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} a_1.$$

Note that

$$a_9 = \frac{-13}{9 \cdot 8} a_7 = \frac{13 \cdot 9 \cdot 5}{9!} a_1.$$

We conjecture that

$$a_{2n+1} = \frac{(-1)^n(5 \cdot 9 \cdot 13 \cdots (4n-3))}{(2n+1)!} a_1, \quad n > 1.$$

Thus,

$$y(x) = a_0 \left[1 + \frac{1}{2 \cdot 1} x^2 - \frac{3}{4 \cdot 3 \cdot 2 \cdot 1} x^4 + \cdots \right] + a_1 \left[x - \frac{1}{3 \cdot 2} x^3 + \frac{5}{5 \cdot 4 \cdot 3 \cdot 2} x^5 - \cdots \right].$$

Choose $y_1(x)$ with $1 = y_1(0) = a_0$ and $0 = y_1'(0) = a_1$.

$$y_1(x) = 1 + \frac{1}{2 \cdot 1} x^2 - \frac{3}{4 \cdot 3 \cdot 2 \cdot 1} x^4 + \cdots = 1 + \frac{1}{2} x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} (3 \cdot 7 \cdot 11 \cdots (4n-5))}{(2n)!} x^{2n}$$

Choose $y_2(x)$ with $0 = y_2(0) = a_0$ and $1 = y_2'(0) = a_1$.

$$y_2(x) = x - \frac{1}{3 \cdot 2} x^3 + \frac{5}{5 \cdot 4 \cdot 3 \cdot 2} x^5 - \cdots = x - \frac{1}{6} x^3 + \sum_{n=2}^{\infty} \frac{(-1)^n (5 \cdot 9 \cdot 12 \cdots (4n-3))}{(2n+1)!} x^{2n+1}.$$

Note that the solutions are chosen so that

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so the solutions are independent.

18. The coefficients of y' and y are $p(x) = x$ and $q(x) = 2$, both polynomials and analytic at $x = 0$. Thus, $x = 0$ is an ordinary point. According to Theorem 2.29, all solutions have radius of convergence $R = \infty$. We seek a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$\begin{aligned} 0 = y'' + x y' + 2y &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n. \end{aligned}$$

Shifting the index of the first term and noting that the second term

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

we write

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n+2) a_n] x^n.$$

Setting coefficients equal to zero,

$$(n+2)(n+1) a_{n+2} + (n+2) a_n = 0 \quad \text{or} \quad a_{n+2} = \frac{-a_n}{n+1}, \quad n \geq 0.$$

Thus,

$$a_2 = -a_0, \quad a_4 = \frac{-a_2}{3} = \frac{a_0}{3}, \quad a_6 = \frac{-a_4}{5} = \frac{-a_0}{5 \cdot 3}, \quad \text{and} \quad a_8 = \frac{-a_6}{7} = \frac{a_0}{7 \cdot 5 \cdot 3}.$$

Note that

$$a_8 = \frac{6 \cdot 4 \cdot 2a_0}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{2^3(3 \cdot 2 \cdot 1)a_0}{7!} = \frac{2^3 3!}{7!} a_0.$$

We conjecture that

$$a_{2n} = \frac{(-1)^n 2^{n-1} (n-1)!}{(2n-1)!} a_0, \quad n \geq 1.$$

Similarly,

$$a_3 = \frac{-a_1}{2}, \quad a_5 = \frac{-a_3}{4} = \frac{a_1}{4 \cdot 2}, \quad a_7 = \frac{-a_5}{6} = \frac{-a_1}{6 \cdot 4 \cdot 2}.$$

Note that

$$a_7 = \frac{-a_1}{2^3(3 \cdot 2 \cdot 1)} = \frac{-a_1}{2^3 3!}.$$

We conjecture that

$$a_{2n+1} = \frac{(-1)^n a_1}{2^n n!}, \quad n \geq 0.$$

Thus,

$$y(x) = a_0 \left[1 - x^2 + \frac{1}{3}x^4 - \frac{1}{5 \cdot 3}x^6 + \cdots \right] + a_1 \left[x - \frac{1}{2}x^3 + \frac{1}{4 \cdot 2}x^5 - \cdots \right].$$

Choose $y_1(x)$ with $1 = y_1(0) = a_0$ and $0 = y_1'(0) = a_1$.

$$y_1(x) = 1 - x^2 + \frac{1}{3}x^4 - \frac{1}{5 \cdot 3}x^6 + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{n-1} (n-1)! x^{2n}}{(2n-1)!}.$$

Choose $y_2(x)$ with $0 = y_2(0) = a_0$ and $1 = y_2'(0) = a_1$.

$$y_2(x) = x - \frac{1}{2}x^3 + \frac{1}{4 \cdot 2}x^5 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n!}.$$

Note that the solutions are chosen so that

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so the solutions are independent.

19. In the equation $y'' + xy' - y$, the coefficient of y' is $p(x) = x$. The coefficient of y is $q(x) = -1$. Both p and q are analytic at $x = 0$, so $x = 0$ is an ordinary point. According to Theorem 2.29, the solutions will have an infinite radius of convergence. We look for a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, with derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$\begin{aligned} 0 &= y'' + xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Note that the second term is

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

and shift the index of the first term.

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n-1)a_n] x^n.$$

Setting the coefficients equal to zero,

$$(n+2)(n+1)a_{n+2} + (n-1)a_n = 0 \quad \text{or} \quad a_{n+2} = \frac{1-n}{(n+2)(n+1)} a_n.$$

Thus,

$$a_2 = \frac{1}{2 \cdot 1} a_0, \quad a_4 = \frac{-1}{4 \cdot 3} a_2 = \frac{-1}{4 \cdot 3 \cdot 2 \cdot 1} a_0, \quad \text{and} \quad a_6 = \frac{-3}{6 \cdot 5} a_4 = \frac{1 \cdot 3}{6!} a_0.$$

Note that

$$a_8 = \frac{-5}{8 \cdot 7} a_6 = \frac{-1 \cdot 3 \cdot 5}{8!} a_0.$$

We conjecture that

$$a_{2n} = \frac{(-1)^{n+1}(1 \cdot 3 \cdot 5 \cdots (2n-3))}{(2n)!} a_0, \quad n \geq 2.$$

Similarly,

$$a_3 = 0a_1 = 0$$

and $a_{2n+1} = 0$ for $n \geq 1$. Thus,

$$y(x) = a_0 \left[1 + \frac{1}{1 \cdot 2} x^2 - \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} x^4 + \cdots \right] + a_1 x.$$

Choose $y_1(x)$ with $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$. Then

$$y_1(x) = 1 + \frac{1}{2 \cdot 1} x^2 - \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} x^4 + \cdots = 1 + \frac{1}{2} x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(1 \cdot 3 \cdot 5 \cdots (2n-3))}{(2n)!} x^{2n}.$$

Choose $y_2(x)$ with $a_0 = y_2(0) = 0$ and $a_1 = y_2'(0) = 1$. Then

$$y_2(x) = x.$$

Note that the solutions were chosen so that

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so the solutions are independent.

20. Dividing by the lead coefficient,

$$y'' + \frac{2x}{1+x^2} y' - \frac{2}{1+x^2} y = 0.$$

The coefficients of y' and y ,

$$P(x) = \frac{2x}{1+x^2} \quad \text{and} \quad Q(x) = \frac{-2}{1+x^2},$$

respectively are analytic at $x = 0$. Therefore, $x = 0$ is an ordinary point. Since the series for $1/(1+x^2)$ has radius of convergence 1, Theorem 2.29 implies that all solutions have radius of convergence $R \geq 1$. We seek now a solution having form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$\begin{aligned} 0 &= (1+x^2)y'' + 2xy' - 2y \\ &= (1+x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n. \end{aligned}$$

Note that sums in the second and third terms are unaffected if we start the index at $n = 0$. We shift the index in the first term to obtain

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + n(n-1) a_n + 2n a_n - 2a_n] x^n. \end{aligned}$$

Setting coefficients equal to zero,

$$\begin{aligned} (n+2)(n+1) a_{n+2} + [n(n-1) + 2n - 2] a_n &= 0 \\ (n+2)(n+1) a_{n+2} + (n+2)(n-1) a_n &= 0 \\ a_{n+2} &= \frac{1-n}{n+1} a_n, \quad n \geq 0. \end{aligned}$$

Thus,

$$a_2 = a_0, \quad a_4 = \frac{-1}{3} a_2 = -\frac{1}{3} a_0, \quad a_6 = \frac{-3}{5} a_4 = \frac{1}{5} a_0, \quad \text{and} \quad a_8 = \frac{-5}{7} a_6 = -\frac{1}{7} a_0.$$

We conjecture that

$$a_{2n} = \frac{(-1)^{n+1}}{2n-1} a_0, \quad n \geq 1.$$

Similarly,

$$a_3 = 0, a_1 = 0.$$

Thus, $a_{2n+1} = 0$ for $n \geq 1$. Thus,

$$y(x) = a_0 \left[1 + x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6 - \cdots \right] + a_1x.$$

Choose $y_1(x)$ so that $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$.

$$y_1(x) = 1 + x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6 - \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} x^{2n}.$$

Choose $y_2(x)$ so that $a_0 = y_2(0) = 0$ and $a_1 = y_2'(0) = 1$.

$$y_2(x) = x$$

Note that y_1 and y_2 are chosen so that

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so y_1 and y_2 are independent solutions.

21. When we put the equation in the form $y'' + y/(1+x) = 0$, we see that the coefficient $1/(1+x)$ is analytic at $x_0 = 0$ and has radius of convergence 1. Then $x_0 = 0$ is an ordinary point, and according to Theorem 2.29, all solutions have radius of convergence $R \geq 1$. We compute that a solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ must satisfy the recurrence formula

$$a_{n+2} = -\frac{n(n+1)a_{n+1} + a_n}{(n+1)(n+2)}, \quad \text{for } n \geq 0.$$

For the first solution we choose $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$. Then $a_2 = -1/2$, $a_3 = 1/6$, and $a_4 = 1/72$, so

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{72}x^4 + \cdots.$$

For the second solution we choose $a_0 = y_1(0) = 0$ and $a_1 = y_1'(0) = 1$. Then $a_2 = 0$, $a_3 = -1/6$, and $a_4 = 1/24$, so

$$y_2(x) = x - \frac{x^3}{6} + \frac{1}{24}x^4 + \cdots.$$

These solutions are linearly independent by Proposition 2.14.

22. When we put the equation in the form $y'' + y'/(1+x^2) - 2y/(1+x^2) = 0$, we see that the coefficients are analytic at $x_0 = 0$ and have radius of convergence 1. Then $x_0 = 0$ is an ordinary point, and according to Theorem 2.29, all solutions have radius of convergence $R \geq 1$. We compute that a solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ must satisfy the recurrence formula

$$a_{n+2} = -\frac{a_{n+1} + (n-2)a_n}{n+2}, \quad \text{for } n \geq 0.$$

For the first solution we choose $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$. Then $a_2 = 1$, $a_3 = -1/3$, and $a_4 = 1/12$, so

$$y_1(x) = 1 + x^2 - \frac{x^3}{3} + \frac{x^4}{12} + \cdots.$$

For the second solution we choose $a_0 = y_1(0) = 0$ and $a_1 = y_1'(0) = 1$. Then $a_2 = -1/2$, $a_3 = 1/2$, $a_4 = -1/8$, so

$$y_2(x) = x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^4}{8} + \cdots.$$

These solutions are linearly independent by Proposition 2.14.

23. The power series for the coefficient $\cos x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$ converges for all x . Then

$x_0 = 0$ is an ordinary point, and according to Theorem 2.29, all solutions have an infinite radius of convergence. If $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution, then

$$\begin{aligned} 0 &= y'' - (\cos x)y \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n \\ &\quad - \left(1 - x^2/2 + \cdots\right) \sum_{n=0}^{\infty} a_n x^n \\ &= [2a_2 - a_0] + [6a_3 - a_1]x \\ &\quad + [12a_4 - a_2 + a_0/2]x^2 + \cdots \end{aligned}$$

Setting the coefficients equal to 0, we see that $a_2 = a_0/2$, $a_3 = a_1/6$, and $a_4 = (a_2 - a_0/2)/12$. For the first solution we choose $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$. Then $a_2 = 1/2$, $a_3 = 0$, and $a_4 = 0$ so

$$S_4(x) = 1 + \frac{x^2}{2} \quad \text{and} \quad y_1(x) = 1 + \frac{x^2}{2} + \cdots$$

For the second solution we choose $a_0 = y_1(0) = 0$ and $a_1 = y_1'(0) = 1$. Then $a_2 = 0$, $a_3 = 1/6$, and $a_4 = 0$, so

$$S_4(x) = x + \frac{x^3}{6} \quad \text{and} \quad y_2(x) = x + \frac{x^3}{6} + \cdots$$

These solutions are linearly independent by Proposition 2.14.

24. The power series for the coefficient $\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$ converges for all x . Then $x_0 = 0$ is an ordinary point, and according to Theorem 2.29, all solutions have an infinite radius of convergence. If $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution, then

$$\begin{aligned} 0 &= y'' + (\sin x)y' + y \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n \\ &\quad + (x - x^3/6 + \cdots) \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= [2a_2 + a_0] + [6a_3 + 2a_1]x \\ &\quad + [12a_4 + 3a_2]x^2 + \cdots \end{aligned}$$

Setting the coefficients equal to 0, we see that $a_2 = -a_0/2$, $a_3 = -a_1/3$, and $a_4 = -a_2/4$. For the first solution we choose $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$. Then $a_2 = -1/2$, $a_3 = 0$, and $a_4 = 1/8$, so

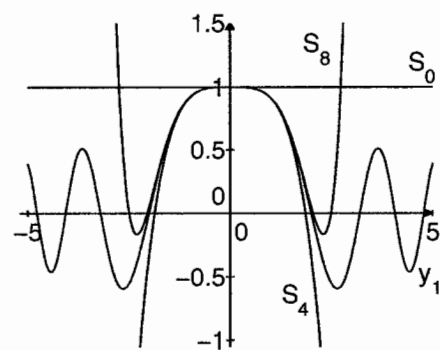
$$S_4(x) = 1 - \frac{x^2}{2} + \frac{1}{8}x^4 \quad \text{and} \quad y_1(x) = 1 - \frac{x^2}{2} + \cdots$$

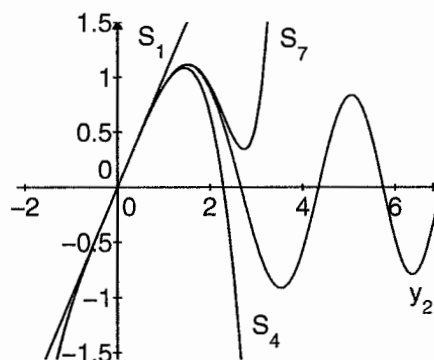
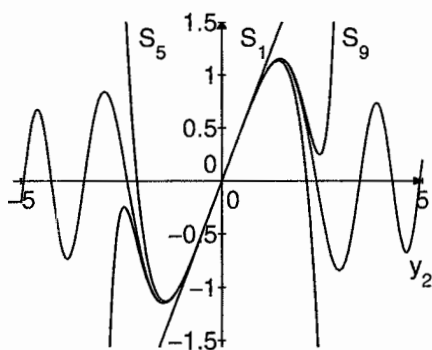
For the second solution we choose $a_0 = y_1(0) = 0$ and $a_1 = y_1'(0) = 1$. Then $a_2 = 0$, $a_3 = -1/3$, and $a_4 = 0$, so

$$S_4(x) = x - \frac{x^3}{3} \quad \text{and} \quad y_2(x) = x - \frac{x^3}{3} + \cdots$$

These solutions are linearly independent by Proposition 2.14. Additional terms are found in a similar manner.

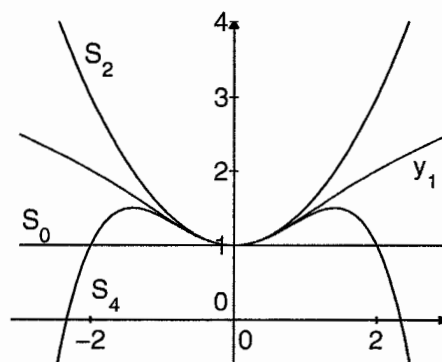
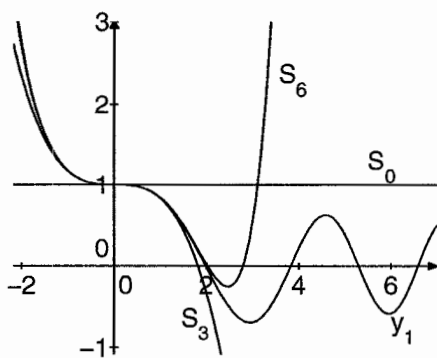
25. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.



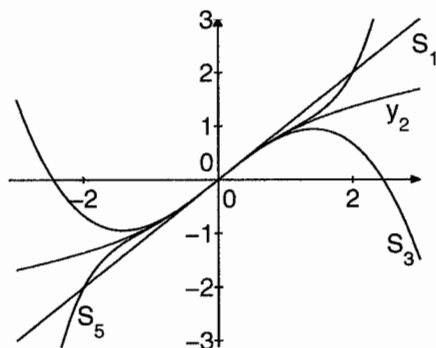


26. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.

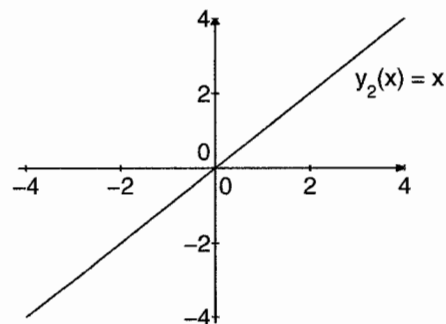
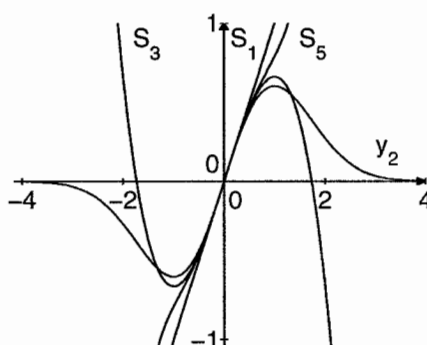
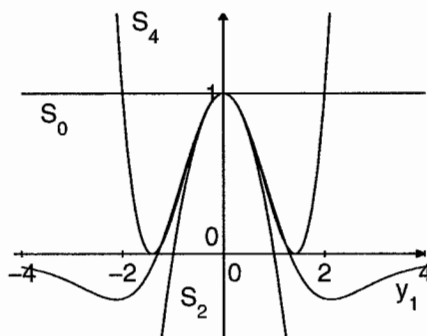
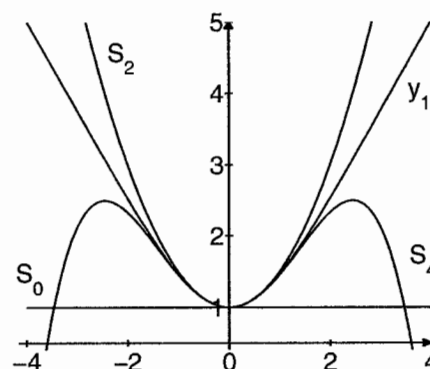
27. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.



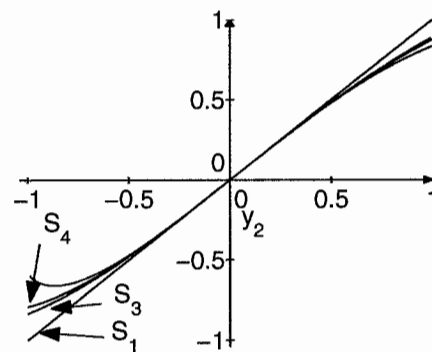
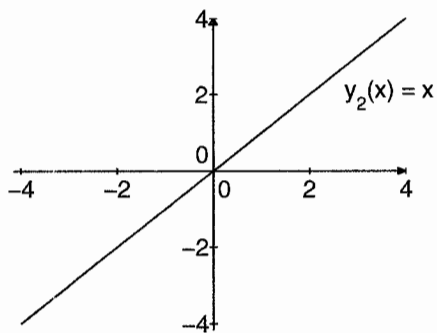
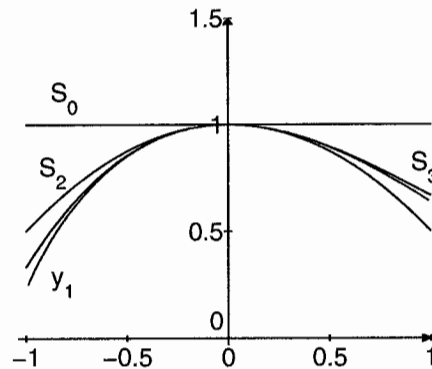
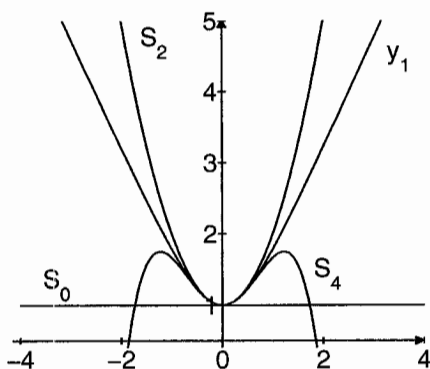
29. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.



28. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.

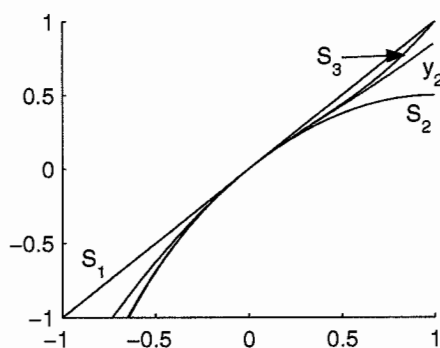
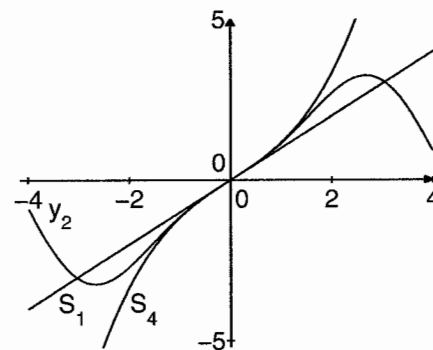
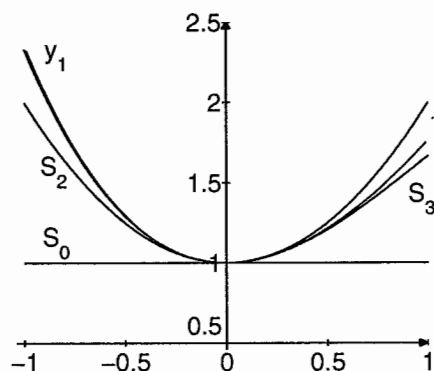


30. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.



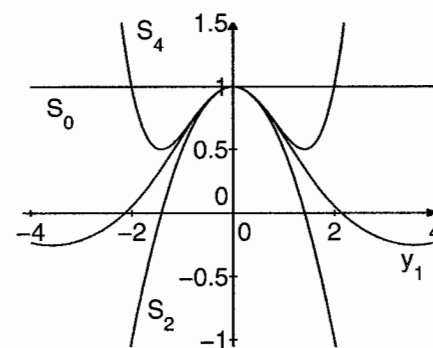
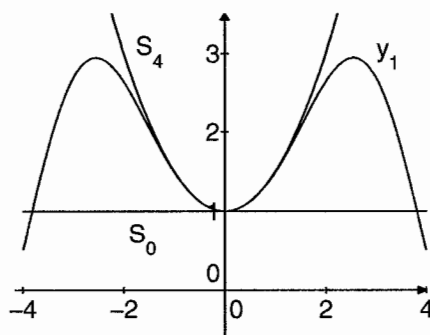
31. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.

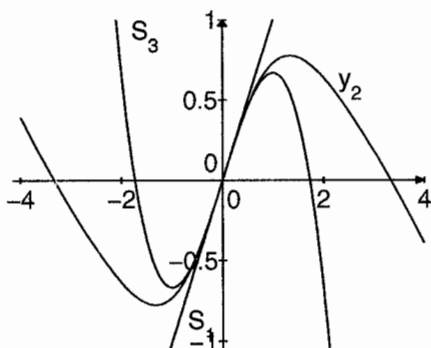
32. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.



34. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.

33. The first figure contains the plot of y_1 and three partial sums, the second the plot of y_2 and three partial sums.





35. Certainly, if $y(x) = (1+x)^p$, $y(0) = 1$. Further, if $p \neq 0$ (the equation is trivially satisfied if $p = 0$), then

$$\begin{aligned}(1+x)y' - py &= (1+x)p(1+x)^{p-1} - p(1+x)^p \\ &= p(1+x)^p - p(1+x)^p \\ &= 0,\end{aligned}$$

showing that $y(x) = (1+x)^p$ is the unique solution of $(1+x)y' = py$, $y(0) = 1$. We now look for a solution having form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substituting,

$$\begin{aligned}0 &= (1+x)y' - py \\ &= (1+x) \sum_{n=1}^{\infty} n a_n x^{n-1} - p \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} p a_n x^n.\end{aligned}$$

Note that the second term

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

and shift the index of the first term.

$$\begin{aligned}0 &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} p a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+1) a_{n+1} + n a_n - p a_n] x^n.\end{aligned}$$

Setting coefficients equal to zero,

$$(n+1) a_{n+1} + (n-p) a_n = 0,$$

or

$$a_{n+1} = \frac{p-n}{n+1} a_n, \quad n \geq 0.$$

Thus, with $a_0 = y(0) = 1$,

$$\begin{aligned}a_1 &= p a_0 = p, \\ a_2 &= \frac{p-1}{2} a_1 = \frac{p(p-1)}{2}, \\ a_3 &= \frac{p-2}{3} a_2 = \frac{p(p-1)(p-2)}{3 \cdot 2}.\end{aligned}$$

We conjecture that

$$a_n = \frac{p(p-1)(p-2) \cdots (p-(n-1))}{n!}.$$

Thus,

$$\begin{aligned}y(x) &= 1 + px + \frac{p(p-1)}{2} x^2 + \cdots \\ &\quad + \frac{p(p-1)(p-2) \cdots (p-n+1)}{n!} x^n + \cdots.\end{aligned}$$

The radius of convergence is

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| &= \lim_{n \rightarrow \infty} \left| \frac{p(p-1) \cdots (p-n+1)}{n!} \cdot \frac{(n+1)!}{p(p-1) \cdots (p-n)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{p-n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1+1/n}{p/n-1} \right| \\ &= 1\end{aligned}$$

Thus, the series converges for $|x| < 1$.

36. The solution in the example is given as

$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k}$$

However,

$$\frac{(-1)^k}{2^k k!} x^{2k} = \frac{(-1)^k}{2^k k!} (x^2)^k = \frac{1}{k!} \left(-\frac{x^2}{2} \right)^k.$$

Thus,

$$y_1(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{x^2}{2} \right)^k = e^{-x^2/2}.$$

————— × —————

Section 11.3. Legendre's Equation

1. (a) Note that

$$\begin{aligned} \frac{1}{\sqrt{1-z}} &= (1-z)^{-1/2} \\ &= 1 + \frac{1}{2}z + \frac{1 \cdot 3}{2^2 2!} z^2 + \frac{1 \cdot 3 \cdot 5}{2^3 3!} z^3 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} z^n + \cdots \end{aligned}$$

provided $|z| < 1$. Thus,

$$\begin{aligned} \frac{1}{\sqrt{1-2xt+t^2}} &= \frac{1}{\sqrt{1+t(2x-t)}} \\ &= 1 + \frac{1}{2}t(2x-t) + \frac{1 \cdot 3}{2^2 \cdot 2!} t^2(2x-t)^2 + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!} t^3(2x-t)^3 + \\ &\quad \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} t^n(2x-t)^n + \cdots \end{aligned}$$

Now, the idea is to gather together like powers of t . For example,

$$\begin{aligned} \frac{1}{\sqrt{1-2xt+t^2}} &= 1 + xt + \left(-\frac{1}{2} + \frac{1 \cdot 3}{2^2 \cdot 2!} (2^2 x^2) \right) t^2 + \left(\frac{1 \cdot 3}{2^2 \cdot 2!} (-4x) + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!} (8x^3) \right) t^3 + \cdots \\ &= 1 + xt + \left(-\frac{1}{2} + \frac{3}{2} x^2 \right) t^2 + \left(-\frac{3}{2} x + \frac{5}{2} x^3 \right) t^3 + \cdots \end{aligned}$$

Note that the coefficients of t are the Legendre polynomials.

(b) If $g(t, x) = (1 - 2xt + t^2)^{-1/2}$, then

$$\frac{\partial g}{\partial x} = t(1 - 2xt + t^2)^{-3/2}.$$

Then,

$$\begin{aligned}\frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial g}{\partial x} \right] &= \frac{\partial}{\partial x} t(1-x^2)(1-2xt+t^2)^{-3/2} \\ &= 3t^2(1-x^2)(1-2xt+t^2)^{-5/2} - 2tx(1-2xt+t^2)^{-3/2} \\ &= t(1-2xt+t^2)^{-5/2}(3t(1-x^2) - 2x(1-2xt+t^2)) \\ &= t(1-2xt+t^2)^{-5/2}(3t-2x+tx^2-2t^2x).\end{aligned}$$

Next,

$$\begin{aligned}\frac{\partial}{\partial t}[tg] &= \frac{\partial}{\partial t} t(1-2xt+t^2)^{-1/2} \\ &= (1-2xt+t^2)^{-1/2} + t(x-t)(1-2xt+t^2)^{-3/2}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\partial^2}{\partial t^2}[tg] &= -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) + t(x-t) \left[-\frac{3}{2}(1-2xt+t^2)^{-5/2}(-2x+2t) \right] \\ &\quad + (x-2t)(1-2xt+t^2)^{-3/2} \\ &= (x-t)(1-2xt+t^2)^{-3/2} + 3t(x-t)^2(1-2xt+t^2)^{-5/2} + (x-2t)(1-2xt+t^2)^{-3/2} \\ &= (2x-3t)(1-2xt+t^2)^{-3/2} + 3t(x-t)^2(1-2xt+t^2)^{-5/2} \\ &= (1-2xt+t^2)^{-5/2}[3t(x-t)^2 + (2x-3t)(1-2xt+t^2)] \\ &= (1-2xt+t^2)^{-5/2}[-3t+2x-tx^2+2t^2x].\end{aligned}$$

Thus,

$$\frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial g}{\partial x} \right] + t \frac{\partial^2}{\partial t^2}[tg] = 0.$$

(c) If $g(t, x) = \sum_{n=0}^{\infty} Q_n(x)t^n$, then

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x} \sum_{n=0}^{\infty} Q_n(x)t^n = \sum_{n=0}^{\infty} Q'_n(x)t^n \\ (1-x^2) \frac{\partial g}{\partial x} &= \sum_{n=0}^{\infty} (1-x^2) Q'_n(x)t^n.\end{aligned}$$

Taking another derivative,

$$\frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial g}{\partial x} \right] = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} [(1-x^2) Q'_n(x)] t^n = \sum_{n=0}^{\infty} [(1-x^2) Q'_n(x)]' t^n.$$

On the other hand,

$$\frac{\partial}{\partial t} tg = \frac{\partial}{\partial t} \sum_{n=0}^{\infty} Q_n(x)t^{n+1} = \sum_{n=0}^{\infty} (n+1) Q_n(x)t^n.$$

Taking another derivative,

$$\frac{\partial^2}{\partial t^2} tg = \sum_{n=0}^{\infty} n(n+1) Q_n(x)t^{n-1}.$$

Thus,

$$t \frac{\partial^2}{\partial t^2} t g = \sum_{n=0}^{\infty} n(n+1) Q_n(x) t^n$$

and

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial g}{\partial x} \right] + t \frac{\partial^2}{\partial t^2} [t g] = \sum_{n=0}^{\infty} [(1-x^2) Q'_n(x)]' t^n + \sum_{n=0}^{\infty} n(n+1) Q_n(x) t^n \\ &= \sum_{n=0}^{\infty} \left\{ [(1-x^2) Q'_n(x)]' + n(n+1) Q_n(x) \right\} t^n. \end{aligned}$$

Thus, for each $n \geq 0$,

$$\begin{aligned} [(1-x^2) Q'_n(x)]' + n(n+1) Q_n(x) &= 0 \\ (1-x^2) Q''_n(x) - 2x Q'_n(x) + n(n+1) Q_n(x) &= 0. \end{aligned}$$

(d) Therefore, $Q_n(x)$ is a solution of

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

of order n .

(e) When $x = 1$,

$$g(t, 1) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n.$$

Hence, $Q_n(1) = 1$. Thus, $Q_n(x)$ is a polynomial solution of Legendre's equation of order n with $Q_n(1) = 1$. Therefore, $Q_n(x) = P_n(x)$, the Legendre polynomial of order n .

2. (a) Start with

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

and differentiate with respect to t .

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$$

Thus,

$$\begin{aligned} (x-t)(1-2xt+t^2)^{-1/2} &= (1-2xt+t^2) \sum_{n=1}^{\infty} n P_n(x) t^{n-1} \\ (x-t) \sum_{n=0}^{\infty} P_n(x) t^n &= (1-2xt+t^2) \sum_{n=1}^{\infty} n P_n(x) t^{n-1} \end{aligned}$$

(b) First,

$$(x-t) \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} x P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} x P_n(x) t^n - \sum_{n=1}^{\infty} P_{n-1}(x) t^n.$$

Secondly,

$$\begin{aligned}(1 - 2xt + t^2) \sum_{n=1}^{\infty} n P_n(x) t^{n-1} &= \sum_{n=1}^{\infty} n P_n(x) t^{n-1} - \sum_{n=1}^{\infty} 2nx P_n(x) t^n + \sum_{n=1}^{\infty} n P_n(x) t^{n+1} \\ &= \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n - \sum_{n=1}^{\infty} 2nx P_n(x) t^n + \sum_{n=2}^{\infty} (n-1) P_{n-1}(x) t^n\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} x P_n(x) t^n - \sum_{n=1}^{\infty} P_{n-1}(x) t^n = \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n - \sum_{n=1}^{\infty} 2nx P_n(x) t^n + \sum_{n=2}^{\infty} (n-1) P_{n-1}(x) t^n.$$

Therefore,

$$\begin{aligned}0 &= [x P_0(x) - P_1(x)] t^0 + [x P_1(x) - P_0(x) - 2P_2(x) + 2x P_1(x)] t^1 \\ &\quad + \sum_{n=2}^{\infty} [x P_n(x) - P_{n-1}(x) - (n+1) P_{n+1}(x) + 2nx P_n(x) - (n-1) P_{n-1}(x)] t^n\end{aligned}$$

Thus, each coefficient must equal zero. First,

$$P_1(x) = x P_0(x).$$

Secondly,

$$2P_2(x) = 3x P_1(x) - P_0(x).$$

And, for $n \geq 2$,

$$(n+1) P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x).$$

Note that the second case ($n = 1$) can be absorbed in this last case. Thus,

$$(n+1) P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x)$$

for $n \geq 1$.

(c) Assume that $P_0(x) = 1$. Then,

$$P_1(x) = x P_0(x) = x.$$

Next, by the recurrence formula,

$$(n+1) P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x),$$

we have

$$2P_2(x) = 3x P_1(x) - P_0(x) = 3x^2 - 1.$$

Thus, $P_2(x) = (3x^2 - 1)/2$. Next, for $n = 2$ we have

$$3P_3(x) = 5x P_2(x) - 2P_1(x) = 5x \left(\frac{3x^2 - 1}{2} \right) - 2x = \frac{15x^3}{2} - \frac{9x}{2}$$

Thus, $P_3(x) = (5x^3 - 3x)/2$. The remaining polynomials are found in a similar manner.

3. Given the recurrence formula

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x),$$

for $n \geq 2$, letting $x = 0$ produces the relation

$$nP_n(0) = -(n-1)P_{n-2}(0), \quad n \geq 2.$$

Replacing n with $2n$ provides

$$\begin{aligned} 2nP_{2n}(0) &= (1-2n)P_{2n-2}(0) \\ P_{2n}(0) &= \frac{1-2n}{2n} P_{2n-2}(0), \quad n \geq 1. \end{aligned}$$

Thus,

$$P_2(0) = \frac{1-2}{2} P_0(0), \quad P_4(0) = \frac{1-4}{4} P_2(0), \quad \text{and} \quad P_6(0) = \frac{1-6}{6} P_4(0),$$

etc., so that

$$P_2(0)P_4(0)P_6(0)\cdots P_{2n}(0) = \frac{1-2}{2} \cdot \frac{1-4}{4} \cdot \frac{1-6}{6} \cdots \frac{1-2n}{2n} P_0(0)P_2(0)P_4(0)\cdots P_{2n-2}(0).$$

Cancelling,

$$P_{2n}(0) = \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} P_0(0).$$

Of course, $P_0(0) = 1$, and

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n-1)(2n)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{(2n)!}{2^n n!}.$$

Thus,

$$P_{2n}(0) = \frac{(-1)^n \frac{(2n)!}{2^n n!}}{2^n n!} = (-1)^n \frac{(2n)!}{2^{2n} (n!)^2}.$$

4. Because $P_n(x)$ is a solution of Legendre's equation, we have

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$$

Substituting $x = 1$,

$$(1-(1)^2)P_n''(1) - 2(1)P_n'(1) + n(n+1)P_n(1) = 0,$$

or equivalently,

$$-2P_n'(1) + n(n+1)P_n(1) = 0.$$

Thus, the coefficient of $P_n''(1)$ is zero and further, $P_n(1) = 1$. Thus

$$P_n'(1) = n(n+1)/2.$$

In a similar manner, with $x = -1$,

$$(1-1)P_n''(-1) - 2(-1)P_n'(-1) + n(n+1)P_n(-1) = 0$$

and

$$2P'_n(-1) + n(n+1)P_n(-1) = 0.$$

However, if n is even, $P_n(-1) = P_n(1)$ and if n is odd, $P_n(-1) = -P_n(1)$. Thus, $P_n(-1) = (-1)^n$, and

$$2P'_n(-1) + n(n+1)(-1)^n = 0$$

$$P'_n(-1) = -(-1)^n \frac{n(n+1)}{2},$$

or equivalently,

$$(-1)^{n+1} P'_n(-1) = \frac{n(n+1)}{2}.$$

5. We seek a solution having form $\sum_{n=0}^{\infty} a_n x^n$, with derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$0 = y'' + xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1}.$$

We shift the index of each term to obtain a common power of x .

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

Thus,

$$0 = 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + a_{n-1}] x^n.$$

Setting coefficients equal to zero, $a_2 = 0$ and

$$0 = (n+2)(n+1) a_{n+2} + a_{n-1} \quad \text{or} \quad a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)}, \quad n \geq 1.$$

Thus,

$$a_3 = \frac{-a_0}{3 \cdot 2}, \quad a_6 = \frac{-a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, \quad \text{and} \quad a_9 = \frac{-a_6}{9 \cdot 8} = \frac{-a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}.$$

We conjecture that

$$a_{3n} = \frac{(-1)^n a_0}{(2 \cdot 3) \cdot (5 \cdot 6) \cdot \dots \cdot (3n-1)(3n)}, \quad n \geq 1.$$

Similarly,

$$a_4 = \frac{-a_1}{4 \cdot 3}, \quad a_7 = \frac{-a_4}{7 \cdot 6} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \quad \text{and} \quad a_{10} = \frac{-a_7}{10 \cdot 9} = \frac{-a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}.$$

We conjecture

$$a_{3n+1} = \frac{(-1)^n a_1}{(3 \cdot 4) \cdot (6 \cdot 7) \cdots (3n)(3n+1)}, \quad n \geq 1.$$

Finally, because $a_2 = 0$, $a_{3n+2} = 0$, for $n \geq 1$. Thus,

$$y(x) = a_0 \left[1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \cdots \right] + a_1 \left[x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \cdots \right]$$

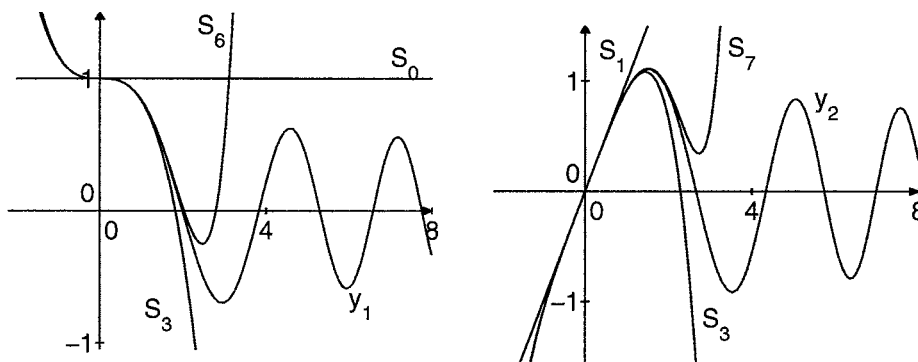
Choose $y_1(x)$ so that $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$.

$$y_1(x) = 1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{(2 \cdot 3) \cdot (5 \cdot 6) \cdots (3n-1)(3n)}.$$

Choose $y_2(x)$ so that $a_0 = y_2(0) = 0$ and $a_1 = y_2'(0) = 1$.

$$y_2(x) = x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n+1}}{(3 \cdot 4) \cdot (6 \cdot 7) \cdots (3n)(3n+1)}.$$

In $y'' + xy = 0$, the coefficients of y is $Q(x) = x$. The function Q is analytic for all x . By Theorem 2.29, the series converges for all x . The solutions and partial sums are plotted below.



6. (a) We seek a solution having form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, with derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$\begin{aligned} 0 &= (1-x^2)y'' - xy' + p^2y = (1-x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + p^2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} p^2 a_n x^n. \end{aligned}$$

Note that the second and third terms are unaffected if we start the index at $n = 0$. We shift the index of the first term to obtain

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_nx^n - \sum_{n=0}^{\infty} na_nx^n + \sum_{n=0}^{\infty} p^2a_nx^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - [n(n-1) + n - p^2]a_n]x^n \end{aligned}$$

Setting coefficients equal to zero,

$$(n+2)(n+1)a_{n+2} - (n^2 - p^2)a_n = 0 \quad \text{or} \quad a_{n+2} = \frac{n^2 - p^2}{(n+2)(n+1)}a_n, \quad n \geq 0.$$

Thus,

$$a_2 = \frac{-p^2}{2 \cdot 1}a_0, \quad a_4 = \frac{4 - p^2}{4 \cdot 3}a_2 = \frac{-p^2(2^2 - p^2)}{4!}a_0,$$

and

$$a_6 = \frac{16 - p^2}{6 \cdot 5}a_4 = \frac{-p^2(2^2 - p^2)(4^2 - p^2)}{6!}a_0.$$

We conjecture that

$$a_{2n} = \frac{-p^2(2^2 - p^2)(4^2 - p^2) \cdots ((2n-2)^2 - p^2)}{(2n)!}a_0, \quad n \geq 1.$$

Similarly,

$$a_3 = \frac{1^2 - p^2}{3 \cdot 2}a_1, \quad a_5 = \frac{3^2 - p^2}{5 \cdot 4}a_3 = \frac{(1^2 - p^2)(3^2 - p^2)}{5!}a_1,$$

and

$$a_7 = \frac{5^2 - p^2}{7 \cdot 6}a_5 = \frac{(1^2 - p^2)(3^2 - p^2)(5^2 - p^2)}{7!}a_1.$$

We conjecture that

$$a_{2n+1} = \frac{(1^2 - p^2)(3^2 - p^2) \cdots ((2n-1)^2 - p^2)}{(2n+1)!}a_1, \quad n \geq 1.$$

Therefore,

$$\begin{aligned} y(x) &= a_0 \left[1 + \frac{-p^2}{2!}x^2 + \frac{-p^2(2^2 - p^2)}{4!}x^4 + \cdots \right] \\ &\quad + a_1 \left[x + \frac{1^2 - p^2}{3!}x^3 + \frac{(1^2 - p^2)(3^2 - p^2)}{5!}x^5 + \cdots \right]. \end{aligned}$$

Choose $y_1(x)$ so that $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$.

$$y_1(x) = 1 + \frac{-p^2}{2!}x^2 + \frac{-p^2(2^2 - p^2)}{4!}x^4 + \cdots = 1 + \sum_{n=1}^{\infty} \frac{-p^2(2^2 - p^2) \cdots ((2n-2)^2 - p^2)}{(2n)!}x^{2n}.$$

Choose $y_2(x)$ so that $a_0 = y_2(0) = 0$ and $a_1 = y_2'(0) = 1$.

$$\begin{aligned} y_2(x) &= x + \frac{1^2 - p^2}{3!} x^3 + \frac{(1^2 - p^2)(3^2 - p^2)}{5!} x^5 + \cdots \\ &= x + \sum_{n=1}^{\infty} \frac{(1^2 - p^2)(3^2 - p^2) \cdots ((2n-1)^2 - p^2)}{(2n+1)!} x^{2n+1}. \end{aligned}$$

If we divide by $1 - x^2$, then

$$y'' - \frac{x}{1-x^2} y' + \frac{p^2}{1-x^2} y = 0.$$

By Theorem 2.30, the radius of convergence for a power series about $x = 0$ is at least as large as the minimum of the radii of convergence for the power series for

$$P(x) = \frac{-x}{1-x^2} \quad \text{and} \quad Q(x) = \frac{p^2}{1-x^2}$$

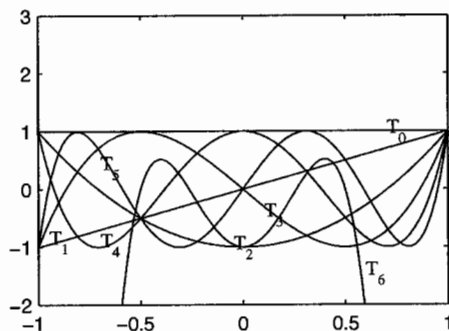
about $x = 0$. Therefore, $R \geq 1$.

- (b) If p is a nonnegative integer, there are two possibilities, p is even or p is odd. In the case that p is even, p^2 is also even and there exists a positive integer n such that $(2n-2)^2 - p^2 = 0$. Hence, beyond the value of n , all terms of $y_1(x)$ are zero and $y_1(x)$ is a polynomial. In the case that p is odd, then there exists a positive integer n so that $(2n-1)^2 - p^2 = 0$. Hence, beyond this value of n , all terms of $y_2(x)$ are zero and $y_2(x)$ is a polynomial.

- (c) The first six Chebychev polynomials.

- First $T_0(x) = 1$.
- Next, with $p = 1$, $y_2(x) = x$. Note that the coefficient of x is $2^{p-1} = 2^{1-1} = 2^0 = 1$. Thus, $T_1(x) = x$.
- With $p = 2$, $y_1(x) = 1 - 2x^2$. Multiply by -1 to get $T_2(x) = 2x^2 - 1$.
- With $p = 3$, $y_2(x) = x - (4/3)x^3$. We want the leading coefficient to be $2^{p-1} = 2^{3-1} = 2^2 = 4$, so multiply by -3 to obtain $T_3(x) = 4x^3 - 3x$.
- With $p = 4$, $y_1(x) = 1 - 8x^2 + 8x^4$. We want the leading coefficient to be $2^{p-1} = 2^{4-1} = 2^3 = 8$, so we're set. $T_4(x) = 8x^4 - 8x^2 + 1$.
- With $p = 5$, $y_2(x) = x - 4x^3 + (16/5)x^5$. We want $2^{p-1} = 2^{5-1} = 16$, so multiply by 5 to obtain $T_5(x) = 16x^5 - 20x^3 + 5x$.
- With $p = 6$, $y_1(x) = 1 - 18x^2 + 48x^4 - 32x^6$. We want $2^{p-1} = 2^{6-1} = 32$, so multiply by -1 . Let $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$.

A plot follows.



7. (a) Consider $y'' - 2xy' + py = 0$. Set

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute,

$$\begin{aligned} 0 &= y'' - 2xy' + py = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + p \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} p a_n x^n. \end{aligned}$$

We will shift indicies to the lowest common power of x .

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=3}^{\infty} 2(n-2) a_{n-2} x^{n-2} + \sum_{n=2}^{\infty} p a_{n-2} x^{n-2} \\ &= (2a_2 + p a_0) x^0 + \sum_{n=3}^{\infty} \{n(n-1) a_n - (2n-4-p) a_{n-2}\} x^{n-2} \end{aligned}$$

Thus,

$$2a_2 + p a_0 = 0 \quad \text{or} \quad a_2 = -\frac{p}{2} a_0,$$

and

$$n(n-1) a_n - (2n-p-4) a_{n-2} = 0, \quad n \geq 3.$$

Equivalently,

$$a_n = \frac{2n-p-4}{n(n-1)} a_{n-2}, \quad n \geq 3.$$

Thus,

$$a_4 = \frac{4-p}{4 \cdot 3} a_2, \quad a_6 = \frac{8-p}{6 \cdot 5} a_4, \quad \dots, \quad a_{2n} = \frac{4n-p-4}{2n(2n-1)} a_{2n-2}.$$

Thus,

$$a_2 a_4 a_6 \cdots a_{2n} = -\frac{p(4-p)(8-p) \cdots (4n-p-4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (2n-1)(2n)} a_0 a_2 a_4 \cdots a_{2n-2},$$

and

$$a_{2n} = -\frac{p(4-p)(8-p) \cdots (4n-p-4)}{(2n)!} a_0.$$

In a similar manner,

$$a_3 = \frac{2-p}{3 \cdot 2} a_1, \quad a_5 = \frac{6-p}{5 \cdot 4} a_3, \quad \dots, \quad a_{2n+1} = \frac{4n-p-2}{(2n+1)(2n)} a_{2n-1}.$$

Thus,

$$a_{2n+1} = \frac{(2-p)(6-p) \cdots (4n-p-2)}{(2n+1)!} a_1.$$

Hence,

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where

$$y_1(x) = 1 - \sum_{n=1}^{\infty} \frac{p(4-p)(8-p) \cdots (4n-p-4)}{(2n)!} x^{2n} \quad (3.1)$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(2-p)(6-p) \cdots (4n-p-2)}{(2n+1)!} x^{2n+1}. \quad (3.2)$$

Using the ratio test on $y_1(x)$,

$$\begin{aligned} \left| \frac{A_{n+1}}{A_n} \right| &= \frac{p(4-p) \cdots (4n-p)}{(2n+2)!} \cdot \frac{(2n)!}{p(4-p) \cdots (4n-p-4)} x^2 \\ &= \frac{4n-p}{(2n+2)(2n+1)} x^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore the series y_1 converges for all x . For $y_2(x)$,

$$\begin{aligned} \left| \frac{A_{n+1}}{A_n} \right| &= \frac{(2-p) \cdots (4n-p+2)}{(2n+3)!} \cdot \frac{(2n+1)!}{(2-p) \cdots (4n-p-2)} x^2 \\ &= \frac{4n-p+2}{(2n+3)(2n+2)} x^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore the series y_2 converges for all x .

(b) We now consider the case when p is a nonnegative integer. In the case that p is odd, neither

$$\frac{-p(4-p)(8-p) \cdots (4n-p-4)}{(2n)!}$$

nor

$$\frac{(2-p)(6-p) \cdots (4n-p-2)}{(2n+1)!}$$

can equal zero. Let's examine the case where p is even. There are two possibilities. Either $p = 2, 6, 10, \dots$ or $p = 4, 8, 12, \dots$, etc. We examine each case separately. Consider the case where $p = 2 + 4l$, l a nonnegative integer. Then

$$\begin{aligned}4n - p - 2 &= 0 \\4n - (2 + 4l) - 2 &= 0 \\4n &= 4l + 4 \\n &= l + 1.\end{aligned}$$

Thus, for $n \geq l + 1$,

$$a_{2n+1} = \frac{(2-p)(6-p)\cdots(4n-p-2)}{(2n+1)!} = 0$$

and the series $y_2(x)$ terminates in a polynomial having degree

$$2n + 1 = 2l + 1 = p/2,$$

which is odd degree. It is easy to show that $y_1(x)$ does not terminate in this case. In the case where $p = 4 + 4l$, l a nonnegative integer,

$$\begin{aligned}4n - p - 4 &= 0 \\4n - (4 + 4l) - 4 &= 0 \\4n &= 4l + 8 \\n &= l + 2.\end{aligned}$$

Thus, for $n \geq l + 2$,

$$a_{2n} = -\frac{p(4-p)\cdots(4n-p-4)}{(2n)!} = 0$$

and the series $y_1(x)$ terminates in a polynomial having degree

$$2n = 2(l + 1) = 2l + 2 = p/2,$$

which is even degree.

(c) Using equations (3.1) and (3.2), we arrive at

$$\begin{aligned}p = 0 &\Rightarrow 1 \\p = 2 &\Rightarrow x \\p = 4 &\Rightarrow 1 - 2x^2 \\p = 6 &\Rightarrow x - \frac{2}{3}x^3 \\p = 8 &\Rightarrow 1 - 4x^2 + \frac{4}{3}x^4 \\p = 10 &\Rightarrow x - \frac{4}{3}x^3 + \frac{4}{15}x^5.\end{aligned}$$

Each of these must be multiplied by an appropriate constant so that the coefficient of x^p is 2^p .

$$H_0(x) = 1$$

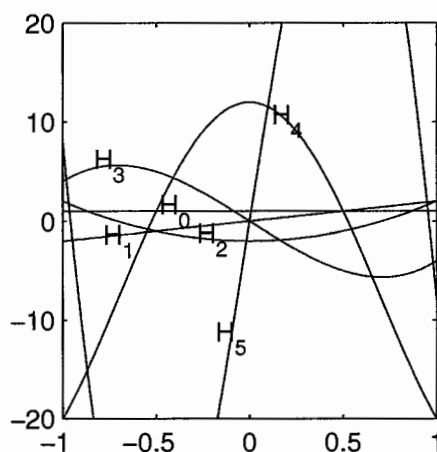
$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$



8. Dividing by $1 - x^2$, Legendre's equation becomes

$$y'' - \frac{2x}{1-x^2} y' + \frac{n(n+1)}{1-x^2} y = 0.$$

Comparing this with $y'' + P(x)y' + Q(x)y = 0$,

$$P(x) = -\frac{2x}{1-x^2} \quad \text{and} \quad Q(x) = \frac{n(n+1)}{1-x^2},$$

both of which are analytic at $x = 0$. Thus, both P and Q have power series expansions about $x = 0$. By Theorem 1.34, the radius of convergence about $x = 0$ is $R = |x_0 - z_0|$, where z_0 is the complex root of $1 - x^2$ closest to $x = 0$. This would be either of the roots 1 or -1 . Thus, the series representations of P and Q about $x = 0$ have radius of convergence $R = 1$. Therefore, by Theorem 2.29, the interval of convergence for solutions of Legendre's equation about $x = 1$ is at least $R = 1$.

Section 11.4. Types of Singular Points—Euler's Equation

1. Divide by the leading coefficient.

$$x^2 y'' + 4xy' - 2xy = 0$$

$$y'' + \frac{4}{x}y' - \frac{2}{x}y = 0$$

Hence, $x = 0$ is a singular point. Thus,

$$P(x) = \frac{4}{x} \quad \text{and} \quad Q(x) = -\frac{2}{x},$$

so

$$xP(x) = 4 \quad \text{and} \quad x^2Q(x) = -2x,$$

both of which are analytic at $x = 0$. Thus, the point $x = 0$ is a regular singular point.

2. Divide by the leading coefficient.

$$y'' + \frac{x-2}{x^2(1-x)}y' - \frac{3}{x(1-x)}y = 0$$

Thus,

$$P(x) = \frac{x-2}{x^2(1-x)} \quad \text{and} \quad Q(x) = \frac{-3}{x(1-x)}$$

are analytic everywhere, except where their denominators equal zero. Thus, $x = 0$ and $x = 1$ are singular points. For $x = 0$,

$$xP(x) = \frac{x-2}{x(1-x)}$$

is not analytic near $x = 0$. Hence, $x = 0$ is an irregular singular point. For $x = 1$,

$$(x-1)P(x) = \frac{2-x}{x^2}$$

and

$$(x-1)^2Q(x) = \frac{-3(1-x)}{x}$$

are both analytic near $x = 1$. Hence, $x = 1$ is a regular singular point.

3. Divide by the leading coefficient.

$$y'' + \frac{4}{x(2-x)^2}y' + \frac{5}{x^2(2-x)^2}y = 0.$$

The functions

$$P(x) = \frac{4}{x(2-x)^2} \quad \text{and} \quad Q(x) = \frac{5}{x^2(2-x)^2}$$

are analytic everywhere, except where their denominators equal zero. Thus, $x = 0$ and $x = 2$ are singular points. For $x = 0$,

$$xP(x) = \frac{4}{(2-x)^2} \quad \text{and} \quad x^2Q(x) = \frac{5}{(2-x)^2}$$

are both analytic near $x = 0$. Thus, $x = 0$ is a regular singular point. For $x = 2$,

$$(x-2)P(x) = \frac{4}{x(2-x)}$$

is not analytic near $x = 2$, so $x = 2$ is an irregular singular point.

4. Divide by the leading coefficient.

$$(x^2 - 4)y'' - (x - 2)y' + 2y = 0$$

$$y'' - \frac{1}{x+2}y' + \frac{2}{x^2-4}y = 0.$$

Thus,

$$P(x) = \frac{-1}{x+2} \quad \text{and} \quad Q(x) = \frac{2}{x^2-4}.$$

Both P and Q are analytic everywhere except where their denominators are zero. Thus, $x = 2$ and $x = -2$ are singular points. For $x = 2$, we have

$$(x-2)P(x) = -\frac{x-2}{x+2}$$

and

$$(x-2)^2Q(x) = \frac{2(x-2)}{x+2},$$

both of which are analytic near $x = 2$. Therefore, $x = 2$ is a regular singular point. For $x = -2$,

$$(x+2)P(x) = -1$$

and

$$(x+2)^2Q(x) = \frac{2(x+2)}{x-2},$$

both of which are analytic near $x = -2$. Therefore, $x = -2$ is a regular singular point.

5. Divide by the leading coefficient.

$$(t^2 - t - 6)y'' + (t^2 - 9)y' + 2y = 0$$

$$y'' + \frac{t+3}{t+2}y' + \frac{2}{(t-3)(t+2)}y = 0$$

Thus,

$$P(t) = \frac{t+3}{t+2} \quad \text{and} \quad Q(t) = \frac{2}{(t-3)(t+2)}$$

are analytic everywhere, except where their denominators are equal to zero. Thus, $t = -2$ and $t = 3$ are singular points. For $t = -2$,

$$(t+2)P(t) = t+3 \quad \text{and} \quad (t+2)^2Q(t) = \frac{2(t+2)}{t-3},$$

both of which are analytic near $t = -2$. Thus, $t = -2$ is a regular singular point. For $t = 3$,

$$(t-3)P(t) = \frac{(t+3)(t-3)}{t+2}$$

and

$$(t-3)^2Q(t) = \frac{2(t-3)}{t+2},$$

both of which are analytic near $t = 3$. Thus, $t = 3$ is a regular singular point.

6. Divide by the leading coefficient.

$$y'' + \frac{x}{(2+x)^2(2-x)}y' + \frac{1}{(2+x)(2-x)^2}y = 0$$

The functions

$$P(x) = \frac{x}{(2+x)^2(2-x)}$$

and

$$Q(x) = \frac{1}{(2+x)(2-x)^2}$$

are analytic everywhere, except where their denominators equal zero. Thus, $x = -2$ and $x = 2$ are singular points. For $x = -2$,

$$(x+2)P(x) = \frac{x}{(2+x)(2-x)}$$

is not analytic near $x = -2$. Hence, $x = -2$ is an irregular singular point. For $x = 2$,

$$(x-2)P(x) = \frac{-x}{(2+x)^2} \quad \text{and} \quad (x-2)^2Q(x) = \frac{1}{2+x}$$

are both analytic near $x = 2$. Thus, $x = 2$ is a regular singular point.

7. Divide by the leading coefficient.

$$(t^2 - 1)y'' + (t - 2)y' + y = 0$$

$$y'' + \frac{t-2}{(t+1)(t-1)}y' + \frac{1}{(t+1)(t-1)}y = 0$$

Thus,

$$P(t) = \frac{t-2}{(t+1)(t-1)} \quad \text{and} \quad Q(t) = \frac{1}{(t+1)(t-1)}$$

are analytic everywhere, except where their denominators equal zero. Thus, $t = -1$ and $t = 1$ are singular points. For $t = -1$,

$$(t+1)P(t) = \frac{t-2}{t-1} \quad \text{and} \quad (t+1)^2Q(t) = \frac{t+1}{t-1},$$

both of which are analytic near $t = -1$. Thus, $t = -1$ is a regular singular point. For $t = 1$,

$$(t-1)P(t) = \frac{t-2}{t+1} \quad \text{and} \quad (t-1)^2Q(t) = \frac{t-1}{t+1},$$

both of which are analytic near $t = 1$. Thus, $t = 1$ is a regular singular point.

8. Divide by the leading coefficient.

$$y'' + \frac{\sin x}{(x-\pi)^2}y' + \frac{\cos x}{(x-\pi)^2}y = 0$$

The functions

$$P(x) = \frac{\sin x}{(x-\pi)^2} \quad \text{and} \quad Q(x) = \frac{\cos x}{(x-\pi)^2}$$

are analytic everywhere, except where the denominators equal zero. Thus, $x = \pi$ is a singular point. However,

$$(x-\pi)^2Q(x) = \cos x$$

is clearly analytic, but what about

$$(x-\pi)P(x) = \frac{\sin x}{x-\pi}?$$

Since

$$\sin x = -(x - \pi) + \frac{(x - \pi)^3}{3!} - \frac{(x - \pi)^5}{5!} + \dots,$$

we can write

$$\begin{aligned}(x - \pi)P(x) &= \frac{\sin x}{x - \pi} \\ &= -1 + \frac{(x - \pi)^2}{3!} - \frac{(x - \pi)^4}{5!} + \dots\end{aligned}$$

Thus, $(x - \pi)P(x)$, having a power series expansion about $x = \pi$, is analytic at $x = \pi$. Therefore, $x = \pi$ is a regular singular point.

9. The equation can be written as $y'' + P(t)y' + Q(t)y = 0$, where $P(t) = t/\sin^2 t$ and $Q(t) = -3/\sin^2 t$. Both P and Q are analytic except where $\sin t = 0$, or where $t = k\pi$ for some integer k . Hence the singular points are $t = k\pi$.

First let $t_0 = 0$. Then we have

$$tP(t) = \left(\frac{t}{\sin t}\right)^2, \quad \text{and} \quad t^2Q(t) = -3\left(\frac{t}{\sin t}\right)^2.$$

Since $t/\sin t \rightarrow 1$ as $t \rightarrow t_0 = 0$, both functions are analytic at $t_0 = 0$. Hence 0 is a regular singular point.

Now suppose that $t_0 = k\pi$, where k is a nonzero integer. Then

$$(t - k\pi)P(t) = \frac{t(t - k\pi)}{\sin^2 t} = t \cdot \frac{t - k\pi}{\sin t} \cdot \frac{1}{\sin t}.$$

Since, by l'Hopital's rule,

$$\lim_{t \rightarrow k\pi} \frac{t - k\pi}{\sin t} = \lim_{t \rightarrow k\pi} \frac{1}{\cos t} = \frac{1}{\cos k\pi} = (-1)^k,$$

the first two factors in are analytic at $t_0 = k\pi$ and have nonzero values there. However, the third factor has no limit as $t \rightarrow k\pi$, so $(t - k\pi)P(t)$ is not analytic at $t_0 = k\pi$. Therefore $k\pi$ is an irregular singular point if k is a nonzero integer.

10. Consider $y = x^s$ and its derivatives.

$$y' = sx^{s-1} \quad \text{and} \quad y'' = s(s-1)x^{s-2}.$$

Substitute.

$$\begin{aligned}0 &= x^2 y'' - 3xy' - 5y \\ &= x^2 s(s-1)x^{s-2} - 3xsx^{s-1} - 5x^s \\ &= [s(s-1) - 3s - 5]x^s.\end{aligned}$$

The indicial equation

$$0 = I(s) = s(s-1) - 3s - 5 = (s-5)(s+1)$$

has roots $s = 5$ and $s = -1$. These provide independent solutions $y_1(x) = x^5$ and $y_2(x) = x^{-1}$ and the general solution

$$y(x) = C_1 x^5 + C_2 x^{-1}.$$

11. Consider $y = x^s$ and its derivatives.

$$y' = sx^{s-1} \quad \text{and} \quad y'' = s(s-1)x^{s-2}$$

Substitute.

$$\begin{aligned}0 &= x^2 y'' + 5xy' + 3y \\ &= x^2 s(s-1)x^{s-2} + 5xsx^{s-1} + 3x^s \\ &= [s(s-1) + 5s + 3]x^s.\end{aligned}$$

The indicial equation

$$0 = I(s) = s(s-1) + 5s + 3 = (s+3)(s+1)$$

has roots $s = -3$ and $s = -1$. These provide independent solutions $y_1(x) = x^{-3}$ and $y_2(x) = x^{-1}$ and the general solution

$$y(x) = C_1 x^{-3} + C_2 x^{-1}.$$

12. Consider $y = x^s$ and its derivatives.

$$y' = sx^{s-1} \quad \text{and} \quad y'' = s(s-1)x^{s-2}$$

Substitute.

$$\begin{aligned}0 &= x^2 y'' + 3xy' + y \\ &= x^2 s(s-1)x^{s-2} + 3xsx^{s-1} + x^s \\ &= [s(s-1) + 3s + 1]x^s\end{aligned}$$

The indicial equation

$$0 = I(s) = s(s-1) + 3s + 1 = (s+1)^2$$

has repeated root $s = -1$. This provides independent solutions $y_1(x) = x^{-1}$ and $y_2(x) = x^{-1} \ln x$ and general solution

$$y(x) = C_1 x^{-1} + C_2 x^{-1} \ln x.$$

13. Consider $y = x^s$ and its derivatives.

$$y' = sx^{s-1} \quad \text{and} \quad y'' = s(s-1)x^{s-2}$$

Negate both sides of the equation.

$$x^2 y'' - xy' + y = 0$$

Substitute.

$$\begin{aligned} 0 &= x^2 s(s-1)x^{s-2} - xsx^{s-1} + x^s \\ &= [s(s-1) - s + 1]x^s \end{aligned}$$

the indicial equation

$$0 = I(s) = s(s-1) - s + 1 = (s-1)^2$$

has repeated root $s = 1$. This provides independent solutions $y_1(x) = x$ and $y_2(x) = x \ln x$ and general solution

$$y(x) = C_1 x + C_2 x \ln x.$$

14. The equation

$$x^2 y'' - 2xy' - 4y = 0$$

has indicial equation

$$0 = I(s) = s(s-1) - 2s - 4 = (s-4)(s+1),$$

which has roots $s = 4$ and $s = -1$. These provide independent solutions $y_1(x) = x^4$ and $y_2(x) = x^{-1}$ and general solution

$$y(x) = C_1 x^4 + C_2 x^{-1}.$$

15. The equation $x^2 y'' + 9xy' + 16y = 0$ has indicial equation

$$0 = I(s) = s(s-1) + 9s + 16 = (s+4)^2.$$

This equation has repeated root $s = -4$. This provides independent solutions $y_1(x) = x^{-4}$ and $y_2(x) = x^{-4} \ln x$ and general solution

$$y(x) = C_1 x^{-4} + C_2 x^{-4} \ln x.$$

16. Divide both sides of the equation $2x^2 y'' + 10xy' + 8y = 0$ by 2.

$$x^2 y'' + 5xy' + 4y = 0$$

The indicial equation

$$0 = I(s) = s(s-1) + 5s + 4 = (s+2)^2$$

has repeated zero $s = -2$. This provides independent solutions $y_1(x) = x^{-2}$ and $y_2(x) = x^{-2} \ln x$ and general solution

$$y(x) = C_1 x^{-2} + C_2 x^{-2} \ln x.$$

17. The equation

$$x^2 y'' - 5xy' + 9y = 0$$

has indicial equation

$$0 = I(s) = s(s-1) - 5s + 9 = (s-3)^2,$$

which has repeated zero $s = 3$. This provides independent solutions $y_1(x) = x^3$ and $y_2(x) = x^3 \ln x$ and general solution

$$y(x) = C_1 x^3 + C_2 x^3 \ln x.$$

18. Let $\tilde{x} = -x$. Then, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{d\tilde{x}} \frac{d\tilde{x}}{dx} = \frac{dy}{d\tilde{x}} (-1) = -\frac{dy}{d\tilde{x}}.$$

Thus,

$$\frac{d}{dx} = -\frac{d}{d\tilde{x}},$$

and we can write

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = -\frac{d}{d\tilde{x}} \left(-\frac{dy}{d\tilde{x}} \right) = \frac{d^2 y}{d\tilde{x}^2}.$$

Substitute these results in $x^2 y'' + pxy' = qy = 0$ to obtain

$$(-\tilde{x})^2 \frac{d^2 y}{d\tilde{x}^2} + p(-\tilde{x}) \left(-\frac{dy}{d\tilde{x}} \right) + qy = 0$$

$$\tilde{x}^2 \frac{d^2 y}{d\tilde{x}^2} + p\tilde{x} \frac{dy}{d\tilde{x}} + qy = 0.$$

19. Consider
- $y = x^s$
- and its derivatives

$$y' = sx^{s-1} \quad \text{and} \quad y'' = s(s-1)x^{s-2}.$$

Substitute.

$$\begin{aligned} 0 &= x^2 y'' + 2x y' - 6y \\ &= x^2 s(s-1)x^{s-2} + 2xsx^{s-1} - 6x^s \\ &= [s(s-1) + 2s - 6]x^s \end{aligned}$$

The indicial equation

$$0 = I(s) = s(s-1) + 2s - 6 = (s+3)(s-2)$$

has roots $s = -3$ and $s = 2$. These provide independent solutions $y_1(x) = x^{-3}$ and $y_2(x) = x^2$ and general solution

$$y(x) = C_1 x^{-3} + C_2 x^2.$$

The initial condition $y(1) = 3$ provides

$$3 = C_1 + C_2.$$

Differentiate.

$$y'(x) = -3C_1 x^{-3} + 2C_2 x$$

The initial condition $y'(1) = 1$ provides

$$1 = -3C_1 + 2C_2.$$

Solving, $C_1 = 1$ and $C_2 = 2$. Thus,

$$y(x) = x^{-3} + 2x^2.$$

20. Consider
- $y = x^s$
- and its derivatives

$$y' = sx^{s-1} \quad \text{and} \quad y'' = s(s-1)x^{s-2}.$$

Substitute.

$$\begin{aligned} 0 &= x^2 y'' - 2y \\ &= x^2 s(s-1)x^{s-2} - 2x^s \\ &= [s(s-1) - 2]x^s \end{aligned}$$

The indicial equation

$$0 = I(s) = s(s-1) - 2 = (s-2)(s+1)$$

has roots $s = 2$ and $s = -1$. These provide independent solutions $y_1(x) = x^2$ and $y_2(x) = x^{-1}$ and general solution

$$y(x) = C_1 x^2 + C_2 x^{-1}.$$

The initial condition $y(1) = 1$ provides

$$1 = C_1 + C_2.$$

Differentiate.

$$y'(x) = 2C_1 x - C_2 x^{-2}$$

The initial condition $y'(1) = 3$ provides

$$3 = 2C_1 - C_2.$$

Solving, $C_1 = 4/3$, $C_2 = -1/3$, and

$$y(x) = \frac{4}{3}x^2 - \frac{1}{3}x^{-1}$$

21. Consider
- $y = (x-3)^s$
- and its derivatives

$$y' = s(x-3)^{s-1} \quad \text{and} \quad y'' = s(s-1)(x-3)^{s-2}.$$

Substitute.

$$\begin{aligned} 0 &= (x-3)^2 y'' + 5(x-3)y' + 4y \\ &= (x-3)^2 s(s-1)(x-3)^{s-2} \\ &\quad + 5(x-3)s(x-3)^{s-1} + 4(x-3)^s \\ &= [s(s-1) + 5s + 4](x-3)^s \end{aligned}$$

The indicial equation

$$0 = I(s) = s(s-1) + 5s + 4 = (s+2)^2$$

has repeated zero $s = -2$. Using Theorem 4.21, this provides independent solutions $y_1(x) = |x-3|^{-2}$ and $y_2(x) = |x-3|^{-2} \ln |x-3|$. The general solution is

$$y(x) = C_1 |x-3|^{-2} + C_2 |x-3|^{-2} \ln |x-3|.$$

No solution can contain the point at which $x = 3$. Because the initial conditions use $x = 4$, the solution of interest is to the right of $x = 3$ and we can write

$$y(x) = C_1 (x-3)^{-2} + C_2 (x-3)^{-2} \ln(x-3).$$

The initial condition $y(4) = 1$ provides $C_1 = 1$. Differentiate.

$$y'(x) = -2C_1(x-3)^{-3} + C_2(x-3)^{-3} - 2C_2(x-3)^{-3} \ln(x-3)$$

The initial condition $y'(4) = 1$ provides

$$1 = -2C_1 + C_2.$$

Solving, $C_1 = 1$, $C_2 = 3$, and

$$y(x) = (x-3)^{-2} + 3(x-3)^{-2} \ln(x-3).$$

22. Consider $y = (x-1)^s$ and its derivatives

$$y' = s(x-1)^{s-1} \quad \text{and} \quad y'' = s(s-1)(x-1)^{s-2}.$$

Substitute.

$$\begin{aligned} 0 &= (x-1)^2 y'' - 6y \\ &= (x-1)^2 s(s-1)(x-1)^{s-2} - 6(x-1)^s \\ &= [s(s-1) - 6](x-1)^s \end{aligned}$$

The indicial equation

$$0 = I(s) = s(s-1) - 6 = (s-3)(s+2)$$

has zeros $s = 3$ and $s = -2$. Using Theorem 4.21, these provide independent solutions $y_1(x) = |x-1|^3$ and $y_2(x) = |x-1|^{-2}$. The general solution is

$$y(x) = C_1|x-1|^3 + C_2|x-1|^{-2}.$$

No solution can contain the point at which $x = 1$. Because the initial conditions are evaluated at $x = 0$, the solution of interest is to the left of $x = 1$ and we can write

$$y(x) = C_1(1-x)^3 + C_2(1-x)^{-2}.$$

The initial condition $y(0) = 1$ provides

$$1 = C_1 + C_2.$$

Differentiate.

$$y'(x) = -3C_1(1-x)^2 + 2C_2(1-x)^{-3}.$$

The initial condition $y'(0) = 1$ provides

$$1 = -3C_1 + 2C_2.$$

Solving, $C_1 = 1/5$ and $C_2 = 4/5$, and

$$y(x) = \frac{1}{5}(1-x)^3 + \frac{4}{5}(1-x)^{-2}.$$

23. (a) Substitute $s = \alpha + i\beta$ in the indicial equation $I(s) = s(s-1) + ps + q$.

$$\begin{aligned} 0 &= I(\alpha + i\beta) \\ &= (\alpha + i\beta)(\alpha + i\beta - 1) + p(\alpha + i\beta) + q \\ &= (\alpha^2 - \alpha - \beta^2 + \alpha p + q) \\ &\quad + i(2\alpha\beta - \beta + p\beta) \end{aligned}$$

Thus,

$$\begin{aligned} \alpha p + q &= \beta^2 + \alpha - \alpha^2, \quad \text{and} \\ p\beta &= \beta - 2\alpha\beta. \end{aligned}$$

If $\beta = 0$, then the root $s = \alpha + i\beta$ is real and handled by Theorem 4.21. Assuming $\beta \neq 0$, the second equation becomes

$$p = 1 - 2\alpha.$$

Substitute this result in the first equation.

$$\begin{aligned} \alpha(1 - 2\alpha) + q &= \beta^2 + \alpha - \alpha^2 \\ \alpha - 2\alpha^2 + q &= \beta^2 + \alpha - \alpha^2 \\ q &= \alpha^2 + \beta^2. \end{aligned}$$

(b) With $x > 0$,

$$y_1(x) = x^\alpha \cos(\beta \ln x).$$

The proof with $x < 0$ is similar. Differentiate.

$$y_1'(x) = \alpha x^{\alpha-1} \cos(\beta \ln x) - \beta x^{\alpha-1} \sin(\beta \ln x)$$

Differentiate a second time.

$$\begin{aligned} y_1''(x) &= [\alpha(\alpha-1) - \beta^2]x^{\alpha-2} \cos(\beta \ln x) \\ &\quad - [\alpha\beta + \beta(\alpha-1)]x^{\alpha-2} \sin(\beta \ln x) \end{aligned}$$

Substitute in $x^2 y'' + pxy' + qy = 0$.

$$\begin{aligned} 0 &= x^2 \left\{ [\alpha(\alpha-1) - \beta^2]x^{\alpha-2} \cos(\beta \ln x) \right. \\ &\quad \left. - [\alpha\beta + \beta(\alpha-1)]x^{\alpha-2} \sin(\beta \ln x) \right\} \\ &\quad + px[\alpha x^{\alpha-1} \cos(\beta \ln x) \\ &\quad - \beta x^{\alpha-1} \sin(\beta \ln x)] + qx^\alpha \cos(\beta \ln x) \\ &= \{(\alpha^2 - \alpha - \beta^2 + p\alpha + q) \cos(\beta \ln x) \\ &\quad - (2\alpha\beta - \beta + \beta p) \sin(\beta \ln x)\} x^\alpha \\ &= 0, \end{aligned}$$

by the development shown in part (a). In a similar manner, one can show that

$$y_2 = x^\alpha \sin(\beta \ln x)$$

is also a solution of $x^2 y'' + pxy' + qy = 0$.

24. The equation $x^2 y'' + 2xy' + y = 0$ has indicial equation

$$0 = I(s) = s(s-1) + 2s + 1 = s^2 + s + 1.$$

This equation has complex zeros $s = -1/2 \pm \sqrt{3}i/2$. These provide independent solutions $y_1(x) = x^{-1/2} \cos(\sqrt{3} \ln x/2)$ and $y_2(x) = x^{-1/2} \sin(\sqrt{3} \ln x/2)$. The general solution, on $x > 0$, is

$$y(x) = C_1 x^{-1/2} \cos\left(\frac{\sqrt{3} \ln x}{2}\right) + C_2 x^{-1/2} \sin\left(\frac{\sqrt{3} \ln x}{2}\right).$$

25. The equation $3x^2 y'' + y = 0$ has indicial equation

$$0 = I(s) = 3s(s-1) + 1 = 3s^2 - 3s + 1.$$

This equation has complex zeros $s = 1/2 \pm \sqrt{3}i/6$. These provide two independent solutions $y_1(x) = x^{1/2} \cos(\sqrt{3} \ln x/6)$ and $y_2(x) = x^{1/2} \sin(\sqrt{3} \ln x/6)$ on $x > 0$. The general solution on $x > 0$ is

$$y(x) = C_1 x^{1/2} \cos\left(\frac{\sqrt{3} \ln x}{6}\right) + C_2 x^{1/2} \sin\left(\frac{\sqrt{3} \ln x}{6}\right).$$

26. The equation $2x^2 y'' + xy' + y = 0$ has indicial equation

$$0 = I(s) = 2s(s-1) + s + 1 = 2s^2 - s + 1.$$

This equation has complex zeros $s = 1/4 \pm \sqrt{7}i/4$. These zeros provide independent solutions $y_1(x) = x^{1/4} \cos(\sqrt{7} \ln x/4)$ and $y_2(x) =$

$x^{1/4} \sin(\sqrt{7} \ln x/4)$ on $x > 0$. The general solution is:

$$y = C_1 x^{1/4} \cos\left(\frac{\sqrt{7} \ln x}{4}\right) + C_2 x^{1/4} \sin\left(\frac{\sqrt{7} \ln x}{4}\right)$$

27. The equation $x^2 y'' + xy' + 4y = 0$ has indicial equation

$$0 = I(s) = s(s-1) + s + 4 = s^2 + 4.$$

This equation has complex zeros $s = \pm 2i$. These provide independent solutions $y_1(x) = \cos(2 \ln x)$ and $y_2(x) = \sin(2 \ln x)$ on $x > 0$. The general solution is

$$y(x) = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x).$$

28. The equation $x^2 y'' + xy' + 4y = 0$ has indicial equation

$$0 = I(s) = s(s-1) + s + 4 = s^2 + 4.$$

This equation has complex zeros $s = \pm 2i$. Hence the general solution is

$$y(x) = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x).$$

The initial condition $y(1) = -2$ provides $C_1 = -2$. Differentiate.

$$y'(x) = -2C_1 x^{-1} \sin(2 \ln x) + 2C_2 x^{-1} \cos(2 \ln x)$$

The initial condition $y'(1) = 2$ provides $C_2 = 1$. Thus, on $x > 0$,

$$y(x) = -2 \cos(2 \ln x) + \sin(2 \ln x).$$

29. The equation $x^2 y'' + 3xy' + 2y = 0$ has indicial equation

$$0 = I(s) = s(s-1) + 3s + 2 = s^2 + 2s + 2.$$

This equation has complex zeros $s = -1 \pm i$. Thus, the general solution on $x > 0$ is

$$y(x) = C_1 x^{-1} \cos(\ln x) + C_2 x^{-1} \sin(\ln x).$$

The initial condition $y(1) = 3$ provides $C_1 = 3$. Differentiate.

$$y' = -C_1 x^{-2} \cos(\ln x) - C_1 x^{-2} \sin(\ln x) - C_2 x^{-2} \sin(\ln x) + C_2 x^{-2} \cos(\ln x)$$

The initial condition $y'(1) = -2$ leads to

$$-2 = -C_1 + C_2.$$

Thus, $C_2 = 1$ and the solution on $x > 0$ is

$$y(x) = 3x^{-1} \cos(\ln x) + x^{-1} \sin(\ln x).$$

30. The equation $x^2 y'' + 2xy' + y = 0$ has indicial equation

$$0 = I(s) = s(s-1) + 2s + 1 = s^2 + s + 1.$$

This equation has complex zeros $s = -1/2 \pm \sqrt{3}i/2$. The general solution is

$$y(x) = C_1 x^{-1/2} \cos(\sqrt{3} \ln x / 2) + C_2 x^{-1/2} \sin(\sqrt{3} \ln x / 2).$$

The initial condition $y(1) = -1$ gives $C_1 = -1$. Differentiate.

$$\begin{aligned} y' &= -\frac{1}{2} C_1 x^{-3/2} \cos(\sqrt{3} \ln x / 2) \\ &\quad - \frac{\sqrt{3} C_1}{2} x^{-3/2} \sin(\sqrt{3} \ln x / 2) \\ &\quad - \frac{1}{2} C_2 x^{-3/2} \sin(\sqrt{3} \ln x / 2) \\ &\quad + \frac{\sqrt{3} C_2}{2} x^{-3/2} \cos(\sqrt{3} \ln x / 2) \end{aligned}$$

The initial condition $y'(1) = (1 + \sqrt{3}/2)$ leads to

$$\frac{1 + \sqrt{3}}{2} = -\frac{1}{2} C_1 + \frac{\sqrt{3}}{2} C_2.$$

Solving, $C_2 = 1$ and the solution is

$$y(x) = -x^{-1/2} \cos\left(\frac{\sqrt{3} \ln x}{2}\right) + x^{-1/2} \sin\left(\frac{\sqrt{3} \ln x}{2}\right).$$

31. The equation $x^2 y'' - xy' + 6y = 0$ has indicial equation

$$0 = I(s) = s(s-1) - s + 6 = s^2 - 2s + 6.$$

This equation has complex roots $s = 1 \pm \sqrt{5}i$. Thus, the general solution is

$$y(x) = C_1 x \cos(\sqrt{5} \ln x) + C_2 x \sin(\sqrt{5} \ln x).$$

The initial condition $y(1) = 2$ provides $C_1 = 2$. Differentiate.

$$\begin{aligned} y'(x) &= C_1 \cos(\sqrt{5} \ln x) - \sqrt{5} C_1 \sin(\sqrt{5} \ln x) \\ &\quad + C_2 \sin(\sqrt{5} \ln x) + \sqrt{5} C_2 \cos(\sqrt{5} \ln x) \end{aligned}$$

The initial condition $y'(1) = 2 - 3\sqrt{5}$ provides

$$2 - 3\sqrt{5} = C_1 + \sqrt{5} C_2.$$

Thus, $C_2 = -3$ and

$$y(x) = 2x \cos(\sqrt{5} \ln x) - 3x \sin(\sqrt{5} \ln x).$$

32. (a) Let $y_1(x) = x^{s_0}$ be one solution of $x^2 y'' + px y' + qy = 0$ and form a second solution $y_2(x) = u(x) y_1(x)$ and its derivatives

$$\begin{aligned} y_2' &= u y_1' + u' y_1, \quad \text{and} \\ y_2'' &= u y_1'' + 2u' y_1' + u'' y_1. \end{aligned}$$

Substituting,

$$\begin{aligned} 0 &= x^2 (u y_1'' + 2u' y_1' + u'' y_1) \\ &\quad + p x (u y_1' + u' y_1) + q u y_1 \\ 0 &= (x^2 y_1'' + p x y_1' + q y_1) u + 2x^2 u' y_1' \\ &\quad + x^2 u'' y_1 + p x u' y_1. \end{aligned}$$

But y_1 satisfies $x^2 y'' + p x y' + q y = 0$, so

$$\begin{aligned} 0 &= 2x^2 u' y_1' + x^2 u'' y_1 + p x u' y_1 \\ x^2 u'' y_1 &= (-2x^2 y_1' - p x y_1) u' \\ \frac{u''}{u'} &= -\frac{2y_1'}{y_1} - \frac{p}{x}. \end{aligned}$$

Integrating,

$$\ln u' = -2 \ln y_1 - p \ln x + D,$$

where D is a constant of integration. Recall that $y_1 = x^{s_0}$, so

$$\begin{aligned} \ln u' &= -2 \ln x^{s_0} - p \ln x + D \\ \ln u' &= \ln x^{-2s_0} + \ln x^{-p} + D \\ \ln u' &= \ln x^{-2s_0-p} + D. \end{aligned}$$

Exponentiating,

$$u' = e^D x^{-2s_0-p},$$

or

$$u' = Cx^{-(p+2s_0)},$$

where we've replaced the constant of integration e^D with another arbitrary constant C .

- (b) Because $s = s_0$ is a double root, the indicial equation can be written

$$\begin{aligned} s(s-1) + ps + q &= (s-s_0)^2 \\ s^2 + (p-1)s + q &= s^2 - 2s_0s + s_0^2. \end{aligned}$$

Hence,

$$\begin{aligned} p-1 &= -2s_0 \\ p &= 1-2s_0. \end{aligned}$$

- (c) Substitute $p = 1 - 2s_0$ in

$$\begin{aligned} u' &= Cx^{-(p+2s_0)} \\ u' &= Cs^{-(1-2s_0+2s_0)} \\ u' &= Cx^{-1}. \end{aligned}$$

Integrating,

$$u = C \ln x,$$

and

$$\begin{aligned} y_2 &= uy_1 \\ &= (C \ln x)x^{s_0} \\ &= Cx^{s_0} \ln x. \end{aligned}$$

With $C = 1$, we have a second independent solution

$$y_2(x) = x^{s_0} \ln x.$$

33. (a) Consider $x^2 y'' + pxy' + qy = 0$ and let $x = e^t$ and $Y(t) = y(e^t)$. Note that $t = \ln x$ and

$$dt = \frac{1}{x} dx.$$

Thus,

$$\frac{d}{dt} = x \frac{d}{dx} \quad \text{and} \quad \frac{d}{dx} = \frac{1}{x} \frac{d}{dt}.$$

Note further that $y(x) = y(e^t) = Y(t) = Y(\ln x)$. Assume that $y(x)$ is a solution; i.e.,

$$x^2 y''(x) + pxy'(x) + qy(x) = 0.$$

Now,

$$Y'(t) = \frac{d}{dt} Y(t) = \frac{d}{dt} y(x) = x \frac{d}{dx} y(x).$$

Thus, $Y'(t) = xy'(x)$ and a second derivative is

$$Y''(t) = \frac{d}{dt} Y'(t) = \frac{d}{dt} xy'(x) = x \frac{d}{dx} xy'(x).$$

Thus,

$$\begin{aligned} Y''(t) &= x(xy''(x) + y'(x)) \\ &= x^2 y''(x) + xy'(x). \end{aligned}$$

Therefore,

$$\begin{aligned} Y''(t) + (p-1)Y'(t) + qY(t) &= [x^2 y''(x) + xy'(x)] + (p-1)xy'(x) \\ &\quad + qy(x) \\ &= x^2 y''(x) + pxy'(x) + qy(x) \\ &= 0. \end{aligned}$$

Conversely, assume that $Y(t)$ is a solution of

$$Y''(t) + (p-1)Y'(t) + qY(t) = 0.$$

But,

$$y'(x) = \frac{d}{dx} y(x) = \frac{d}{dx} Y(t) = \frac{1}{x} \frac{d}{dt} Y(t).$$

Thus, $y'(x) = (1/x) Y'(t)$ and a second derivative is

$$\begin{aligned} y''(x) &= \frac{d}{dx} y'(x) \\ &= \frac{d}{dx} \left(\frac{1}{x} Y'(t) \right) \\ &= \frac{1}{x} \frac{d}{dx} Y'(t) - \frac{1}{x^2} Y'(t) \\ &= \frac{1}{x} \left(\frac{1}{x} \frac{d}{dt} Y'(t) \right) - \frac{1}{x^2} Y'(t) \\ &= \frac{1}{x^2} Y''(t) - \frac{1}{x^2} Y'(t). \end{aligned}$$

Thus,

$$\begin{aligned} & x^2 y''(x) + pxy'(x) + qy(x) \\ &= x^2 \left[\frac{1}{x^2} Y''(t) - \frac{1}{x^2} Y'(t) \right] + px \left[\frac{1}{x} Y'(t) \right] \\ &\quad + qY(t) \\ &= Y''(t) - Y'(t) + pY'(t) + qY(t) \\ &= Y''(t) + (p-1)Y'(t) + qY(t) \\ &= 0. \end{aligned}$$

Hence, $y(x)$ is a solution of $x^2 y'' + pxy' + qy = 0$ if and only if $Y(t)$ [or $Y(\ln x)$] is a solution of $Y''(t) + (p-1)Y'(t) + qY(t) = 0$.

- (b) The characteristic polynomial of $Y'' + (p-1)Y' + qY = 0$ is found by substituting $Y(t) = e^{\lambda t}$.

$$e^{\lambda t}(\lambda^2 + (p-1)\lambda + q) = 0$$

Thus, the characteristic equation is

$$\lambda^2 + (p-1)\lambda + q = 0.$$

To find the indicial equation of $x^2 y'' + pxy' + qy = 0$, substitute $y = x^s$.

$$\begin{aligned} x^2 s(s-1)x^{s-2} + pxsx^{s-1} + qx^s &= 0 \\ x^s[s(s-1) + ps + q] &= 0 \end{aligned}$$

Thus, the indicial equation is

$$\begin{aligned} s(s-1) + ps + q &= 0 \\ s^2 + (p-1)s + q &= 0, \end{aligned}$$

which matches the characteristic equation of $Y'' + (p-1)Y' + qY = 0$.

- (c) If $Y''(t) + (p-1)Y'(t) + qY(t) = 0$ has solution

$$Y(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t},$$

then $x^2 y'' + pxy' + qy = 0$ has solution

$$\begin{aligned} y(x) &= Y(\ln x) = C_1 e^{s_1 \ln x} + C_2 e^{s_2 \ln x} \\ &= C_1 x^{s_1} + C_2 x^{s_2}. \end{aligned}$$

- (d) If $Y''(t) + (p-1)Y'(t) + qY(t) = 0$ has solution

$$Y(t) = C_1 e^{s_0 t} + C_2 t e^{s_0 t}$$

then $x^2 y'' + pxy' + qy = 0$ has solution

$$\begin{aligned} y(x) &= Y(\ln x) = C_1 e^{s_0 \ln x} + C_2 (\ln x) e^{s_0 \ln x} \\ &= C_1 x^{s_0} + C_2 x^{s_0} \ln x. \end{aligned}$$

- (e) If $Y''(t) + (p-1)Y'(t) + qY(t) = 0$ has solution

$$Y(t) = e^{\alpha t} (C_1 \cos \beta t + C_2 \sin \beta t),$$

then $x^2 y'' + pxy' + qy = 0$ has solution

$$\begin{aligned} y(x) &= Y(\ln x) \\ &= e^{\alpha \ln x} (C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)) \\ &= x^\alpha (C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)). \end{aligned}$$

—X—

Section 11.5. Series Solutions Near Regular Singular Points

1. To see if there is a nontrivial solution $y = \sum_n a_n x^n$, we differentiate y :

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

and then insert these expressions into the given differential equation:

$$\begin{aligned} 0 &= x^2 y'' + 4xy' + 2y = x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= 2a_0 + 6a_1 x + \sum_{n=2}^{\infty} (n(n-1) + 4n + 2)a_n x^n. \end{aligned}$$

Equating the coefficient of each power of x on the right with zero (since the left side is zero), we obtain $2a_0 = 0$, $6a_1 = 0$, and

$$(n(n-1) + 4n + 2)a_n = 0, \quad n \geq 2.$$

Clearly, a_0 and a_1 must be zero. Since $n(n-1) + 4n + 2 = n^2 + 3n + 2$ is never zero for $n \geq 2$, we conclude that all the $a_n = 0$, for $n \geq 2$, as well. Thus the only power series solution $\sum_{n=0}^{\infty} a_n x^n$ to this differential equation is the trivial one where all the a_n equal zero.

2. (a) Comparing $x^2 y'' - y = 0$ with $x^2 y'' + xp(x)y' + q(x)y = 0$ reveals $p(x) = 0$ and $q(x) = -1$ are both analytic at $x = 0$. Hence, $x = 0$ is a regular singular point of $x^2 y'' - y = 0$.
On the other hand, dividing $x^3 y'' - y = 0$ by x gives

$$x^2 y'' - \frac{1}{x} y = 0.$$

Comparing this equation with $x^2 y'' + xp(x)y' + q(x)y = 0$ reveals $p(x) = 0$ and $q(x) = -1/x$. However, q is not analytic at $x = 0$. Hence, $x = 0$ is an irregular singular point of $x^3 y'' - y = 0$.

(b) Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2 y'' - y = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} - \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=0}^{\infty} [(s+n)(s+n-1) - 1] a_n x^{s+n} \\ &= [s(s-1) - 1] a_0 x^s + \sum_{n=1}^{\infty} [(s+n)(s+n-1) - 1] a_n x^{s+n}. \end{aligned}$$

Thus, the indicial equation is $I(s) = s^2 - s - 1$ with roots $s_2 = (1 - \sqrt{5})/2$ and $s_1 = (1 + \sqrt{5})/2$. The remaining terms provide the recurrence relation

$$[(s+n)(s+n-1) - 1]a_n = 0 \quad \text{or} \quad I(s+n)a_n = 0, \quad n \geq 1.$$

Because $I(s+n)$ cannot equal zero for either s_1 or s_2 and any n , a_n must equal zero for all $n \geq 1$. Hence

$$y(x) = a_0 x^s$$

is a Frobenius solution for $s = (1 \pm \sqrt{5})/2$.

On the other hand, in the case of equation $x^3 y'' - y = 0$,

$$x^3 y' = x \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} = \sum_{n=1}^{\infty} (s+n-1)(s+n-2)a_{n-1} x^{s+n},$$

where we've shifted the index of summation. Substituting,

$$\begin{aligned} 0 = x^3 y'' - y &= \sum_{n=1}^{\infty} (s+n-1)(s+n-2)a_{n-1} x^{s+n} - \sum_{n=0}^{\infty} a_n x^{s+n} \\ &= -a_0 x^s + \sum_{n=1}^{\infty} [(s+n-1)(s+n-2)a_{n-1} - a_n] x^{s+n}. \end{aligned}$$

Thus, $a_0 = 0$ and the remaining terms provide the recurrence relation

$$(s+n-1)(s+n-2)a_{n-1} = a_n, \quad n \geq 1.$$

Thus, regardless of the values of s ,

$$a_1 = s(s-1)a_0 = 0, \quad a_2 = (s+1)sa_1 = 0, \quad \dots$$

Thus, $a_n = 0$ for all n and there is no Frobenius solution.

3. Compare $x^2 y'' + 4xy' + 2y = 0$ with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 4$ and $q(x) = 2$ are both analytic at $x = 0$. Hence $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = 4$ and $q_0 = q(0) = 2$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 + 3s + 2.$$

Alternatively, let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} \quad \text{and} \quad 4xy' = 4 \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} = \sum_{n=0}^{\infty} 4(s+n)a_n x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 = x^2 y'' + 4xy' + 2y &= \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} 4(s+n)a_n x^{s+n} + \sum_{n=0}^{\infty} 2a_n x^{s+n} \\ &= [s(s-1) + 4s + 2]a_0 x^s + \sum_{n=1}^{\infty} [(s+n)(s+n-1) + 4(s+n) + 2]a_n x^{s+n} \\ &= (s^2 + 3s + 2)a_0 x^s + \sum_{n=1}^{\infty} [(s+n)^2 + 3(s+n) + 2]a_n x^{s+n}. \end{aligned}$$

Again, this gives the indicial polynomial $I(s) = s^2 + 3s + 2$.

4. We have to write the equation in the form in 5.2 with $x_0 = -1$. To do so we divide by $(x - 1)^2$, to obtain

$$(x + 1)^2 y'' + \frac{1}{(x - 1)^2} (x + 1) y' - \frac{1}{(x - 1)^2} y = 0.$$

Thus, the coefficients are $p(x) = 1/(x - 1)^2$ and $q(x) = -1/(x - 1)^2$. According to 6.8, the indicial equation is $I(s) = s^2 + (p_0 - 1)s + q_0 = 0$, where $p_0 = p(-1) = 1/4$, and $q_0 = q(-1) = -1/4$. Hence the indicial equation is $s^2 - 3s/4 - 1/4 = 0$.

5. We have to write the equation in the form in 5.2 with $x_0 = 0$. To do so we multiply by $x/\sin x$, to obtain

$$x^2 y'' - \frac{3x}{\sin x} y = 0.$$

Thus, the coefficients are $p(x) = 0$ and $q(x) = -3x/\sin x$. According to 6.8, the indicial equation is $I(s) = s^2 + (p_0 - 1)s + q_0 = 0$, where $p_0 = p(0) = 0$, and $q_0 = q(0)$. By l'Hôpital's rule, $q(0) = -3$. Hence the indicial equation is $s^2 - s - 3 = 0$.

6. We have to write the equation in the form in 5.2 with $x_0 = 0$. To do so we multiply by $x^2/(\cos x - 1)$, to obtain

$$x^2 y'' + \frac{x^2}{\cos x - 1} x y' + \frac{x^2}{\cos x - 1} y = 0.$$

Thus, the coefficients are $p(x) = q(x) = x^2/(\cos x - 1)$. According to 6.8, the indicial equation is $I(s) = s^2 + (p_0 - 1)s + q_0 = 0$, where $p_0 = p(0)$, and $q_0 = q(0)$. By two applications of l'Hôpital's rule, $p(0) = q(0) = -2$. Hence the indicial equation is $s^2 - 3s - 2 = 0$.

7. Set

$$y(x) = x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$2x(x - 1)y'' + 3(x - 1)y' - y = [2x^2 y'' + 3xy' - y] + [-2xy'' - 3y'].$$

But,

$$2x^2 y'' + 3xy' - y = \sum_{n=0}^{\infty} [2(s + n)(s + n - 1) + 3(s + n) - 1] a_n x^{s+n}.$$

Next,

$$\begin{aligned} -2xy'' - 3y' &= -2x \sum_{n=0}^{\infty} (s + n)(s + n - 1) a_n x^{s+n-2} - 3 \sum_{n=0}^{\infty} (s + n) a_n x^{s+n-1} \\ &= - \sum_{n=0}^{\infty} 2(s + n)(s + n - 1) a_n x^{s+n-1} - \sum_{n=0}^{\infty} 3(s + n) a_n x^{s+n-1}. \end{aligned}$$

Thus,

$$\begin{aligned}
 0 &= [2x^2y'' + 3xy' - y] + [-2xy'' - 3y'] \\
 &= \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 3(s+n) - 1]a_n x^{s+n} \\
 &\quad - \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 3(s+n)]a_n x^{s+n-1} \\
 &= \sum_{n=1}^{\infty} [2(s+n-1)(s+n-2) + 3(s+n-1) - 1]a_{n-1} x^{s+n-1} \\
 &\quad - \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 3(s+n)]a_n x^{s+n-1} \\
 &= -[2s(s-1) + 3s]a_0 x^{s-1} + \sum_{n=1}^{\infty} \{[2(s+n-1)(s+n-2) + 3(s+n-1) - 1]a_{n-1} \\
 &\quad - [2(s+n)(s+n-1) + 3(s+n)]a_n\} x^{s+n-1}.
 \end{aligned}$$

Thus, the indicial equation is

$$I(s) = -[2s^2 + s] = -s(2s + 1) = 0,$$

with roots $s_2 = -1/2$ and $s_1 = 0$. The remaining terms provide the recurrence relation

$$[2(s+n-1)(s+n-2) + 3(s+n-1) - 1]a_{n-1} = [2(s+n)(s+n-1) + 3(s+n)]a_n, \quad n \geq 1.$$

With $s = s_1 = 0$, this becomes

$$\begin{aligned}
 [2n(n-1) + 3n]a_n &= [2(n-1)(n-2) + 3(n-1) - 1]a_{n-1} \\
 (2n^2 + n)a_n &= [2n^2 - 6n + 4 + 3n - 3 - 1]a_{n-1} \\
 n(2n+1)a_n &= n(2n-3)a_{n-1} \\
 a_n &= \frac{2n-3}{2n+1} a_{n-1},
 \end{aligned}$$

for $n \geq 1$.

8. Multiply by x to obtain

$$x^2y'' + xy' - 2xy = 0,$$

then compare with $x^2y'' + xp(x)y' + q(x)y = 0$ to get $p(x) = 1$ and $q(x) = -2x$, both analytic at $x = 0$. Hence, $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = 1$ and $q_0 = q(0) = 0$, so the polynomial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n}, \quad xy' = \sum_{n=0}^{\infty} (s+n)a_n x^{s+n},$$

and

$$-2xy = -2x \sum_{n=0}^{\infty} a_n x^{s+n} = -\sum_{n=1}^{\infty} 2a_{n-1} x^{s+n},$$

where we've shifted the index of summation. Substituting,

$$\begin{aligned} 0 &= x^2 y'' + xy' - 2xy \\ &= \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} - \sum_{n=1}^{\infty} 2a_{n-1} x^{s+n} \\ &= [s(s-1) + s] a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) + (s+n)] a_n - 2a_{n-1}\} x^{s+n} \\ &= s^2 a_0 x^s + \sum_{n=1}^{\infty} [(s+n)^2 a_n - 2a_{n-1}] x^{s+n}. \end{aligned}$$

Thus, $I(s) = s^2$ and the remaining terms provide the recurrence relation

$$(s+n)^2 a_n - 2a_{n-1} = 0, \quad n \geq 1.$$

At $s = 0$, this becomes

$$n^2 a_n = 2a_{n-1} \quad \text{or} \quad a_n = \frac{2a_{n-1}}{n^2}, \quad n \geq 1.$$

9. Multiply by x to obtain

$$x^2 y'' + x(1-x)y' - xy = 0,$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$ to get $p(x) = 1-x$ and $q(x) = -x$, both analytic at $x = 0$. Hence, $x = 0$ is a regular singular point. Further, $p_0 = p(0) = 1$ and $q_0 = q(0) = 0$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n},$$

and

$$x(1-x)y' = (1-x) \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} = \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1} x^{s+n},$$

where we've shifted the index of summation. Finally,

$$-xy = -x \sum_{n=0}^{\infty} a_n x^{s+n} = -\sum_{n=1}^{\infty} a_{n-1} x^{s+n},$$

where we've again shifted the index of summation. Substituting,

$$\begin{aligned}
 0 &= x^2 y'' + x(1-x)y' - xy \\
 &= \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1} x^{s+n} - \sum_{n=1}^{\infty} a_{n-1} x^{s+n} \\
 &= [s(s-1) + s]a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) + (s+n)]a_n - [(s+n-1) + 1]a_{n-1}\} x^{s+n} \\
 &= s^2 a_0 x^s + \sum_{n=1}^{\infty} [(s+n)^2 a_n - (s+n)a_{n-1}] x^{s+n}.
 \end{aligned}$$

Thus, the indicial equation is $I(s) = s^2 = 0$, with repeated roots $s_2 = s_1 = 0$. The remaining terms provide the recurrence relation,

$$(s+n)^2 a_n - (s+n)a_{n-1} = 0,$$

which for $s = 0$ becomes

$$n^2 a_n = n a_{n-1} \quad \text{or} \quad a_n = \frac{a_{n-1}}{n}, \quad n \geq 1.$$

10. Compare $x^2 y'' - xy' + x^2 y = 0$ with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = -1$ and $q(x) = x^2$ are both analytic at $x = 0$. Thus, $x = 0$ is a regular singular point. Further, $p_0 = p(0) = -1$ and $q_0 = q(0) = 0$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 2s.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n}, \quad -xy' = -\sum_{n=0}^{\infty} (s+n)a_n x^{s+n},$$

and

$$x^2 y = x^2 \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=2}^{\infty} a_{n-2} x^{s+n},$$

where we've shifted the index of summation. Substituting,

$$\begin{aligned}
 0 &= x^2 y'' - xy' + x^2 y \\
 &= \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} - \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} + \sum_{n=2}^{\infty} a_{n-2} x^{s+n} \\
 &= [s(s-1) - s]a_0 x^s + [(s+1)s - (s+1)]a_1 x^{s+1} + \sum_{n=2}^{\infty} \{[(s+n)(s+n-1) - (s+n)]a_n + a_{n-2}\} x^{s+n} \\
 &= (s^2 - 2s)a_0 x^s + (s^2 - 1)a_1 x^{s+1} + \sum_{n=2}^{\infty} \{[(s+n)^2 - 2(s+n)]a_n + a_{n-2}\} x^{s+n}.
 \end{aligned}$$

Therefore, the indicial equation is $I(s) = s^2 - 2s = s(s - 2) = 0$ with roots $s_2 = 0$ and $s_1 = 2$. The second term must also equal zero,

$$(s^2 - 1)a_1 = 0,$$

but $s^2 - 1$ is not zero for either root of the indicial equation. Hence, $a_1 = 0$. Finally, the remaining terms provide the recurrence relation

$$[(s + n)^2 - 2(s + n)]a_n + a_{n-2} = 0 \quad \text{or} \quad (s + n)(s + n - 2)a_n = -a_{n-2}, \quad n \geq 2.$$

For the larger root, $s_1 = 2$, this becomes

$$(n + 2)na_n = -a_{n-2} \quad \text{or} \quad a_n = \frac{-a_{n-2}}{n(n + 2)}, \quad n \geq 2.$$

11. Multiplying both sides by $x/4$, we obtain

$$x^2 y'' + \frac{x}{2} y' + \frac{x}{4} y = 0.$$

Comparing with $x^2 y'' + xp(x)y' + q(x)y = 0$, we see that $p(x) = 1/2$ and $q(x) = x/4$, both of which are analytic near $x = 0$. We seek a solution having form

$$y = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n},$$

$a_0 \neq 0$, with derivatives

$$y' = \sum_{n=0}^{\infty} a_n (s + n) x^{s+n-1} \quad \text{and} \quad y' = \sum_{n=0}^{\infty} a_n (s + n)(s + n - 1) x^{s+n-2}.$$

Therefore,

$$4x^2 y'' = \sum_{n=0}^{\infty} 4a_n (s + n)(s + n - 1) x^{s+n}, \quad 2xy' = \sum_{n=0}^{\infty} 2a_n (s + n) x^{s+n},$$

and

$$xy = \sum_{n=0}^{\infty} a_n x^{s+n-1} = \sum_{n=1}^{\infty} a_{n-1} x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= 4x^2 y'' + 2xy' + xy \\ &= \sum_{n=0}^{\infty} 4a_n (s + n)(s + n - 1) x^{s+n} + \sum_{n=0}^{\infty} 2a_n (s + n) x^{s+n} + \sum_{n=1}^{\infty} a_{n-1} x^{s+n} \\ &= [4s(s - 1) + 2s]a_0 x^s + \sum_{n=1}^{\infty} \{[4(s + n)(s + n - 1) + 2(s + n)]a_n + a_{n-1}\} x^{s+n}. \end{aligned}$$

Setting coefficients equal to zero,

$$[4s(s - 1) + 2s]a_0 = 0.$$

But $a_0 \neq 0$, so the indicial equation

$$4s(s-1) + 2s = 0 \quad \text{or} \quad 2s(2s-1) = 0$$

has roots $s = 0$ and $s = 1/2$. Further,

$$[4(s+n)(s+n-1) + 2(s+n)]a_n + a_{n-1} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-1}}{(s+n)(4s+4n-2)},$$

for $n \geq 1$. With $s = 0$,

$$a_n = \frac{-a_{n-1}}{n(4n-2)} = \frac{-a_{n-1}}{2n(2n-1)}.$$

Indeed,

$$a_1 = \frac{-a_0}{2 \cdot 1}, \quad a_2 = \frac{-a_1}{4 \cdot 3} = \frac{a_0}{4!}, \quad a_3 = \frac{-a_2}{6 \cdot 5} = \frac{-a_0}{6!}, \quad \dots, \quad a_n = \frac{a_0(-1)^n}{(2n)!},$$

for $n \geq 1$. Hence, with $a_0 = 1$, we have solution

$$y_1(x) = x^0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n)!} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}.$$

With $s = 1/2$,

$$a_n = \frac{-a_{n-1}}{(1/2+n)4n} = \frac{-a_{n-1}}{2n(2n+1)},$$

for $n \geq 1$. Hence,

$$a_1 = \frac{-a_0}{2 \cdot 3}, \quad a_2 = \frac{-a_1}{4 \cdot 5} = \frac{-a_0}{5!}, \quad a_3 = \frac{-a_2}{6 \cdot 7} = \frac{a_0}{7!}, \quad \dots, \quad a_n = \frac{a_0(-1)^n}{(2n+1)!},$$

for $n \geq 1$. With $a_0 = 1$, we have solution

$$y_2(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n+1)!} \right] = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!},$$

12. We seek a solution

$$y = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n},$$

$a_0 \neq 0$, with derivatives

$$y' = \sum_{n=0}^{\infty} (s+n) a_n x^{s+n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{s+n-2}.$$

Thus,

$$\begin{aligned} 2x(x+1)y'' &= (2x^2 + 2x) \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{s+n-2} \\ &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1) a_n x^{s+n} + \sum_{n=0}^{\infty} 2(s+n)(s+n-1) a_n x^{s+n-1}. \end{aligned}$$

Next,

$$3(x+1) = (3x+3) \sum_{n=0}^{\infty} (s+n)a_n x^{s+n-1} = \sum_{n=0}^{\infty} 3(s+n)a_n x^{s+n} + \sum_{n=0}^{\infty} 3(s+n)a_n x^{s+n-1}$$

Substituting,

$$\begin{aligned} 0 &= 2x(x+1)y'' + 3(x+1)y' - y \\ &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n-1} + \sum_{n=0}^{\infty} 3(s+n)a_n x^{s+n} \\ &\quad + \sum_{n=0}^{\infty} 3(s+n)a_n x^{s+n-1} - \sum_{n=0}^{\infty} a_n x^{s+n} \\ &= \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 3(s+n) - 1]a_n x^{s+n} + \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 3(s+n)]a_n x^{s+n-1}. \end{aligned}$$

Shift the index of the first term.

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} [2(s+n-1)(s+n-2) + 3(s+n-1) - 1]a_{n-1} x^{s+n-1} \\ &\quad + \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 3(s+n)]a_n x^{s+n-1} \\ &= \sum_{n=1}^{\infty} [(s+n-1)(2s+2n-1) - 1]a_{n-1} x^{s+n-1} + \sum_{n=1}^{\infty} [(s+n)(2s+2n+1)]a_n x^{s+n-1} \\ &\quad + [2s(s-1) + 3s]a_0 x^{s-1} \\ &= \sum_{n=1}^{\infty} \{ [(s+n-1)(2s+2n-1) - 1]a_{n-1} + [(s+n)(2s+2n+1)]a_n \} x^{s+n-1} + (2s^2 + s)a_0 x^{s-1}. \end{aligned}$$

The indicial equation

$$0 = I(s) = 2s^2 + s = s(2s+1)$$

has roots $s = 0$ and $s = -1/2$. Setting the remaining coefficients equal to zero,

$$[(s+n-1)(2s+2n-1) - 1]a_{n-1} + (s+n)(2s+2n+1)a_n = 0,$$

or

$$a_n = \frac{1 - (s+n-1)(2s+2n-1)}{(s+n)(2s+2n+1)}, \quad n \geq 1.$$

In the case that $s = 0$, the recurrence equation becomes

$$a_n = \frac{1 - (n-1)(2n-1)}{n(2n+1)} a_{n-1} = \frac{n(3-2n)}{n(2n+1)} a_{n-1} = \frac{3-2n}{2n+1} a_{n-1}.$$

Thus,

$$a_1 = \frac{1}{3} a_0, \quad a_2 = \frac{-1}{5} a_1 = -\frac{1}{3} \cdot \frac{1}{5} a_0, \quad \text{and} \quad a_3 = \frac{-3}{7} a_2 = \frac{1}{3} \cdot \frac{1}{5} \cdot \frac{3}{7} a_0.$$

We conjecture that

$$a_n = \frac{1 \cdot (-1) \cdot (-3) \cdot (-5) \cdots (3-2n)}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n+1)} = \frac{(-1)^{n+1}}{(2n-1)(2n+1)} = \frac{(-1)^{n+1}}{4n^2-1}, \quad n \geq 1.$$

Thus,

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = x_0 \left\{ a_0 \left(1 + \frac{1}{3}x - \frac{1}{15}x^2 + \frac{1}{35}x^3 - \cdots + \frac{(-1)^{n+1}}{4n^2-1}x^n + \cdots \right) \right\}.$$

With $a_0 = 1$, we have one solution,

$$y_1(x) = 1 + \frac{1}{3}x - \frac{1}{15}x^2 + \frac{1}{35}x^3 - \cdots + \frac{(-1)^{n+1}}{4n^2-1}x^n + \cdots.$$

Next, with $s = -1/2$, the recurrence relation becomes

$$a_n = \frac{1 - (n-3/2)(2n-2)}{(n-1/2)(2n)} a_{n-1} = \frac{(1-2n)(2-n)}{n(1-2n)} a_{n-1} = \frac{2-n}{n} a_{n-1}.$$

Thus, $a_1 = a_0$, $a_2 = 0$, and $a_n = 0$ for all $n \geq 2$. Thus,

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = x^{-1/2} \{a_0(1+x)\}.$$

With $a_0 = 1$,

$$y_2(x) = x^{-1/2}(1+x) = x^{-1/2} + x^{1/2}.$$

13. Dividing by 2 produces

$$x^2 y'' - \frac{3}{2} x y' + (1+x)y = 0,$$

which upon comparison with $x^2 y'' + x p(x) y' + q(x) y = 0$, reveal $p(x) = -3/2$ and $q(x) = 1+x$, both of which are analytic at $x = 0$. We seek solution

$$y = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n},$$

$a_0 \neq 0$, with derivatives

$$y' = \sum_{n=0}^{\infty} a_n (s+n) x^{s+n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} a_n (s+n)(s+n-1) x^{s+n-2}.$$

Thus,

$$2x^2 y'' = \sum_{n=0}^{\infty} 2a_n (s+n)(s+n-1) x^{s+n}, \quad -3xy' = -\sum_{n=0}^{\infty} 3a_n (s+n) x^{s+n},$$

and

$$(2+2x)y = \sum_{n=0}^{\infty} 2a_n x^{s+n} + \sum_{n=0}^{\infty} 2a_n x^{s+n+1} = \sum_{n=0}^{\infty} 2a_n x^{s+n} + \sum_{n=1}^{\infty} 2a_{n-1} x^{s+n}.$$

Thus,

$$\begin{aligned}
 0 &= 2x^2 y'' - 3xy' + (2 + 2x)y \\
 &= \sum_{n=0}^{\infty} 2a_n(s+n)(s+n-1)x^{s+n} - \sum_{n=0}^{\infty} 3a_n(s+n)x^{s+n} + \sum_{n=0}^{\infty} 2a_n x^{s+n} + \sum_{n=1}^{\infty} 2a_{n-1} x^{s+n} \\
 &= [2s(s-1) - 3s + 2]a_0 x^s + \sum_{n=1}^{\infty} \{[2(s+n)(s+n-1) - 3(s+n) + 2]a_n + 2a_{n-1}\} x^{s+n}
 \end{aligned}$$

Setting coefficients equal to zero,

$$[2s(s-1) - 3s + 2]a_0 = 0.$$

But, $a_0 \neq 0$, so the indicial equation

$$0 = 2s(s-1) - 3s + 2 = (2s-1)(s-2)$$

has zeros $s = 1/2$ and $s = 2$. Further,

$$[2(s+n)(s+n-1) - 3(s+n) + 2]a_n + 2a_{n-1} = 0,$$

so

$$a_n = \frac{-2a_{n-1}}{2(s+n)(s+n-1) - 3(s+n) + 2}, \quad n \geq 1.$$

With $s = 2$,

$$a_n = \frac{-2a_{n-1}}{2(n+2)(n+1) - 3(n+2) + 2} = \frac{-2a_{n-1}}{n(2n+3)}, \quad n \geq 1.$$

Thus,

$$a_1 = \frac{-2a_0}{1 \cdot 5}, \quad a_2 = \frac{-2a_1}{2 \cdot 7} = \frac{2^2 a_0}{(1 \cdot 2)(5 \cdot 7)}, \quad a_3 = \frac{-2a_2}{3 \cdot 9} = \frac{-2^3 a_0}{(1 \cdot 2 \cdot 3)(5 \cdot 7 \cdot 9)},$$

and in general,

$$a_n = \frac{(-1)^n 2^n a_0}{n!(5 \cdot 7 \cdots (2n+3))}, \quad n \geq 1.$$

With $a_0 = 1$, we have solution

$$y_1(x) = x^2 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{n!(5 \cdot 7 \cdots (2n+3))} \right].$$

With $s = 1/2$,

$$a_n = \frac{-2a_{n-1}}{2(1/2+n)(n-1/2) - 3(1/2+n) + 2} = \frac{-2a_{n-1}}{n(2n-3)}.$$

Hence,

$$a_1 = \frac{-2a_0}{1 \cdot (-1)}, \quad a_2 = \frac{-2a_1}{2 \cdot 1} = \frac{2^2 a_0}{(1 \cdot 2)(-1 \cdot 1)}, \quad a_3 = \frac{-2a_2}{3 \cdot 3} = \frac{-2^3 a_0}{(1 \cdot 2 \cdot 3)(-1 \cdot 1 \cdot 3)},$$

and in general,

$$a_n = \frac{(-1)^{n+1} 2^n a_0}{n!(1 \cdot 3 \cdots (2n-3))}, \quad n \geq 2.$$

With $a_0 = 1$, we have a second solution

$$y_2(x) = x^{1/2} \left[1 + 2x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2^n x^n}{n!(1 \cdot 3 \cdots (2n-3))} \right].$$

14. Comparing

$$x^2 y'' + x y' - \frac{4x^2 + 1}{4} y = 0$$

with $x^2 y'' + x p(x) y' + q(x) y = 0$ gives $p(x) = 1$ and $q(x) = -1/4 - x^2$, both of which are analytic at $x = 0$. We seek a solution

$$y = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n},$$

$a_0 \neq 0$, with derivatives

$$y' = \sum_{n=0}^{\infty} (s+n) a_n x^{s+n-1}, \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{s+n-2}.$$

Thus,

$$4x^2 y'' = \sum_{n=0}^{\infty} 4(s+n)(s+n-1) a_n x^{s+n}, \quad 4x y' = \sum_{n=0}^{\infty} 4(s+n) a_n x^{s+n},$$

and,

$$(4x^2 + 1)y = (4x^2 + 1) \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=0}^{\infty} 4a_n x^{s+n+2} + \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= 4x^2 y'' + 4x y' - (4x^2 + 1)y \\ &= \sum_{n=0}^{\infty} 4(s+n)(s+n-1) a_n x^{s+n} + \sum_{n=0}^{\infty} 4(s+n) a_n x^{s+n} - \sum_{n=0}^{\infty} 4a_n x^{s+n+2} - \sum_{n=0}^{\infty} a_n x^{s+n}. \end{aligned}$$

Shift the index of the third term.

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} 4(s+n)(s+n-1) a_n x^{s+n} + \sum_{n=0}^{\infty} 4(s+n) a_n x^{s+n} - \sum_{n=2}^{\infty} 4a_{n-2} x^{s+n} - \sum_{n=0}^{\infty} a_n x^{s+n} \\ &= \sum_{n=0}^{\infty} [4(s+n)(s+n-1) + 4(s+n) - 1] a_n x^{s+n} - \sum_{n=2}^{\infty} 4a_{n-2} x^{s+n} \\ &= [4s(s-1) + 4s - 1] a_0 x^s + [4(s+1)s + 4(s+1) - 1] a_1 x^{s+1} \\ &\quad + \sum_{n=2}^{\infty} \{ [4(s+n)(s+n-1) + 4(s+n) - 1] a_n - 4a_{n-2} \} x^{s+n} \end{aligned}$$

Setting coefficients equal to zero,

$$[4s(s-1) + 4s - 1] a_0 = 0.$$

But $a_0 \neq 0$, so the indicial equation

$$0 = 4s(s-1) + 4s - 1 = (2s+1)(2s-1)$$

has zeros $s = -1/2$ and $s = 1/2$. Further,

$$[4(s+1)s + 4(s+1) - 1]a_1 = (4s^2 + 8s + 3)a_1 = 0,$$

but $4s^2 + 8s + 3$ is not zero when $s_1 = -1/2$ or $s = 1/2$. Thus, $a_1 = 0$. Finally,

$$[4(s+n)(s+n-1) + 4(s+n) - 1]a_n - 4a_{n-2} = 0,$$

so

$$a_n = \frac{4a_{n-2}}{4(s+n)(s+n-1) + 4(s+n) - 1} = \frac{4a_{n-2}}{(2s+2n+1)(2s+2n-1)},$$

for $n \geq 2$. With $s_1 = 1/2$, this becomes

$$a_n = \frac{4a_{n-2}}{(2n+2)(2n)} = \frac{a_{n-2}}{n(n+1)},$$

for $n \geq 2$. Thus,

$$a_2 = \frac{a_0}{2 \cdot 3}, \quad a_4 = \frac{a_2}{4 \cdot 5}, \quad a_6 = \frac{a_4}{6 \cdot 7}, \quad \dots, \quad a_{2n} = \frac{a_{2n-2}}{2n(2n+1)}.$$

Multiplying,

$$a_2 a_4 a_6 \dots a_{2n} = \frac{a_0 a_2 a_4 \dots a_{2n-2}}{[2 \cdot 4 \cdot 6 \dots 2n][3 \cdot 5 \cdot 7 \dots (2n+1)]}.$$

Cancelling,

$$a_{2n} = \frac{a_0}{2^n n! (3 \cdot 5 \cdot 7 \dots (2n+1))}, \quad n \geq 1.$$

Note that

$$3 \cdot 5 \cdot 7 \dots (2n+1) = \frac{(2n+1)!}{2 \cdot 4 \cdot 6 \dots 2n} = \frac{(2n+1)!}{2^n n!},$$

so

$$a_{2n} = \frac{a_0}{(2n+1)!}, \quad n \geq 1.$$

Because $a_1 = 0$, it is easily seen that $a_{2n+1} = 0$, for $n \geq 0$. Thus, with $s_1 = 1/2$ and $a_0 = 1$, we have

$$y_1(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n+1)!} \right].$$

In similar fashion, with $s_2 = -1/2$, the recurrence relation becomes

$$a_n = \frac{4a_{n-2}}{2n(2n-2)} = \frac{a_{n-2}}{n(n-1)}, \quad n \geq 2.$$

Thus,

$$a_2 = \frac{a_0}{2 \cdot 1}, \quad a_4 = \frac{a_2}{4 \cdot 3}, \quad \dots, \quad a_{2n} = \frac{a_{2n-2}}{2n(2n-1)}.$$

Multiplying,

$$a_2 a_4 \cdots a_{2n} = \frac{a_0 a_2 \cdots a_{2n-2}}{[2 \cdot 4 \cdots 2n][1 \cdot 3 \cdots (2n-1)]}.$$

Cancelling,

$$a_{2n} = \frac{a_0}{2^n n! [1 \cdot 3 \cdots (2n-1)]}.$$

Since

$$1 \cdot 3 \cdots (2n-1) = \frac{(2n)!}{2 \cdot 4 \cdots 2n} = \frac{(2n)!}{2^n n!},$$

we can write

$$a_{2n} = \frac{a_0}{(2n)!}.$$

Because $a_1 = 0$, $a_{2n+1} = 0$ for $n \geq 0$. Thus, with $s_2 = -1/2$ and $a_0 = 1$,

$$y_2(x) = x^{-1/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \right]$$

15. Compare $x^2 y'' + xy' + (x^2 - 1/16)y = 0$ with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 1$ and $q(x) = x^2 - 1/16$, both of which are analytic at $x = 0$. Thus, $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = 1$ and $q_0 = q(0) = -1/16$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{1}{16}.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{s+n}, \quad xy' = \sum_{n=0}^{\infty} (s+n) a_n x^{s+n},$$

and

$$\left(x^2 - \frac{1}{16}\right)y = \left(x^2 - \frac{1}{16}\right) \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=2}^{\infty} a_{n-2} x^{s+n} - \frac{1}{16} \sum_{n=0}^{\infty} a_n x^{s+n},$$

where we've shifted the first index of summation. Substituting,

$$\begin{aligned} 0 &= x^2 y'' + xy' + \left(x^2 - \frac{1}{16}\right)y \\ &= \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n) a_n x^{s+n} + \sum_{n=2}^{\infty} a_{n-2} x^{s+n} - \frac{1}{16} \sum_{n=0}^{\infty} a_n x^{s+n} \\ &= \left[s(s-1) + s - \frac{1}{16}\right] a_0 x^s + \left[(s+1)s + (s+1) - \frac{1}{16}\right] a_1 x^{s+1} \\ &\quad + \sum_{n=2}^{\infty} \left\{ \left[(s+n)(s+n-1) + (s+n) - \frac{1}{16}\right] a_n + a_{n-2} \right\} x^{s+n}. \end{aligned}$$

The first term gives the indicial equation

$$I(s) = s^2 - \frac{1}{16} = \left(s + \frac{1}{4}\right)\left(s - \frac{1}{4}\right),$$

with roots $s_2 = -1/4$ and $s_1 = 1/4$. Setting the second term equal to zero,

$$\left(s^2 + 2s + \frac{15}{16}\right)a_1 = 0,$$

provides $a_1 = 0$ as the coefficient of a_1 is not zero for either root of the indicial equation. The remaining terms provide the recurrence relation,

$$\left[(s+n)^2 - \frac{1}{16}\right]a_n + a_{n-2} = 0, \quad n \geq 2.$$

Thus,

$$a_n = \frac{-a_{n-2}}{(s+n+1/4)(s+n-1/4)} = \frac{-16a_{n-2}}{(4s+4n+1)(4s+4n-1)}$$

for $n \geq 2$. With $s_1 = 1/4$, this becomes

$$a_n = \frac{-16a_{n-2}}{(4n+2)(4n)} = \frac{-2a_{n-2}}{(2n+1)n}, \quad n \geq 2.$$

Thus,

$$a_2 = \frac{-2a_0}{5 \cdot 2}, \quad a_4 = \frac{-2a_2}{9 \cdot 4}, \quad \dots, \quad a_{2n} = \frac{-2a_{2n-2}}{(4n+1)(2n)}.$$

Multiplying the left and right-hand sides of these equations,

$$a_2 a_4 \cdots a_{2n} = \frac{(-2)^n}{[5 \cdot 9 \cdots (4n+1)][2 \cdot 4 \cdots (2n)]} a_0 a_2 \cdots a_{2n-2}.$$

Cancelling,

$$a_{2n} = \frac{(-1)^n}{[5 \cdot 9 \cdots (4n+1)][1 \cdot 2 \cdots n]} a_0 \quad \text{or} \quad a_{2n} = \frac{(-1)^n}{[5 \cdot 9 \cdots (4n+1)]n!} a_0, \quad n \geq 1.$$

Because $a_1 = 0$, it is easily seen that $a_{2n+1} = 0$, for $n \geq 0$. That is, all odd coefficients are zero. Thus, with $s_1 = 1/4$ and $a_0 = 1$, we have

$$y_1(x) = x^{1/4} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{[5 \cdot 9 \cdots (4n+1)]n!} \right].$$

In similar fashion, with $s_2 = -1/4$,

$$a_n = \frac{-a_{n-2}}{n(n-\frac{1}{2})} = \frac{-2a_{n-2}}{n(2n-1)}, \quad n \geq 2.$$

Again, because $a_1 = 0$, all odd coefficients equal zero, but

$$a_2 = \frac{-2a_0}{2 \cdot 3}, \quad a_4 = \frac{-2a_2}{4 \cdot 7}, \quad \dots, \quad a_{2n} = \frac{-2a_{2n-2}}{2n(4n-1)}, \quad n \geq 1.$$

Multiplying left and right-hand sides,

$$a_2 a_4 \cdots a_{2n} = \frac{(-2)^n}{[2 \cdot 4 \cdots 2n][3 \cdot 7 \cdots (4n-1)]} a_0 a_2 \cdots a_{2n-2}.$$

Cancelling,

$$a_{2n} = \frac{(-1)^n}{[1 \cdot 2 \cdots n][3 \cdot 7 \cdots (4n-1)]} a_0 = \frac{(-1)^n}{n![3 \cdot 7 \cdots (4n-1)]} a_0.$$

Thus, with $s_2 = -1/4$ and $a_0 = 1$, we have

$$y_2(x) = x^{-1/4} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n![3 \cdot 7 \cdots (4n-1)]} \right].$$

16. Divide by 3,

$$x^2 y'' + \frac{2}{3} x y' + \frac{x^2}{3} y = 0,$$

then compare with $x^2 y'' + x p(x) y' + q(x) y = 0$ and note that $p(x) = 2/3$ and $q(x) = x^2/3$, both of which are analytic at $x = 0$. Thus, $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = 2/3$ and $q_0 = q(0) = 0$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{1}{3}s.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We'll find it easier to work with $3x^2 y'' + 2xy' + x^2 y = 0$. Then,

$$3x^2 y'' = \sum_{n=0}^{\infty} 3(s+n)(s+n-1)a_n x^{s+n}, \quad 2xy' = \sum_{n=0}^{\infty} 2(s+n)a_n x^{s+n},$$

and

$$x^2 y = x^2 \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=2}^{\infty} a_{n-2} x^{s+n},$$

where we've shifted the index of summation. Substituting,

$$\begin{aligned} 0 &= 3x^2 y'' + 2xy' + x^2 y = \sum_{n=0}^{\infty} 3(s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} 2(s+n)a_n x^{s+n} + \sum_{n=2}^{\infty} a_{n-2} x^{s+n} \\ &= [3s(s-1) + 2s]a_0 x^s + [3(s+1)s + 2(s+1)]a_1 x^{s+1} \\ &\quad + \sum_{n=2}^{\infty} \{[3(s+n)(s+n-1) + 2(s+n)]a_n + a_{n-2}\} x^{s+n}. \end{aligned}$$

The first term yields the indicial equation,

$$I(s) = 3s^2 - s = s(3s - 1) = 0,$$

with roots $s_2 = 0$ and $s_1 = 1/3$. Setting the second coefficient equal to zero,

$$(s+1)(3s+2)a_1 = 0,$$

we see that $a_1 = 0$ because its coefficient is not zero for either root of the indicial equation. The remaining terms provide the recurrence relation

$$[3(s+n)^2 - (s+n)]a_n + a_{n-2} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-2}}{(s+n)(3s+3n-1)}, \quad n \geq 2.$$

Because $a_1 = 0$, it's easy to see that $a_{2n+1} = 0$ for all $n \geq 0$. With $s_1 = 1/3$, the recurrence relation becomes

$$a_n = \frac{-a_{n-2}}{(\frac{1}{3}+n)(3n)} = \frac{-a_{n-2}}{n(3n+1)}, \quad n \geq 2.$$

Thus,

$$a_2 = \frac{-a_0}{2 \cdot 7}, \quad a_4 = \frac{-a_2}{4 \cdot 13}, \quad \dots, \quad a_{2n} = \frac{-a_{2n-2}}{2n(6n+1)}.$$

Multiplying left and right-hand sides of these results,

$$a_2 a_4 \cdots a_{2n} = \frac{(-1)^n}{[2 \cdot 4 \cdots (2n)][7 \cdot 13 \cdots (6n+1)]} a_0 a_2 \cdots a_{2n-2}.$$

Cancelling,

$$a_{2n} = \frac{(-1)^n}{2^n n! [7 \cdot 13 \cdots (6n+1)]} a_0.$$

Hence, with $s_1 = 1/3$ and $a_0 = 1$, we have

$$y_1(x) = x^{1/3} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n! [7 \cdot 13 \cdots (6n+1)]} \right].$$

With $s_2 = 0$, the recursion equation becomes

$$a_n = \frac{-a_{n-2}}{n(3n-1)}, \quad n \geq 2.$$

Again, all odd terms equal zero. But,

$$a_2 = \frac{-a_0}{2 \cdot 5}, \quad a_4 = \frac{-a_2}{4 \cdot 11}, \quad \dots, \quad a_{2n} = \frac{-a_{2n-2}}{2n(6n-1)}.$$

Multiplying left and right-hand sides of these results,

$$a_2 a_4 \cdots a_{2n} = \frac{(-1)^n}{[2 \cdot 4 \cdots 2n][5 \cdot 11 \cdots (6n-1)]} a_0 a_2 \cdots a_{2n-2}.$$

Cancelling,

$$a_{2n} = \frac{(-1)^n}{2^n n! [5 \cdot 11 \cdots (6n-1)]} a_0.$$

With $s_2 = 0$ and $a_0 = 1$, we obtain

$$y_2(x) = x^0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n! [5 \cdot 11 \cdots (6n-1)]} \right].$$

17. Dividing by 2,

$$x^2 y'' - \frac{x}{2} y' + \frac{x+1}{2} y = 0,$$

then comparing with $x^2 y'' + xp(x)y' + q(x)y = 0$ reveals $p(x) = -1/2$ and $q(x) = (x+1)/2$, both of which are analytic at $x = 0$. Hence, $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = -1/2$ and $q_0 = q(0) = 1/2$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{3}{2}s + \frac{1}{2}.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We'll find the equation $2x^2 y'' - xy' + (x+1)y = 0$ easier to deal with. Thus,

$$2x^2 y'' = \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n}, \quad -xy' = -\sum_{n=0}^{\infty} (s+n)a_n x^{s+n},$$

and

$$(x+1)y = (x+1) \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=1}^{\infty} a_{n-1} x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n},$$

where we've shifted the index of the first summation. Substituting,

$$\begin{aligned} 0 &= 2x^2 y'' - xy' + (x+1)y \\ &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n} - \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} + \sum_{n=1}^{\infty} a_{n-1} x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n} \\ &= [2s(s-1) - s + 1]a_0 x^s + \sum_{n=1}^{\infty} \{[2(s+n)(s+n-1) - (s+n) + 1]a_n + a_{n-1}\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = 2s^2 - 3s + 1 = (2s-1)(s-1) = 0,$$

with roots $s_1 = 1/2$ and $s_1 = 1$. The remaining terms provide the recurrence relation

$$(2s+2n-1)(s+n-1)a_n + a_{n-1} = 0, \quad n \geq 1.$$

With $s_1 = 1$, this becomes

$$n(2n+1)a_n = -a_{n-1} \quad \text{or} \quad a_n = \frac{-a_{n-1}}{n(2n+1)}.$$

Thus,

$$a_1 = \frac{-a_0}{1 \cdot 3}, \quad a_2 = \frac{-a_1}{2 \cdot 5}, \quad \dots, \quad a_n = \frac{-a_{n-1}}{n(2n+1)}.$$

Multiplying left and right-hand sides of these equations,

$$a_1 a_2 \cdots a_n = \frac{(-1)^n}{[1 \cdot 2 \cdots n][3 \cdot 5 \cdots (2n+1)]} a_0 a_1 \cdots a_{n-1}.$$

Cancelling,

$$a_n = \frac{(-1)^n}{n![3 \cdot 5 \cdot \dots \cdot (2n+1)]} a_0.$$

We can simplify a bit further by noticing that

$$3 \cdot 5 \cdot \dots \cdot (2n+1) = \frac{2 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)}{2 \cdot 4 \cdot \dots \cdot (2n)} = \frac{(2n+1)!}{2^n \cdot n!}$$

Thus,

$$a_n = \frac{(-1)^n 2^n n!}{n!(2n+1)!} a_0 = \frac{(-1)^n 2^n}{(2n+1)!} a_0.$$

Thus, with $s_1 = 1$, and $a_0 = 1$, we have one solution

$$y_1(x) = x^1 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n.$$

With $s_2 = 1/2$, the recurrence relation becomes

$$2n \left(n - \frac{1}{2} \right) a_n + a_{n-1} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-1}}{n(2n-1)}, \quad n \geq 1.$$

Thus,

$$a_1 = \frac{-a_0}{1 \cdot 1}, \quad a_2 = \frac{-a_1}{2 \cdot 3}, \quad \dots \quad a_n = \frac{-a_{n-1}}{n(2n-1)}.$$

Multiplying left and right-hand sides together, we get

$$a_1 a_2 \cdots a_n = \frac{(-1)^n}{[1 \cdot 2 \cdot \dots \cdot n][1 \cdot 3 \cdot \dots \cdot (2n-1)]} a_0 a_1 \cdots a_{n-1}.$$

Cancelling,

$$a_n = \frac{(-1)^n}{n![1 \cdot 3 \cdot \dots \cdot (2n-1)]} a_0.$$

We can simplify a bit further by noticing that

$$1 \cdot 3 \cdot \dots \cdot (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot (2n-2)} = \frac{(2n-1)!}{2^{n-1}[1 \cdot 2 \cdot \dots \cdot (n-1)]} = \frac{(2n-1)!}{2^{n-1}(n-1)!}.$$

Thus,

$$a_n = \frac{(-1)^n 2^{n-1} (n-1)!}{n!(2n-1)!} a_0 = \frac{(-1)^n 2^{n-1}}{n(2n-1)!} a_0.$$

Thus, with $s_2 = 1/2$ and $a_0 = 1$, we have a second independent solution

$$y_2(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{n-1}}{n(2n-1)!} x^n \right].$$

18. Divide by 3,

$$x^2 y'' + \frac{1}{3} x(1-x)y' - \frac{1}{3} y = 0$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$. Note that $p(x) = (1-x)/3$ and $q(x) = -1/3$, both analytic at $x = 0$, so $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = 1/3$ and $q_0 = q(0) = -1/3$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{2}{3}s - \frac{1}{3}.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We'll find the equation $3x^2 y'' + (x - x^2)y' - y = 0$ easier to deal with. Thus,

$$3x^2 y'' = \sum_{n=0}^{\infty} 3(s+n)(s+n-1)a_n x^{s+n},$$

and

$$x(1-x)y' = (1-x) \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} = \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1} x^{s+n},$$

where we've shifted the index of the second summation. Substituting,

$$\begin{aligned} 0 &= 3x^2 y'' + x(1-x)y' - y \\ &= \sum_{n=0}^{\infty} 3(s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1} x^{s+n} - \sum_{n=0}^{\infty} a_n x^{s+n} \\ &= [3s(s-1) + s - 1]a_0 x^s + \sum_{n=1}^{\infty} \{[3(s+n)(s+n-1) + (s+n) - 1]a_n - (s+n-1)a_{n-1}\} x^{s+n} \\ &= (3s^2 - 2s - 1)a_0 x^s + \sum_{n=1}^{\infty} \{[3(s+n)^2 - 2(s+n) - 1]a_n - (s+n-1)a_{n-1}\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = 3s^2 - 2s - 1 = (3s+1)(s-1) = 0,$$

with roots $s_2 = -1/3$ and $s_1 = 1$. The remaining terms provide the recurrence relation

$$(3s+3n+1)(s+n-1)a_n - (s+n-1)a_{n-1} = 0, \quad n \geq 1.$$

With $s_1 = 1$, this becomes

$$n(3n+4)a_n - na_{n-1} = 0 \quad \text{or} \quad a_n = \frac{a_{n-1}}{3n+4}.$$

Thus,

$$a_1 = \frac{a_0}{7}, \quad a_2 = \frac{a_1}{10}, \quad \dots, \quad a_n = \frac{a_{n-1}}{3n+4}.$$

Multiplying left and right-hand sides of these equations,

$$a_1 a_2 \cdots a_n = \frac{a_0 a_1 \cdots a_{n-1}}{7 \cdot 10 \cdots (3n+4)}.$$

Cancelling,

$$a_n = \frac{a_0}{7 \cdot 10 \cdots (3n+4)}.$$

Thus, with $s_1 = 1$ and $a_0 = 1$, one solution is

$$y_1(x) = x \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{7 \cdot 10 \cdots (3n+4)} \right].$$

With $s_2 = -1/3$, the recurrence equation becomes

$$3n \left(n - \frac{4}{3} \right) a_n - \left(n - \frac{4}{3} \right) a_{n-1} = 0 \quad \text{or} \quad a_n = \frac{a_{n-1}}{3n}.$$

Thus,

$$a_1 = \frac{a_0}{3}, \quad a_2 = \frac{a_1}{6}, \quad \dots, \quad a_n = \frac{a_{n-1}}{3n}.$$

Multiplying left and right-hand sides of these equations,

$$a_1 a_2 \cdots a_n = \frac{a_0 a_1 \cdots a_{n-1}}{3 \cdot 6 \cdots 3n}.$$

Cancelling,

$$a_n = \frac{a_0}{3^n [1 \cdot 2 \cdots n]} = \frac{a_0}{3^n n!}.$$

Thus, with $s_2 = -1/3$ and $a_0 = 1$, we have a second independent solution,

$$y_2(x) = x^{-1/3} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{3^n n!} \right] = x^{-1/3} \sum_{n=0}^{\infty} \frac{(x/3)^n}{n!} = x^{-1/3} e^{x/3}.$$

19. Multiply by $x/(2(1+x))$,

$$x^2 y'' + \frac{x}{2(1+x)} y' - \frac{x}{2(1+x)} y = 0,$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 1/(2(1+x))$ and $q(x) = -x/(2(1+x))$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 1/2$ and $q_0 = q(0) = 0$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{1}{2}s.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We'll find the equation $2(x + x^2)y'' + y' - y = 0$ easier to deal with. Thus,

$$\begin{aligned} 0 &= 2(x + x^2)y'' + y' - y \\ &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n-1} + \sum_{n=1}^{\infty} 2(s+n-1)(s+n-2)a_{n-1} x^{s+n-1} \\ &\quad + \sum_{n=0}^{\infty} (s+n)a_n x^{s+n-1} - \sum_{n=1}^{\infty} a_{n-1} x^{s+n-1}, \end{aligned}$$

where we've shifted the index of the second summation and fourth summations. Thus,

$$\begin{aligned} 0 &= [2s(s-1) + s]a_0 x^{s-1} \\ &\quad + \sum_{n=1}^{\infty} \{[2(s+n)(s+n-1) + (s+n)]a_n + [2(s+n-1)(s+n-2) - 1]a_{n-1}\} x^{s+n-1}, \end{aligned}$$

and the indicial equation is

$$I(s) = 2s^2 - s = s(2s - 1) = 0,$$

with roots $s_2 = 0$ and $s_1 = 1/2$. The remaining terms provide the recurrence relation

$$(s+n)(2s+2n-1)a_n + [2(s+n-1)(s+n-2) - 1]a_{n-1} = 0,$$

for $n \geq 1$. With $s_1 = 1/2$, this becomes

$$a_n = -\frac{2n^2 - 4n + 1/2}{n(2n+1)} a_{n-1}, \quad n \geq 1.$$

Thus,

$$a_1 = -\frac{3/2}{1 \cdot 3} a_0, \quad a_2 = -\frac{1/2}{2 \cdot 5} a_1, \quad a_3 = -\frac{13/2}{3 \cdot 7} a_2, \quad \dots$$

etc. Thus,

$$a_1 a_2 a_3 \dots a_n = (-1)^n \frac{(-3) \cdot \frac{1}{2} \cdot \frac{13}{2} \dots \frac{4n^2 - 8n + 1}{2}}{n!(3 \cdot 5 \cdot 7 \dots (2n+1))} a_0 a_1 a_2 \dots a_{n-1}.$$

Cancelling,

$$a_n = (-1)^n \frac{(-3) \cdot 1 \cdot 13 \dots (4n^2 - 8n + 1)}{2^n n! (3 \cdot 5 \cdot 7 \dots (2n+1))} a_0, \quad n \geq 1.$$

But,

$$3 \cdot 5 \cdot 7 \dots (2n+1) = \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \dots 2n(2n+1)}{2 \cdot 4 \cdot 6 \dots 2n} = \frac{(2n+1)!}{2^n n!}.$$

Thus,

$$a_n = (-1)^n \frac{(-3) \cdot 1 \cdot 13 \dots (4n^2 - 8n + 1)}{(2n+1)!} a_0.$$

With $a_0 = 1$ and $s_1 = 1/2$,

$$y_1(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (-3) \cdot 1 \cdot 13 \dots (4n^2 - 8n + 1)}{(2n+1)!} x^n \right].$$

With $s_2 = 0$, the recurrence relation becomes

$$a_n = -\frac{2n^2 - 6n + 3}{n(2n - 1)}, \quad n \geq 1.$$

Thus,

$$a_1 = -\frac{-1}{1 \cdot 1} a_0, \quad a_2 = -\frac{-1}{2 \cdot 3} a_1, \quad a_3 = -\frac{3}{3 \cdot 5} a_2, \quad \dots$$

etc. Thus,

$$a_1 a_2 a_3 \cdots a_n = (-1)^n \frac{(-1)(-1)(3) \cdots (2n^2 - 6n + 3)}{n!(1 \cdot 3 \cdot 5 \cdots (2n - 1))} a_0 a_1 a_2 \cdots a_{n-1}.$$

Cancelling,

$$a_n = (-1)^n \frac{(-1) \cdot (-1) \cdot 3 \cdot 11 \cdots (2n^2 - 6n + 3)}{n!(1 \cdot 3 \cdot 5 \cdots (2n - 1))} a_0.$$

But $1 \cdot 3 \cdot 5 \cdots (2n - 1) = (2n)!/(2^n n!)$, so

$$a_n = \frac{(-1)^n 2^n ((-1) \cdot (-1) \cdot 3 \cdot 11 \cdots (2n^2 - 6n + 3))}{(2n)!} a_0.$$

Thus, with $s_2 = 0$ and $a_0 = 1$,

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n \cdot (-1) \cdot (-1) \cdot 3 \cdot 11 \cdots (2n^2 - 6n + 3)}{(2n)!} x^n.$$

20. Multiply by $x/2$,

$$x^2 y'' + \frac{1}{2} x y' + \frac{1}{2} x^2 y = 0,$$

then compare with $x^2 y'' + x p(x) y' + q(x) y = 0$. Note that $p(x) = 1/2$ and $q(x) = x^2/2$, both analytic at $x = 0$, so $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = 1/2$ and $q_0 = q(0) = 0$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{1}{2}s.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We'll find the equation $2x^2 y'' + x y' + x^2 y = 0$ easier to deal with. Thus,

$$2x^2 y'' = \sum_{n=0}^{\infty} 2(s+n)(s+n-1) a_n x^{s+n}, \quad x y' = \sum_{n=0}^{\infty} (s+n) a_n x^{s+n},$$

and

$$x^2 y = x^2 \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=2}^{\infty} a_{n-2} x^{s+n},$$

where we've shifted the index of the last summation. Substituting,

$$\begin{aligned}
 0 &= 2x^2 y'' + xy' + x^2 y \\
 &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} + \sum_{n=2}^{\infty} a_{n-2} x^{s+n} \\
 &= [2s(s-1) + s]a_0 x^s + [2(s+1)s + (s+1)]a_1 x^{s+1} \\
 &\quad + \sum_{n=2}^{\infty} \{[2(s+n)(s+n-1) + (s+n)]a_n + a_{n-2}\} x^{s+n} \\
 &= (2s^2 - s)a_0 x^s + (2s+1)(s+1)a_1 x^{s+1} + \sum_{n=2}^{\infty} \{[2(s+n)(s+n-1) + (s+n)]a_n + a_{n-2}\} x^{s+n}.
 \end{aligned}$$

Thus, the indicial equation is

$$I(s) = 2s^2 - s = s(2s - 1) = 0,$$

with roots $s_2 = 0$ and $s_1 = 1/2$. The remaining terms provide the recurrence relation

$$(s+n)(2s+2n-1)a_n + a_{n-2} = 0.$$

With $s_1 = 1/2$, this becomes

$$n(2n+1)a_n = -a_{n-2} \quad \text{or} \quad a_n = \frac{-a_{n-2}}{n(2n+1)}.$$

Thus,

$$a_2 = \frac{-a_0}{2 \cdot 5}, \quad a_4 = \frac{-a_2}{4 \cdot 9}, \quad \dots \quad a_{2n} = \frac{-a_{2n-2}}{2n(4n+1)}.$$

Multiplying left and right-hand sides of these equations,

$$a_2 a_4 \cdots a_{2n} = \frac{(-1)^n a_0 a_2 \cdots a_{2n-2}}{[2 \cdot 4 \cdots 2n][5 \cdot 9 \cdots (4n+1)]}.$$

Cancelling,

$$a_{2n} = \frac{(-1)^n a_0}{2^n n! [5 \cdot 9 \cdots (4n+1)]}, \quad n \geq 1.$$

Next, the coefficient of x^{s+1} must equal zero,

$$(2s+1)(s+1)a_1 = 0.$$

However, with $s = s_1 = 1/2$, this forces $a_1 = 0$, then the recurrence relation assures that $a_{2n+1} = 0$ for all n . Thus, with $s_1 = 1/2$ and $a_0 = 1$, one solution is

$$y_1(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n! [5 \cdot 9 \cdots (4n+1)]} \right].$$

Next, with $s_2 = 0$, the recurrence relation becomes

$$n(2n-1)a_n + a_{n-2} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-2}}{n(2n-1)}.$$

Thus,

$$a_2 = \frac{-a_0}{2 \cdot 3}, \quad a_4 = \frac{-a_2}{4 \cdot 7}, \quad \dots, \quad a_{2n} = \frac{-a_{2n-2}}{2n(4n-1)}.$$

Multiplying left and right-hand sides of these equations,

$$a_2 a_4 \cdots a_{2n} = \frac{(-1)^n a_0 a_2 \cdots a_{2n-2}}{[2 \cdot 4 \cdots 2n][3 \cdot 7 \cdots (4n-1)]}.$$

Cancelling,

$$a_{2n} = \frac{(-1)^n a_0}{2^n n! [3 \cdot 7 \cdots (4n-1)]}, \quad n \geq 1.$$

Again, $(2s+1)(s+1)a_1 = 0$ and $s = 0$ insures that $a_1 = 0$, and the recurrence relation gives $a_{2n+1} = 0$ for all n . Thus, with $s = 0$ and $a_0 = 1$, a second independent solution is

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n! [3 \cdot 7 \cdots (4n-1)]}.$$

21. Multiply by $x/2$,

$$x^2 y'' + \frac{1}{2} x y' + \frac{1}{2} x y = 0,$$

then compare with $x^2 y'' + x p(x) y' + q(x) y = 0$. Note that $p(x) = 1/2$ and $q(x) = x/2$, both analytic at $x = 0$, so $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = 1/2$ and $q_0 = q(0) = 0$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{1}{2}s.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We'll find the equation $2x^2 y'' + x y' + x y = 0$ easier to deal with. Thus,

$$2x^2 y'' = \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n}, \quad x y' = \sum_{n=0}^{\infty} (s+n)a_n x^{s+n},$$

and

$$x y = x \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=1}^{\infty} a_{n-1} x^{s+n},$$

where we've shifted the index of the last summation. Substituting,

$$\begin{aligned} 0 &= 2x^2 y'' + x y' + x y \\ &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} + \sum_{n=1}^{\infty} a_{n-1} x^{s+n} \\ &= [2s(s-1) + s]a_0 x^s + \sum_{n=1}^{\infty} \{[2(s+n)(s+n-1) + (s+n)]a_n + a_{n-1}\} x^{s+n} \\ &= (2s^2 - s)a_0 x^s + \sum_{n=1}^{\infty} \{[2(s+n)^2 - (s+n)]a_n + a_{n-1}\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = 2s^2 - s = s(2s - 1) = 0$$

with roots $s_2 = 0$ and $s_1 = 1/2$. The remaining terms provide the recurrence relation

$$(s + n)(2s + 2n - 1)a_n + a_{n-1} = 0, \quad n \geq 1.$$

With $s_1 = 1/2$, this becomes

$$n(2n + 1)a_n = -a_{n-1} \quad \text{or} \quad a_n = \frac{-a_{n-1}}{n(2n + 1)}.$$

Thus,

$$a_1 = \frac{-a_0}{1 \cdot 3}, \quad a_2 = \frac{-a_1}{2 \cdot 5}, \quad \dots \quad a_n = \frac{-a_{n-1}}{n(2n + 1)}.$$

Multiplying left and right-hand sides of these equations,

$$a_1 a_2 \cdots a_n = \frac{(-1)^n a_0 a_1 \cdots a_{n-1}}{[1 \cdot 2 \cdots n][3 \cdot 5 \cdots (2n + 1)]}.$$

Cancelling,

$$a_n = \frac{(-1)^n a_0}{n![3 \cdot 5 \cdots (2n + 1)]}.$$

We can further simplify by noticing

$$\begin{aligned} 3 \cdot 5 \cdots (2n + 1) &= \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n + 1)}{2 \cdot 4 \cdots (2n)} \\ &= \frac{(2n + 1)!}{2^n n!}. \end{aligned}$$

Thus,

$$a_n = \frac{(-1)^n n! 2^n a_0}{n!(2n + 1)!} = \frac{(-1)^n 2^n a_0}{(2n + 1)!}.$$

Thus, with $s_1 = 1/2$ and $a_0 = 1$, one solution is

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{(2n + 1)!}.$$

Next, with $s_2 = 0$, the recurrence relation becomes

$$n(2n - 1)a_n + a_{n-1} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-1}}{n(2n - 1)}, \quad n \geq 1.$$

Thus,

$$a_1 = \frac{-a_0}{1 \cdot 1}, \quad a_2 = \frac{-a_1}{2 \cdot 3}, \quad \dots \quad a_n = \frac{-a_{n-1}}{n(2n - 1)}.$$

Multiply the left and right-hand sides of these equation,

$$a_1 a_2 \cdots a_n = \frac{(-1)^n a_0 a_1 \cdots a_{n-1}}{[1 \cdot 2 \cdots n][1 \cdot 3 \cdots (2n - 1)]}.$$

Cancelling,

$$a_n = \frac{(-1)^n a_0}{n! [1 \cdot 3 \cdot \dots \cdot (2n-1)]}.$$

We can simplify further by noticing that

$$1 \cdot 3 \cdot \dots \cdot (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (2n-1) \cdot 2n}{2 \cdot 4 \cdot \dots \cdot 2n} = \frac{(2n)!}{2^n n!}.$$

Thus,

$$a_n = \frac{(-1)^n 2^n n! a_0}{n! (2n)!} = \frac{(-1)^n 2^n a_0}{(2n)!}.$$

Thus, with $s_1 = 0$ and $a_0 = 1$, we get a second solution

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{(2n)!}.$$

22. Divide by 2,

$$x^2 y'' - \frac{1}{2} x y' + \frac{1+x}{2} y = 0,$$

then compare with $x^2 y'' + x p(x) y' + q(x) y = 0$. Note that $p(x) = -1/2$ and $q(x) = (1+x)/2$, both analytic at $x = 0$, so $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = -1/2$ and $q_0 = q(0) = 1/2$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{3}{2}s + \frac{1}{2}.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We'll find the equation $2x^2 y'' - xy' + (1+x)y = 0$ easier to deal with. Thus,

$$2x^2 y'' = \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n}, \quad -xy' = -\sum_{n=0}^{\infty} (s+n)a_n x^{s+n},$$

and

$$(1+x)y = (1+x) \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=0}^{\infty} a_n x^{s+n} + \sum_{n=1}^{\infty} a_{n-1} x^{s+n},$$

where we've shifted the index on the second summation. Substituting,

$$\begin{aligned} 0 &= 2x^2 y'' - xy' + (1+x)y \\ &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n} - \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n} + \sum_{n=1}^{\infty} a_{n-1} x^{s+n} \\ &= [2s(s-1) - s + 1]a_0 x^s + \sum_{n=1}^{\infty} \{[2(s+n)(s+n-1) - (s+n) + 1]a_n + a_{n-1}\} x^{s+n} \\ &= (2s^2 - 3s + 1)a_0 x^s + \sum_{n=1}^{\infty} \{[2(s+n)^2 - 3(s+n) + 1]a_n + a_{n-1}\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = 2s^2 - 3s + 1 = (2s - 1)(s - 1) = 0,$$

with roots $s_2 = 1/2$ and $s_1 = 1$. The remaining terms provide the recurrence relation

$$(2s + 2n - 1)(s + n - 1)a_n + a_{n-1} = 0, \quad n \geq 1.$$

With $s_1 = 1$, this becomes

$$(2n + 1)na_n + a_{n-1} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-1}}{n(2n + 1)}.$$

Thus,

$$a_1 = \frac{-a_0}{1 \cdot 3}, \quad a_2 = \frac{-a_1}{2 \cdot 5}, \quad \dots \quad a_n = \frac{-a_{n-1}}{n(2n + 1)}.$$

Multiplying the left and right-hand sides of these equations,

$$a_1 a_2 \cdots a_n = \frac{(-1)^n a_0 a_1 \cdots a_{n-1}}{[1 \cdot 2 \cdots n][3 \cdot 5 \cdots (2n + 1)]}.$$

Cancelling,

$$a_n = \frac{(-1)^n a_0}{n![3 \cdot 5 \cdots (2n + 1)]}.$$

We can simplify further by noticing

$$3 \cdot 5 \cdots (2n + 1) = \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n + 1)}{2 \cdot 4 \cdots (2n)} = \frac{(2n + 1)!}{2^n n!}.$$

Thus,

$$a_n = \frac{(-1)^n 2^n n! a_0}{n!(2n + 1)!} = \frac{(-1)^n 2^n a_0}{(2n + 1)!}.$$

Thus, with $s_1 = 1$ and $a_0 = 1$, one solution is

$$y_1(x) = x \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{(2n + 1)!} \right].$$

With $s_2 = 1/2$, the recurrence relation becomes

$$n(2n - 1)a_n + a_{n-1} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-1}}{n(2n - 1)}, \quad n \geq 1.$$

Thus,

$$a_1 = \frac{-a_0}{1 \cdot 1}, \quad a_2 = \frac{-a_1}{2 \cdot 3}, \quad \dots \quad a_n = \frac{-a_{n-1}}{n(2n - 1)}.$$

Multiplying the left and right-hand sides of these equations,

$$a_1 a_2 \cdots a_n = \frac{(-1)^n a_0 a_1 \cdots a_{n-1}}{[1 \cdot 2 \cdots n][1 \cdot 3 \cdots (2n - 1)]}.$$

Cancelling,

$$a_n = \frac{(-1)^n a_0}{n! [1 \cdot 3 \cdot \dots \cdot (2n-1)]}.$$

We can simplify further by noting that

$$1 \cdot 3 \cdot \dots \cdot (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (2n-1)(2n)}{2 \cdot 4 \cdot \dots \cdot (2n)} = \frac{(2n)!}{2^n n!}.$$

Thus,

$$a_n = \frac{(-1)^n 2^n n! a_0}{n! (2n)!} = \frac{(-1)^n 2^n a_0}{(2n)!}.$$

Thus, with $s_1 = 1/2$ and $a_0 = 1$, a second independent solution is

$$y_2(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{(2n)!} \right].$$

23. Divide by 2,

$$x^2 y'' - \frac{1}{2} x y' + \frac{1-x^2}{2} y = 0,$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$. Note that $p(x) = -1/2$ and $q(x) = (1-x^2)/2$, both analytic at $x = 0$, so $x = 0$ is a regular singular point. Further, note that $p_0 = p(0) = -1/2$ and $q_0 = q(0) = 1/2$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{3}{2}s + \frac{1}{2}.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We'll find the equation $2x^2 y'' - xy' + (1-x^2)y = 0$ easier to deal with. Thus,

$$2x^2 y'' = \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n}, \quad -xy' = -\sum_{n=0}^{\infty} (s+n)a_n x^{s+n},$$

and

$$(1-x^2)y = (1-x^2) \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=0}^{\infty} a_n x^{s+n} - \sum_{n=2}^{\infty} a_{n-2} x^{s+n},$$

where we've shifted the index on the second summation. Substituting,

$$\begin{aligned} 0 &= 2x^2 y'' - xy' + (1-x^2)y \\ &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n} - \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n} - \sum_{n=2}^{\infty} a_{n-2} x^{s+n} \\ &= [2s(s-1) - s + 1]a_0 x^s + [2(s+1)s - (s+1) + 1]a_1 x^{s+1} \\ &\quad + \sum_{n=2}^{\infty} \{2(s+n)(s+n-1) - (s+n) + 1\} a_n - a_{n-2} \} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = 2s^2 - 3s + 1 = (2s - 1)(s - 1) = 0,$$

with roots $s_1 = 1/2$ and $s_1 = 1$. The remaining terms provide the recurrence relation

$$(2s + 2n - 1)(s + n - 1)a_n - a_{n-2} = 0, \quad n \geq 2.$$

With $s_1 = 1$, this becomes

$$(2n + 1)na_n - a_{n-2} = 0 \quad \text{or} \quad a_n = \frac{a_{n-2}}{n(2n + 1)}.$$

Thus,

$$a_2 = \frac{a_0}{2 \cdot 5}, \quad a_4 = \frac{a_2}{4 \cdot 9}, \quad \dots \quad a_{2n} = \frac{a_{2n-2}}{2n(4n + 1)}.$$

Multiplying left and right-hand sides of these equations,

$$a_2 a_4 \cdots a_{2n} = \frac{a_0 a_2 \cdots a_{2n-2}}{[2 \cdot 4 \cdots 2n][5 \cdot 9 \cdots (4n + 1)]}.$$

Cancelling,

$$a_{2n} = \frac{a_0}{2^n n! [5 \cdot 9 \cdots (4n + 1)]}.$$

The coefficient of $[2(s + 1)s - (s + 1) + 1]a_1$ must also equal zero, but with $s_1 = 1$, this becomes $3a_1 = 0$, so $a_1 = 0$. With the recurrence relation, this implies that $a_{2n+1} = 0$ for all n . Thus, with $s_1 = 1$ and $a_0 = 1$, one solution is

$$y_1(x) = x \left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n! [5 \cdot 9 \cdots (4n + 1)]} \right].$$

Next, with $s_2 = 1/2$, the recurrence relation becomes

$$n(2n - 1)a_n - a_{n-2} = 0 \quad \text{or} \quad a_n = \frac{a_{n-2}}{n(2n - 1)}, \quad n \geq 2.$$

Thus,

$$a_2 = \frac{a_0}{2 \cdot 3}, \quad a_4 = \frac{a_2}{4 \cdot 7}, \quad \dots, \quad a_{2n} = \frac{a_{2n-2}}{2n(4n - 1)}.$$

Multiplying the left and right-hand sides of these equations,

$$a_2 a_4 \cdots a_{2n} = \frac{a_0 a_2 \cdots a_{2n-2}}{[2 \cdot 4 \cdots 2n][3 \cdot 7 \cdots (4n - 1)]}.$$

Cancelling,

$$a_{2n} = \frac{a_0}{2^n n! [3 \cdot 7 \cdots (4n - 1)]}.$$

Again, the coefficient of $[2(s + 1)s - (s + 1) + 1]a_1$ must equal zero, but with $s_2 = 1/2$, this becomes $a_1 = 0$. With the recurrence relation, this implies that $a_{2n+1} = 0$ for all n . Thus, with $s_2 = 1/2$ and $a_0 = 1$, a second independent solution is

$$y_2(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n! [3 \cdot 7 \cdots (4n - 1)]} \right].$$

24. Divide by 2,

$$x^2 y'' + \frac{3}{2} x y' - \frac{x^2 + 1}{2} y = 0,$$

then compare with $x^2 y'' + x p(x) y' + q(x) y = 0$ and note that $p(x) = 3/2$ and $q(x) = -(x^2 + 1)/2$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. We will find the first equation easier to deal with, gathering together the fundamental expressions times a fixed power of x .

$$0 = [2x^2 y'' + 3xy' - y] - x^2[y]$$

Set $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substitute series for the fundamental expressions, obtaining

$$0 = \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 3(s+n) - 1] a_n x^{s+n} - x^2 \sum_{n=0}^{\infty} a_n x^{s+n}.$$

The coefficient of a_n in the first term can be written $I(s+n)$, where the indicial polynomial is

$$I(s) = 2s(s-1) + 3s - 1 = 2s^2 + s - 1 = (2s-1)(s+1).$$

We shift the index of the second sum and combine terms. In doing so, we isolate terms for $n = 0$ and $n = 1$.

$$0 = I(s)a_0 x^s + I(s+1)a_1 x^{s+1} + \sum_{n=2}^{\infty} [I(s+n)a_n - a_{n-2}] a_n x^{s+n}$$

The first term reveals the indicial equation, $I(s) = (2s-1)(s+1)$, with roots $s_1 = 1/2$ and $s_2 = -1$. The coefficient of x^{s+1} is $I(s+1)a_1$, which must equal zero. Note that $I(s+1) \neq 0$ if $s = s_1 = 1/2$ or $s = s_2 = -1$. Hence, $a_1 = 0$. The remaining terms provide the recurrence relation

$$\begin{aligned} I(s+n)a_n &= a_{n-2} \\ (2s+2n-1)(s+n+1)a_n &= a_{n-2}, \quad n \geq 2. \end{aligned}$$

As $a_1 = 0$, this relation implies that all odd-numbered coefficients equal zero. Rewriting the recurrence relation for even integers,

$$a_{2n} = \frac{a_{2n-2}}{(2s+4n-1)(s+2n+1)}, \quad n \geq 1.$$

Setting $s = s_1 = 1/2$,

$$a_{2n} = \frac{a_{2n-2}}{2n(4n+3)}, \quad n \geq 1.$$

Thus,

$$a_2 = \frac{a_0}{2 \cdot 1 \cdot 7}, \quad a_4 = \frac{a_2}{2 \cdot 2 \cdot 11}, \quad a_6 = \frac{a_4}{2 \cdot 3 \cdot 15}.$$

The product of these n recurrence formulae is

$$a_2 a_4 a_6 \dots a_{2n} = \frac{a_0 a_2 a_4 \dots a_{2n-2}}{2^n \cdot (1 \cdot 2 \dots n)(7 \cdot 11 \cdot 15 \dots (4n+3))}, \quad n \geq 1.$$

Cancelling common factors,

$$a_{2n} = \frac{a_0}{2^n n! (7 \cdot 11 \cdot 15 \dots (4n+3))}, \quad n \geq 1.$$

If we set $a_0 = 1$ and remember that $s = s_1 = 1/2$, then our first solution is

$$y_1(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n! (7 \cdot 11 \cdot 15 \cdots (4n+3))} \right].$$

With $s_2 = -1$, the recurrence relation becomes

$$a_{2n} = \frac{a_{2n-2}}{2n(4n-3)}, \quad n \geq 1.$$

Thus,

$$a_2 = \frac{a_0}{2 \cdot 1 \cdot 1}, \quad a_4 = \frac{a_2}{2 \cdot 2 \cdot 5}, \quad a_6 = \frac{a_4}{2 \cdot 3 \cdot 9}$$

The product of these n recurrence formulae is

$$a_2 a_4 a_6 \cdots a_{2n} = \frac{a_0 a_2 a_4 \cdots a_{2n-2}}{2^n (1 \cdot 2 \cdot 3 \cdots n)(1 \cdot 5 \cdot 9 \cdots (4n-3))}.$$

Cancelling common factors,

$$a_{2n} = \frac{a_0}{2^n n! (1 \cdot 5 \cdot 9 \cdots (4n-3))}, \quad n \geq 1.$$

If we set $a_0 = 1$ and remember that $s = s_2 = -1$, then our second solution is

$$y_2(x) = x^{-1} \left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n! (1 \cdot 5 \cdot 9 \cdots (4n-3))} \right].$$

25. Multiply by $x/2$,

$$x^2 y'' + \frac{1}{2} x y' + \frac{1}{2} x^3 y = 0,$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$. Note that $p(x) = 1/2$ and $q(x) = x^3/2$, both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, note that $p_0 = p(0) = 1/2$ and $q_0 = q(0) = 0$. Thus, the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - \frac{1}{2}s.$$

It will be easier to work with the equation $2x^2 y'' + xy' + x^3 y = 0$. Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$2x^2 y'' = \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_n x^{s+n}, \quad xy' = \sum_{n=0}^{\infty} (s+n)a_n x^{s+n},$$

and

$$x^3 y = x^3 \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=3}^{\infty} a_{n-3} x^{s+n},$$

where we've shifted the index of summation. Substituting,

$$\begin{aligned}
 0 &= 2x^2y' + xy' + x^3y \\
 &= \sum_{n=0}^{\infty} 2(s+n)(s+n-1)a_nx^{s+n} + \sum_{n=0}^{\infty} (s+n)a_nx^{s+n} + \sum_{n=3}^{\infty} a_{n-3}x^{s+n} \\
 &= [2s(s-1) + s]a_0x^s + [2(s+1)s + (s+1)]a_1x^{s+1} + [2(s+2)(s+1) + (s+2)]a_2x^{s+2} \\
 &\quad + \sum_{n=3}^{\infty} \{[2(s+n)(s+n-1) + (s+n)]a_n + a_{n-3}\}x^{s+n}.
 \end{aligned}$$

Thus, the indicial equation is

$$I(s) = 2s^2 - s = s(2s - 1) = 0,$$

with roots $s_2 = 0$ and $s_1 = 1/2$. The remaining terms provide the recurrence relation

$$(s+n)(2s+2n-1)a_n + a_{n-3} = 0, \quad n \geq 3.$$

With $s_1 = 1/2$, this becomes

$$n(2n+1)a_n + a_{n-3} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-3}}{n(2n+1)}.$$

Thus,

$$a_3 = \frac{-a_0}{3 \cdot 7}, \quad a_6 = \frac{-a_3}{6 \cdot 13}, \quad \dots, \quad a_{3n} = \frac{-a_{3n-3}}{3n(6n+1)}.$$

Multiplying left and right-hand sides,

$$a_3 a_6 \cdots a_{3n} = \frac{(-1)^n a_0 a_3 \cdots a_{3n-3}}{[3 \cdot 6 \cdots 3n][7 \cdot 13 \cdots (6n+1)]}.$$

Cancelling,

$$a_{3n} = \frac{(-1)^n a_0}{3^n n! [7 \cdot 13 \cdots (6n+1)]}.$$

We also need $[2(s+1)s + (s+1)]a_1 = 0$, but with $s = 1/2$, this becomes $3a_1 = 0$ so $a_1 = 0$. Thus, coupled with the recurrence relation means that $a_{3n+1} = 0$ for all n .

Similarly, $[2(s+2)(s+1) + (s+2)]a_2 = 0$ becomes $10a_2 = 0$, so $a_2 = 0$. The recurrence relation then reveals that $a_{3n+2} = 0$ for all n . Thus, with $s_1 = 1/2$ and $a_0 = 1$, one solution is

$$y_1(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{3^n n! [7 \cdot 13 \cdots (6n+1)]} \right].$$

Next, with $s_2 = 0$, the recurrence relation becomes

$$n(2n-1)a_n + a_{n-3} = 0 \quad \text{or} \quad a_n = \frac{-a_{n-3}}{n(2n-1)}, \quad n \geq 3.$$

Thus,

$$a_3 = \frac{-a_0}{3 \cdot 5}, \quad a_6 = \frac{-a_3}{6 \cdot 11}, \quad \dots, \quad a_{3n} = \frac{-a_{3n-3}}{3n(6n-1)}.$$

Multiplying left and right-hand sides,

$$a_3 a_6 \cdots a_{3n} = \frac{(-1)^n a_0 a_3 \cdots a_{3n-3}}{[3 \cdot 6 \cdots 3n][5 \cdot 11 \cdots (6n-1)]}.$$

Cancelling,

$$a_{3n} = \frac{(-1)^n a_0}{3^n n! [5 \cdot 11 \cdots (6n-1)]}.$$

With $s_2 = 0$, $[2(s+1)s + (s+1)]a_1 = 0$, becomes $a_1 = 0$. The recurrence relation then gives $a_{3n+1} = 0$ for all n . With $s_2 = 0$, $[2(s+2)(s+1) + (s+2)]a_2 = 0$ becomes $6a_2 = 0$. Thus, $a_2 = 0$ and the recurrence relation then shows that $a_{3n+2} = 0$ for all n . Hence, with $s_2 = 0$ and $a_0 = 1$, a second independent solution is

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{3^n n! [5 \cdot 11 \cdots (6n-1)]}.$$

26. From Example 5.26 we know that the recurrence formula is

$$(s+n)(s+n-1)a_n = -a_{n-1}.$$

Setting $s = s_1 = 1$, this becomes

$$a_n = -\frac{a_{n-1}}{n(n+1)}.$$

In the usual way we find that $a_n = (-1)^n a_0 / [n!(n+1)!]$, so, setting $a_0 = 1$, the solution is

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^n.$$

27. Compare

$$x^2 y'' + xy' + x^2 y = 0$$

with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 1$ and $q(x) = x^2$, both of which are analytic at $x = 0$. Hence, $x = 0$ is a regular singular point. Further, $p_0 = p(0) = 1$ and $q_0 = q(0) = 0$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2.$$

Let

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n}, \quad xy' = \sum_{n=0}^{\infty} (s+n)a_n x^{s+n},$$

and

$$x^2 y = x^2 \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=2}^{\infty} a_{n-2} x^{s+n},$$

where we've shifted the index of summation. Substituting,

$$\begin{aligned} 0 &= x^2 y'' + x y' + x^2 y \\ &= \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n) a_n x^{s+n} + \sum_{n=2}^{\infty} a_{n-2} x^{s+n} \\ &= [s(s-1) + s] a_0 x^s + [(s+1)s + (s+1)] a_1 x^{s+1} + \sum_{n=2}^{\infty} \{[(s+n)(s+n-1) + (s+n)] a_n + a_{n-2}\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 = 0,$$

with roots $s_2 = s_1 = 0$. We also need $[(s+1)s + (s+1)] a_1 = 0$, but with $s = 0$, this becomes $a_1 = 0$. The remaining terms provide the recurrence relation

$$(s+n)^2 a_n + a_{n-2} = 0, \quad n \geq 2.$$

With $s = 0$, this becomes

$$n^2 a_n + a_{n-2} = 0 \quad \text{or} \quad a_n = -\frac{a_{n-2}}{n^2}, \quad n \geq 2.$$

Thus,

$$a_2 = \frac{-a_0}{2^2}, \quad a_4 = \frac{-a_2}{4^2}, \quad \dots, \quad a_{2n} = \frac{-a_{2n-2}}{(2n)^2}$$

Multiplying left and right-hand sides of these equations,

$$a_2 a_4 \cdots a_{2n} = \frac{(-1)^n a_0 a_2 \cdots a_{2n-2}}{2^2 \cdot 4^2 \cdots (2n)^2}.$$

Cancelling,

$$a_{2n} = \frac{(-1)^n a_0}{(2 \cdot 4 \cdots 2n)^2} = \frac{(-1)^n a_0}{[2^n (1 \cdot 2 \cdots n)]^2} = \frac{(-1)^n a_0}{2^{2n} (n!)^2}.$$

Finally, because $a_1 = 0$, the recurrence relation gives that $a_{2n+1} = 0$ for all n . Thus, with $s_1 = 0$ and $a_0 = 1$, we have one solution,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}.$$

28. Collecting the fundamental expressions, we can write the equation as

$$[x^2 y'' + xy - y] + x^2 y = 0.$$

Using the series for the fundamental expressions in (5.4) and (5.8), this becomes

$$0 = \sum_{n=0}^{\infty} [(n+s)(n+s-1) + (n+s) - 1] a_n x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n+2}.$$

Shifting the index in the second series and isolating the extra terms in the first series, we get

$$0 = (s^2 - 1) a_0 x^s + [(s+1)^2 - 1] a_1 x^{s+1} + \sum_{n=2}^{\infty} \{[(s+n)^2 - 1] a_n + a_{n-2}\} x^{s+n}.$$

Since $a_0 \neq 0$, the first term gives the indicial equation $s^2 - 1 = 0$. Hence the roots are $s_1 = 1$ and $s_2 = -1$.

Setting $s = s_1 = 1$, the second term becomes $3a_1 = 0$, so $a_1 = 0$. The coefficient of the general term gives the recurrence formula

$$[(n+1)^2 - 1]a_n = -a_{n-2} \quad \text{or} \quad n(n+2)a_n = -a_{n-2}$$

Since $a_1 = 0$, we conclude that $a_{2n+1} = 0$ for all n . For even subscripts, the recurrence formula becomes

$$a_{2n} = -\frac{a_{2(n-1)}}{2^2 n(n+1)}.$$

From this we conjecture that

$$a_{2n} = \frac{(-1)^n}{n!(n+1)!} a_0.$$

Since $s_1 = 1$, the Frobenius solution is

$$y(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n+1)!n!} x^{2n}.$$

Next look at $s = s_2 = -1$. The second term in the series is $-a_1 = 0$, so $a_1 = 0$. Setting the coefficient of the third term ($n = 2$) equal to 0, we get $0 \cdot a_2 = -a_0$. Since $a_0 \neq 0$, we cannot solve this equation for a_2 , so there is no second Frobenius solution.

29. Collecting the fundamental expressions, we can write the equation as

$$[x^2 y'' + xy - y/4] + x^2 y = 0.$$

Using the series for the fundamental expressions in (5.4) and (5.8), this becomes

$$0 = \sum_{n=0}^{\infty} [(n+s)(n+s-1) + (n+s) - 1/4] a_n x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n+2}.$$

Shifting the index in the second series and isolating the extra terms in the first series, we get

$$0 = (s^2 - 1/4)a_0 x^s + [(s+1)^2 - 1/4]a_1 x^{s+1} + \sum_{n=2}^{\infty} \{[(s+n)^2 - 1/4]a_n + a_{n-2}\} x^{s+n}.$$

Since $a_0 \neq 0$, the first term gives the indicial equation $s^2 - 1/4 = 0$. Hence the roots are $s_1 = 1/2$ and $s_2 = -1/2$.

Setting $s = s_1 = 1/2$, the second term becomes $2a_1 = 0$, so $a_1 = 0$. The coefficient of the general term gives the recurrence formula

$$[(n+1/2)^2 - 1/4]a_n = -a_{n-2} \quad \text{or} \quad n(n+1)a_n = -a_{n-2}$$

Since $a_1 = 0$, we conclude that $a_{2n+1} = 0$ for all n . For even subscripts, the recurrence formula becomes

$$a_{2n} = -\frac{a_{2(n-1)}}{2n(2n+1)}.$$

From this we conjecture that

$$a_{2n} = \frac{(-1)^n}{(2n+1)!} a_0.$$

Since $s_1 = 1/2$, the Frobenius solution is

$$y(x) = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}.$$

Next look at $s = s_2 = -1/2$. The second term in the series is $0 \cdot a_1 = 0$, so a_1 can be given any value. The coefficient of the general term gives us the recurrence formula $n(n-1)a_n = -a_{n-2}$. Since $n(n-1) \neq 0$ for $n \geq 2$, we can solve for all n . Hence there is a second Frobenius solution.

30. Divide by $(1-x^2)$, obtaining

$$y'' - \frac{x}{1-x^2} y' + \frac{p^2}{1-x^2} y = 0,$$

and compare with $y'' + P(x)y' + Q(x)y = 0$, noting that

$$P(x) = \frac{x}{x^2-1} \quad \text{and} \quad Q(x) = \frac{p^2}{1-x^2}.$$

Note that

$$(x-1)P(x) = \frac{x}{x+1} \quad \text{and} \quad (x-1)^2 Q(x) = -p^2$$

are both analytic near $x = 1$, so $x = 1$ is a regular singular point. Next, note that

$$(x+1)P(x) = \frac{x}{x-1} \quad \text{and} \quad (x+1)^2 Q(x) = \frac{p^2(x+1)^2}{1-x}$$

are both analytic near $x = -1$, so $x = -1$ is also a regular singular point. We first seek a solution about the point $x = 1$. Set $t = x - 1$, then

$$\frac{d}{dt} = \frac{d}{dx} \quad \text{and} \quad x = t + 1.$$

With these translations, $(x^2-1)y'' + xy' - p^2y = 0$ becomes

$$t(t+2)y'' + (t+1)y' - p^2y = 0,$$

where the prime now denotes differentiation with respect to t . Set $y(t) = \sum_{n=0}^{\infty} a_n t^{s+n}$. Then, substituting,

$$\begin{aligned} 0 &= [t^2 y'' + t y' - p^2 y] + [2t y'' + y'] \\ &= \sum_{n=0}^{\infty} [(s+n)(s+n-1) + (s+n) - p^2] a_n t^{s+n} + \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + (s+n)] a_n t^{s+n-1}. \end{aligned}$$

Shifting the index of the *first* summation,

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} [(s+n-1)(s+n-2) + (s+n-1) - p^2] a_{n-1} t^{s+n-1} \\ &\quad + \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + (s+n)] a_n t^{s+n-1}. \end{aligned}$$

Isolate the $n = 0$ term and combine.

$$0 = [2s(s-1) + s]a_0t^{s-1} + \sum_{n=1}^{\infty} \{[(s+n-1)(s+n-2) + (s+n-1) - p^2]a_{n-1} + [2(s+n)(s+n-1) + (s+n)]a_n\}t^{s+n-1}$$

The first term gives the indicial equation,

$$I(s) = 2s(s-1) + s = 2s^2 - s = s(2s-1) = 0,$$

with roots $s_1 = 1/2$ and $s_2 = 0$. The remaining terms provide the recurrence relation

$$I(s+n)a_n = -[(s+n-1)(s+n-2) + (s+n-1) - p^2]a_{n-1},$$

for $n \geq 1$. This can be used with $s_1 = 1/2$ and $s_2 = 0$ (depending on p) to find two independent solutions of the form

$$y(t) = t^s \sum_{n=0}^{\infty} a_n t^n.$$

The translation $t = x - 1$ will provide Frobenius series in terms of $x - 1$.

31. Multiply by x ,

$$x^2 y'' + x(1-x)y' + pxy = 0,$$

then comparison with $x^2 y'' + xp(x)y' + q(x)y = 0$ reveals that $p(x) = 1 - x$ and $q(x) = px$ are both analytic at $x = 0$. Further, $p_0 = p(0) = 1$ and $q_0 = q(0) = 0$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2.$$

Let

$$y = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n},$$

and

$$x(1-x)y' = (1-x) \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} = \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1} x^{s+n},$$

where we've shifted the index of summation. Finally,

$$pxy = px \sum_{n=0}^{\infty} a_n x^{s+n} = \sum_{n=1}^{\infty} p a_{n-1} x^{s+n}.$$

Substituting,

$$\begin{aligned}
 0 &= x^2 y'' + x(1-x)y' + pxy \\
 &= \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} + \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1} x^{s+n} + \sum_{n=1}^{\infty} p a_{n-1} x^{s+n} \\
 &= [s(s-1) + s]a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) + (s+n)]a_n - [(s+n-1) - p]a_{n-1}\} x^{s+n} \\
 &= s^2 a_0 + \sum_{n=1}^{\infty} [(s+n)^2 a_n - (s+n-1-p)a_{n-1}] x^{s+n}.
 \end{aligned}$$

Thus, the indicial equation is $I(s) = s^2$, with equal roots $s_1 = s_2 = 0$. The remaining terms provide the recurrence relation

$$(s+n)^2 a_n - (s+n-1-p)a_{n-1} = 0, \quad n \geq 1.$$

With $s = 0$, this becomes

$$n^2 a_n - (n-1-p)a_{n-1} = 0 \quad \text{or} \quad a_n = \frac{n-1-p}{n^2} a_{n-1}, \quad n \geq 1.$$

Now, if p is a positive integer, the numerator of this recurrence relation will equal zero when

$$n-1-p=0 \quad \text{or} \quad n=p+1.$$

Thus, if $N = p+1$, then $a_n = 0$ for all $n \geq N$. That is, the Frobenius series terminates and becomes a polynomial of degree $N-1$.

32. Suppose that y_1 is a solution of $y'' + p(x)y' + q(x)y = 0$, which we will write in the form

$$y'' + py' + qy = 0,$$

where $p = p(x)$ and $q = q(x)$. Let $y_2 = uy_1$, where u , y_1 , and y_2 are functions of x , with u and y_2 to be determined. Then

$$\begin{aligned}
 y_2' &= uy_1' + u'y_1 \\
 y_2'' &= uy_1'' + 2u'y_1' + u''y_1.
 \end{aligned}$$

Subbing into $y'' + py' + qy = 0$,

$$(uy_1'' + 2u'y_1' + u''y_1) + p(uy_1' + u'y_1) + quy_1 = 0.$$

Rearranging terms,

$$u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0.$$

Since y_1 is a solution of $y'' + py' + qy = 0$, we know that $y_1'' + py_1' + qy_1 = 0$ and

$$u''y_1 + u'(2y_1' + py_1) = 0.$$

The equation is separable.

$$\frac{u''}{u'} = -\frac{2y_1' + py_1}{y_1} = -\frac{2y_1'}{y_1} - p$$

Integrating,

$$\ln u' = -2 \ln y_1 - \int p(x) dx.$$

Exponentiating,

$$u' = e^{\ln y_1^{-2}} e^{-\int p(x) dx} = \frac{e^{-\int p(x) dx}}{y_1^2}.$$

Another integration provides

$$u = \int \frac{e^{-\int p(x) dx}}{[y_1(x)]^2} dx.$$

Finally, $y_2 = uy_1$, so

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x) dx}}{[y_1(x)]^2} dx.$$

33. (a) If we multiply the equation by $x/(1-x)$ we get

$$x^2 y'' + x \frac{\gamma - (1 + \alpha + \beta)x}{1-x} y' + \frac{-\alpha\beta x}{1-x} y = 0.$$

This equation is in the form of (6.1), with $p(x) = [\gamma - (1 + \alpha + \beta)x]/(1-x)$ and $q(x) = -\alpha\beta x/(1-x)$. Both of these are analytic except at $x = 1$, so $x = 0$ is a regular singular point. The indicial equation is

$$s^2 + [p(0) - 1]s + q(0) = s^2 + [\gamma - 1]s = 0.$$

This equation has roots 0 and $1 = \gamma$.

- (b) If we multiply the equation by $(1-x)/x$ we get

$$(x-1)^2 y'' - \frac{\gamma - (1 + \alpha + \beta)x}{x} (x-1)y' - \alpha\beta(1-x)y = 0.$$

This equation has the form of (6.1), with x replaced by $x-1$, and coefficients $p(x) = -[\gamma - (1 + \alpha + \beta)x]/x$, and $q(x) = -\alpha\beta(1-x)x$. Both are analytic except at $x = 0$, so $x = 1$ is a regular singular point. The indicial polynomial is

$$s^2 + [p(1) - 1]s + q(1) = s^2 + [\alpha + \beta - \gamma]s = 0.$$

This equation has roots 0 and $\gamma - \alpha - \beta$.

- (c) Multiplying the equation by x and separating the powers of x times fundamental expressions, we get

$$\begin{aligned} 0 &= x^2(1-x)y'' + x[\gamma - (1 + \alpha + \beta)x]y' - \alpha\beta xy \\ &= [x^2 y'' + \gamma xy'] - x[x^2 y'' + (1 + \alpha + \beta)xy' + \alpha\beta y]. \end{aligned}$$

Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^{s+n}$, and similar formulas for the derivatives of y and using a little algebra, this becomes

$$\sum_{n=0}^{\infty} (n+s)(n+s+\gamma-1)a_n x^{s+n} - x \sum_{n=0}^{\infty} (n+s+\alpha)(n+s+\beta)a_n x^{s+n} = 0.$$

Shifting the index in the second sum, and isolating the first term we get

$$0 = s(s + \gamma - 1)a_0x^s + \sum_{n=1}^{\infty} [(n+s)(n+s+\gamma-1)a_n - (n+s+\alpha-1)(n+s+\beta-1)a_{n-1}]x^{s+n}.$$

The first term gives us the indicial equation and its roots $s = 0$ and $s = 1 - \gamma$. If $1 - \gamma$ is not a positive integer, then Theorem ?? says that we get a solution corresponding to $s = 0$. The general term of the series gives us the recurrence formula

$$a_n = \frac{(n+s+\alpha-1)(n+s+\beta-1)}{(n+s)(n+s+\gamma-1)}a_{n-1} \quad \text{for } n \geq 1.$$

For $s = 0$ this becomes

$$a_n = \frac{(n+\alpha-1)(n+\beta-1)}{n(n+\gamma-1)}a_{n-1} \quad \text{for } n \geq 1.$$

For $n = 1$, this becomes $a_1 = \alpha \cdot \beta / a \cdot \gamma$, and for $n = 2$,

$$a_2 = \frac{(\alpha+1)(\beta+1)}{2(\gamma+1)}a_1 = \frac{\alpha \cdot (\alpha+1) \cdot \beta \cdot (\beta+1)}{2! \cdot \gamma \cdot (\gamma+1)}.$$

- (d) If $1 - \gamma$ is not an integer, then the two roots, 0 and $1 - \gamma$ do not differ by an integer, so, by Theorem ??, we get a Frobenius solution for each roots. If we set $s = 1 - \gamma$, the recurrence formula becomes

$$a_n = \frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{(n+1-\gamma)n}a_{n-1} \quad \text{for } n \geq 1.$$

For $n = 1$ we get

$$a_1 = \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{(2-\gamma) \cdot 1}a_0,$$

and for $n = 2$,

$$a_2 = \frac{(2+\alpha-\gamma)(2+\beta-\gamma)}{(3-\gamma) \cdot 2}a_1 = \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{(2-\gamma)(3-\gamma) \cdot 2!}a_0$$

————— × —————

Section 11.6. Series Solutions Near Regular Singular Points — The General Case

1. Compare $x^2y'' + 3xy' + (1 - 2x)y = 0$ with $x^2y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 3$ and $q(x) = 1 - 2x$ are both analytic at $x = 0$, making this point a regular singular point. Further, $p_0 = p(0) = 3$ and $q_0 = q(0) = 1$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 + 2s + 1 = (s + 1)^2.$$

Thus, we seek a solution having form

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n.$$

Hence,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n(s) x^{s+n}, \quad 3x y' = \sum_{n=0}^{\infty} 3(s+n) a_n(s) x^{s+n},$$

and

$$(1-2x)y = (1-2x) \sum_{n=0}^{\infty} a_n(s) x^{s+n} = \sum_{n=0}^{\infty} a_n(s) x^{s+n} - \sum_{n=1}^{\infty} 2a_{n-1}(s) x^{s+n},$$

where we've shifted the index of the second summation. Substituting, we get

$$0 = x^2 y'' + 3x y' + (1-2x)y = (s+1)^2 a_0 x^s + \sum_{n=1}^{\infty} [(s+n+1)^2 a_n(s) - 2a_{n-1}(s)] x^{s+n}.$$

Thus, the indicial equation is $I(s) = (s+1)^2 = 0$ with equal roots $s_2 = s_1 = -1$. The remaining terms provide the recurrence relation

$$a_n(s) = \frac{2a_{n-1}(s)}{(s+n+1)^2}, \quad n \geq 1.$$

Since the roots of the indicial equation are equal, we solve the recurrence relation for $a_n(s)$ as a function of s .

$$a_1(s) = \frac{2a_0}{(s+2)^2}, \quad a_2(s) = \frac{2a_1(s)}{(s+3)^2}, \quad \dots, \quad a_n(s) = \frac{2a_{n-1}(s)}{(s+n+1)^2}.$$

Multiplying left and right-hand sides of these equations,

$$a_1(s) a_2(s) \cdots a_n(s) = \frac{2^n}{(s+2)^2 (s+2)^2 \cdots (s+n+1)^2} a_0 a_1(s) \cdots a_{n-1}(s).$$

Cancelling,

$$a_n(s) = \frac{2^n}{(s+2)^2 (s+3)^2 \cdots (s+n+1)^2} a_0.$$

Evaluating at $s = -1$, we get

$$a_n(-1) = \frac{2^n}{1^2 \cdot 2^2 \cdots n^2} a_0.$$

With $a_0 = 1$, this becomes

$$a_n(-1) = \frac{2^n}{(n!)^2},$$

giving us the solution

$$y_1(x) = y(-1, x) = x^{-1} \sum_{n=0}^{\infty} a_n(-1) x^n = \sum_{n=0}^{\infty} \frac{2^n}{(n!)^2} x^{n-1}.$$

To find a second solution, we use logarithmic differentiation and write

$$\ln |a_n(s)| = n \ln 2 + \ln |a_0| - 2 \sum_{k=2}^{n+1} \ln(s+k).$$

Thus,

$$\frac{1}{a_n(s)} a'_n(s) = -2 \sum_{k=2}^{n+1} \frac{1}{s+k}.$$

Evaluating at $s = -1$,

$$a'_n(-1) = -2a_n(-1) \sum_{k=2}^{n+1} \frac{1}{k-1} = -\frac{2^{n+1}}{(n!)^2} \sum_{k=1}^n \frac{1}{k},$$

where we've shifted the index of summation. Hence, our second fundamental solution is

$$y_2(x) = y_1(x) \ln x + x^{-1} \sum_{n=1}^{\infty} a'_n(-1)x^n = y_1(x) \ln x - \sum_{n=1}^{\infty} \frac{2^{n+1}H(n)}{(n!)^2} x^{n-1},$$

where we've let $H(n) = \sum_{k=1}^n 1/k$.

2. Compare $x^2y'' - x(1+x)y' + y = 0$ with $x^2y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = -(1+x)$ and $q(x) = 1$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = -1$ and $q_0 = q(0) = 1$, so the indicial polynomial should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 2s + 1 = (s - 1)^2.$$

Let

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s)x^n = \sum_{n=0}^{\infty} a_n(s)x^{s+n}.$$

Then

$$x^2y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n(s)x^{s+n},$$

and

$$-x(1+x)y' = -(1+x) \sum_{n=0}^{\infty} (s+n)a_n(s)x^{s+n} = -\sum_{n=0}^{\infty} (s+n)a_n(s)x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1}(s)x^{s+n},$$

where we've shifted the index of the second summation. Substituting, we get

$$0 = x^2y'' - x(1+x)y' + y = (s-1)^2a_0x^s + \sum_{n=1}^{\infty} [(s+n-1)^2a_n(s) - (s+n-1)a_{n-1}(s)]x^{s+n}.$$

Therefore, the indicial equation is $I(s) = (s-1)^2 = 0$, with equal roots $s_2 = s_1 = 1$. The remaining terms provide the recurrence relation

$$(s+n-1)^2a_n(s) - (s+n-1)a_{n-1}(s) = 0, \quad n \geq 1.$$

Since the roots of the indicial equation are equal, we solve the recurrence relation for $a_n(s)$ as a function of s .

$$a_1(s) = \frac{a_0}{s}, \quad a_2(s) = \frac{a_1(s)}{s+1}, \quad \dots, \quad a_n(s) = \frac{a_{n-1}(s)}{s+n-1}.$$

Multiplying left and right-hand sides of these equations, we have

$$a_1(s) a_2(s) \cdots a_n(s) = \frac{a_0 a_1(s) \cdots a_{n-1}(s)(s)}{s(s+1) \cdots (s+n-1)}.$$

Cancelling,

$$a_n(s) = \frac{a_0}{s(s+1) \cdots (s+n-1)}.$$

Evaluating at $s = 1$,

$$a_n(1) = \frac{a_0}{n!}.$$

Choosing $a_0 = 1$, our first solution is

$$y_1(x) = y(1, x) = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = x e^x.$$

To find a second solution, we must differentiate $a_n(s)$. Using logarithmic differentiation,

$$\ln |a_n(s)| = \ln |a_0| - \sum_{k=1}^n \ln |s+k-1|.$$

Thus,

$$\frac{1}{a_n(s)} a'_n(s) = - \sum_{k=1}^n \frac{1}{s+k-1}.$$

Evaluating at $s = 1$,

$$a'_n(1) = -a_n(1) \sum_{k=1}^n \frac{1}{k} = -\frac{a_0}{n!} \sum_{k=1}^n \frac{1}{k}.$$

We set $a_0 = 1$ and let $H(n) = \sum_{k=1}^n (1/k)$. Then,

$$a'_n(1) = -\frac{1}{n!} H(n).$$

Thus,

$$y_2(x) = y_1(x) \ln x + x \sum_{n=1}^{\infty} a'_n(1) x^n = x e^x \ln x - x \sum_{n=1}^{\infty} \frac{H(n) x^n}{n!}.$$

3. Multiply by x .

$$x^2 y'' + x(1-x)y' - xy = 0,$$

then comparison with $x^2 y'' + xp(x)y' + q(x)y = 0$ reveals that $p(x) = 1-x$ and $q(x) = -x$ are both analytic at $x = 0$. Thus, $x = 0$ is a regular singular point. Note that $p_0 = p(0) = 1$ and $q_0 = q(0) = 0$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2.$$

Thus, we seek a solution of the form

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}.$$

Thus,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n(s)x^{s+n},$$

and

$$x(1-x)y' = (1-x) \sum_{n=0}^{\infty} (s+n)a_n(s)x^{s+n} = \sum_{n=0}^{\infty} (s+n)a_n(s)x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1}(s)x^{s+n}$$

where we've shifted the index of the second summation. Finally,

$$-xy = -x \sum_{n=0}^{\infty} a_n(s)x^{s+n} = - \sum_{n=1}^{\infty} a_{n-1}(s)x^{s+n},$$

where again we have shifted the index of summation. Substituting, we get

$$0 = x^2 y'' + x(1-x)y' - xy = s^2 a_0 x^n + \sum_{n=1}^{\infty} [(s+n)^2 a_n(s) - (s+n)a_{n-1}(s)] x^{s+n}.$$

Thus, the indicial equation is $I(s) = s^2 = 0$ with equal roots $s_2 = s_1 = 0$. The remaining terms provide the recurrence relation

$$(s+n)^2 a_n(s) = (s+n)a_{n-1}(s), \quad n \geq 1.$$

Since the roots of the indicial equation are equal, we solve the recurrence relation for $a_n(s)$ as a function of s . Thus,

$$a_1(s) = \frac{a_0}{s+1}, \quad a_2(s) = \frac{a_1(s)}{s+2}, \quad \dots, \quad a_n(s) = \frac{a_{n-1}(s)}{s+n}.$$

Multiplying left and right-hand sides of these equations, we have

$$a_1(s)a_2(s) \cdots a_n(s) = \frac{1}{(s+1)(s+2) \cdots (s+n)} a_0 a_1(s) \cdots a_{n-1}(s).$$

Cancelling,

$$a_n(s) = \frac{1}{(s+1)(s+2) \cdots (s+n)} a_0.$$

Evaluating at $s = 0$, we get

$$a_n(0) = \frac{1}{n!} a_0.$$

Choosing $a_0 = 1$, our first solution is

$$y_1(x) = y(0, x) = x^0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

To find our second solution, we must differentiate $a_n(s)$. We'll use logarithmic differentiation and write

$$\ln |a_n(s)| = \ln |a_0| - \sum_{k=1}^n \ln |s+k|.$$

Thus,

$$\frac{1}{a_n(s)} a'_n(s) = - \sum_{k=1}^n \frac{1}{s+k}.$$

Evaluating at $s = 0$,

$$a'_n(0) = -a_n(0) \sum_{k=1}^n \frac{1}{k} = -\frac{a_0}{n!} \sum_{k=1}^n \frac{1}{k}.$$

We set $a_0 = 1$ and let $H(n) = \sum_{k=1}^n (1/k)$. Then,

$$a'_n(0) = -\frac{1}{n!} H(n).$$

Thus,

$$y_2(x) = y_1(x) \ln x + x^0 \sum_{n=0}^{\infty} a'_n(0) x^n = e^x \ln x - \sum_{n=1}^{\infty} \frac{H(n)x^n}{n!}$$

4. Compare

$$x^2 y'' + x(x-3)y' + 4y = 0$$

with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = x-3$ and $q(x) = 4$ are both analytic at zero, making $x = 0$ a regular singular point. Note that $p_0 = p(0) = -3$ and $q_0 = q(0) = 4$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 4s + 4 = (s-2)^2.$$

Thus, we seek a solution having form

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}.$$

Thus,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n(s) x^{s+n},$$

and

$$x(x-3)y' = (x-3) \sum_{n=0}^{\infty} (s+n) a_n(s) x^{s+n} = \sum_{n=1}^{\infty} (s+n-1) a_{n-1}(s) x^{s+n} - \sum_{n=0}^{\infty} 3(s+n) a_n(s) x^{s+n},$$

where we've shifted the index of the first summation. Finally,

$$4y = \sum_{n=0}^{\infty} 4a_n(s) x^{s+n}.$$

Substituting, we get

$$\begin{aligned}
 0 &= x^2 y'' + x(x-3)y' + 4y \\
 &= \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n(s)x^{s+n} + \sum_{n=1}^{\infty} (s+n-1)a_{n-1}(s)x^{s+n} \\
 &\quad - \sum_{n=0}^{\infty} 3(s+n)a_n(s)x^{s+n} + \sum_{n=0}^{\infty} 4a_n(s)x^{s+n} \\
 &= [s(s-1) - 3s + 4]a_0x^s + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) - 3(s+n) + 4]a_n(s) + (s+n-1)a_{n-1}(s)\}x^{s+n} \\
 &= (s-2)^2a_0x^s + \sum_{n=1}^{\infty} [(s+n-2)^2a_n(s) + (s+n-1)a_{n-1}(s)]x^{s+n}.
 \end{aligned}$$

Thus, the indicial equation is indeed $I(s) = (s-2)^2 = 0$ with equal roots $s_2 = s_1 = 2$. The remaining terms provide the recurrence relation

$$(s+n-2)^2a_n(s) + (s+n-1)a_{n-1}(s) = 0, \quad n \geq 1.$$

Since the roots of the indicial equation are equal, we solve the recurrence relation for $a_n(s)$ as a function of s . Thus

$$a_1(s) = -\frac{s}{(s-1)^2}a_0, \quad a_2(s) = -\frac{s+1}{s^2}a_1(s), \quad \dots, \quad a_n(s) = -\frac{s+n-1}{(s+n-2)^2}a_{n-1}(s).$$

Multiplying left and right-hand sides of the equations, we get

$$a_1(s)a_2(s) \cdots a_n(s) = (-1)^n \frac{s(s+1) \cdots (s+n-1)}{(s-1)^2s^2 \cdots (s+n-2)^2} a_0a_1(s) \cdots a_{n-1}(s).$$

Cancelling, we get

$$a_n(s) = \frac{(-1)^n(s+n-1)}{(s-1)^2s(s+1) \cdots (s+n-2)} a_0.$$

Evaluating at $s = 2$, we get

$$a_n(2) = \frac{(-1)^n(n+1)}{1 \cdot 2 \cdot 3 \cdots n} a_0 = \frac{(-1)^n(n+1)}{n!} a_0.$$

Choosing $a_0 = 1$, our first solution is

$$y_1(x) = y(2, x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{n!} x^n.$$

Next, we use logarithmic differentiation to find $a'_n(s)$.

$$\ln |a_n(s)| = \ln |a_0| + \ln |s+n-1| - 2 \ln |s-1| - \sum_{k=0}^{n-2} \ln |s+k|$$

Thus,

$$\begin{aligned}\frac{1}{a_n(s)} a'_n(s) &= \frac{1}{s+n-1} - \frac{2}{s-1} - \sum_{k=0}^{n-2} \frac{1}{s+k} \\ a'_n(s) &= a_n(s) \left[\frac{1}{s+n-1} - \frac{2}{s-1} - \sum_{k=0}^{n-2} \frac{1}{s+k} \right],\end{aligned}$$

Substitute $s = 2$. Then

$$\begin{aligned}a'_n(2) &= a_n(2) \left[\frac{1}{n+1} - 2 - \sum_{k=0}^{n-2} \frac{1}{k+2} \right] \\ &= \frac{(-1)^n(n+1)}{n!} \left[\frac{1}{n+1} - 2 - \sum_{k=2}^n \frac{1}{k} \right] \\ &= \frac{(-1)^n(n+1)}{n!} \left[\frac{1}{n+1} - 1 - \sum_{k=1}^n \frac{1}{k} \right] \\ &= \frac{(-1)^n}{n!} \left[1 - (n+1) - (n+1) \sum_{k=1}^n \frac{1}{k} \right] \\ &= \frac{(-1)^{n+1}}{n!} [n + (n+1)H(n)],\end{aligned}$$

where we've let $H(n) = \sum_{k=1}^n (1/k)$. Thus,

$$y_2(x) = y_1(x) \ln x + x^2 \sum_{n=1}^{\infty} a'_n(2)x^n = y_1(x) \ln x + x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} [n + (n+1)H(n)] x^n.$$

5. Multiply by x ,

$$x^2 y'' - x(4+x)y' + 2xy = 0,$$

then compare this with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = -4 - x$ and $q(x) = 2x$ are both analytic at $x = 0$, making this a regular singular point. Further, $p_0 = p(0) = -4$ and $q_0 = q(0) = 0$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 5s.$$

We seek a solution having form

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s)x^n.$$

Thus,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n(s)x^{s+n},$$

and

$$-x(4+x)y' = -(4+x) \sum_{n=0}^{\infty} (s+n)a_n(s)x^{s+n} = -\sum_{n=0}^{\infty} 4(s+n)a_n(s)x^{s+n} - \sum_{n=1}^{\infty} (s+n-1)a_{n-1}(s)x^{s+n},$$

where we've shifted the index of the second summation. Finally,

$$2xy = 2x \sum_{n=0}^{\infty} a_n(s) s^{s+n} = \sum_{n=1}^{\infty} 2a_{n-1}(s) x^{s+n},$$

where we've shifted the index of summation. Substituting, we get

$$\begin{aligned} 0 &= x^2 y'' - x(4+x)y' + 2xy \\ &= (s^2 - 5s)a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)^2 - 5(s+n)]a_n(s) - (s+n-3)a_{n-1}(s)\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is $I(s) = s(s-5)$ with roots $s_1 = 5$ and $s_2 = 0$. The remaining terms provide the recurrence relation

$$(s+n)(s+n-5)a_n(s) = (s+n-3)a_{n-1}(s), \quad n \geq 1.$$

Because the roots differ by a positive integer, we use the smaller root, $s_2 = 0$, first. We get

$$n(n-5)a_n(0) = (n-3)a_{n-1}(0), \quad n \geq 1.$$

Hence

$$\begin{aligned} n=1: \quad & -4a_1(0) = -2a_0 \Rightarrow a_1(0) = \frac{1}{2}a_0 \\ n=2: \quad & -6a_2(0) = -1a_1(0) \Rightarrow a_2(0) = \frac{1}{6}a_1(0) = \frac{1}{12}a_0 \\ n=3: \quad & -6a_3(0) = 0 \cdot a_2(0) \Rightarrow a_3(0) = 0 \\ n=4: \quad & -4a_4(0) = a_3(0) \Rightarrow a_4(0) = 0. \end{aligned}$$

Finally, if $n = 5$, then

$$0a_5(0) = 2a_4(0) = 0,$$

so $a_5(0) = a_5$ is arbitrary, and

$$a_n(0) = \frac{n-3}{n(n-5)} a_{n-1}(0), \quad n \geq 6.$$

Next,

$$a_6(0) = \frac{3}{6 \cdot 1} a_5, \quad a_7(0) = \frac{4}{7 \cdot 2} a_6(0), \quad \dots, \quad a_n(0) = \frac{n-3}{n(n-5)} a_{n-1}(0).$$

Multiplying left and right-hand sides of the preceding equations,

$$a_6(0) a_7(0) \cdots a_n(0) = \frac{3 \cdot 4 \cdots (n-3)}{[6 \cdot 7 \cdots n][1 \cdot 2 \cdots (n-5)]} a_5 a_6(0) \cdots a_{n-1}(0).$$

Cancelling, for $n \geq 6$,

$$a_n(0) = \frac{3 \cdot 4 \cdots (n-3)}{[6 \cdot 7 \cdots n](n-5)!} a_5 = \frac{3 \cdot 4 \cdot 5 a_5}{(n-2)(n-1)n(n-5)!} = \frac{60a_5}{(n-2)(n-1)n(n-5)!}$$

Hence,

$$y = a_0 \left(1 + \frac{1}{2}x + \frac{1}{12}x^2 \right) + a_5 \left[x^5 + \sum_{n=6}^{\infty} \frac{60x^n}{(n-2)(n-1)n(n-5)!} \right].$$

By letting $a_0 = 1$ and $a_5 = 0$, then $a_0 = 0$ and $a_5 = 1$, we obtain a fundamental set of solutions

$$y_1(x) = 1 + \frac{1}{2}x + \frac{1}{12}x^2, \quad \text{and} \quad y_2(x) = x^5 + \sum_{n=6}^{\infty} \frac{60x^n}{(n-2)(n-1)n(n-5)!}.$$

6. Compare $x^2y'' + x^2y' - 2y = 0$ with $x^2y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = x$ and $q(x) = -2$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 0$ and $q_0 = q(0) = -2$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - s - 2 = (s - 2)(s + 1).$$

Let

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s)x^n = \sum_{n=0}^{\infty} a_n(s)x^{s+n}.$$

Then,

$$x^2y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n(s)x^{s+n},$$

and

$$x^2y' = x \sum_{n=0}^{\infty} (s+n)a_n(s)x^{s+n} = \sum_{n=1}^{\infty} (s+n-1)a_{n-1}(s)x^{s+n},$$

where we've shifted the index of summation. Substituting,

$$\begin{aligned} 0 &= x^2y'' + x^2y' - 2y \\ &= (s^2 - s - 2)a_0x^s + \sum_{n=1}^{\infty} \{[(s+n)^2 - (s+n) - 2]a_n(s) + (s+n-1)a_{n-1}(s)\}x^{s+n}. \end{aligned}$$

Thus, the indicial equation is $I(s) = (s - 2)(s + 1) = 0$, with roots $s_2 = -1$ and $s_1 = 2$. The remaining terms provide the recurrence relation

$$(s + n - 2)(s + n + 1)a_n(s) + (s + n - 1)a_{n-1}(s) = 0, \quad n \geq 1.$$

Using the smaller root, $s_2 = -1$, this becomes

$$n(n - 3)a_n(-1) = -(n - 2)a_{n-1}(-1), \quad n \geq 1.$$

With $n = 1$

$$(1)(-2)a_1(-1) = 1a_0 \quad \text{or} \quad a_1(-1) = -\frac{1}{2}a_0.$$

With $n = 2$,

$$(2)(-1)a_2(-1) = 0a_1(-1) \quad \text{or} \quad a_2(-1) = 0.$$

With $n = 3$,

$$3(0)a_3(-1) = -1a_2(-1) \quad \text{or} \quad 0a_3(-1) = 0.$$

Thus, we are free to choose any value for $a_3(-1) = a_3$. Then, with $n \geq 4$,

$$a_4(-1) = \frac{-2a_3}{4 \cdot 1}, \quad a_5 = \frac{-3a_4(-1)}{5 \cdot 2}, \quad \dots, \quad a_n(-1) = \frac{-(n-2)a_{n-1}(-1)}{n(n-3)}.$$

Multiplying left and right-hand sides of these equations, we get

$$a_4(-1)a_5(-1)\cdots a_n(-1) = \frac{(-1)^{n-3}(2 \cdot 3 \cdot \dots \cdot (n-2))}{(4 \cdot 5 \cdot \dots \cdot n)(1 \cdot 2 \cdot \dots \cdot (n-3))} a_3 a_4(-1) \cdots a_{n-1}(-1).$$

Cancelling,

$$a_n(-1) = \frac{(-1)^{n-3}(n-2)}{4 \cdot 5 \cdot \dots \cdot n} a_3 = \frac{(-1)^{n-3}6(n-2)}{n!} a_3, \quad n \geq 4.$$

Thus,

$$y(x) = a_0 x^{-1} \left[1 - \frac{1}{2}x \right] + a_3 x^{-1} \left[1 + \sum_{n=4}^{\infty} \frac{(-1)^{n-3}6(n-2)}{n!} x^n \right].$$

By choosing $a_0 = 1$ and $a_3 = 0$, then $a_0 = 0$ and $a_3 = 1$, you get two independent solutions.

7. Multiply by x ,

$$x^2 y'' - x(3+x)y' + 2xy = 0,$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = -(3+x)$ and $q(x) = 2x$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = -3$ and $q_0 = q(0) = 0$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 4s = s(s - 4).$$

Let

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s)x^n = \sum_{n=0}^{\infty} a_n(s)x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2 y'' - x(3+x)y' + 2xy \\ &= (s^2 - 4s)a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)^2 - 4(s+n)]a_n(s) - (s+n-3)a_{n-1}(s)\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is $I(s) = s(s - 4) = 0$ with roots $s_2 = 0$ and $s_1 = 4$. The remaining terms provide the recurrence relation

$$(s+n)(s+n-4)a_n(s) - (s+n-3)a_{n-1}(s) = 0, \quad n \geq 1.$$

Starting with the smaller root, $s_2 = 0$, the recurrence relation becomes

$$n(n-4)a_n(0) = (n-3)a_{n-1}(0), \quad n \geq 1.$$

For $n = 1, 2$, and 3 , this becomes

$$a_1(0) = \frac{-2}{1 \cdot (-3)} a_0 = \frac{2}{3} a_0, \quad a_2(0) = \frac{-1}{2(-2)} a_1(0) = \frac{1}{6} a_0, \quad \text{and} \quad a_3(0) = 0, a_4(0) = 0.$$

For $n = 4$, the recurrence relation becomes

$$4 \cdot 0 \cdot a_4(0) - 1a_3(0) = 0 \quad \text{or} \quad 0a_4(0) = a_3(0) = 0.$$

Thus, we are free to choose $a_4(0) = a_4$, after which

$$a_n(0) = \frac{n-3}{n(n-4)} a_{n-1}(0), \quad n \geq 5.$$

Thus,

$$a_5(0) = \frac{2}{5 \cdot 1} a_4(0), \quad a_6(0) = \frac{3}{6 \cdot 2} a_5(0), \quad \dots, \quad a_n(0) = \frac{n-3}{n(n-4)} a_{n-1}(0).$$

Multiplying left and right-hand sides of these equations, we get

$$a_5(0) a_6(0) \cdots a_n(0) = \frac{2 \cdot 3 \cdots (n-3) \cdot a_4(0) a_5(0) \cdots a_{n-1}(0)}{(5 \cdot 6 \cdots n)(1 \cdot 2 \cdots (n-4))}.$$

Cancelling,

$$a_n(0) = \frac{(n-3)a_4}{5 \cdot 6 \cdots n} = \frac{24(n-3)a_4}{n!}, \quad n \geq 5.$$

Thus,

$$y(x) = a_0 \left[1 + \frac{2}{3}x + \frac{1}{6}x^2 \right] + a_4 \left[x^4 + \sum_{n=5}^{\infty} \frac{24(n-3)}{n!} x^n \right].$$

By choosing $a_0 = 1$ and $a_4 = 0$, then $a_0 = 0$ and $a_4 = 1$, you get two independent solutions.

8. Compare $x^2 y'' + x(6+x)y' + xy = 0$ with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 6+x$ and $q(x) = x$, both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 6$ and $q_0 = q(0) = 0$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 + 5s = s(s + 5).$$

Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}.$$

Then,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n(s) x^{s+n},$$

and

$$x(6+x)y' = (6+x) \sum_{n=0}^{\infty} (s+n) a_n(s) x^{s+n} = \sum_{n=0}^{\infty} 6(s+n) a_n(s) x^{s+n} + \sum_{n=1}^{\infty} (s+n-1) a_{n-1}(s) x^{s+n},$$

where we've shifted the index of the second summation. Finally,

$$xy = x \sum_{n=0}^{\infty} a_n(s) x^{s+n} = \sum_{n=1}^{\infty} a_{n-1}(s) x^{s+n},$$

where again we've shifted the index of the last summation. Substituting,

$$\begin{aligned} 0 &= x^2 y'' + x(6+x)y' + xy \\ &= [s(s-1) + 6s] a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) + 6(s+n)] a_n(s) + [(s+n-1) + 1] a_{n-1}(s)\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 + 5s = s(s + 5) = 0,$$

with roots $s_2 = -5$ and $s_1 = 0$. The remaining terms provide the recurrence relation

$$(s + n)(s + n + 5)a_n(s) + (s + n)a_{n-1}(s) = 0, \quad n \geq 1.$$

Working with the smaller root, $s = s_2 = -5$, this becomes

$$n(n - 5)a_n(-5) = -(n - 5)a_{n-1}(-5), \quad n \geq 1.$$

However, for $N = s_1 - s_2 = 0 - (-5) = 5$, we have

$$0a_5(-5) = 0,$$

so this is a false degeneracy. Providing $n \neq 5$, we have

$$a_n(-5) = \frac{-a_{n-1}(-5)}{n},$$

so

$$a_1(-5) = -a_0, \quad a_2(-5) = \frac{-a_1(-5)}{2} = \frac{a_0}{2}, \quad a_3(-5) = \frac{-a_2(-5)}{3} = -\frac{a_0}{6}, \quad a_4(-5) = \frac{-a_3(-5)}{4} = \frac{a_0}{24}.$$

Letting $a_5(-5) = a_5$ be an arbitrary constant,

$$a_6(-5) = \frac{-a_5}{6}, \quad a_7(-5) = \frac{-a_6(-5)}{7}, \quad \dots,$$

etc., and

$$a_6(-5) a_7(-5) \cdots a_n(-5) = \frac{(-1)^{n-5} a_5(-5) a_6(-5) \cdots a_{n-1}(-5)}{6 \cdot 7 \cdots n}.$$

Cancelling,

$$a_n(-5) = \frac{(-1)^{n-5} a_5 5!}{n!} = \frac{(-1)^{n-5} 120 a_5}{n!},$$

for $n \geq 6$. Thus,

$$y(x) = a_0 x^{-5} \left[1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{24} x^4 \right] + a_5 x^{-5} \left[x^5 + 120 \sum_{n=6}^{\infty} \frac{(-1)^{n-5} x^n}{n!} \right]$$

Choosing $a_0 = 1$ and $a_5 = 0$ gives one solution.

$$y_1(x) = x^{-5} \left[1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{24} x^4 \right]$$

Choosing $a_0 = 0$ and $a_5 = 1$ gives a second independent solution.

$$y_2(x) = 1 + 120 \sum_{n=6}^{\infty} \frac{(-1)^{n-5} x^{n-5}}{n!} = 1 + 120 \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n+5)!},$$

where we've shifted the index of the second summation.

9. Multiply by x , $x^2 y'' + xy = 0$, then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 0$ and $q(x) = x$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 0$ and $q_0 = q(0) = 0$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - s = s(s - 1) = 0.$$

Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}.$$

Substituting,

$$0 = x^2 y'' + xy = s(s-1)a_0 x^s + \sum_{n=1}^{\infty} [(s+n)(s+n-1)a_n(s) + a_{n-1}(s)] x^{s+n}.$$

Thus, the indicial equation is

$$I(s) = s(s-1) = 0,$$

with roots $s_2 = 0$ and $s_1 = 1$. The remaining terms provide the recurrence relation

$$(s+n)(s+n-1)a_n(s) + a_{n-1}(s) = 0, \quad n \geq 1.$$

Working with the smaller root, $s = s_2 = 0$, this becomes

$$n(n-1)a_n(0) = -a_{n-1}(0), \quad n \geq 1.$$

However, for $N = s_1 - s_2 = 1 - 0 = 1$, this becomes

$$0a_1 = -a_0 \neq 0,$$

so we have a real degeneracy. Solve the recurrence relation for $a_n(s)$ as a function of s .

$$a_n(s) = \frac{-a_{n-1}(s)}{(s+n)(s+n-1)}, \quad n \geq 1.$$

Thus,

$$a_1(s) = \frac{-a_0}{(s+1)s}, \quad a_2(s) = \frac{-a_1(s)}{(s+2)(s+1)},$$

etc., and

$$a_1(s) a_2(s) \cdots a_n(s) = \frac{(-1)^n a_0 a_1(s) \cdots a_{n-1}(s)}{[(s+1)(s+2) \cdots (s+n)][s(s+1) \cdots (s+n-1)]}.$$

Cancelling,

$$a_n(s) = \frac{(-1)^n a_0}{[(s+1)(s+2) \cdots (s+n)][s(s+1) \cdots (s+n-1)]}, \quad n \geq 1.$$

Because the smaller root is $s = s_2 = 0$, define

$$Y(s, x) = sy(s, x) = x^s \sum_{n=0}^{\infty} sa_n(s)x^n = x^s \sum_{n=0}^{\infty} b_n(s)x^n,$$

where $b_n(s) = sa_n(s)$. For example, $b_0(s) = sa_0$, $b_1(s) = sa_1(s) = -a_0/(s+1)$, and

$$b_n(s) = \frac{(-1)^n a_0}{[(s+1)(s+2) \cdots (s+n)][(s+1)(s+2) \cdots (s+n-1)]}, \quad n \geq 2.$$

Evaluating at $s = s_2 = 0$, $b_0(0) = 0$, $b_1(0) = -a_0$, and

$$b_n(0) = \frac{(-1)^n a_0}{(1 \cdot 2 \cdots n)(1 \cdot 2 \cdots (n-1))} = \frac{(-1)^n a_0}{n!(n-1)!},$$

for $n \geq 2$. Thus, remembering that

$$Y(s, x) = x^s \sum_{n=0}^{\infty} b_n(s) x^n,$$

and choosing $a_0 = 1$ with $s = s_2 = 0$,

$$y_1(x) = Y(0, x) = x^0 \sum_{n=0}^{\infty} b_n(0) x^n = -x + \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n!(n-1)!}.$$

We can absorb the first term into the summation and write

$$y_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(n-1)!}.$$

Take the derivative of

$$Y(s, x) = x^s \sum_{n=0}^{\infty} b_n(s) x^n$$

with respect to s .

$$\frac{\partial Y}{\partial s}(s, x) = x^s \ln x \sum_{n=0}^{\infty} b_n(s) x^n + x^s \sum_{n=0}^{\infty} b'_n(s) x^n$$

Then, a second solution is

$$\frac{\partial Y}{\partial s}(0, x) = x^0 \ln x \sum_{n=0}^{\infty} b_n(0) x^n + x^0 \sum_{n=0}^{\infty} b'_n(0) x^n = y_1(x) \ln x + \sum_{n=0}^{\infty} b'_n(0) x^n$$

Thus, we need to differentiate $b_n(s)$. First, $b'_0(s) = a_0$ and $b'_1(s) = a_0/(s+1)^2$, so $b'_0(0) = a_0$ and $b'_1(0) = a_0$. Next, using logarithmic differentiation, for $n \geq 2$, we can write

$$b'_n(s) = b_n(s) \left[-\frac{1}{s+1} - \frac{1}{s+2} - \cdots - \frac{1}{s+n} - \frac{1}{s+1} - \frac{1}{s+2} - \cdots - \frac{1}{s+n-1} \right],$$

and

$$b'_n(0) = -b_n(0) \left[1 + \frac{1}{2} + \cdots + \frac{1}{n} + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right] = \frac{(-1)^{n+1} a_0}{n!(n-1)!} [H(n) + H(n-1)],$$

where $H(n) = \sum_{k=1}^n 1/k$. Thus, a second independent solution is

$$y_2(x) = y_1(x) \ln x + \left[a_0 + a_0 x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} a_0 x^n}{n!(n-1)!} [H(n) + H(n-1)] \right].$$

With $a_0 = 1$, this becomes

$$y_2(x) = y_1(x) \ln x + 1 + x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} [H(n) + H(n-1)]}{n!(n-1)!} x^n.$$

10. Compare $x^2 y'' - x(6+x)y' + 10y$ with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = -(6+x)$ and $q(x) = 10$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = -6$ and $q_0 = q(0) = 10$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 7s + 10 = (s-2)(s-5).$$

Set

$$y(x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n},$$

then, substituting,

$$\begin{aligned} 0 &= x^2 y'' - x(6+x)y' + 10y \\ &= [s(s-1) - 6s + 10]a_0 x^s + \\ &\quad \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) - 6(s+n) + 10]a_n(s) - (s+n-1)a_{n-1}(s)\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 - 7s + 10 = (s-2)(s-5) = 0,$$

with roots $s_2 = 2$ and $s_1 = 5$. The remaining terms provide the recurrence relation

$$(s+n-2)(s+n-5)a_n(s) = (s+n-1)a_{n-1}(s), \quad n \geq 1.$$

Using the smaller root, $s_2 = 2$, this becomes

$$n(n-3)a_n(2) = (n+1)a_{n-1}(2).$$

With $a_0 \neq 0$, we compute $-2a_1(2) = 2a_0$, or $a_1(2) = -a_0$. Next, $-2a_2(2) = 3a_1(2)$, or $a_2(2) = -3a_1(2)/2 = 3a_0/2$. But with $n = s_1 - s_2 = 3$, $0 \cdot a_3(2) = 4a_2(2) \neq 0$, so this equation cannot be solved. Hence, we have a real degeneracy and we compute $a_n(s)$ as a function of s .

$$a_n(s) = \frac{(s+n-1)a_{n-1}(s)}{(s+n-2)(s+n-5)}.$$

Consequently,

$$a_1(s) = \frac{sa_0}{(s-1)(s-4)}, \quad a_2(s) = \frac{(s+1)a_1(s)}{s(s-3)}, \quad a_3(s) = \frac{(s+2)a_2(s)}{(s+1)(s-2)},$$

etc., and

$$a_1(s) a_2(s) \cdots a_n(s) = \frac{s(s+1) \cdots (s+n-1)a_0 a_1(s) \cdots a_{n-1}(s)}{[(s-1)s \cdots (s+n-2)][(s-4)(s-3) \cdots (s+n-5)]},$$

Cancelling,

$$a_n(s) = \frac{(s+n-1)a_0}{(s-1)[(s-4)(s-3) \cdots (s+n-5)]}, \quad n \geq 4.$$

Next, working with the smaller root, set

$$Y(s, x) = (s-2)y(s, x) = x^s \sum_{n=0}^{\infty} (s-2)a_n(s)x^n = x^s \sum_{n=0}^{\infty} b_n(s)x^n,$$

where $b_n(s) = (s-2)a_n(s)$. Now, $b_0(s) = (s-2)a_0$,

$$b_1(s) = (s-2)a_1(s) = \frac{s(s-2)a_0}{(s-1)(s-4)} \quad \text{and} \quad b_2(s) = (s-2)a_2(s) = \frac{s(s+1)(s-2)a_0}{(s-1)s(s-4)(s-3)}.$$

Also,

$$b_3(s) = (s-2)a_3(s) = \frac{(s+2)a_0}{(s-1)(s-3)(s-4)}.$$

Finally, for $n \geq 3$,

$$b_n(s) = (s-2)a_n(s) = \frac{(s+n-1)a_0}{(s-1)[(s-4)(s-3)(s-1) \cdots (s+n-5)]}.$$

At $s = 2$, $b_0(2) = b_1(2) = b_2(2) = 0$, and $b_3(2) = 2a_0$. But, for $n \geq 4$,

$$b_n(2) = \frac{(n+1)a_0}{1 \cdot [(-2)(-1)(1)(2) \cdots (n-3)]} = \frac{(n+1)a_0}{2(n-3)!}.$$

Thus, with $s = 2$ and $a_0 = 1$,

$$\begin{aligned} y_1(x) &= Y(2, x) = x^2 \sum_{n=0}^{\infty} b_n(2)x^n = x^2 \left[0 + 0 \cdot x + 0x^2 + 2x^3 + \sum_{n=4}^{\infty} \frac{n+1}{2(n-3)!} x^n \right] \\ &= 2x^5 + \sum_{n=4}^{\infty} \frac{n+1}{2(n-3)!} x^{n+2}. \end{aligned}$$

We differentiate with respect to s .

$$\frac{\partial Y}{\partial s} = \frac{\partial}{\partial s} x^s \sum_{n=0}^{\infty} b_n(s)x^n = x^s \ln x \sum_{n=0}^{\infty} b_n(s)x^n + x^s \sum_{n=0}^{\infty} b'_n(s)x^n$$

Then

$$\frac{\partial Y}{\partial s}(2, x) = x^2 \ln x \sum_{n=0}^{\infty} b_n(2)x^n + x^2 \sum_{n=0}^{\infty} b'_n(2)x^n = y_1(x) \ln x + x^2 \sum_{n=0}^{\infty} b'_n(2)x^n$$

is a second independent solution. But, $b'_0(s) = a_0$, and using logarithmic differentiation,

$$\begin{aligned} b'_1(s) &= \frac{a_0 s(s-2)}{(s-1)(s-4)} \left[\frac{1}{s} + \frac{1}{s-2} - \frac{1}{s-1} - \frac{1}{s-4} \right] \\ &= \frac{a_0 s}{(s-1)(s-4)} \left[\frac{s-2}{s} + 1 - \frac{s-2}{s-1} - \frac{s-2}{s-4} \right]. \end{aligned}$$

Hence, $b'_1(2) = -a_0$. Similarly,

$$b'_2(s) = \frac{s(s+1)a_0}{(s-1)s(s-4)(s-3)} \left[\frac{s-2}{s} + \frac{s-2}{s+1} + 1 - \frac{s-2}{s-1} - \frac{s-2}{s} - \frac{s-2}{s-4} - \frac{s-2}{s-3} \right].$$

Hence, $b'_2(2) = 3a_0/2$. Similarly,

$$b'_3(s) = b_3(s) \left[\frac{1}{s+2} - \frac{1}{s-1} - \frac{1}{s-3} - \frac{1}{s-4} \right],$$

and $b'_3(2) = 3a_0/2$. Finally,

$$b'_n(s) = b_n(s) \left[\frac{1}{s+n-1} - \frac{1}{s-1} - \frac{1}{s-4} - \frac{1}{s-3} - \frac{1}{s-1} - \cdots - \frac{1}{s+n-5} \right].$$

Thus, for $n \geq 4$,

$$\begin{aligned} b'_n(2) &= b_n(2) \left[\frac{1}{n+1} - 1 + \frac{1}{2} + 1 - 1 - \frac{1}{2} - \cdots - \frac{1}{n-3} \right] \\ &= \frac{(n+1)a_0}{2(n-3)!} \left[\frac{1}{n+1} + \frac{1}{2} - \sum_{k=1}^{n-3} \frac{1}{k} \right] \\ &= \frac{(n+1)a_0}{2(n-3)!} \left[\frac{1}{2} + \frac{1 - (n+1)H(n-3)}{n+1} \right], \end{aligned}$$

where $H(n) = \sum_{k=1}^n 1/k$. Thus, with $s = 2$,

$$\begin{aligned} y_2(x) &= y_1(x) \ln x + x^2 \sum_{n=0}^{\infty} b'_n(2)x^n \\ &= y_1(x) \ln x + x^2 \left[a_0 - a_0x + \frac{3}{2}a_0x^2 + \frac{3}{2}a_0x^3 \right] + x^2 \sum_{n=4}^{\infty} \frac{(n+1)a_0}{2(n-3)!} \left[\frac{1}{2} + \frac{1 - (n+1)H(n-3)}{n+1} \right] x^n \end{aligned}$$

and with $a_0 = 1$,

$$y_2(x) = y_1(x) \ln x + x^2 - x^3 + \frac{3}{2}x^4 + \frac{3}{2}x^5 + \sum_{n=4}^{\infty} \frac{(n+1)a_0}{2(n-3)!} \left[\frac{1}{2} + \frac{1 - (n+1)H(n-3)}{n+1} \right] x^{n+2}$$

11. Compare $x^2y'' - 3xy' + (3+4x)y = 0$ with $x^2y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = -3$ and $q(x) = 3+4x$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = -3$ and $q_0 = q(0) = 3$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 4s + 3 = (s-1)(s-3) = 0.$$

Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2 y'' - 3xy' + (3 + 4x)y \\ &= [s(s-1) - 3s + 3]a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) - 3(s+n) + 3]a_n(s) + 4a_{n-1}(s)\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 - 4s + 3 = (s-1)(s-3) = 0,$$

with roots $s_2 = 1$ and $s_1 = 3$. The remaining terms provide the recurrence relation

$$(s+n-1)(s+n-3)a_n(s) + 4a_{n-1}(s) = 0, \quad n \geq 1.$$

Working with the smaller root, $s = s_2 = 1$, this becomes

$$n(n-2)a_n(1) = -4a_{n-1}(1), \quad n \geq 1.$$

Thus, $-a_1(1) = -4a_0$ or $a_1(1) = 4a_0 \neq 0$. However, for $N = s_1 - s_2 = 3 - 1 = 2$, this becomes

$$0a_2(1) = -4a_1(1) \neq 0,$$

so we have a real degeneracy. Compute $a_n(s)$ as a function of s .

$$a_n(s) = \frac{-4a_{n-1}(s)}{(s+n-1)(s+n-3)}, \quad n \geq 1.$$

Thus,

$$a_1(s) = \frac{-4a_0}{s(s-2)}, \quad a_2(s) = \frac{-4a_1(s)}{(s+1)(s-1)},$$

etc., and

$$a_1(s) a_2(s) \cdots a_n(s) = \frac{(-4)^n a_0 a_1(s) \cdots a_{n-1}(s)}{[s(s+1) \cdots (s+n-1)][(s-2)(s-1) \cdots (s+n-3)]}.$$

Cancelling,

$$a_n(s) = \frac{(-4)^n a_0}{[s(s+1) \cdots (s+n-1)][(s-2)(s-1) \cdots (s+n-3)]}$$

for $n \geq 1$. Because the smaller roots is $s = s_2 = 1$, define

$$Y(s, x) = (s-1)y(s, x) = x^s \sum_{n=0}^{\infty} (s-1)a_n(s)x^n = x^s \sum_{n=0}^{\infty} b_n(s)x^n,$$

where $b_n(s) = (s-1)a_n(s)$. For example, $b_0(s) = (s-1)a_0$, $b_1(s) = -4a_0(s-1)/(s(s-2))$, $b_2(s) = -4a_0/(s+1)$, and

$$b_n(s) = \frac{(-4)^n a_0}{[s(s+1) \cdots (s+n-1)][(s-2)s(s+1) \cdots (s+n-3)]},$$

for $n \geq 3$. Evaluating at $s = s_2 = 1$, $b_0(1) = 0$, $b_1(1) = 0$, $b_2(1) = -2a_0$, and

$$b_n(1) = \frac{(-4)^n a_0}{(1 \cdot 2 \cdot \dots \cdot n)(-1)(1 \cdot 2 \cdot \dots \cdot (n-2))} = \frac{(-1)^{n+1} 4^n a_0}{n!(n-2)!}, \quad n \geq 3.$$

Remembering that

$$Y(s, x) = x^s \sum_{n=0}^{\infty} b_n(s) x^n,$$

then with $s = s_2 = 1$,

$$y_1(x) = x^1 \sum_{n=0}^{\infty} b_n(1) x^n = x \left[0 + 0x - 2a_0 x^2 + \sum_{n=3}^{\infty} \frac{(-1)^{n+1} 4^n a_0}{n!(n-2)!} x^n \right].$$

With $a_0 = -1$, this becomes

$$y_1(x) = 2x^3 + \sum_{n=3}^{\infty} \frac{(-1)^n 4^n}{n!(n-2)!} x^{n+1}.$$

A second solution is found by taking the derivative of $Y(s, x) = x^s \sum_{n=0}^{\infty} b_n(s) x^n$ with respect to s .

$$\frac{\partial Y}{\partial s} = x^s \ln x \sum_{n=0}^{\infty} b_n(s) x^n + x^s \sum_{n=0}^{\infty} b'_n(s) x^n = y_1(x) \ln x + x^s \sum_{n=0}^{\infty} b'_n(s) x^n$$

Thus, we need to differentiate $b_n(s)$. First, $b'_0(s) = a_0$,

$$b'_1(s) = b_1(s) \left[\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s-2} \right] = \frac{-4a_0}{s(s-2)} \left[1 - \frac{s-1}{s} - \frac{s-1}{s-2} \right],$$

and

$$b'_2(s) = \frac{4a_0}{(s+1)^2}.$$

Thus, $b'_0(1) = a_0$, $b'_1(1) = 4a_0$, and $b'_2(1) = a_0$. Using more logarithmic differentiation, for $n \geq 3$, we have

$$b'_n(s) = b_n(s) \left[-\frac{1}{s} - \frac{1}{s+1} - \dots - \frac{1}{s+n-1} - \frac{1}{s-2} - \frac{1}{s} - \frac{1}{s+1} - \dots - \frac{1}{s+n-3} \right].$$

At $s = 1$,

$$b'_n(1) = -b_n(1) \left[1 + \frac{1}{2} + \dots + \frac{1}{n} - 1 + 1 + \frac{1}{2} + \dots + \frac{1}{n-2} \right] = \frac{(-1)^n 4^n a_0}{n!(n-2)!} [H(n) - 1 + H(n-2)],$$

for $n \geq 3$ and $H(n) = \sum_{k=1}^n (1/k)$. Hence,

$$\begin{aligned} y_2(x) &= y_1(x) \ln x + x^1 \sum_{n=0}^{\infty} b'_n(1) x^n \\ &= y_1(x) \ln x + x \left[a_0 + 4a_0 x + a_0 x^2 + \sum_{n=3}^{\infty} \frac{(-1)^n 4^n a_0}{n!(n-2)!} [H(n) - 1 + H(n-2)] x^n \right]. \end{aligned}$$

With $a_0 = -1$,

$$y_2(x) = y_1(x) \ln x - x - x^2 - x^3 - \sum_{n=3}^{\infty} \frac{(-4)^n [H(n) - 1 + H(n-2)]}{n!(n-2)!} x^{n+1}.$$

12. Multiply by x ,

$$x^2 y'' + x(3 + 2x)y' + 8xy = 0,$$

then comparison with $x^2 y'' + xp(x)y' + q(x)y = 0$ reveals that $p(x) = 3 + 2x$ and $q(x) = 8x$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 3$ and $q_0 = q(0) = 0$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 + 2s = s(s + 2) = 0.$$

Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2 y'' + x(3 + 2x)y' + 8xy \\ &= [s(s-1) + 3s]a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) + 3(s+n)]a_n(s) + [2(s+n-1) + 8]a_{n-1}(s)\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 + 2s = s(s + 2) = 0,$$

with roots $s_2 = -2$ and $s_1 = 0$. The remaining terms provide the recurrence relation

$$(s+n)(s+n+2)a_n(s) = -(2s+2n+6)a_{n-1}(s), \quad n \geq 1.$$

Using the smaller root, $s = s_2 = -2$, this becomes

$$(n-2)na_n(-2) = -2(n+1)a_{n-1}(-2), \quad n \geq 1.$$

With $a_0 \neq 0$, we compute $-a_1(-2) = -4a_0$, $a_1(-2) = 4a_0$. But, with $n = s_1 - s_2 = 2$,

$$0a_2(-2) = -6a_1(-2) \neq 0,$$

so this equation cannot be solved and we have a real degeneracy. We find $a_n(s)$ as a function of s .

$$a_n(s) = \frac{-2(s+n+3)a_{n-1}(s)}{(s+n)(s+n+2)}, \quad n \geq 1.$$

Consequently,

$$a_1(s) = \frac{-2(s+4)a_0}{(s+1)(s+3)}, \quad a_2(s) = \frac{-2(s+5)a_1}{(s+2)(s+4)}, \quad a_3(s) = \frac{-2(s+6)a_2(s)}{(s+3)(s+5)},$$

etc. Therefore,

$$a_1(s) a_2(s) \dots a_n(s) = \frac{(-2)^n (s+4)(s+5) \dots (s+n+3) a_0 a_1(s) \dots a_{n-1}(s)}{[(s+1)(s+2) \dots (s+n)][(s+3)(s+4) \dots (s+n+2)]},$$

and cancelling,

$$a_n(s) = \frac{(-2)^n(s+n+3)a_0}{[(s+1)(s+2)\cdots(s+n)](s+3)}.$$

Next, working with the smaller root, $s_2 = -2$, set

$$Y(s, x) = (s+2)y(s, x) = x^s \sum_{n=0}^{\infty} (s+2)a_n(s)x^n = x^s \sum_{n=0}^{\infty} b_n(s)x^n.$$

Now, $b_0(s) = (s+2)a_0$,

$$b_1(s) = (s+2)a_1(s) = \frac{-2(s+2)(s+4)}{(s+1)(s+3)}a_0 \quad \text{and} \quad b_2(s) = (s+2)a_2(s) = \frac{4(s+5)a_0}{(s+1)(s+3)}.$$

Finally, for $n \geq 3$,

$$b_n(s) = (s+2)a_n(s) = \frac{(-2)^n(s+n+3)a_0}{[(s+1)(s+3)\cdots(s+n)](s+3)}.$$

At $s = -2$, $b_0(-2) = b_1(-2) = 0$ and $b_2(-2) = -12a_0$. But, for $n \geq 3$,

$$b_n(-2) = \frac{(-2)^n(n+1)a_0}{[(-1)(1)\cdots(n-2)](1)} = \frac{(-1)^{n+1}2^n(n+1)a_0}{(n-2)!}$$

Thus, with $s = -2$ and $a_0 = 1$,

$$\begin{aligned} y_1(x) &= Y(-2, x) = x^{-2} \sum_{n=0}^{\infty} b_n(-2)x^n = x^{-2} \left[0 + 0x - 12x^2 + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}2^n(n+1)}{(n-2)!} x^n \right] \\ &= -12 + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}2^n(n+1)x^{n-2}}{(n-2)!}. \end{aligned}$$

We differentiate with respect to s to find a second independent solution.

$$\frac{\partial Y}{\partial s} = \frac{\partial}{\partial s} x^s \sum_{n=0}^{\infty} b_n(s)x^n = x^s \ln x \sum_{n=0}^{\infty} b_n(s)x^n + x^s \sum_{n=0}^{\infty} b'_n(s)x^n = y_1 \ln x + x^s \sum_{n=0}^{\infty} b'_n(s)x^n.$$

But, $b'_0(s) = a_0$, and using logarithmic differentiation,

$$\begin{aligned} b'_1(s) &= \frac{-2a_0(s+2)(s+4)}{(s+1)(s+3)} \left[\frac{1}{s+2} + \frac{1}{s+4} - \frac{1}{s+1} - \frac{1}{s+3} \right] \\ &= \frac{-2a_0(s+4)}{(s+1)(s+3)} \left[1 + \frac{s+2}{s+4} - \frac{s+2}{s+1} - \frac{s+2}{s+3} \right]. \end{aligned}$$

Similarly,

$$b'_2(s) = b_2(s) \left[\frac{1}{s+2} + \frac{1}{s+4} - \frac{1}{s+1} - \frac{1}{s+3} \right].$$

Thus, $b'_0(-2) = a_0$, $b'_1(-2) = 4a_0$, and $b'_2(-2) = b_2(-2)(1/3 + 1 - 1) = -4a_0$. Finally, using logarithmic differentiation,

$$b'_n(s) = b_n(s) \left[\frac{1}{s+n+3} - \frac{1}{s+1} - \frac{1}{s+3} - \cdots - \frac{1}{s+n} - \frac{1}{s+3} \right],$$

for $n \geq 3$. Therefore,

$$\begin{aligned} b'_n(-2) &= b_n(-2) \left[\frac{1}{n+1} + 1 - 1 - \frac{1}{2} - \cdots - \frac{1}{n-2} - 1 \right] \\ &= \frac{(-1)^{n+1} 2^n (n+1) a_0}{(n-2)!} \left[\frac{1}{n+1} - \sum_{k=1}^{n-2} \frac{1}{k} \right] \\ &= \frac{(-1)^{n+1} 2^n (n+1) a_0}{(n-2)!} \left[\frac{1 - (n+1)H(n-2)}{n+1} \right], \end{aligned}$$

where $H(n) = \sum_{k=1}^n 1/k$. Thus, with $s = -2$,

$$\begin{aligned} y_2(x) &= y_1(x) \ln x + x^{-2} \sum_{n=0}^{\infty} b'_n(-2) x^n \\ &= y_1(x) \ln x + x^{-2} \left[a_0 + 4a_0x - 4a_0x^2 + \sum_{n=3}^{\infty} \frac{(-1)^{n+1} 2^n (n+1) a_0}{(n-2)!} \left[\frac{1 - (n+1)H(n-2)}{n+1} \right] x^n \right] \end{aligned}$$

Now, with $a_0 = 1$,

$$\begin{aligned} y_2(x) &= y_1(x) \ln x + x^{-2} \left[1 + 4x - 4x^2 + \sum_{n=3}^{\infty} \frac{(-1)^{n+1} 2^n [1 - (n+1)H(n-2)]}{(n-2)!} x^n \right] \\ &= y_1(x) \ln x + x^{-2} + 4x^{-1} - 4 + \sum_{n=3}^{\infty} \frac{(-2)^n [(n+1)H(n-2) - 1]}{(n-2)!} x^{n-2}. \end{aligned}$$

13. Compare $x^2 y'' + x(2+3x)y' - 2y = 0$ with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 2+3x$ and $q(x) = -2$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 2$ and $q_0 = q(0) = -2$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 + s - 2 = (s+2)(s-1) = 0.$$

Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2 y'' + x(2+3x)y' - 2y \\ &= [(s(s-1) + 2s - 2)a_0 x^s + \sum_{n=1}^{\infty} \{(s+n)(s+n-1) + 2(s+n) - 2\} a_n(s) + 3(s+n-1)a_{n-1}(s)] x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 + s - 2 = (s+2)(s-1) = 0,$$

with roots $s_2 = -2$ and $s_1 = 1$. The remaining terms provide the recurrence relation.

$$(s+n+2)(s+n-1)a_n(s) + 3(s+n-1)a_{n-1}(s) = 0, \quad n \geq 1.$$

Working with the smaller root, $s = s_2 = -2$, this becomes

$$n(n-3)a_n(-2) = -3(n-3)a_{n-1}(-2), \quad n \geq 1.$$

However, for $N = s_1 - s_2 = 1 - (-2) = 3$, we have $0a_3(-2) = 0$, so this is a false degeneracy. Providing $n \neq 3$, we have

$$a_n(-2) = \frac{-3a_{n-1}(-2)}{n}.$$

Thus,

$$a_1(-2) = -3a_0 \quad \text{and} \quad a_2(-2) = \frac{-3a_1(-2)}{2} = \frac{9}{2}a_0.$$

Letting a_3 be an arbitrary constant,

$$a_4(-2) = \frac{-3a_3}{4}, \quad a_5(-2) = \frac{-3a_4(-2)}{5},$$

etc., and

$$a_4(-2)a_5(-2)\cdots a_n(-2) = \frac{(-3)^{n-3}a_3a_4(-2)\cdots a_{n-1}(-2)}{4 \cdot 5 \cdots n}.$$

Cancelling,

$$a_n(-2) = \frac{6(-3)^{n-3}a_3}{n!}, \quad n \geq 4.$$

Thus, for $s = s_2 = -2$,

$$y(x) = x^s \sum_{n=0}^{\infty} a_n(s)x^n = x^{-2} \left[a_0 - 3a_0x + \frac{9}{2}a_0x^2 \right] + x^{-2} \left[a_3x^3 + \sum_{n=4}^{\infty} \frac{6(-3)^{n-3}a_3}{n!} x^n \right].$$

With $a_0 = 1$ and $a_3 = 0$, one solution is

$$y_1(x) = x^{-2} \left(1 - 3x + \frac{9}{2}x^2 \right) = x^{-2} - 3x^{-1} + \frac{9}{2}.$$

With $a_0 = 0$ and $a_3 = 1$, a second independent solution is

$$y_2(x) = x^{-2} \left[x^3 + \sum_{n=4}^{\infty} \frac{6(-3)^{n-3}}{n!} x^n \right] = x + 6 \sum_{n=4}^{\infty} \frac{(-3)^{n-3}}{n!} x^{n-2}.$$

14. Multiply by $x/2$,

$$x^2y'' + 3xy' + \frac{1}{2}xy = 0,$$

then compare with $x^2y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 3$ and $q(x) = x/2$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 3$ and $q_0 = q(0) = 0$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 + 2s = s(s + 2) = 0.$$

Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2 y'' + 3xy' + \frac{1}{2}xy \\ &= [s(s-1) + 3s]a_0 x^s + \sum_{n=1}^{\infty} \left\{ [(s+n)(s+n-1) + 3(s+n)]a_n(s) + \frac{1}{2}a_{n-1}(s) \right\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 + 2s = s(s+2) = 0,$$

with roots $s_2 = -2$ and $s_1 = 0$. The remaining terms provide the recurrence relation

$$(s+n)(s+n+2)a_n(s) + \frac{1}{2}a_{n-1}(s) = 0, \quad n \geq 1.$$

Starting with the smaller root, $s = s_2 = -2$, this becomes

$$n(n-2)a_n(-2) = -\frac{1}{2}a_{n-1}(-2), \quad n \geq 1.$$

Thus, $-a_1(-2) = -a_0/2$, or $a_1(-2) = a_0/2$. However, for $N = s_1 - s_2 = 0 - (-2) = 2$, this becomes

$$0a_2(-2) = -\frac{1}{2}a_1(-2) \neq 0,$$

so this equation has no solution and we have a real degeneracy. Compute $a_n(s)$ as a function of s .

$$a_n(s) = \frac{-a_{n-1}(s)}{2(s+n)(s+n+2)}, \quad n \geq 1.$$

Thus,

$$a_1(s) = \frac{-a_0}{2(s+1)(s+3)}, \quad a_2(s) = \frac{-a_1(s)}{2(s+2)(s+4)},$$

etc., and

$$a_1(s)a_2(s)\dots a_n(s) = \frac{(-1)^n a_0 a_1(s) \dots a_{n-1}(s)}{2^n [(s+1)(s+2)\dots(s+n)][(s+3)(s+4)\dots(s+n+2)]}.$$

Cancelling,

$$a_n(s) = \frac{(-1)^n a_0}{2^n [(s+1)(s+2)\dots(s+n)][(s+3)(s+4)\dots(s+n+2)]},$$

for $n \geq 1$. With the smaller root, $s = s_2 = -2$, define

$$Y(s, x) = (s+2)y(s, x) = x^s \sum_{n=0}^{\infty} (s+2)a_n(s)x^n = x^s \sum_{n=0}^{\infty} b_n(s)x^n,$$

where $b_n(s) = (s+2)a_n(s)$. For example, $b_0(s) = (s+2)a_0$,

$$b_1(s) = \frac{-a_0(s+2)}{2(s+1)(s+3)} \quad \text{and} \quad b_2(s) = \frac{a_0}{4(s+1)(s+3)(s+4)}.$$

For $n \geq 3$,

$$b_n(s) = \frac{(-1)^n a_0}{2^n [(s+1)(s+3) \cdots (s+n)] [(s+3)(s+4) \cdots (s+n+2)]}.$$

Evaluating at $s = s_2 = -2$, $b_0(-2) = 0$, $b_1(-2) = 0$, $b_2(-2) = -a_0/8$, and

$$b_n(-2) = \frac{(-1)^n a_0}{2^n (-1)[1 \cdot 2 \cdots (n-2)][1 \cdot 2 \cdots n]} = \frac{(-1)^{n+1} a_0}{2^n (n-2)! n!}, \quad n \geq 3.$$

Thus, remembering

$$Y(s, x) = x^s \sum_{n=0}^{\infty} b_n(s) x^n,$$

Then with $s = s_2 = -2$,

$$y_1(x) = Y(-2, x) = x^{-2} \sum_{n=0}^{\infty} b_n(-2) x^n = x^{-2} \left[-\frac{1}{8} a_0 x^2 + \sum_{n=3}^{\infty} \frac{(-1)^{n+1} a_0}{2^n (n-2)! n!} x^n \right].$$

With $a_0 = 1$, this becomes

$$y_1(x) = -\frac{1}{8} x^2 + \sum_{n=3}^{\infty} \frac{(-1)^{n+1} x^{n-2}}{2^n (n-2)! n!}.$$

A second solution is found by taking the derivative of $Y(s, x) = x^s \sum_{n=0}^{\infty} b_n(s) x^n$ with respect to s .

$$\frac{\partial Y}{\partial s} = x^s \ln x \sum_{n=0}^{\infty} b_n(s) x^n + x^s \sum_{n=0}^{\infty} b'_n(s) x^n = y_1(x) \ln x + x^s \sum_{n=0}^{\infty} b'_n(s) x^n.$$

Thus, we need to differentiate $b_n(s)$. First, $b'_0(s) = a_0$, and logarithmic differentiation provides

$$\begin{aligned} b'_1(s) &= b_1(s) \left[\frac{1}{s+2} - \frac{1}{s+1} - \frac{1}{s+3} \right] \\ &= \frac{-a_0}{2(s+1)(s+3)} \left[1 - \frac{s+2}{s+1} - \frac{s+2}{s+3} \right]. \end{aligned}$$

Similarly,

$$b'_2(s) = b_2(s) \left[-\frac{1}{s+1} - \frac{1}{s+3} - \frac{1}{s+4} \right].$$

Thus, $b'_0(-2) = a_0$, $b'_1(-2) = a_0/2$, and $b'_2(-2) = b_2(-2)(1 - 1 - 1/2) = a_0/16$. In addition, for $n \geq 3$,

$$b'_n(s) = b_n(s) \left[-\frac{1}{s+1} - \frac{1}{s+3} - \cdots - \frac{1}{s+n} - \frac{1}{s+3} - \frac{1}{s+4} - \cdots - \frac{1}{s+n+2} \right].$$

At $s = -2$,

$$\begin{aligned} b'_n(-2) &= -b_n(-2) \left[-1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n-2} + 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right] \\ &= \frac{(-1)^{n+2} a_0}{2^n (n-2)! n!} [-1 + H(n-2) + H(n)], \end{aligned}$$

where $H(n) = \sum_{k=1}^n 1/k$. Hence,

$$\begin{aligned} y_2(x) &= y_1(x) \ln x + x^{-2} \sum_{n=0}^{\infty} b'_n (-2)x^n \\ &= y_1(x) \ln x + x^{-2} \left[a_0 + \frac{a_0}{2}x + \frac{a_0}{16}x^2 + \sum_{n=3}^{\infty} \frac{(-1)^{n+2}a_0}{2^n(n-2)!n!} [H(n) + H(n-2) - 1]x^n \right]. \end{aligned}$$

With $a_0 = 1$,

$$y_2(x) = y_1(x) \ln x + x^{-2} + \frac{1}{2}x^{-1} + \frac{1}{16} + \sum_{n=3}^{\infty} \frac{(-1)^{n+2}[H(n) + H(n-2) - 1]}{2^n(n-2)!n!} x^{n-2}$$

15. Multiply by x ,

$$x^2 y'' + x^2 y' + xy = 0,$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = x$ and $q(x) = x$, both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 0$ and $q_0 = q(0) = 0$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - s = s(s - 1).$$

Set

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2 y'' + x^2 y' + xy \\ &= s(s-1)a_0 x^s + \sum_{n=1}^{\infty} \{(s+n)(s+n-1)a_n + [(s+n-1)+1]a_{n-1}(s)\}x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s(s-1) = 0,$$

with roots $s_2 = 0$ and $s_1 = 1$. The remaining terms provide the recurrence relation

$$(s+n)(s+n-1)a_n = -(s+n)a_{n-1}(s), \quad n \geq 1.$$

Working with the smaller root, $s = s_2 = 0$, this becomes

$$n(n-1)a_n = -na_{n-1}(s).$$

However, with $N = s_1 - s_2 = 1 - 0 = 1$,

$$0a_1 = -a_0.$$

Because $a_0 \neq 0$, this equation cannot be solved and we have a real degeneracy. We start by setting

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s)x^n = \sum_{n=0}^{\infty} a_n(s)x^{s+n},$$

and solve for a_n as a function of s .

$$a_n = \frac{-a_{n-1}(s)}{s+n-1}, \quad n \geq 1.$$

Thus,

$$a_1 = \frac{-a_0}{s}, \quad a_2 = \frac{-a_1}{s+1}, \quad \dots, \quad a_n = \frac{-a_{n-1}(s)}{s+n-1}.$$

Multiplying these equation together,

$$a_1 a_2 \cdots a_n = \frac{(-1)^n a_0 a_1 \cdots a_{n-1}(s)}{s(s+1) \cdots (s+n-1)}.$$

Cancelling,

$$a_n(s) = \frac{(-1)^n a_0}{s(s+1) \cdots (s+n-1)}.$$

Define

$$\begin{aligned} Y(s, x) &= sy(s, x) \\ &= s \sum_{n=0}^{\infty} a_n(s) x^{s+n} \\ &= \sum_{n=0}^{\infty} b_n(s) x^{s+n}, \end{aligned}$$

where $b_n(s) = sa_n(s)$. Thus, for example,

$$b_0 = sa_0 \quad \text{and} \quad b_1(s) = -a_0.$$

For $n \geq 2$,

$$b_n(s) = sa_n(s) = \frac{(-1)^n a_0}{(s+1) \cdots (s+n-1)}.$$

Evaluating at $s = 0$,

$$b_0(0) = 0 \quad \text{and} \quad b_1(0) = -a_0.$$

Then,

$$b_n(0) = \frac{(-1)^n a_0}{(n-1)!},$$

for $n \geq 2$. Thus, remembering

$$Y(s, x) = x^s \sum_{n=0}^{\infty} b_n(s) x^n,$$

and choosing $a_0 = -1$ with $s = s_2 = 0$,

$$y_1(x) = Y(0, x) = x^0 \left[x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^n}{(n-1)!} \right] = x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(n-1)!} x^n.$$

A second solution is found by taking the derivative of

$$Y(s, x) = sy(s, x) = x^s \sum_{n=0}^{\infty} b_n(s) x^n$$

with respect to s .

$$\begin{aligned}\frac{\partial Y}{\partial s} &= x^s \ln x \sum_{n=0}^{\infty} b_n(s) x^n + x^s \sum_{n=0}^{\infty} b'_n(s) x^n \\ &= y_1(x) \ln x + x^s \sum_{n=0}^{\infty} b'_n(s) x^n\end{aligned}$$

Thus, we need to differentiate $b_n(s)$. First, $b'_0(s) = a_0$ and $b'_1(s) = 0$, so $b'_0(0) = a_0$ and $b'_1(0) = 0$. Next, using logarithmic differentiation,

$$b'_n(s) = b_n(s) \left[-\frac{1}{s+1} - \cdots - \frac{1}{s+n-1} \right]$$

and

$$b'_n(0) = -b_n(0) \sum_{k=1}^{n-1} \frac{1}{k} = \frac{(-1)^{n+1} a_0}{(n-1)!} H(n-1),$$

where $H(n) = \sum_{k=1}^n 1/k$. Choosing $a_0 = 1$, a second independent solution is, remembering that $s = s_2 = 0$,

$$\begin{aligned}y_2(x) &= y_1(x) \ln x + x^0 \sum_{n=0}^{\infty} b'_n(0) x^n \\ &= y_1(x) \ln x + 1 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} H(n-1)}{(n-1)!} x^n.\end{aligned}$$

16. Multiply by x ,

$$x^2 y'' - x(1+x^2)y' - 2x^2 y = 0,$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$, and note that $p(x) = -(1+x^2)$ and $q(x) = -2x^2$, both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = -1$ and $q_0 = q(0) = 0$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 2s = s(s-2) = 0.$$

Set

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{s+n},$$

and

$$\begin{aligned}-x(1+x^2)y' &= -(1+x^2) \sum_{n=0}^{\infty} (s+n) a_n x^{s+n} \\ &= -\sum_{n=0}^{\infty} (s+n) a_n x^{s+n} - \sum_{n=2}^{\infty} (s+n-2) a_{n-2} x^{s+n},\end{aligned}$$

where we've shifted the index of the second summation. Finally,

$$-2x^2y = -2x^2 \sum_{n=0}^{\infty} a_n x^{s+n} = - \sum_{n=2}^{\infty} 2a_{n-2} x^{s+n},$$

where again we've shifted the index of the last summation. Substituting,

$$\begin{aligned} 0 &= x^2 y'' - x(1+x^2)y' - 2x^2 y \\ &= [s(s-1) - s]a_0 x^s + [(s+1)s - (s+1)]a_1 x^{s+1} \\ &\quad + \sum_{n=2}^{\infty} \{[(s+n)(s+n-1) - (s+n)]a_n - [(s+n-2) + 2]a_{n-2}\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 - 2s = s(s-2) = 0,$$

with roots $s_2 = 0$ and $s_1 = 2$. Setting the coefficient of the second term equal to zero,

$$(s+1)(s-1)a_1 = 0,$$

because the coefficient of a_1 does not equal zero for either $s = s_2 = 0$ or $s = s_1 = 2$, we must have $a_1 = 0$. The remaining terms provide the recurrence relation

$$(s+n)(s+n-2)a_n - (s+n)a_{n-2} = 0, \quad n \geq 2.$$

Starting with the smaller root, $s = s_2 = 0$, this becomes

$$n(n-2)a_n = na_{n-2}, \quad n \geq 2.$$

However, for $N = s_1 - s_2 = 2 - 0 = 2$, this becomes

$$0a_2 = 2a_0 \neq 0,$$

so this equation has no solution and we have a real degeneracy. Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}$$

and compute a_n as a function of s .

$$a_n = \frac{a_{n-2}}{s+n-2}, \quad n \geq 2.$$

Because $a_1 = 0$, this gives us that $a_{2k+1}(s) = 0$ for $k \geq 0$. Starting with $a_0 \neq 0$,

$$a_{2n} = \frac{a_{2n-2}}{s+2n-2}, \quad n \geq 1.$$

Thus,

$$\begin{aligned} a_2 &= \frac{a_0}{s} \\ a_4 &= \frac{a_2}{s+2}, \end{aligned}$$

and

$$a_2 a_4 \cdots a_{2n} = \frac{a_0 a_2 \cdots a_{2n-2}}{s(s+2) \cdots (s+2n-2)}, \quad n \geq 1.$$

Cancelling,

$$a_{2n}(s) = \frac{a_0}{s(s+2) \cdots (s+2n-2)}, \quad n \geq 1.$$

As the smaller root is $s = s_2 = 0$, define

$$Y(s, x) = sy(s, x) = x^s \sum_{n=0}^{\infty} s a_{2n}(s) x^{2n} = x^s \sum_{n=0}^{\infty} b_{2n}(s) x^{2n},$$

where $b_{2n}(s) = s a_{2n}(s)$. For example, $b_0(s) = s a_0$, $b_2(s) = a_0$, and

$$b_{2n}(s) = \frac{a_0}{(s+2)(s+4) \cdots (s+2n-2)}, \quad n \geq 2.$$

Evaluating at $s = s_2 = 0$, $b_0(0) = 0$, $b_2(0) = a_0$, and

$$b_{2n}(0) = \frac{a_0}{2 \cdot 4 \cdots (2n-2)} = \frac{a_0}{2^{n-1}(n-1)!}, \quad n \geq 2.$$

Remembering

$$Y(s, x) = x^s \sum_{n=0}^{\infty} b_{2n}(s) x^{2n},$$

and choosing $a_0 = 1$ with $s = s_2 = 0$,

$$y_1(x) = Y(0, x) = x^2 + \sum_{n=2}^{\infty} \frac{x^{2n}}{2^{n-1}(n-1)!}.$$

A second solution is found by taking the derivative of

$$Y(s, x) = sy(s, x) = x^s \sum_{n=0}^{\infty} b_{2n}(s) x^{2n}$$

with respect to s .

$$\begin{aligned} \frac{\partial Y}{\partial s} &= x^s \ln x \sum_{n=0}^{\infty} b_{2n}(s) x^{2n} + x^s \sum_{n=0}^{\infty} b'_{2n}(s) x^{2n} \\ &= y_1(x) \ln x + x^s \sum_{n=0}^{\infty} b'_{2n}(s) x^{2n}. \end{aligned}$$

Thus, we need to differentiate $b_{2n}(s)$. First, $b'_0(s) = a_0$, $b'_2(s) = 0$, so $b'_0(0) = a_0$ and $b'_2(0) = 0$. Next, using logarithmic differentiation,

$$b'_{2n}(s) = b_{2n}(s) \left[-\frac{1}{s+2} - \frac{1}{s+4} - \cdots - \frac{1}{s+2n-2} \right],$$

and

$$\begin{aligned}
 b'_{2n}(0) &= -b_{2n}(0) \left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n-2} \right] \\
 &= \frac{-b_{2n}(0)}{2} \left[1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right] \\
 &= \frac{-a_0}{2^n(n-1)!} \sum_{k=1}^{n-1} \frac{1}{k} \\
 &= \frac{-a_0}{2^n(n-1)!} H(n-1),
 \end{aligned}$$

where $H(n) = \sum_{k=1}^n 1/k$. Choosing $a_0 = 1$, remembering that $s = s_2 = 0$, a second independent solution is

$$\begin{aligned}
 y_2(x) &= y_1(x) \ln x + x^0 \sum_{n=0}^{\infty} b'_{2n}(0) x^{2n} \\
 &= y_1(x) \ln x + 1 - \sum_{n=2}^{\infty} \frac{H(n-2)}{2^n(n-1)!} x^{2n}.
 \end{aligned}$$

17. Multiply by x .

$$x^2 y'' + x y' - x y = 0,$$

then compare with $x^2 y'' + x p(x) y' + q(x) y = 0$ and note that $p(x) = 1$ and $q(x) = -x$, both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, note that $p_0 = p(0) = 1$ and $q_0 = q(0) = 0$, making the indicial equation

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2.$$

Thus, we seek a solution of the form

$$y(x, s) = x^s \sum_{n=0}^{\infty} a_n(s) x^n = \sum_{n=0}^{\infty} a_n(s) x^{s+n}.$$

Substituting,

$$\begin{aligned}
 0 &= x^2 y'' + x y' - x y \\
 &= s^2 a_0 x^2 + \sum_{n=1}^{\infty} [(s+n)^2 a_n - a_{n-1}(s)] x^{s+n}.
 \end{aligned}$$

Thus, the indicial equation is $I(s) = s^2 = 0$ with equal roots $s_2 = s_1 = 0$. The remaining terms provide the recurrence relation

$$\begin{aligned}
 (s+n)^2 a_n &= a_{n-1}(s) \\
 a_n &= \frac{a_{n-1}(s)}{(s+n)^2}, \quad n \geq 1.
 \end{aligned}$$

Since the roots of the indicial equation are equal, we solve the indicial equation for a_n as a function of s . Thus,

$$\begin{aligned} a_1 &= \frac{a_0}{(s+1)^2} \\ a_2 &= \frac{a_1}{(s+2)^2} \\ &\vdots \\ a_n &= \frac{a_{n-1}(s)}{(s+n)^2}. \end{aligned}$$

Multiplying left and right-hand sides of these equations, we have

$$a_1 a_2 \cdots a_n = \frac{1}{[(s+1)(s+2) \cdots (s+n)]^2} a_0 a_1 \cdots a_{n-1}(s).$$

Cancelling,

$$a_n(s) = \frac{a_0}{[(s+1)(s+2) \cdots (s+n)]^2}.$$

Evaluating at $s = 0$,

$$a_n(0) = \frac{a_0}{(n!)^2},$$

and choosing $a_0 = 1$ gives

$$a_n(0) = \frac{1}{(n!)^2},$$

and a first solution

$$y_1 = y(0, x) = x^0 \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.$$

We now use logarithmic differentiation to find $a'_n(s)$.

$$\begin{aligned} \ln |a_n(s)| &= \ln |a_0| - 2 \sum_{k=1}^n \ln |s+k| \\ \frac{1}{a_n(s)} a'_n(s) &= -2 \sum_{k=1}^n \frac{1}{s+k} \end{aligned}$$

With $s = 0$,

$$a'_n(0) = -2a_n(0) \sum_{k=1}^n \frac{1}{k} = -\frac{2}{n!} H(n),$$

where we let $H(n) = \sum_{k=1}^n 1/k$. Thus, a second independent solution is

$$\begin{aligned} y_2 &= y_1 \ln x + x^0 \sum_{n=1}^{\infty} a'_n(0) x^n \\ &= y_1 \ln x - 2 \sum_{n=1}^{\infty} \frac{H(n)}{n!} x^n. \end{aligned}$$

18. Divide by 4,

$$x^2 y'' + \frac{1-2x}{4} y = 0,$$

then compare with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 0$ and $q(x) = (1-2x)/4$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 0$ and $q_0 = q(0) = 1/4$, so the indicial polynomial is

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - s + \frac{1}{4} = \left(s - \frac{1}{2}\right)^2.$$

Let

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

We will find it easier to work with the equation $4x^2 y'' + (1-2x)y = 0$. Thus,

$$4x^2 y'' = \sum_{n=0}^{\infty} 4(s+n)(s+n-1)a_n x^{s+n},$$

and

$$\begin{aligned} (1-2x)y &= (1-2x) \sum_{n=0}^{\infty} a_n x^{s+n} \\ &= \sum_{n=0}^{\infty} a_n x^{s+n} - \sum_{n=1}^{\infty} 2a_{n-1}(s)x^{s+n}, \end{aligned}$$

where we've shifted the index of the second summation. Substituting,

$$\begin{aligned} 0 &= 4x^2 y'' + (1-2x)y \\ &= (2s-1)^2 a_0 x^s + \sum_{n=1}^{\infty} \{[2(s+n)-1]^2 a_n - 2a_{n-1}(s)\} x^{s+n} \end{aligned}$$

Thus, the indicial equation is $I(s) = (2s-1)^2 = 0$, with equal roots $s_2 = s_1 = 1/2$. The remaining terms provide the recurrence relation

$$[2(s+n)-1]^2 a_n - 2a_{n-1}(s) = 0,$$

or, equivalently,

$$a_n(s) = \frac{2a_{n-1}(s)}{(2s+2n-1)^2}, \quad n \geq 1.$$

We solve the recurrence relation for a_n as a function of s .

$$\begin{aligned} a_1 &= \frac{2a_0}{(2s+1)^2} \\ a_2 &= \frac{2a_1}{(2s+3)^2} \\ &\vdots \\ a_n &= \frac{2a_{n-1}(s)}{(2s+2n-1)^2}. \end{aligned}$$

Multiplying left and right-hand sides of these equations, we get

$$a_1 a_2 \cdots a_n = \frac{2^n a_0 a_1 \cdots a_{n-1}(s)}{(2s+1)^2(2s+3)^2 \cdots (2s+2n-1)^2}.$$

Cancelling,

$$a_n(s) = \frac{2^n a_0}{(2s+1)^2(2s+3)^2 \cdots (2s+2n-1)^2}.$$

Evaluating at $s = 1/2$, we get

$$\begin{aligned} a_n(1/2) &= \frac{2^n a_0}{(2)^2(4)^2 \cdots (2n)^2} \\ &= \frac{2^n a_0}{2^{2n}(n!)^2} \\ &= \frac{a_0}{2^n (n!)^2}. \end{aligned}$$

Choosing $a_0 = 1$, our first solution is

$$\begin{aligned} y_1(x) &= y(1/2, x) \\ &= x^{1/2} \sum_{n=0}^{\infty} \frac{x^n}{2^n (n!)^2}. \end{aligned}$$

To find our second solution, we must differentiate $a_n(s)$. Use logarithmic differentiation and write

$$\ln |a_n(s)| = \ln |2^n a_0| - 2 \sum_{k=1}^n \ln |2s+2k-1|.$$

Differentiate.

$$\frac{1}{a_n(s)} a'_n(s) = -4 \sum_{k=1}^n \frac{1}{2s+2k-1}$$

Evaluating at $s = 1/2$,

$$\begin{aligned} a'_n(1/2) &= -4a_n(1/2) \sum_{k=1}^n \frac{1}{2k} \\ &= \frac{-2a_0}{2^n (n!)^2} \sum_{k=1}^n \frac{1}{k}. \end{aligned}$$

We set $a_0 = 1$ and let $H(n) = \sum_{k=1}^n 1/k$. Then

$$a'_n(1/2) = -\frac{1}{2^{n-1}(n!)^2} H(n).$$

Thus, a second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln x + x^{1/2} \sum_{n=1}^{\infty} a'_n(1/2) x^n \\ &= y_1(x) \ln x - x^{1/2} \sum_{n=1}^{\infty} \frac{H(n)x^n}{2^{n-1}(n!)^2}. \end{aligned}$$

19. Compare $x^2y'' + xy' - (1 + x^2)y = 0$ with $x^2y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 1$ and $q(x) = -(1 + x^2)$, both of which are analytic at $x = 0$, making $x = 0$ a regular singular point. Further, note that $p_0 = p(0) = 1$, and $q_0 = q(0) = -1$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 1 = (s + 1)(s - 1) = 0.$$

Set

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2y'' + xy' - (1 + x^2)y \\ &= [s(s-1) + s - 1]a_0x^s + [(s+1)s + (s+1) - 1]a_1x^{s+1} \\ &\quad + \sum_{n=2}^{\infty} \{[(s+n)(s+n-1) + (s+n) - 1]a_n - a_{n-2}\}x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 - 1 = (s + 1)(s - 1) = 0,$$

with roots $s_2 = -1$ and $s_1 = 1$. Setting the coefficient of the second term equal to zero,

$$s(s+2)a_1 = 0,$$

and because the coefficient of a_1 does not equal zero for either $s = s_2 = -1$ or $s = s_1 = 1$, we must have $a_1 = 0$. The remaining terms provide the recurrence relation

$$(s+n+1)(s+n-1)a_n - a_{n-2} = 0, \quad n \geq 2.$$

Starting with the smaller root, $s = s_2 = -1$, this becomes

$$n(n-2)a_n = a_{n-2}, \quad n \geq 2.$$

However, for $N = s_1 - s_2 = 1 - (-1) = 2$, this becomes

$$0a_2 = a_0 \neq 0,$$

so this equation has no solution and we have a real degeneracy. Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s)x^n = \sum_{n=0}^{\infty} a_n(s)x^{s+n}$$

and compute a_n as a function of s .

$$a_n = \frac{a_{n-2}}{(s+n+1)(s+n-1)}, \quad n \geq 2.$$

Because $a_1 = 0$, this gives us that $a_{2k+1}(s) = 0$ for all $k \geq 0$. Starting with $a_0 \neq 0$,

$$a_{2n} = \frac{a_{2n-2}}{(s+2n+1)(s+2n-1)}, \quad n \geq 1.$$

Thus,

$$a_2 = \frac{a_0}{(s+3)(s+1)}$$

$$a_4 = \frac{a_2}{(s+5)(s+3)},$$

etc., and

$$a_2 a_4 \cdots a_{2n} = \frac{a_0 a_2 \cdots a_{2n-2}}{[(s+3)(s+5) \cdots (s+2n+1)][(s+1)(s+3) \cdots (s+2n-1)]}.$$

Cancelling,

$$a_{2n}(s) = \frac{a_0}{[(s+3)(s+5) \cdots (s+2n+1)][(s+1)(s+3) \cdots (s+2n-1)]}.$$

As the smaller roots is $s = s_2 = -1$, define

$$Y(s, x) = (s+1)y(s, x) = x^s \sum_{n=0}^{\infty} (s+1)a_{2n}(s)x^{2n} = x^s \sum_{n=0}^{\infty} b_{2n}(s)x^{2n},$$

where $b_{2n}(s) = (s+1)a_{2n}(s)$. For example, $b_0(s) = (s+1)a_0$, $b_2(s) = a_0/(s+3)$, and

$$b_{2n}(s) = \frac{a_0}{[(s+3)(s+5) \cdots (s+2n+1)][(s+3) \cdots (s+2n-1)]}.$$

Evaluating at $s = s_2 = -1$, $b_0(-1) = 0$, $b_2(-1) = a_0/2$, and

$$\begin{aligned} b_{2n}(-1) &= \frac{a_0}{[2 \cdot 4 \cdots 2n][2 \cdot 4 \cdots (2n-2)]} \\ &= \frac{a_0}{[2^n n!][2^{n-1} (n-1)!]} \\ &= \frac{a_0}{2^{2n-1} n! (n-1)!}. \end{aligned}$$

Thus, remembering

$$Y(s, x) = x^s \sum_{n=0}^{\infty} b_{2n}(s)x^{2n},$$

and choosing $a_0 = 1$ with $s = s_2 = -1$,

$$\begin{aligned} y_1(x) &= Y(-1, x) = x^{-1} \left[\frac{1}{2} x^2 + \sum_{n=2}^{\infty} \frac{x^{2n}}{2^{2n-1} n! (n-1)!} \right] \\ &= \frac{1}{2} x + \sum_{n=2}^{\infty} \frac{x^{2n-1}}{2^{2n-1} n! (n-1)!}. \end{aligned}$$

A second solution is found by taking the derivative of

$$Y(s, x) = (s+1)y(s, x) = x^s \sum_{n=0}^{\infty} b_{2n}(s)x^{2n}$$

with respect to s .

$$\begin{aligned}\frac{\partial Y}{\partial s} &= x^s \ln x \sum_{n=0}^{\infty} b_{2n}(s)x^{2n} + x^s \sum_{n=0}^{\infty} b'_{2n}(s)x^{2n} \\ &= y_1(x) \ln x + x^s \sum_{n=0}^{\infty} b'_{2n}(s)x^{2n}.\end{aligned}$$

Thus, we need to differentiate $b_{2n}(s)$. First, $b'_0(s) = a_0$, $b'_2(s) = -a_0/(s+3)^2$, so $b'_0(-1) = a_0$ and $b'_2(-1) = -a_0/4$. Next, using logarithmic differentiation,

$$b'_{2n}(s) = b_{2n}(s) \left[-\frac{1}{s+3} - \frac{1}{s+5} - \cdots - \frac{1}{s+2n+1} - \frac{1}{s+3} - \frac{1}{s+5} - \cdots - \frac{1}{s+2n-1} \right],$$

and

$$\begin{aligned}b'_{2n}(-1) &= -b_{2n}(-1) \left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n-2} \right] \\ &= -\frac{a_0}{2^{2n-1}n!(n-1)!} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} \right] \\ &= -\frac{a_0}{2^{2n-1}n!(n-1)!} \left[H(n-1) + \frac{1}{2n} \right], \\ &= \frac{-a_0[2nH(n-1) + 1]}{2^{2n}(n!)^2},\end{aligned}$$

where $H(n) = \sum_{k=1}^n 1/k$. Remembering that $s = s_2 = -1$ and choosing $a_0 = 1$, a second independent solution is

$$\begin{aligned}y_2(x) &= y_1(x) \ln x + x^{-1} \sum_{n=0}^{\infty} b'_{2n}(-1)x^{2n} \\ &= y_1(x) \ln x + x^{-1} \left[1 - \frac{1}{4}x^2 - \sum_{n=2}^{\infty} \frac{[2nH(n-1) + 1]}{2^{2n}(n!)^2} x^{2n} \right].\end{aligned}$$

20. Compare

$$x^2 y'' + x(x-3)y' + 4y = 0$$

with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = x-3$ and $q(x) = 4$ are both analytic at zero, making $x = 0$ a regular singular point. Note that $p_0 = p(0) = -3$ and $q_0 = q(0) = 4$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 4s + 4 = (s-2)^2.$$

Thus, we seek a solution having form

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_n(s)x^n = \sum_{n=0}^{\infty} a_n(s)x^{s+n}.$$

Thus,

$$x^2 y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n},$$

and

$$\begin{aligned} x(x-3)y' &= (x-3) \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} \\ &= \sum_{n=1}^{\infty} (s+n-1)a_{n-1}(s)x^{s+n} - \sum_{n=0}^{\infty} 3(s+n)a_n x^{s+n}, \end{aligned}$$

where we've shifted the index of the first summation. Finally,

$$4y = \sum_{n=0}^{\infty} 4a_n x^{s+n}.$$

Substituting, we get

$$\begin{aligned} 0 &= x^2 y'' + x(x-3)y' + 4y \\ &= \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n} + \sum_{n=1}^{\infty} (s+n-1)a_{n-1}(s)x^{s+n} - \sum_{n=0}^{\infty} 3(s+n)a_n x^{s+n} + \sum_{n=0}^{\infty} 4a_n x^{s+n} \\ &= [s(s-1) - 3s + 4]a_0 x^s + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) - 3(s+n) + 4]a_n + (s+n-1)a_{n-1}(s)\} x^{s+n} \\ &= (s-2)^2 a_0 x^s + \sum_{n=1}^{\infty} [(s+n-2)^2 a_n + (s+n-1)a_{n-1}(s)] x^{s+n}. \end{aligned}$$

Thus, the indicial equation is indeed $I(s) = (s-2)^2 = 0$ with equal roots $s_2 = s_1 = 2$. The remaining terms provide the recurrence relation

$$(s+n-2)^2 a_n + (s+n-1)a_{n-1}(s) = 0, \quad n \geq 1.$$

Since the roots of the indicial equation are equal, we solve the recurrence relation for a_n as a function of s . Thus

$$\begin{aligned} a_1(s) &= -\frac{s}{(s-1)^2} a_0 \\ a_2(s) &= -\frac{s+1}{s^2} a_1 \\ a_3(s) &= -\frac{s+2}{(s+1)^2} a_2 \\ &\vdots \\ a_n &= -\frac{s+n-1}{(s+n-2)^2} a_{n-1}(s). \end{aligned}$$

Multiplying left and right-hand sides of the equations, we get

$$a_1 a_2 \cdots a_n = (-1)^n \frac{s(s+1) \cdots (s+n-1)}{(s-1)^2 s^2 \cdots (s+n-2)^2} a_0 a_1 \cdots a_{n-1}(s).$$

Cancelling, we get

$$a_n(s) = \frac{(-1)^n (s+n-1)}{(s-1)^2 s(s+1) \cdots (s+n-2)} a_0.$$

Evaluating at $s = 2$, we get

$$\begin{aligned} a_n(2) &= \frac{(-1)^n(n+1)}{1 \cdot 2 \cdot 3 \cdots n} a_0 \\ &= \frac{(-1)^n(n+1)}{n!} a_0. \end{aligned}$$

Choosing $a_0 = 1$, our first solution is

$$y_1(x) = y(2, x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{n!} x^n.$$

Next, we use logarithmic differentiation to find $a'_n(s)$.

$$\ln |a_n(s)| = \ln |a_0| + \ln |s + n - 1| - 2 \ln |s - 1| - \sum_{k=0}^{n-2} \ln |s + k|$$

Thus,

$$\begin{aligned} \frac{1}{a_n(s)} a'_n(s) &= \frac{1}{s + n - 1} - \frac{2}{s - 1} - \sum_{k=0}^{n-2} \frac{1}{s + k} \\ a'_n(s) &= a_n(s) \left[\frac{1}{s + n - 1} - \frac{2}{s - 1} - \sum_{k=0}^{n-2} \frac{1}{s + k} \right], \end{aligned}$$

Substitute $s = 2$. Then

$$\begin{aligned} a'_n(2) &= a_n(2) \left[\frac{1}{n+1} - 2 - \sum_{k=0}^{n-2} \frac{1}{k+2} \right] \\ &= \frac{(-1)^n(n+1)}{n!} \left[\frac{1}{n+1} - 2 - \sum_{k=2}^n \frac{1}{k} \right] \\ &= \frac{(-1)^n(n+1)}{n!} \left[\frac{1}{n+1} - 1 - \sum_{k=1}^n \frac{1}{k} \right] \\ &= \frac{(-1)^n}{n!} \left[1 - (n+1) - (n+1) \sum_{k=1}^n \frac{1}{k} \right] \\ &= \frac{(-1)^{n+1}}{n!} [n + (n+1)H(n)], \end{aligned}$$

where we've let $H(n) = \sum_{k=1}^n (1/k)$. Thus,

$$\begin{aligned} y_2 &= y_1 \ln x + x^2 \sum_{n=1}^{\infty} a'_n(2) x^n \\ &= y_1 \ln x + x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} [n + (n+1)H(n)] x^n. \end{aligned}$$

21. Compare $x^2 y'' + 2x(x-2)y' + 2(2-3x)y = 0$ with $x^2 y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 2x - 4$ and $q(x) = 4 - 6x$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = -4$ and $q_0 = q(0) = 4$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 - 5s + 4 = (s-1)(s-4) = 0.$$

Set

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Substituting,

$$\begin{aligned} 0 &= x^2 y'' + x(2x-4)y' + (4-6x)y \\ &= [s(s-1) - 4s + 4]a_0 x^s \\ &\quad + \sum_{n=1}^{\infty} \{[(s+n)(s+n-1) - 4(s+n) + 4]a_n + [2(s+n-1) - 6]a_{n-1}(s)\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 - 5s + 4 = (s-4)(s-1) = 0,$$

with roots $s_2 = 1$ and $s_1 = 4$. The remaining terms provide the recurrence relation

$$(s+n-4)(s+n-1)a_n + 2(s+n-4)a_{n-1}(s) = 0, \quad n \geq 1.$$

Working with the smaller root, $s = s_2 = 1$, this becomes

$$n(n-3)a_n = -2(n-3)a_{n-1}(s), \quad n \geq 1.$$

However, for $N = s_1 - s_2 = 4 - 1 = 3$, we have

$$0a_3 = 0,$$

so this is a false degeneracy. Providing $n \neq 3$, we have

$$a_n = \frac{-2a_{n-1}(s)}{n},$$

so

$$a_1 = -2a_0 \quad \text{and} \quad a_2 = \frac{-2a_1}{2} = 2a_0.$$

Letting a_3 be an arbitrary constant,

$$\begin{aligned} a_4 &= \frac{-2a_3}{4}, \\ a_5 &= \frac{-2a_4}{5}, \end{aligned}$$

etc., and

$$a_4 a_5 \dots a_n = \frac{(-2)^{n-3} a_3 a_4 \dots a_{n-1}(s)}{4 \cdot 5 \cdot \dots \cdot n}.$$

Cancelling,

$$\begin{aligned} a_n &= \frac{(-2)^{n-3}a_3}{4 \cdot 5 \cdot \dots \cdot n} \\ &= \frac{6(-2)^{n-3}a_3}{n!}, \quad n \geq 4. \end{aligned}$$

Thus,

$$\begin{aligned} y(x) &= x[a_0 - 2a_0x + 2a_0x^2] + x \left[a_3x^3 + \sum_{n=4}^{\infty} \frac{6(-2)^{n-3}}{n!} x^n \right] \\ &= a_0x[1 - 2x + 2x^2] + a_3x \left[x^3 + \sum_{n=4}^{\infty} \frac{6(-2)^{n-3}}{n!} x^n \right]. \end{aligned}$$

Setting $a_0 = 1$ and $a_3 = 0$ provides

$$y_1(x) = x - 2x^2 + 2x^3.$$

Setting $a_0 = 0$ and $a_3 = 1$ provides a second independent solution

$$y_2(x) = x^4 + \sum_{n=4}^{\infty} \frac{6(-2)^{n-3}}{n!} x^{n+1}.$$

22. Compare $x^2y'' + x(3+x^2)y' + (1+3x^2)y = 0$ with $x^2y'' + xp(x)y' + q(x)y = 0$ and note that $p(x) = 3+x^2$ and $q(x) = 1+3x^2$ are both analytic at $x = 0$, making $x = 0$ a regular singular point. Further, $p_0 = p(0) = 3$ and $q_0 = q(0) = 1$, so the indicial equation should be

$$I(s) = s^2 + (p_0 - 1)s + q_0 = s^2 + 2s + 1 = (s + 1)^2 = 0.$$

Set

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{s+n}.$$

Then,

$$x^2y'' = \sum_{n=0}^{\infty} (s+n)(s+n-1)a_n x^{s+n},$$

and

$$\begin{aligned} x(3+x^2)y' &= (3+x^2) \sum_{n=0}^{\infty} (s+n)a_n x^{s+n} \\ &= \sum_{n=0}^{\infty} 3(s+n)a_n x^{s+n} + \sum_{n=2}^{\infty} (s+n-2)a_{n-2} x^{s+n}, \end{aligned}$$

Where we've shifted the index of the second summation. Finally,

$$\begin{aligned} (1+3x^2)y &= (1+3x^2) \sum_{n=0}^{\infty} a_n x^{s+n} \\ &= \sum_{n=0}^{\infty} a_n x^{s+n} + \sum_{n=2}^{\infty} 3a_{n-2} x^{s+n}, \end{aligned}$$

where we've again shifted the index of the second summation. Substituting,

$$\begin{aligned} 0 &= x^2 y'' + x(3 + x^2)y' + (1 + 3x^2)y \\ &= [s(s-1) + 3s + 1]a_0 x^s + [(s+1)s + 3(s+1) + 1]a_1 x^{s+1} \\ &\quad + \sum_{n=2}^{\infty} \{[(s+n)(s+n-1) + 3(s+n) + 1]a_n + [(s+n-2) + 3]a_{n-2}\} x^{s+n}. \end{aligned}$$

Thus, the indicial equation is

$$I(s) = s^2 + 2s + 1 = (s+1)^2 = 0,$$

with equal roots $s_2 = s_1 = -1$. The coefficient of x^{s+1} must equal zero,

$$(s+2)^2 a_1 = 0,$$

but the coefficient of a_1 is not zero for $s_2 = s_1 = -1$. Thus, $a_1 = 0$. The remaining terms provide the recurrence relation

$$(s+n+1)^2 a_n + (s+n+1)a_{n-2} = 0, \quad n \geq 2.$$

For $s = -1$, this becomes

$$n^2 a_n + n a_{n-2} = 0, \quad n \geq 2.$$

Thus,

$$a_n = -\frac{a_{n-2}}{n},$$

which, because $a_1 = 0$, implies that $a_{2k+1} = 0$ for $k \geq 0$. Thus,

$$a_{2n} = \frac{-a_{2n-2}}{2n},$$

and

$$\begin{aligned} a_2 &= \frac{-a_0}{2}, \\ a_4 &= \frac{-a_2}{4}, \end{aligned}$$

etc, so that

$$a_2 a_4 \cdots a_{2n} = \frac{(-1)^n a_0 a_2 \cdots a_{2n-2}}{2 \cdot 4 \cdots 2n}.$$

Cancelling,

$$a_{2n} = \frac{(-1)^n a_0}{2^n n!}, \quad n \geq 1.$$

Hence, one solution is

$$y_1(x) = x^{-1} \left[a_0 + \sum_{n=1}^{\infty} \frac{(-1)^n a_0}{2^n n!} x^{2n} \right].$$

With $a_0 = 1$, this becomes

$$y_1(x) = x^{-1} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} \right].$$

To find a second independent solution, solve the recurrence relation for a_n as a function of s .

$$a_{2n} = \frac{-a_{2n-2}}{s + 2n + 1}, \quad n \geq 1.$$

Thus,

$$a_2 = \frac{-a_0}{s + 3},$$

$$a_4 = \frac{-a_2}{s + 5},$$

etc., and

$$a_2 a_4 \cdots a_{2n} = \frac{(-1)^n a_0 a_2 \cdots a_{2n-2}}{(s + 3)(s + 5) \cdots (s + 2n + 1)}.$$

Cancelling,

$$a_{2n}(s) = \frac{(-1)^n a_0}{(s + 3)(s + 5) \cdots (s + 2n + 1)}, \quad n \geq 1.$$

Set

$$y(s, x) = x^s \sum_{n=0}^{\infty} a_{2n}(s) x^{2n}$$

and differentiate with respect to s to find a second solution.

$$\begin{aligned} \frac{\partial y}{\partial x} &= x^s \ln x \sum_{n=0}^{\infty} a_{2n}(s) x^{2n} + x^s \sum_{n=1}^{\infty} a'_{2n}(s) x^{2n} \\ &= y_1(x) \ln x + x^s \sum_{n=1}^{\infty} a'_{2n}(s) x^{2n}. \end{aligned}$$

Using logarithmic differentiation,

$$a'_{2n}(s) = a_{2n}(s) \left[-\frac{1}{s + 3} - \frac{1}{s + 5} - \cdots - \frac{1}{s + 2n + 1} \right].$$

At $s = -1$,

$$\begin{aligned} a'_{2n}(-1) &= -a_{2n}(-1) \left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \right] \\ &= \frac{(-1)^{n+1} a_0}{2^n n!} \left[\frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) \right] \\ &= \frac{(-1)^{n+1} a_0}{2^{n+1} n!} H(n), \end{aligned}$$

where $H(n) = \sum_{k=1}^n 1/k$. Thus, using $a_0 = 1$ and $s = -1$, second solution is

$$y_2(x) = y_1(x) \ln x + x^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H(n)}{2^{n+1} n!} x^{2n}.$$

23. (a) Divide by
- $1 - x^2$
- to get the equation

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0.$$

The coefficients are $P(x) = -2x/(1-x^2)$ and $Q(x) = n(n+1)/(1-x^2)$. We see that $(x-1)P(x) = 2x/(x+1)$ and $(x-1)^2Q(x) = -n(n+1)(x-1)/(x+1)$ are both analytic at $x = x_0 = 1$, so $x_0 = 1$ is a regular singular point. Similarly, $(x+1)P(x) = 2x/(x-1)$ and $(x+1)^2Q(x) = -n(n+1)(x+1)/(x-1)$ are both analytic at $x = x_0 = -1$, so $x_0 = -1$ is a regular singular point.

- (b) We make the substitution
- $x = t + 1$
- , or
- $t = x - 1$
- , and the equation becomes

$$-t(2+t)y'' - 2(1+t)y' + n(n+1)y = 0.$$

We multiply by $-t$ so that the fundamental expressions are present, and the equation becomes

$$(2+t)t^2y'' + 2(1+t)ty' - n(n+1)ty = 0.$$

Separating powers of t times fundamental expressions, this becomes

$$[2t^2y'' + 2ty'] + t[t^2y'' + 2ty' - n(n+1)y] = 0.$$

Substituting the series for the fundamental expressions, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 2(s+n)]a_n t^{s+n} \\ & + t \sum_{n=0}^{\infty} [(s+n)(s+n-1) + 2(s+n) - n(n+1)]a_n t^{s+n+1} \\ & = 0. \end{aligned}$$

The indicial polynomial is the coefficient of t^s , and it comes from the first term in the first sum. Thus $I(s) = 2s^2$. We have a double root at $s = 0$, so there is only one Frobenius solution. However, since $s = 0$, this solution has the form $y_1(t) = \sum_{n=0}^{\infty} a_n t^n$, where $a_0 \neq 0$. Hence y_1 is analytic at $t = 0$, and $y_1(0) \neq 0$. In terms of x we have the solution $Y_1(x) = y_1(x-1) = \sum_{n=0}^{\infty} a_n (x-1)^n$, so Y_1 is analytic at $x = x_0 = 1$, and therefore bounded there. By Theorem 6.46, the second solution has the form

$$Y_2(x) = Y_1(x) \ln|x-1| + \sum_{n=1}^{\infty} b_n (x-1)^n.$$

Since $\ln|x-1|$ approaches $-\infty$ as $x \rightarrow 1$ and $Y_1(1) \neq 0$, this solution is unbounded.

- (c) We make the substitution
- $x = t - 1$
- , or
- $t = x + 1$
- , and the equation becomes

$$t(2-t)y'' - 2(t-1)y' + n(n+1)y = 0.$$

We multiply by t so that the fundamental expressions are present, and the equation becomes

$$(2-t)t^2y'' + 2(1-t)ty' + n(n+1)ty = 0.$$

Separating powers of t times fundamental expressions, this becomes

$$[2t^2y'' + 2ty'] + t[-t^2y'' - 2ty' + n(n+1)y] = 0.$$

Substituting the series for the fundamental expressions, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} [2(s+n)(s+n-1) + 2(s+n)] a_n t^{s+n} \\ & + t \sum_{n=0}^{\infty} [-(s+n)(s+n-1) - 2(s+n) + n(n+1)] a_n t^{s+n+1} \\ & = 0. \end{aligned}$$

The indicial polynomial is the coefficient of t^s , and it comes from the first term in the first sum. Thus $I(s) = 2s^2$. We have a double root at $s = 0$, so there is only one Frobenius solution. However, since $s = 0$, this solution has the form $y_1(t) = \sum_{n=0}^{\infty} a_n t^n$, where $a_0 \neq 0$. Hence y_1 is analytic at $t = 0$, and $y_1(0) \neq 0$. In terms of x we have the solution $Y_1(x) = y_1(x+1) = \sum_{n=0}^{\infty} a_n (x+1)^n$, so Y_1 is analytic at $x = x_0 = -1$, and therefore bounded there. By Theorem 6.46, the second solution has the form

$$Y_2(x) = Y_1(x) \ln |x+1| + \sum_{n=1}^{\infty} b_n (x+1)^n.$$

Since $\ln |x+1|$ approaches $-\infty$ as $x \rightarrow -1$ and $Y_1(-1) \neq 0$, this solution is unbounded.

————— × —————

Section 11.7. Bessel's Equation and Bessel Functions

1. Using

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(r+n)!} \left(\frac{x}{2}\right)^{2n+r},$$

we can write

$$\begin{aligned} J_{1/2}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(1/2+n)!} \left(\frac{x}{2}\right)^{2n+1/2} = \left(\frac{x}{2}\right)^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(1/2+n)!} \left(\frac{x}{2}\right)^{2n+1} \\ &= \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! 2^{n+1} (1/2+n)!} \end{aligned}$$

Now,

$$2^n n! = 2^n n(n-1) \cdots 1 = 2n(2n-2) \cdots 2.$$

Also,

$$\begin{aligned}
 2^{n+1} \left(\frac{1}{2} + n \right)! &= 2^{n+1} \Gamma \left(\frac{1}{2} + n + 1 \right) \\
 &= 2^{n+1} \left(\frac{1}{2} + n \right) \Gamma \left(\frac{1}{2} + n \right) \\
 &= 2^{n+1} \left(n + \frac{1}{2} \right) \left(n - \frac{1}{2} \right) \cdots \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{2} \right) \\
 &= (2n+1)(2n-1) \cdots 1 \cdot \sqrt{\pi}.
 \end{aligned}$$

Thus, rearranging terms,

$$2^n n! 2^{n+1} \left(\frac{1}{2} + n \right)! = (2n+1)(2n)(2n-1)(2n-2) \cdots (2)(1)\sqrt{\pi} = (2n+1)!\sqrt{\pi}.$$

Therefore,

$$J_{1/2}(x) = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sqrt{\frac{2}{\pi x}} \sin x.$$

In a similar fashion

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n-1/2)!} \left(\frac{x}{2} \right)^{2n-1/2} = \left(\frac{x}{2} \right)^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n! 2^n (n-1/2)!}.$$

Again,

$$\begin{aligned}
 2^n n! 2^n (n-1/2)! &= 2^n n(n-1) \cdots 1 \cdot 2^n \left(n - \frac{1}{2} \right) \left(n - \frac{3}{2} \right) \cdots \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{2} \right) \\
 &= 2n(2n-2) \cdots 2 \cdot (2n-1)(2n-3) \cdots 1 \cdot \sqrt{\pi} \\
 &= (2n)!\sqrt{\pi}.
 \end{aligned}$$

Thus,

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sqrt{\frac{2}{\pi x}} \cos x.$$

2. By definition, we can write

$$x^p J_p(x) = x^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2} \right)^{2n+p} = x^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \frac{x^{2n+p}}{2^{2n+p}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{n!(n+p)! 2^{2n+p}}.$$

Thus,

$$\begin{aligned}
 (x^p J_p(x))' &= \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+p) x^{2n+2p-1}}{n!(n+p)! 2^{2n+p}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p-1}}{n!(n+p-1)! 2^{2n+p-1}} \\
 &= x^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p-1)!} \left(\frac{x}{2} \right)^{2n+p-1} = x^p J_{p-1}(x).
 \end{aligned}$$

3. By definition, we can write

$$x^{-p} J_p(x) = x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p} = x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{n!(n+p)!2^{2n+p}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!(n+p)!2^{2n+p}}.$$

Thus,

$$[x^{-p} J_p(x)]' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{n!(n+p)!2^{2n+p}} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(n-1)!(n+p)!2^{2n+p-1}},$$

so

$$[x^{-p} J_p(x)]' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{n!(n+p+1)!2^{2n+p+1}} = -x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p+1}}{n!(n+p+1)!2^{2n+p+1}} = -x^{-p} J_{p+1}(x).$$

4. First, the product rule provides

$$[x^p J_p]' = x^p J_p' + p x^{p-1} J_p.$$

Now use the result

$$[x^p J_p]' = x^p J_{p-1}$$

to write

$$\begin{aligned} x^p J_p' + p x^{p-1} J_p &= x^p J_{p-1} \\ x^p J_p' &= x^p J_{p-1} - p x^{p-1} J_p \\ J_p' &= J_{p-1} - \frac{p}{x} J_p. \end{aligned}$$

Secondly, use the product rule to write

$$[x^{-p} J_p]' = x^{-p} J_p' - p x^{-p-1} J_p.$$

Now, use the result

$$[x^{-p} J_p]' = -x^{-p} J_{p+1}$$

and write

$$\begin{aligned} x^{-p} J_p' - p x^{-p-1} J_p &= -x^{-p} J_{p+1} \\ x^{-p} J_p' &= -x^{-p} J_{p+1} + p x^{-p-1} J_p \\ J_p' &= -J_{p+1} + \frac{p}{x} J_p. \end{aligned}$$

5. In Exercise 4, we showed that

$$\begin{aligned} J_p' &= J_{p-1} - \frac{p}{x} J_p \\ J_p' &= -J_{p+1} + \frac{p}{x} J_p. \end{aligned}$$

Thus,

$$\begin{aligned} J_{p-1} - \frac{p}{x} J_p &= -J_{p+1} + \frac{p}{x} J_p \\ J_{p+1} &= \frac{2p}{x} J_p - J_{p-1}. \end{aligned}$$

Similarly, if we add the two original equations, we get

$$2J_p' = J_{p-1} - J_{p+1}.$$

6. In Exercise 1, we established that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Using the result from Exercise 5,

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x),$$

and $p = 1/2$, we write

$$J_{3/2}(x) = J_{1/2+1}(x) = \frac{2(1/2)}{x} J_{1/2}(x) - J_{-1/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x),$$

so,

$$J_{3/2}(x) = J_{1/2+1}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{1}{x} \sin x - \cos x \right) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x - x \cos x}{x} \right).$$

7. Let a and b be consecutive zeros of J_0 . That is, $J_0(a) = J_0(b) = 0$. By Rolle's theorem, there exists a $a < c < b$ so that $J'_0(c) = 0$. Use $J'_p = -J_{p+1} + (p/x)J_p$ with $p = 0$ to obtain

$$J'_0 = -J_1.$$

Thus, $J_1(c) = -J'_0(c) = 0$. Therefore, there exists a zero of J_1 between any consecutive zeros of J_0 .

On the other hand, let α and β be consecutive zeros of J_1 ; i.e., $J_1(\alpha) = J_1(\beta) = 0$. Then α and β are consecutive zeros of $xJ_1(x)$. Let $g(x) = xJ_1(x)$. By Rolle's Theorem, there exists $\alpha < \gamma < \beta$ so that $g'(\gamma) = 0$. However, using $J'_p = J_{p-1} - (p/x)J_p$ with $p = 1$,

$$\begin{aligned} J'_1 &= J_0 - \frac{1}{x} J_1 \\ xJ'_1 &= xJ_0 - J_1 \\ xJ_0 &= xJ'_1 + J_1 \\ xJ_0 &= \frac{d}{dx} \{xJ_1\} \\ xJ_0 &= g'(x). \end{aligned}$$

Hence, $xJ_0(x) = g'(x)$ and

$$\gamma J_0(\gamma) = g'(\gamma) = 0.$$

Because $\gamma > 0$, $J_0(\gamma) = 0$. Hence, between any two consecutive zeros of J_1 lies a zero of J_0 .

8. Consider the Bessel equation of order r .

$$x^2 y'' + xy' + (x^2 - r^2)y = 0$$

Let $y(x) = x^{-1/2}u(x)$. Then,

$$y' = x^{-1/2}u' - \frac{1}{2}x^{-3/2}u,$$

and

$$\begin{aligned} y'' &= x^{-1/2}u'' - \frac{1}{2}x^{-3/2}u' - \frac{1}{2}x^{-3/2}u' + \frac{3}{4}x^{-5/2}u \\ &= x^{-1/2}u'' - x^{-3/2}u' + \frac{3}{4}x^{-5/2}u. \end{aligned}$$

Substituting,

$$\begin{aligned}
 0 &= x^2 \left(x^{-1/2} u'' - x^{-3/2} u' + \frac{3}{4} x^{-5/2} u \right) + x \left(x^{-1/2} u' - \frac{1}{2} x^{-3/2} u \right) + (x^2 - r^2) x^{-1/2} u \\
 &= x^{3/2} u'' + \frac{1}{4} x^{-1/2} u + x^{3/2} u - r^2 x^{-1/2} u \\
 &= x^{3/2} \left[u'' + \left(1 + \frac{1}{4} x^{-2} - r^2 x^{-2} \right) u \right] \\
 &= x^{3/2} \left[u'' + \left(1 - \frac{r^2 - 1/4}{x^2} \right) u \right].
 \end{aligned}$$

Thus, provided $x^{3/2}$ is not identically zero,

$$u'' + \left(1 - \frac{r^2 - 1/4}{x^2} \right) u = 0.$$

Now, when $r = 1/2$, this last equation becomes

$$u'' + u = 0,$$

with $u_1(x) = \cos x$ and $u_2(x) = \sin x$ forming a fundamental set of solutions. With $y(x) = x^{-1/2} u(x)$, we get the following solutions of Bessel's equation with $r = 1/2$.

$$y_1(x) = x^{-1/2} u_1(x) = x^{-1/2} \cos x$$

$$y_2(x) = x^{-1/2} u_2(x) = x^{-1/2} \sin x.$$

9. Consider

$$x^2 y'' + xy' + (\mu^2 x^2 - p^2)y = 0,$$

then let $y(x) = u(\mu x)$. That is, set $t = \mu x$, then

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \mu \frac{dy}{dt},$$

and

$$y'' = \frac{d}{dx} \left(\mu \frac{dy}{dt} \right) = \mu \frac{d^2 y}{dt^2} \frac{dt}{dx} = \mu^2 \frac{d^2 y}{dt^2}.$$

Substituting,

$$\left(\frac{t^2}{\mu^2} \right) (\mu^2 y'') + \left(\frac{t}{\mu} \right) (\mu y') + (t^2 - p^2)y = 0,$$

where y' now means differentiation with respect to t . Thus, our equation becomes

$$t^2 y'' + ty' + (t^2 - p^2)y = 0,$$

which has general solution

$$y(t) = AJ_p(t) + BY_p(t).$$

Of course, with $t = \mu x$, this becomes,

$$y(x) = aJ_p(\mu x) + bY_p(\mu x),$$

where a and b are arbitrary constants.

10. Consider

$$\frac{d^2 y}{dx^2} + xy = 0,$$

known as Airy's equation and set $u = x^{-1/2}y$ and $t = (2/3)x^{3/2}$. Then, $y = x^{1/2}u$ and

$$\frac{dy}{dx} = x^{1/2} \frac{du}{dx} + \frac{1}{2} x^{-1/2} u.$$

A second derivative,

$$\begin{aligned} \frac{d^2 y}{dx^2} &= x^{1/2} \frac{d^2 u}{dx^2} + \frac{1}{2} x^{-1/2} \frac{du}{dx} + \frac{1}{2} x^{-1/2} \frac{du}{dx} - \frac{1}{4} x^{-3/2} u \\ &= x^{1/2} \frac{d^2 u}{dx^2} + x^{-1/2} \frac{du}{dx} - \frac{1}{4} x^{-3/2} u. \end{aligned}$$

Substituting,

$$0 = \frac{d^2 y}{dx^2} + xy = x^{1/2} \frac{d^2 u}{dx^2} + x^{-1/2} \frac{du}{dx} - \frac{1}{4} x^{-3/2} u + x^{3/2} u.$$

However, since $t = (2/3)x^{3/2}$,

$$\frac{du}{dx} = \frac{du}{dt} \frac{dt}{dx} = x^{1/2} \frac{du}{dt}.$$

A second time,

$$\begin{aligned} \frac{d^2 u}{dx^2} &= x^{1/2} \frac{d}{dx} \left(\frac{du}{dt} \right) + \frac{1}{2} x^{-1/2} \frac{du}{dt} \\ &= x^{1/2} \frac{d^2 u}{dt^2} \frac{dt}{dx} + \frac{1}{2} x^{-1/2} \frac{du}{dt} \\ &= x \frac{d^2 u}{dt^2} + \frac{1}{2} x^{-1/2} \frac{du}{dt}. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= \frac{d^2 y}{dx^2} + xy \\ &= x^{1/2} \left(x \frac{d^2 u}{dt^2} + \frac{1}{2} x^{-1/2} \frac{du}{dt} \right) + x^{-1/2} \left(x^{1/2} \frac{du}{dt} \right) - \frac{1}{4} x^{-3/2} u + x^{3/2} u \\ &= x^{3/2} \frac{d^2 u}{dt^2} + \frac{1}{2} \frac{du}{dt} + \frac{du}{dt} - \frac{1}{4} x^{-3/2} u + x^{3/2} u. \end{aligned}$$

But $x^{3/2} = 3t/2$, so

$$0 = \frac{d^2 y}{dx^2} + xy = \frac{3t}{2} \frac{d^2 u}{dt^2} + \frac{3}{2} \frac{du}{dt} - \frac{1}{4} \left(\frac{2}{3t} \right) u + \frac{3t}{2} u.$$

Thus,

$$0 = t \frac{d^2 u}{dt^2} + \frac{du}{dt} - \frac{1}{9t} u + tu,$$

and

$$0 = t^2 \frac{d^2 u}{dt^2} + t \frac{du}{dt} + \left(t^2 - \frac{1}{9} \right) u.$$

Of course, the solution of this equation is

$$y(t) = AJ_{1/3}(t) + BJ_{-1/3}(t).$$

With $t = (2/3)x^{3/2}$, this becomes

$$y(x) = aJ_{1/3}\left(\frac{2}{3}x^{3/2}\right) + bJ_{-1/3}\left(\frac{2}{3}x^{3/2}\right),$$

with a and b arbitrary constants.

11. With $t = ax^b$,

$$dt = abx^{b-1}dx \quad \text{and} \quad \frac{dx}{dt} = \frac{1}{ab}x^{1-b}.$$

Hence, we have the following operator.

$$\frac{d}{dt} = \frac{d}{dx} \frac{dx}{dt} = \frac{1}{ab}x^{1-b} \frac{d}{dx}.$$

Now, we can write

$$t^2 \frac{d^2u}{dt^2} + t \frac{du}{dt} + (t^2 - r^2)u = 0$$

as

$$t \frac{d}{dt} \left(t \frac{du}{dt} \right) + (t^2 - r^2)u = 0.$$

Substituting $t = ax^b$ and using our operator,

$$\begin{aligned} ax^b \left(\frac{1}{ab} x^{1-b} \frac{d}{dx} \left(ax^b \left(\frac{1}{ab} x^{1-b} \frac{du}{dx} \right) \right) \right) + (a^2 x^{2b} - r^2)u &= 0 \\ \frac{1}{b} x \frac{d}{dx} \left(\frac{1}{b} x \frac{du}{dx} \right) + (a^2 x^{2b} - r^2)u &= 0 \\ x \frac{d}{dx} \left(x \frac{du}{dx} \right) + (a^2 b^2 x^{2b} - b^2 r^2)u &= 0. \end{aligned}$$

Now, with $u = x^c y$,

$$x \frac{du}{dx} = x \left[x^c \frac{dy}{dx} + cx^{c-1}y \right] = x^{c+1} \frac{dy}{dx} + cx^c y.$$

Substituting these in our last result,

$$\begin{aligned} x \frac{d}{dx} \left[x^{c+1} \frac{dy}{dx} + cx^c y \right] + (a^2 b^2 x^{2b} - b^2 r^2)x^c y &= 0 \\ x \left[x^{c+1} \frac{d^2y}{dx^2} + (c+1)x^c \frac{dy}{dx} + cx^c \frac{dy}{dx} + c^2 x^{c-1}y \right] + (a^2 b^2 x^{2b} - b^2 r^2)x^c y &= 0 \\ x^{c+2} \frac{d^2y}{dx^2} + x^{c+1} [2c+1] \frac{dy}{dx} + (a^2 b^2 x^{2b} - b^2 r^2 + c^2)x^c y &= 0. \end{aligned}$$

Dividing by x^c ,

$$x^2 \frac{d^2 y}{dx^2} + (2c + 1)x \frac{dy}{dx} + (a^2 b^2 x^{2b} - b^2 r^2 + c^2)y = 0.$$

Now, letting $\alpha = 2c + 1$, $\beta = ab$, and $\gamma = c^2 - r^2 b^2$, this last equation becomes

$$x^2 \frac{d^2 y}{dx^2} + \alpha x \frac{dy}{dx} + (\beta^2 x^{2b} + \gamma)y = 0.$$

12. Consider

$$m \frac{d^2 y}{dt^2} + k e^{-\epsilon t} y = 0.$$

Set $x = e^{-\epsilon t/2}$. Then,

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{-\epsilon}{2} e^{-\epsilon t/2} \frac{dy}{dx}.$$

A second differentiation,

$$\begin{aligned} \frac{d^2 y}{dt^2} &= \frac{d}{dt} \left(-\frac{\epsilon}{2} e^{-\epsilon t/2} \frac{dy}{dx} \right) \\ &= -\frac{\epsilon}{2} e^{-\epsilon t/2} \frac{d}{dt} \left(\frac{dy}{dx} \right) + \frac{\epsilon^2}{4} e^{-\epsilon t/2} \frac{dy}{dx} \\ &= -\frac{\epsilon}{2} e^{-\epsilon t/2} \frac{d^2 y}{dx^2} \frac{dx}{dt} + \frac{\epsilon^2}{4} e^{-\epsilon t/2} \frac{dy}{dx} \\ &= \frac{\epsilon^2}{4} e^{-\epsilon t} \frac{d^2 y}{dx^2} + \frac{\epsilon^2}{4} e^{-\epsilon t/2} \frac{dy}{dx}. \end{aligned}$$

Substituting,

$$\begin{aligned} 0 &= m \frac{d^2 y}{dt^2} + k e^{-\epsilon t} y \\ &= m \left[\frac{\epsilon^2}{4} e^{-\epsilon t} \frac{d^2 y}{dx^2} + \frac{\epsilon^2}{4} e^{-\epsilon t/2} \frac{dy}{dx} \right] + k e^{-\epsilon t} y \\ &= \frac{m\epsilon^2}{4} e^{-\epsilon t} \frac{d^2 y}{dx^2} + \frac{m\epsilon^2}{4} e^{-\epsilon t/2} \frac{dy}{dx} + k e^{-\epsilon t} y. \end{aligned}$$

But, $x = e^{-\epsilon t/2}$, so

$$\begin{aligned} 0 &= \frac{m\epsilon^2}{4} x^2 \frac{d^2 y}{dx^2} + \frac{m\epsilon^2}{4} x \frac{dy}{dx} + k x^2 y \\ 0 &= x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(\frac{4k}{m\epsilon^2} x^2 - 0 \right) y. \end{aligned}$$

Comparing with Exercise 9,

$$x^2 y'' + xy' + (\mu^2 x^2 - p^2)y = 0,$$

we have

$$\mu^2 = \frac{4k}{m\epsilon^2} \quad \text{and} \quad p = 0.$$

Thus, the solution in Exercise 9,

$$y(x) = aJ_p(\mu x) + bY_p(\mu x),$$

becomes, in this case,

$$y(x) = aJ_0(\mu x) + bY_0(\mu x),$$

where

$$\mu = \frac{2\sqrt{k/m}}{\epsilon}.$$

Thus, as a function of t ,

$$y(t) = aJ_0(\mu e^{-\epsilon t/2}) + bY_0(\mu e^{-\epsilon t/2}),$$

where, again $\mu = 2\sqrt{k/m}/\epsilon$.

13. According to the text, the roots of the indicial equation for Bessel's equation of order 2 are $s = \pm 2$. All of the odd-numbered coefficients are equal to 0, and by 7.4, the recurrence formula for the even-numbered terms is

$$a_{2n} = -\frac{a_{2n-2}}{(s+2n-2)(s+2n+2)}.$$

Solving for the first two coefficients as functions of s , we get

$$a_2(s) = -\frac{a_0}{s(s+4)} \quad \text{and} \quad a_4(s) = -\frac{a_2(s)}{(s+2)(s+6)} = \frac{a_0}{s(s+2)(s+4)(s+6)}.$$

Proceeding inductively we find that

$$a_{2n}(s) = \frac{(-1)^n a_0}{s \cdot (s+2) \cdot (s+4) \cdot \dots \cdot (s+2n-2) \cdot (s+4) \cdot \dots \cdot (s+2n+2)}, \quad \text{for } n \geq 1.$$

Notice that a_2 is defined near $s = -2$, but for $n \geq 2$, $a_{2n}(s)$ contains the factor $s+2$ in the denominator, and so is undefined at $s = -1$, reflecting the fact that we have an exceptional case.

Following Section 6, we set $b_{2n}(s) = (s+2)a_{2n}(s)$. Then

$$b_0(s) = (s+2)a_0, \quad b_2(s) = -\frac{(s+2)a_0}{s(s+4)}, \quad \text{and} \quad b_4(s) = \frac{a_0}{s(s+4)(s+6)}.$$

For $n \geq 3$

$$b_{2n}(s) = \frac{(-1)^n a_0}{s \cdot (s+4) \cdot \dots \cdot (s+2n-2) \cdot (s+4) \cdot \dots \cdot (s+2n+2)}.$$

We form

$$Y(s, x) = \sum_{n=0}^{\infty} b_{2n}(s) x^{s+2n}.$$

Our first solution is $Y(-2, x)$. Evaluating the coefficients at $s = -2$, we get $b_0(-2) = b_2(-2) = 0$, and

$$b_{2n}(-2) = \frac{(-1)^{n+1} a_0}{2^{2n-1} (n-2)! n!}, \quad \text{for } n \geq 2.$$

Thus our solution is

$$\begin{aligned}
 Y(-2, x) &= x^{-2} a_0 \sum_{n=2}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2^{2n-1}(n-2)!n!} \\
 &= -a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2^{2n+3}n!(n+2)!} \\
 &= \frac{-a_0}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+2)!} \left(\frac{x}{2}\right)^{2n+2} \\
 &= \frac{-a_0}{2} J_2(x).
 \end{aligned}$$

Our second solution will be

$$\frac{\partial Y}{\partial s}(-2, x) = Y(-2, x) \ln x + \sum_{n=0}^{\infty} b'_{2n}(-2)x^{-1+2n}.$$

We compute that $b'_0 = a_0$ and $b'_2(-2) = a_0/4$. For $n \geq 2$ we have

$$\ln |b_{2n}(s)| = \ln |a_0| - \sum_{k=1}^n \ln |s + 2k + 2| - \sum_{k=1}^{n-2} \ln |s + 2k + 2| - \ln |s|.$$

Hence

$$\begin{aligned}
 b'_{2n}(-2) &= -b_{2n}(-2) \left[\sum_{k=1}^n \frac{1}{2k} + \sum_{k=1}^{n-2} \frac{1}{2k} - \frac{1}{2} \right] \\
 &= \frac{(-1)^n a_0 [\phi(n) + \phi(n-2)]}{2^{2n}(n-2)!n!} + \frac{1}{2} b_{2n}(-2)
 \end{aligned}$$

for $n \geq 2$. Hence, our second solution is

$$\begin{aligned}
 \frac{\partial Y}{\partial s}(-2, x) &= \frac{-a_0}{2} J_2(x) \ln x + a_0 x^{-2} + \frac{a_0}{4} + x^{-2} \sum_{n=2}^{\infty} \frac{(-1)^n a_0 [\phi(n) + \phi(n-2)]}{2^{2n}(n-2)!n!} x^{2n} \\
 &\quad + \frac{1}{2} x^{-2} \sum_{n=0}^{\infty} b_{2n}(-2) x^{2n}
 \end{aligned}$$

The last term will be recognized as $Y(-2, x)/2 = -a_0 J_2(x)/4$. If we choose $a_0 = -2$, our solutions are

$$\begin{aligned}
 y_1(x) &= Y(-2, x) = J_2(x) \quad \text{and} \\
 y_2(x) &= \frac{\partial Y}{\partial s}(-2, x) \\
 &= J_2(x) \ln x - \frac{2}{x^2} - \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n [\phi(n) + \phi(n+2)]}{n!(n+2)!} \left(\frac{x}{2}\right)^{2n+2} + \frac{1}{2} J_2(x)
 \end{aligned}$$

Chapter 12. Fourier Series

Section 12.1. Computation of Fourier Series

1. The function $f(x) = |\sin x|$ is even on the interval $[-\pi, \pi]$. Hence, the Fourier expansion will contain only cosine terms. Its Fourier coefficients are

$$a_n = \frac{2}{\pi} \int_0^\pi |\sin x| \cos nx \, dx.$$

However, $\sin x \geq 0$ on $[0, \pi]$, so

$$a_n = \frac{2}{\pi} \int_0^\pi \sin x \cos nx \, dx.$$

Now, for $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^\pi \sin x \, dx = \frac{2}{\pi} [-\cos x]_0^\pi = \frac{4}{\pi}.$$

For $n = 1$,

$$a_1 = \frac{2}{\pi} \int_0^\pi \sin x \cos x \, dx = 0.$$

Now, for $n > 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi \sin x \cos nx \, dx \\ &= \frac{1}{\pi} \int_0^\pi [\sin(1+n)x + \sin(1-n)x] \, dx \\ &= \frac{1}{\pi} \left[\frac{-1}{1+n} \cos(1+n)x - \frac{1}{1-n} \cos(1-n)x \right]_0^\pi \\ &= \frac{1}{\pi} \left[\frac{-1}{1+n} (-1)^{1+n} - \frac{1}{1-n} (-1)^{1-n} \right. \\ &\quad \left. + \frac{1}{1+n} + \frac{1}{1-n} \right] \\ &= \frac{1}{\pi} \left\{ \frac{1}{1+n} [1 - (-1)^{1+n}] \right. \\ &\quad \left. + \frac{1}{1-n} [1 - (-1)^{1-n}] \right\} \end{aligned}$$

Thus, if n is odd, $a_n = 0$. On the other hand, if n is even,

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[\frac{2}{1+n} + \frac{2}{1-n} \right] \\ &= \frac{1}{\pi} \left[\frac{2-2n+2+2n}{(1+n)(1-n)} \right] \\ &= \frac{4}{\pi(1-n^2)}. \end{aligned}$$

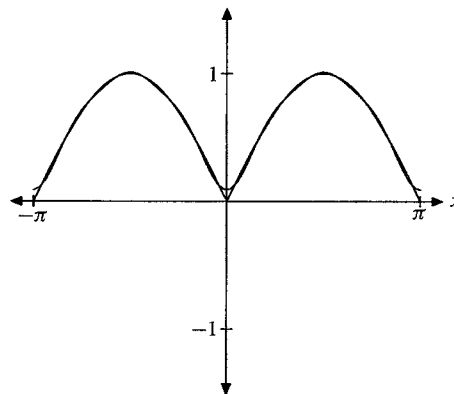
Let us now write

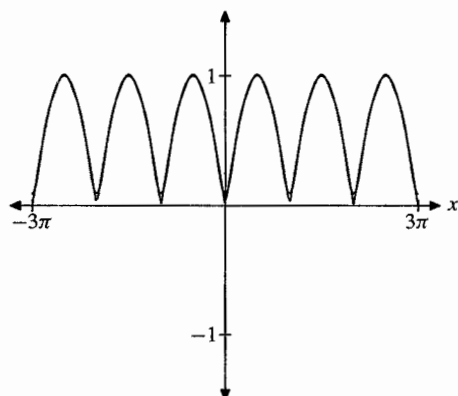
$$a_{2n} = \frac{4}{\pi(1-4n^2)},$$

for $n \geq 1$. Thus, the Fourier representation is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos 2nx.$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.





2. On the interval $-\pi \leq x \leq \pi$, the function $f(x) = |x|$ is even, so only cosine terms will appear in the Fourier series expansion and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} |x| \cos nx \, dx.$$

But $|x| = x$ on $[0, \pi]$. Thus,

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx,$$

and integration by parts provides

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right] \\ &= \frac{2}{n^2\pi} [(-1)^n - 1], \end{aligned}$$

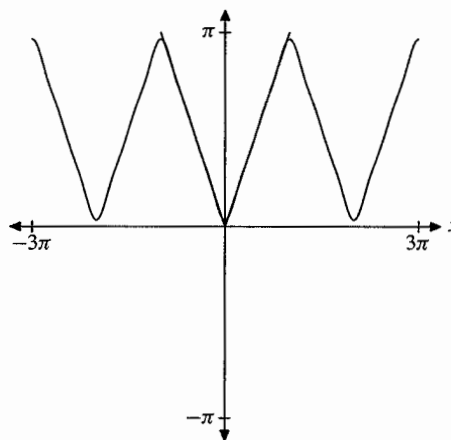
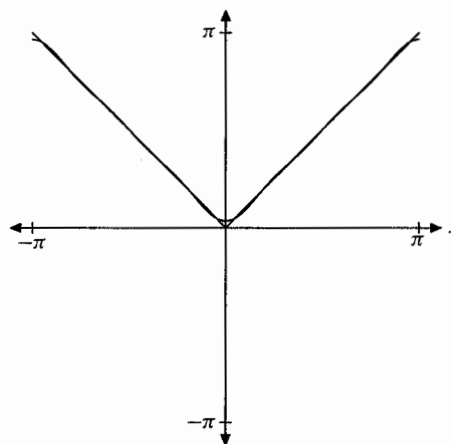
provided $n \neq 0$. In the case that $n = 0$,

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} x \, dx \\ &= \frac{2}{\pi} \left[\frac{1}{2} x^2 \right]_0^{\pi} \\ &= \frac{1}{\pi} (\pi^2) \\ &= \pi. \end{aligned}$$

Thus, $f(x) = |x|$ has Fourier Series

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} [(-1)^n - 1] \cos nx. \end{aligned}$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.



3. On the interval $[-\pi, \pi]$, the function

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0, \\ x, & 0 \leq x \leq \pi, \end{cases}$$

is neither even nor odd, so the Fourier series may have both sine and cosine terms. However, f is identically zero on $[-\pi, 0]$, so

$$a_n = \frac{1}{\pi} \int_0^\pi f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^\pi x \cos nx \, dx.$$

Integration by parts provides

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^\pi \\ &= \frac{1}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right] \\ &= \frac{1}{\pi n^2} [(-1)^n - 1], \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{1}{\pi} \int_0^\pi x \, dx = \frac{1}{\pi} \cdot \frac{1}{2} x^2 \Big|_0^\pi = \frac{\pi}{2}.$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^\pi x \sin nx \, dx. \end{aligned}$$

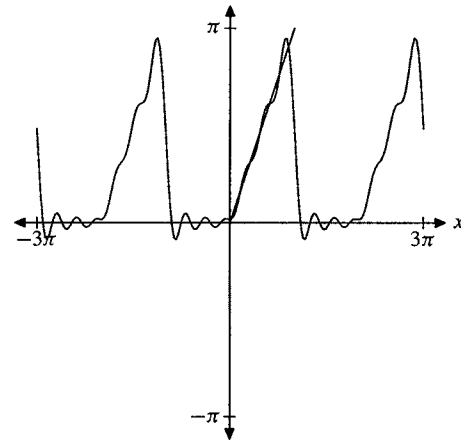
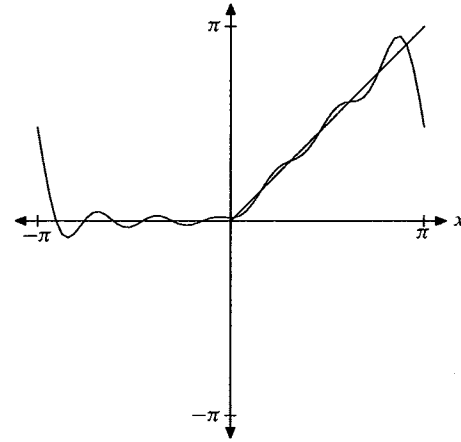
Integration by parts provides

$$\begin{aligned} b_n &= \frac{1}{\pi} \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi \\ &= \frac{1}{\pi} \left[\frac{-\pi \cos n\pi}{n} \right] \\ &= \frac{(-1)^{n+1}}{n}. \end{aligned}$$

Hence, $f(x)$ has Fourier series expansion

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \\ &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{[(-1)^n - 1]}{\pi n^2} \cos nx \right. \\ &\quad \left. + \frac{(-1)^{n+1}}{n} \sin nx \right] \end{aligned}$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.



4. On the interval $[-\pi, \pi]$, the function

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0, \\ \sin x, & 0 \leq x \leq \pi, \end{cases}$$

is neither even nor odd, so the Fourier series may

have both sine and cosine terms.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx \, dx \\ &\quad + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \end{aligned}$$

However, f is identically zero on $[-\pi, 0)$, so

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin x \cos nx \, dx. \end{aligned}$$

However, two integration by parts provides

$$\begin{aligned} a_n &= \frac{n^2}{\pi(n^2 - 1)} \left[\frac{\sin x \sin nx}{n} + \frac{\cos x \cos nx}{n^2} \right]_0^{\pi} \\ &= \frac{n^2}{\pi(n^2 - 1)} \left[\frac{-\cos n\pi}{n^2} - \frac{1}{n^2} \right] \\ &= \frac{n^2}{\pi(n^2 - 1)} \left[\frac{(-1)^{n+1}}{n^2} - \frac{1}{n^2} \right] \\ &= \frac{1}{\pi(n^2 - 1)} [(-1)^{n+1} - 1], \end{aligned}$$

provided $n \neq 1, 0$. In the case where $n = 1$,

$$\begin{aligned} a_1 &= \frac{1}{\pi} \int_0^{\pi} \sin x \cos x \, dx \\ &= \frac{1}{2\pi} \int_0^{\pi} \sin 2x \, dx = 0. \end{aligned}$$

In the case where $n = 0$,

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin x \, dx = \frac{2}{\pi}.$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin x \sin nx \, dx. \end{aligned}$$

This time we use a product to sum identity to write

$$\begin{aligned} b_n &= \frac{1}{2\pi} \int_0^{\pi} [\cos(1-n)x - \cos(1+n)x] \, dx \\ &= \frac{1}{2\pi} \left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n} \right]_0^{\pi} \\ &= 0, \end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

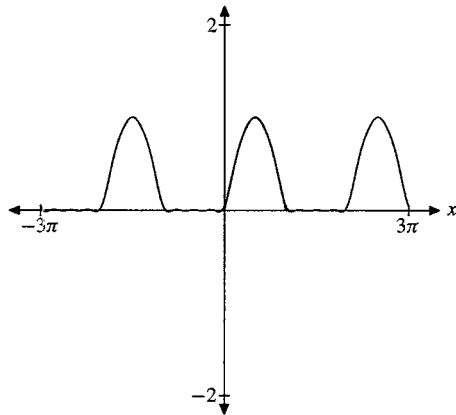
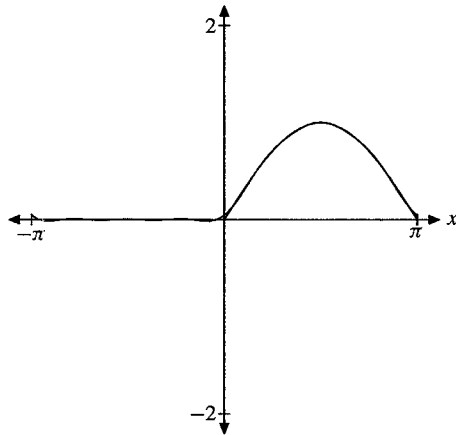
$$\begin{aligned} b_1 &= \frac{1}{\pi} \int_0^{\pi} \sin^2 x \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2\pi} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} \\ &= \frac{1}{2}. \end{aligned}$$

Hence, $f(x)$ has Fourier series expansion

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \\ &= \frac{2/\pi}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} - 1}{\pi(n^2 - 1)} \cos nx + \frac{1}{2} \sin x \\ &= \frac{1}{\pi} + \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^{n+1} - 1}{n^2 - 1} \cos nx + \frac{1}{2} \sin x. \end{aligned}$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ fol-

low.



5. The function $f(x) = x \cos x$ is odd on the interval $[-\pi, \pi]$. Thus, the Fourier expansion will contain only sine terms.

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin nx \, dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \frac{x}{2} [\sin(n-1)x + \sin(n+1)x] \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \sin(n-1)x \, dx \\ &\quad + \frac{1}{\pi} \int_0^{\pi} x \sin(n+1)x \, dx. \end{aligned}$$

Using integration by parts, the first integral is

$$\begin{aligned} &\frac{1}{\pi} \int_0^{\pi} x \sin(n-1)x \, dx \\ &= \frac{1}{\pi} \left[\frac{-x \cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[\frac{-\pi \cos(n-1)\pi}{n-1} \right] \\ &= \frac{-\cos(n-1)\pi}{n-1} \\ &= \frac{(-1)^n}{n-1}, \end{aligned}$$

provided $n \neq 1$. Similarly, the second integral is

$$\frac{1}{\pi} \int_0^{\pi} x \sin(n+1)x \, dx = \frac{(-1)^n}{n+1}.$$

Thus,

$$b_n = (-1)^n \left[\frac{1}{n-1} + \frac{1}{n+1} \right] = (-1)^n \frac{2n}{n^2-1},$$

provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned} b_1 &= \frac{1}{\pi} \int_0^{\pi} x \cos x \sin x \, dx \\ &= \frac{1}{2\pi} \int_0^{\pi} x \sin 2x \, dx. \end{aligned}$$

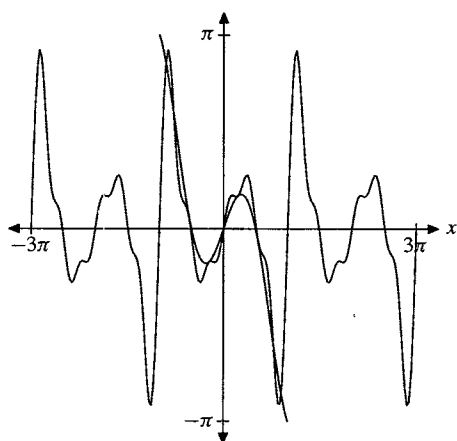
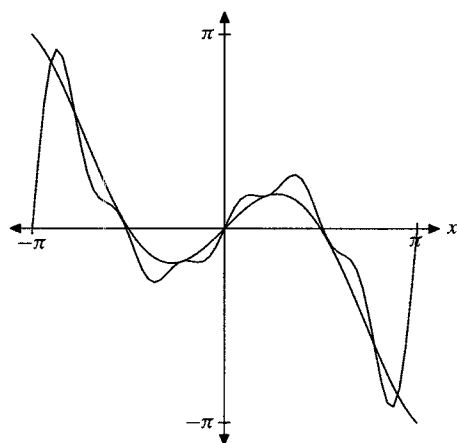
Integrating by parts,

$$\begin{aligned} b_1 &= \frac{1}{2\pi} \left[\frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi} \\ &= \frac{1}{2\pi} \left[\frac{-\pi}{2} \right] \\ &= -\frac{1}{4}. \end{aligned}$$

Thus, the Fourier series expansion is

$$f(x) = -\frac{1}{4} \sin x + \sum_{n=2}^{\infty} (-1)^n \frac{2n}{n^2 - 1} \sin nx.$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.



6. The function $f(x) = x \sin x$ is even in the interval $[-\pi, \pi]$.

$$\begin{aligned} f(-x) &= -x \sin(-x) = -x(-\sin x) = x \sin x \\ &= f(x) \end{aligned}$$

Thus, the Fourier series contains only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \frac{x}{2} [\sin(1-n)x + \sin(1+n)x] \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \sin(1-n)x \, dx \\ &\quad + \frac{1}{\pi} \int_0^{\pi} x \sin(1+n)x \, dx \end{aligned}$$

Integrating by parts, the first integral becomes

$$\begin{aligned} &\frac{1}{\pi} \int_0^{\pi} x \sin(1-n)x \, dx \\ &= -\frac{1}{\pi} \int_0^{\pi} x \sin(n-1)x \, dx \\ &= -\frac{1}{\pi} \left[\frac{-x \cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{\pi} \\ &= -\frac{1}{\pi} \left[\frac{-\pi \cos(n-1)\pi}{n-1} \right] \\ &= \frac{(-1)^{n-1}}{n-1}, \end{aligned}$$

provided $n \neq 1$. Similarly, the second integral is

$$\frac{1}{\pi} \int_0^{\pi} x \sin(1+n)x \, dx = \frac{(-1)^n}{n+1}.$$

Thus,

$$\begin{aligned} a_n &= \frac{(-1)^{n-1}}{n-1} + \frac{(-1)^n}{n+1} \\ &= (-1)^{n-1} \left[\frac{1}{n-1} - \frac{1}{n+1} \right] \\ &= (-1)^{n-1} \frac{2}{n^2 - 1} \end{aligned}$$

Note that this gives us $a_0 = (-1)(-2) = 2$. In the

case where $n = 1$,

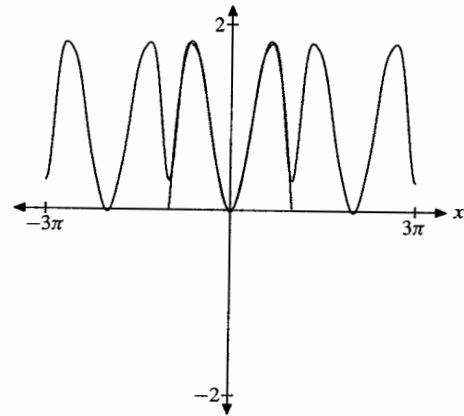
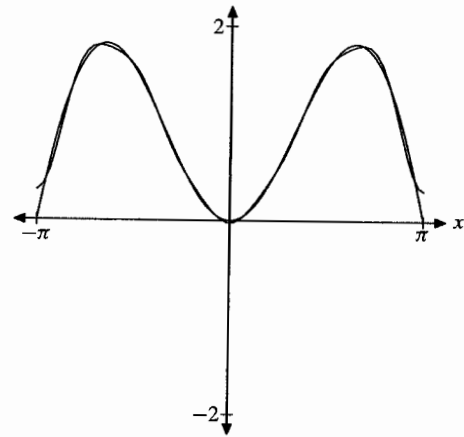
$$\begin{aligned}
 a_1 &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx \\
 &= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx \\
 &= \frac{1}{\pi} \left[\frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi} \\
 &= \frac{1}{\pi} \left[\frac{-\pi}{2} \right] \\
 &= -\frac{1}{2}.
 \end{aligned}$$

Hence the Fourier series expansion is

$$\begin{aligned}
 f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\
 &= 1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 2 \cos nx}{n^2 - 1}.
 \end{aligned}$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ fol-

low.



7. If

$$\begin{cases} 1+x, & -1 \leq x \leq 0, \\ 1, & 0 < x \leq 1, \end{cases}$$

then on the interval $[-1, 1]$

$$\begin{aligned}
 a_n &= \int_{-1}^1 f(x) \cos n\pi x \, dx \\
 &= \int_{-1}^0 (1+x) \cos n\pi x \, dx + \int_0^1 \cos n\pi x \, dx.
 \end{aligned}$$

The second integral is straight forward.

$$\int_0^1 \cos n\pi x \, dx = \frac{1}{n\pi} \sin n\pi x \Big|_0^1 = \frac{1}{n\pi} (0-0) = 0.$$

The first integral requires integration by parts.

$$\begin{aligned} & \int_{-1}^0 (1+x) \cos n\pi x dx \\ &= \left[\frac{(1+x) \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2} \right]_{-1}^0 \\ &= \frac{1}{n^2\pi^2} [1 - \cos n\pi] \end{aligned}$$

Hence, combining integrals, for $n \geq 1$,

$$a_n = \frac{1}{n^2\pi^2} [1 - (-1)^n] = \begin{cases} 0, & n \text{ even,} \\ \frac{2}{n^2\pi^2}, & n \text{ odd.} \end{cases}$$

For $n = 0$,

$$\begin{aligned} a_0 &= \int_{-1}^1 f(x) dx \\ &= \int_{-1}^0 (1+x) dx + \int_0^1 dx \\ &= \left(x + \frac{1}{2}x^2 \right) \Big|_{-1}^0 + 1 \\ &= \frac{3}{2}. \end{aligned}$$

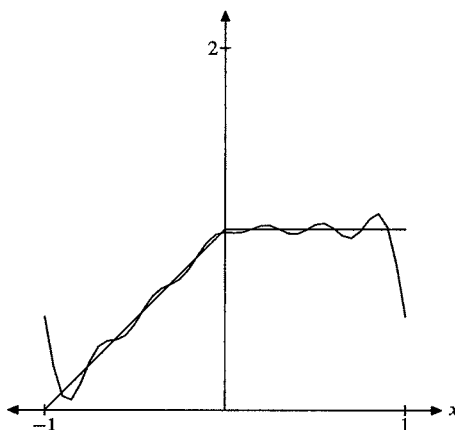
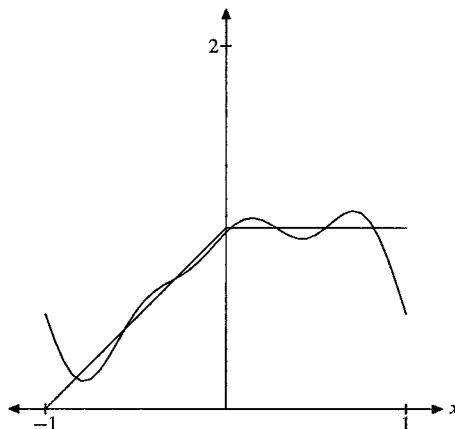
Similarly,

$$\begin{aligned} b_n &= \int_{-1}^1 f(x) \sin n\pi x dx \\ &= \int_{-1}^0 (1+x) \sin n\pi x dx + \int_0^1 \sin n\pi x dx \\ &= -\frac{1}{n\pi} - \frac{1}{n\pi} [\cos n\pi - 1] \\ &= \frac{(-1)^{n+1}}{n\pi}. \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &\sim \frac{3}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n+1)\pi x)}{(2n+1)^2} \\ &\quad - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\pi x. \end{aligned}$$

In the following figures, the first displays S_3 , the second S_6 .



8. Note that $f(x) = 4 - x^2$ is even on the interval $[-2, 2]$, so the Fourier expansion has only cosine terms. Indeed,

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx \\ &= \int_0^2 (4 - x^2) \cos \frac{n\pi x}{2} dx. \end{aligned}$$

Integrating by parts,

$$a_n = \left[\frac{2(4-x^2) \sin(n\pi x/2)}{n\pi} - \frac{8x \cos(n\pi x/2)}{n^2\pi^2} + \frac{16 \sin(n\pi x/2)}{n^3\pi^3} \right]_0^2$$

$$= -\frac{16 \cos n\pi}{n^2\pi^2} = \frac{16(-1)^{n+1}}{n^2\pi^2},$$

provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \int_0^2 (4-x^2)dx = 4x - \frac{1}{3}x^3 = \frac{16}{3}.$$

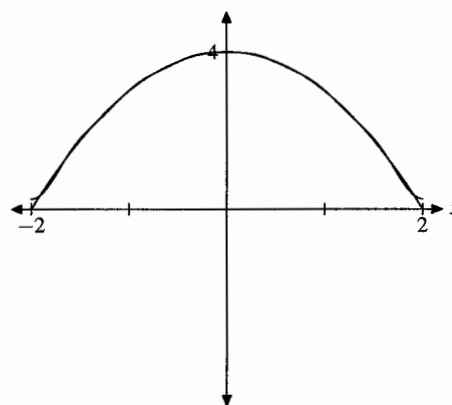
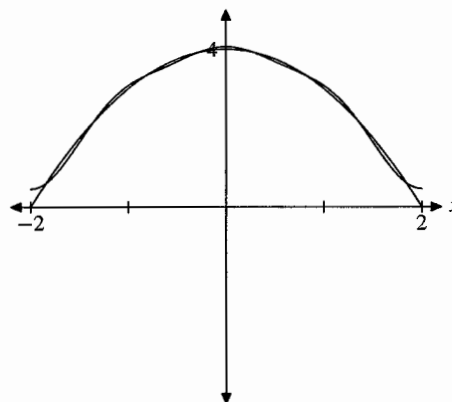
Hence, the Fourier representation is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

$$= \frac{8}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos \frac{n\pi x}{2}.$$

In the following figures, the first displays S_3 , the

second S_6 .



9. Note that $f(x) = x^3$ is odd on the interval $[-1, 1]$. Thus, the Fourier series representation will only contain sine terms with coefficient

$$b_n = 2 \int_0^1 f(x) \sin n\pi x dx$$

$$= 2 \int_0^1 x^3 \sin n\pi x dx.$$

Several applications of integration by parts provides

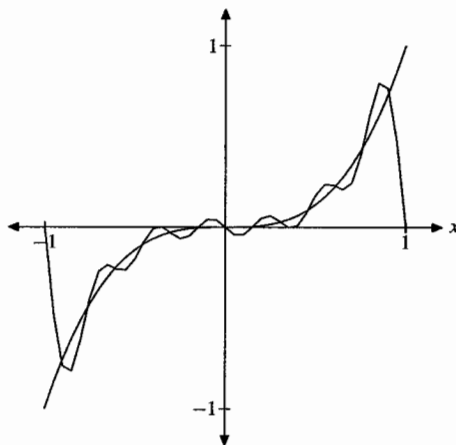
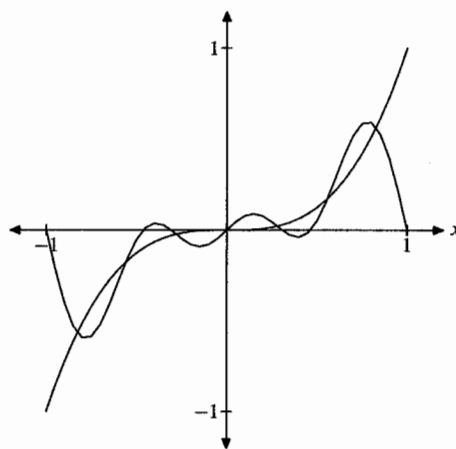
and right S_6 .

$$\begin{aligned}
 b_n &= -\frac{2x^3 \cos n\pi x}{n\pi} + \frac{6x^2 \sin n\pi x}{n^2\pi^2} \\
 &\quad + \frac{12x \cos n\pi x}{n^3\pi^3} - \frac{12 \sin n\pi x}{n^4\pi^4} \Big|_0^1 \\
 &= -\frac{2 \cos n\pi}{n\pi} + \frac{12 \cos n\pi}{n^3\pi^3} \\
 &= (-1)^n \frac{2(6 - n^2\pi^2)}{n^3\pi^3}.
 \end{aligned}$$

Therefore, the Fourier representation is

$$f(x) \sim \sum_{n=1}^{\infty} (-1)^n \frac{2(6 - n^2\pi^2)}{n^3\pi^3} \sin n\pi x.$$

In the following figures, the first displays S_3 , the sec-



10. The function $f(x) = \sin x \cos^2 x$ is odd on the interval $[-\pi, \pi]$, so the Fourier representation should contain only sine terms. Indeed,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin x \cos^2 x \sin nx \, dx.$$

Using the fact that $\sin 2x = 2 \sin x \cos x$,

$$b_n = \frac{1}{\pi} \int_0^{\pi} \cos x \sin 2x \sin nx \, dx.$$

Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^\pi (\cos x) \frac{1}{2} [\cos(n-2)x \\ &\quad - \cos(n+2)x] dx \\ &= \frac{1}{2\pi} \int_0^\pi \cos x \cos(n-2)x dx \\ &\quad - \frac{1}{2\pi} \int_0^\pi \cos x \cos(n+2)x dx. \end{aligned}$$

Using a product to sum identity, the first integral on the right becomes

$$\begin{aligned} &\frac{1}{2\pi} \int_0^\pi \frac{1}{2} [\cos((n-2)+1)x \\ &\quad + \cos((n-2)-1)x] dx \\ &= \frac{1}{4\pi} \left[\frac{\sin(n-1)x}{n-1} + \frac{\sin(n-3)x}{n-3} \right]_0^\pi \\ &= 0, \end{aligned}$$

provided $n \neq 1, 3$. Similarly, the second integral is also zero, provided $n \neq -3, -1$ (which of course cannot happen). It remains to find what happens at $n = 1$ and $n = 3$.

$$\begin{aligned} b_1 &= \frac{2}{\pi} \int_0^\pi \sin^2 x \cos^2 x dx \\ &= \frac{2}{\pi} \cdot \frac{1}{4} \int_0^\pi \sin^2 2x dx \\ &= \frac{1}{2\pi} \int_0^\pi \frac{1 - \cos 4x}{2} dx \\ &= \frac{1}{4\pi} \left[x - \frac{1}{4} \sin 4x \right]_0^\pi \\ &= \frac{1}{4}. \end{aligned}$$

Next, for $n = 3$,

$$\begin{aligned} b_3 &= \frac{2}{\pi} \int_0^\pi \sin x \cos^2 x \sin 3x dx \\ &= \frac{1}{\pi} \int_0^\pi \cos x \sin 2x \sin 3x dx \\ &= \frac{1}{\pi} \int_0^\pi (\cos x) \frac{1}{2} [\cos(3x-2x) \\ &\quad - \cos(3x+2x)] dx \\ &= \frac{1}{2\pi} \int_0^\pi \cos^2 x dx \\ &\quad - \frac{1}{2\pi} \int_0^\pi \cos x \cos 5x dx. \end{aligned}$$

The first integral is

$$\begin{aligned} &\frac{1}{2\pi} \int_0^\pi \frac{1 + \cos 2x}{2} dx \\ &= \frac{1}{4\pi} \left[x + \frac{1}{2} \sin 2x \right]_0^\pi = \frac{1}{4}. \end{aligned}$$

The second integral is

$$\begin{aligned} &\frac{1}{2\pi} \int_0^\pi \frac{1}{2} [\cos(5x+x) + \cos(5x-x)] dx \\ &= \frac{1}{4\pi} \int_0^\pi (\cos 6x + \cos 4x) dx \\ &= \frac{1}{4\pi} \left[\frac{1}{6} \sin 6x + \frac{1}{4} \sin 4x \right]_0^\pi \\ &= 0. \end{aligned}$$

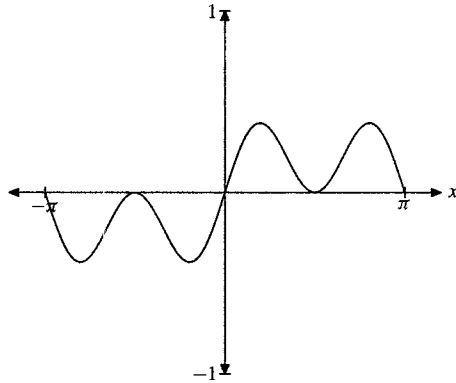
Thus, $b_3 = 1/4$ and the Fourier representation is

$$\begin{aligned} f(x) &\sim b_1 \sin x + b_3 \sin 3x \\ &\sim \frac{1}{4} \sin x + \frac{1}{4} \sin 3x. \end{aligned}$$

It is interesting to attack this problem with trig identities, writing

$$\begin{aligned} \sin x \cos^2 x &= \frac{1}{2} \sin 2x \cos x \\ &= \frac{1}{2} \left[\frac{1}{2} (\sin(2x+x) + \sin(2x-x)) \right] \\ &= \frac{1}{4} \sin 3x + \frac{1}{4} \sin x. \end{aligned}$$

This agrees nicely with the calculus solution, but with much less effort. Note that the partial sum S_3 agrees exactly with the graph of f in the following image.



11. If

$$f(x) = \begin{cases} 0, & -1 \leq x \leq 0, \\ x^2, & 0 < x \leq 1, \end{cases}$$

then on the interval $[-1, 1]$

$$\begin{aligned} a_n &= \int_{-1}^1 f(x) \cos n\pi x dx \\ &= \int_0^1 x^2 \cos n\pi x dx. \end{aligned}$$

With $n = 0$,

$$a_0 = \int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}.$$

For $n \geq 1$, several applications of integration by parts provides

$$\begin{aligned} a_n &= \frac{x^2 \sin n\pi x}{n\pi} + \frac{2x \cos n\pi x}{n^2\pi^2} - \frac{2 \sin n\pi x}{n^3\pi^3} \Big|_0^1 \\ &= \frac{2 \cos n\pi}{n^2\pi^2} \\ &= (-1)^n \frac{2}{n^2\pi^2}. \end{aligned}$$

Next, for $n \geq 1$

$$\begin{aligned} b_n &= \int_{-1}^1 f(x) \sin n\pi x dx \\ &= \int_0^1 x^2 \sin n\pi x dx. \end{aligned}$$

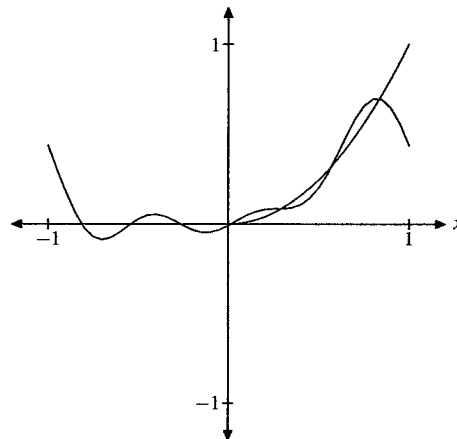
Several applications of integration by parts provides

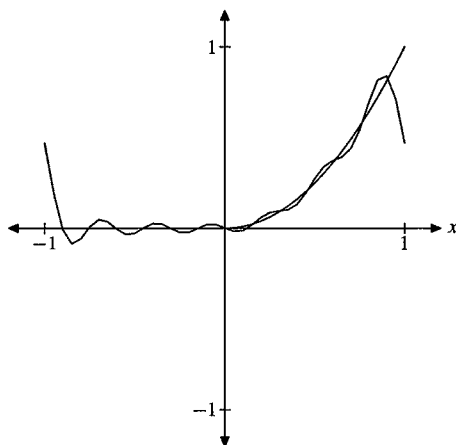
$$\begin{aligned} b_n &= -\frac{x^2 \cos n\pi x}{n\pi} + \frac{2x \sin n\pi x}{n^2\pi^2} + \frac{2 \cos n\pi x}{n^3\pi^3} \Big|_0^1 \\ &= -\frac{\cos n\pi}{n\pi} + \frac{2 \cos n\pi}{n^3\pi^3} - \frac{2}{n^3\pi^3} \\ &= \frac{1}{n^3\pi^3} [2(-1)^n - n^2\pi^2(-1)^n - 2]. \end{aligned}$$

Thus, the Fourier representation for the function is

$$\begin{aligned} f(x) \sim \frac{1}{6} + \sum_{n=1}^{\infty} \left[(-1)^n \frac{2}{n^2\pi^2} \cos n\pi x \right. \\ \left. + \frac{2(-1)^n - n^2\pi^2(-1)^n - 2}{n^3\pi^3} \sin n\pi x \right]. \end{aligned}$$

In the following figures, the first displays S_3 , the second S_6 .





12. If

$$f(x) = \begin{cases} \sin(\pi x/2), & -2 \leq x \leq 0, \\ 0, & 0 < x \leq 2, \end{cases}$$

then on the interval $[-2, 2]$,

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^0 \sin \frac{\pi x}{2} \cos \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^0 \frac{1}{2} \left[\sin \frac{(1+n)\pi x}{2} \right. \\ &\quad \left. + \sin \frac{(1-n)\pi x}{2} \right] dx \\ &= \frac{1}{4} \left[-\frac{2}{\pi(1+n)} \cos \frac{(1+n)\pi x}{2} \right. \\ &\quad \left. - \frac{2}{\pi(1-n)} \cos \frac{(1-n)\pi x}{2} \right]_{-2}^0 \\ &= -\frac{1}{4} \left\{ \frac{4}{\pi(1+n)(1-n)} \right. \\ &\quad \left. + (-1)^n \frac{4}{\pi(1+n)(1-n)} \right\} \\ &= \frac{(-1)^{n+1} - 1}{\pi(1+n)(1-n)}, \end{aligned}$$

provided $n \neq 1$. Note that this implies that $a_0 =$

$-2/\pi$. In the case that $n = 1$,

$$a_1 = \frac{1}{2} \int_{-2}^0 \sin \frac{\pi x}{2} \cos \frac{\pi x}{2} dx = \frac{1}{4} \int_{-2}^0 \sin \pi x dx = 0.$$

Next,

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^0 \sin \frac{\pi x}{2} \sin \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^0 \frac{1}{2} \left[\cos \frac{(1-n)\pi x}{2} - \cos \frac{(1+n)\pi x}{2} \right] dx \\ &= \frac{1}{4} \left[\frac{2}{\pi(1-n)} \sin \frac{(1-n)\pi x}{2} \right. \\ &\quad \left. - \frac{2}{\pi(1+n)} \sin \frac{(1+n)\pi x}{2} \right]_{-2}^0 \\ &= 0, \end{aligned}$$

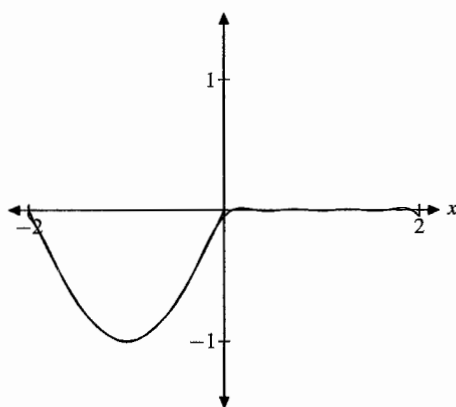
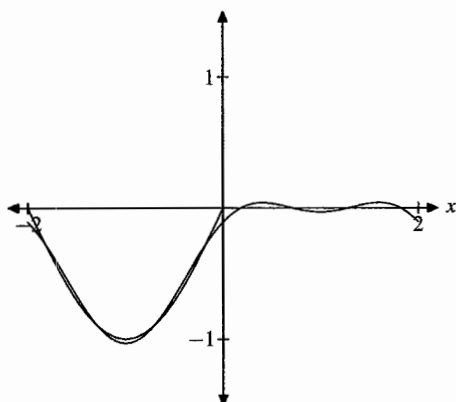
provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned} b_1 &= \frac{1}{2} \int_{-2}^0 \sin^2 \frac{\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^0 \frac{1 - \cos \pi x}{2} dx \\ &= \frac{1}{4} \left[x - \frac{1}{\pi} \sin \pi x \right]_{-2}^0 \\ &= \frac{1}{2}. \end{aligned}$$

Thus, the Fourier representation is

$$\begin{aligned} f(x) &\sim -\frac{1}{\pi} + \frac{1}{2} \sin \frac{\pi x}{2} \\ &\quad + \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^{n+1} - 1}{1 - n^2} \cos \frac{n\pi x}{2}. \end{aligned}$$

In the following figures, the first displays S_3 , the second S_6 .



13. If

$$f(x) = \begin{cases} \cos \pi x, & -1 \leq x \leq 0, \\ 1, & 0 < x \leq 1, \end{cases}$$

then on the interval $[-1, 1]$,

$$\begin{aligned} a_n &= \int_{-1}^1 f(x) \cos n\pi x dx \\ &= \int_{-1}^0 \cos \pi x \cos n\pi x dx + \int_0^1 \cos n\pi x dx. \end{aligned}$$

For $n = 0$,

$$a_0 = \int_{-1}^0 \cos \pi x dx + \int_0^1 dx = 1.$$

For $n = 1$, the first integral becomes

$$\begin{aligned} \int_{-1}^0 \cos^2 \pi x dx &= \int_{-1}^0 \frac{1 + \cos 2\pi x}{2} dx \\ &= \frac{1}{2} \left[x + \frac{1}{2\pi} \sin 2\pi x \right]_{-1}^0 \\ &= \frac{1}{2}. \end{aligned}$$

With $n > 1$,

$$\begin{aligned} \int_{-1}^0 \cos \pi x \cos n\pi x dx &= \frac{1}{2} \int_{-1}^0 [\cos(n-1)\pi x + \cos(n+1)\pi x] dx \\ &= \frac{1}{2} \left[\frac{\sin(n-1)\pi x}{(n-1)\pi} + \frac{\sin(n+1)\pi x}{(n+1)\pi} \right]_{-1}^0 \\ &= 0. \end{aligned}$$

With $n \geq 1$, the second integral becomes

$$\int_0^1 \cos n\pi x dx = \frac{\sin n\pi x}{n\pi} \Big|_0^1 = 0.$$

In summary, $a_0 = 1$, $a_1 = 1/2$, and $a_n = 0$ for all $n > 1$. Next, with $n \geq 1$,

$$\begin{aligned} b_n &= \int_{-1}^1 f(x) \sin n\pi x dx \\ &= \int_{-1}^0 \cos \pi x \sin n\pi x dx + \int_0^1 \sin n\pi x dx. \end{aligned}$$

With $n = 1$, the first integral becomes

$$\int_{-1}^0 \cos \pi x \sin \pi x dx = \frac{1}{2} \int_{-1}^0 \sin 2\pi x dx = 0.$$

With $n > 1$, the first integral becomes

$$\begin{aligned} & \int_{-1}^0 \cos \pi x \sin n \pi x dx \\ &= \frac{1}{2} \int_{-1}^0 [\sin(n-1)\pi x + \sin(n+1)\pi x] dx \\ &= -\frac{1}{2} \left[\frac{\cos(n-1)\pi x}{(n-1)\pi} + \frac{\cos(n+1)\pi x}{(n+1)\pi} \right]_{-1}^0 \\ &= -\frac{1}{2} \left\{ \left[\frac{1}{(n-1)\pi} + \frac{1}{(n+1)\pi} \right] \right. \\ & \quad \left. - \left[\frac{(-1)^{n-1}}{(n-1)\pi} + \frac{(-1)^{n+1}}{(n+1)\pi} \right] \right\}, \end{aligned}$$

which, after some simplification, equals

$$\int_{-1}^0 \cos \pi x \sin n \pi x dx = -n \left[\frac{1 + (-1)^n}{(n-1)(n+1)\pi} \right].$$

The second integral is straight forward.

$$\int_0^1 \sin n \pi x dx = -\frac{\cos n \pi x}{n \pi} \Big|_0^1 = \frac{1}{n \pi} [1 - (-1)^n].$$

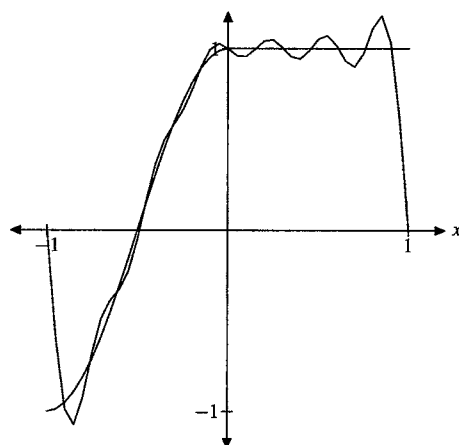
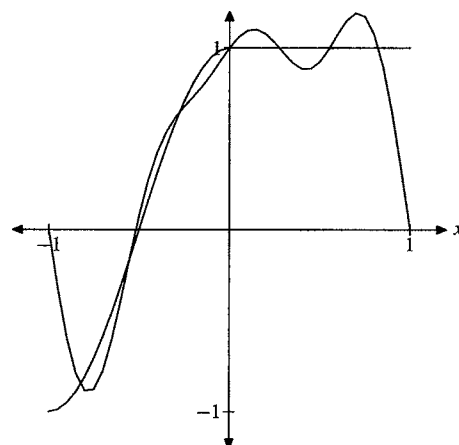
In summary,

$$b_n = \begin{cases} \frac{2}{\pi}, & n = 1 \\ -n \left[\frac{1 + (-1)^n}{(n-1)(n+1)\pi} \right] + \frac{1}{n\pi} [1 - (-1)^n], & n > 1. \end{cases}$$

Thus, the Fourier representation of f is

$$\begin{aligned} f(x) \sim & \frac{1}{2} + \frac{1}{2} \cos \pi x + \frac{2}{\pi} \sin \pi x \\ & + \sum_{n=2}^{\infty} \left\{ -n \left[\frac{1 + (-1)^n}{(n-1)(n+1)\pi} \right] \right. \\ & \left. + \frac{1}{n\pi} [1 - (-1)^n] \right\} \sin n \pi x. \end{aligned}$$

In the following figures, the first S_3 , the second S_6 .



14. The function

$$f(x) = \begin{cases} 1+x, & -1 \leq x \leq 0, \\ 1-x, & 0 < x \leq 1, \end{cases}$$

is even on the interval $[-1, 1]$, so the Fourier repre-

sensation contains only cosine terms.

$$\begin{aligned} a_n &= \int_{-1}^1 f(x) \cos n\pi x \\ &= \int_{-1}^0 (1+x) \cos n\pi x dx \\ &\quad + \int_0^1 (1-x) \cos n\pi x dx. \end{aligned}$$

Integrating by parts, the first integral becomes

$$\begin{aligned} &\int_{-1}^0 (1+x) \cos n\pi x dx \\ &= \left[\frac{(1+x) \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2} \right]_{-1}^0 \\ &= \frac{1}{n^2\pi^2} - \frac{(-1)^n}{n^2\pi^2} \\ &= \frac{1 - (-1)^n}{n^2\pi^2}. \end{aligned}$$

Similarly, the second integral becomes

$$\int_0^1 (1-x) \cos n\pi x dx = \frac{1 - (-1)^n}{n^2\pi^2}.$$

Thus,

$$a_n = \frac{1 - (-1)^n}{n^2\pi^2} + \frac{1 - (-1)^n}{n^2\pi^2} = \frac{2}{n^2\pi^2}(1 - (-1)^n),$$

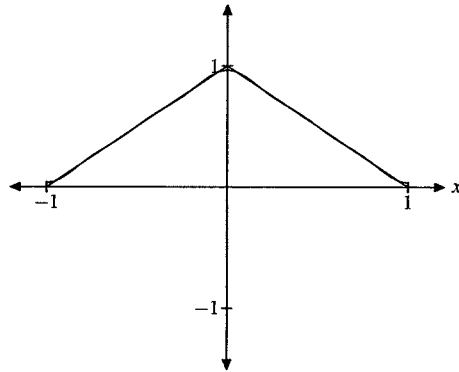
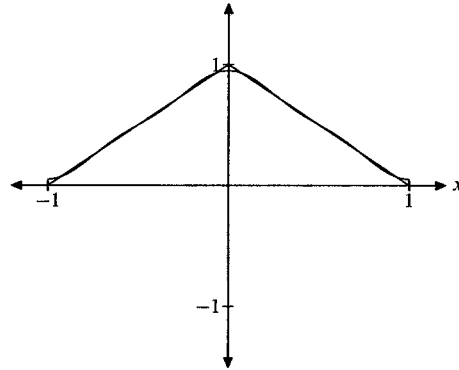
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \int_{-1}^0 (1+x) dx + \int_0^1 (1-x) dx = 1.$$

Thus, the Fourier representation is

$$f(x) = \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \cos n\pi x.$$

In the following figures, the first displays S_3 , the second S_6 .



15. If

$$f(x) = \begin{cases} 2+x, & -2 \leq x \leq 0, \\ -2+x, & 0 < x \leq 2, \end{cases}$$

then f is odd on the interval $[-2, 2]$ and we will need only calculate sine terms.

$$\begin{aligned} b_n &= \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} dx \\ &= \int_0^2 (-2+x) \sin \frac{n\pi x}{2} dx \end{aligned}$$

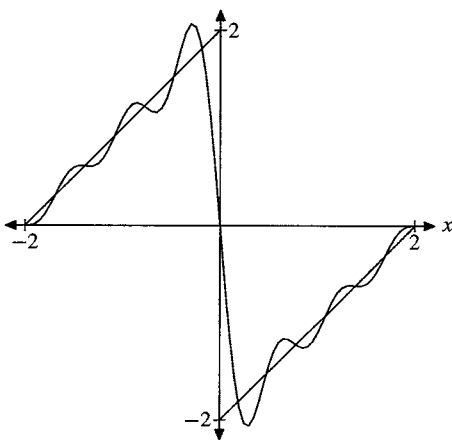
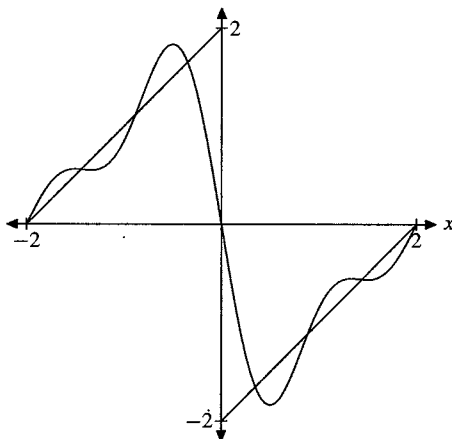
Integration by parts provides

$$\begin{aligned} b_n &= \left[-\frac{2(x-2) \cos(n\pi x/2)}{n\pi} + \frac{4 \sin(n\pi x/2)}{n^2\pi^2} \right]_0^2 \\ &= -\frac{4}{n\pi}. \end{aligned}$$

Hence, the Fourier representation of the function f is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{-4}{n\pi} \sin \frac{n\pi x}{2}.$$

In the following figures, the first displays S_3 , the second S_6 .



16. If

$$f(x) = \begin{cases} 2, & -2 \leq x \leq 0 \\ 2-x, & 0 < x \leq 2, \end{cases}$$

then on the interval $[-2, 2]$,

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^0 2 \cos \frac{n\pi x}{2} dx \\ &\quad + \frac{1}{2} \int_0^2 (2-x) \cos \frac{n\pi x}{2} dx. \end{aligned}$$

The first integral is straightforward,

$$\int_{-2}^0 \cos \frac{n\pi x}{2} dx = \frac{2}{\pi n} \sin \frac{n\pi x}{2} \Big|_{-2}^0 = 0,$$

provided $n \neq 0$. Using integration by parts, the second integral becomes

$$\begin{aligned} &\frac{1}{2} \int_0^2 (2-x) \cos \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \left[\frac{2(2-x) \sin(n\pi x/2)}{n\pi} - \frac{4 \cos(n\pi x/2)}{n^2 \pi^2} \right]_0^2 \\ &= \frac{1}{2} \left[-\frac{4(-1)^n}{n^2 \pi^2} + \frac{4}{n^2 \pi^2} \right] \\ &= \frac{2}{n^2 \pi^2} [1 - (-1)^n], \end{aligned}$$

provided $n \neq 0$. Thus,

$$a_n = \frac{2}{n^2 \pi^2} [1 - (-1)^n],$$

provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{1}{2} \int_{-2}^0 2 dx + \frac{1}{2} \int_0^2 (2-x) dx = 3.$$

Next,

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^0 2 \sin \frac{n\pi x}{2} dx \\ &\quad + \frac{1}{2} \int_0^2 (2-x) \sin \frac{n\pi x}{2} dx. \end{aligned}$$

The first integral is straightforward,

$$\begin{aligned} \int_{-2}^0 \sin \frac{n\pi x}{2} dx &= \frac{-2}{n\pi} \cos \frac{n\pi x}{2} \Big|_{-2}^0 \\ &= -\frac{2}{n\pi} [1 - (-1)^n]. \end{aligned}$$

Integrating by parts, the second integral becomes

$$\begin{aligned}
 & \frac{1}{2} \int_0^2 (2-x) \sin \frac{n\pi x}{2} dx \\
 &= \frac{1}{2} \left[-\frac{2(2-x) \cos(n\pi x/2)}{n\pi} - \frac{4 \sin(n\pi x/2)}{n^2 \pi^2} \right]_0^2 \\
 &= -\frac{1}{2} \left[-\frac{4}{n\pi} \right] \\
 &= \frac{2}{n\pi}.
 \end{aligned}$$

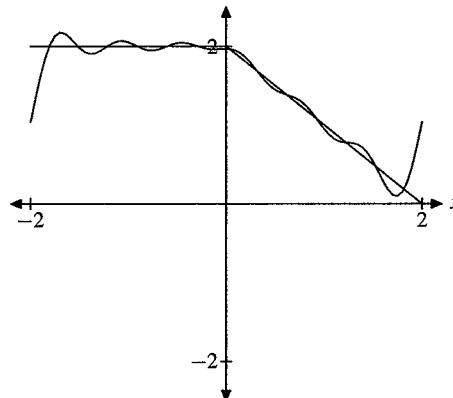
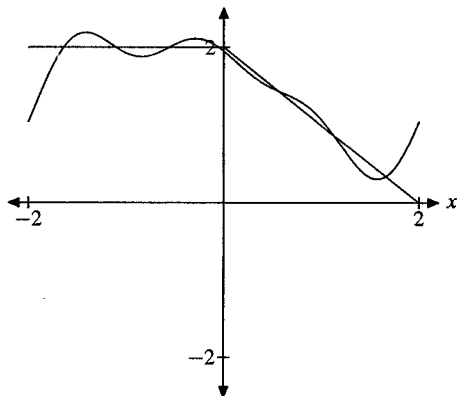
Thus,

$$b_n = -\frac{2}{n\pi} [1 - (-1)^n] + \frac{2}{n\pi} = \frac{2}{n\pi} (-1)^n.$$

Therefore, the Fourier representation is

$$\begin{aligned}
 f(x) \sim & \frac{3}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \cos \frac{n\pi x}{2} \\
 & + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{2}.
 \end{aligned}$$

In the following figures, the first displays S_3 , the second S_6 .



17. Note that $f(x) = x^2$ is even, so the Fourier series will have only cosine terms and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx.$$

Two applications of integration by parts provides

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi} \\
 &= \frac{2}{\pi} \left[\frac{2\pi \cos n\pi}{n^2} \right] \\
 &= \frac{4 \cos n\pi}{n^2} \\
 &= (-1)^n \frac{4}{n^2},
 \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2}{\pi} \cdot \frac{x^3}{3} \Big|_0^{\pi} = \frac{2}{3} \pi^2.$$

Hence, $f(x) = x^2$ has Fourier series

$$\begin{aligned}
 f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\
 &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}.
 \end{aligned}$$

The partial sum S_7 is displayed on $[-\pi, \pi]$ and $[-2\pi, 2\pi]$.

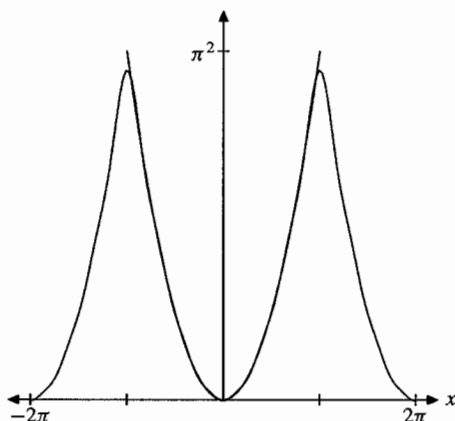
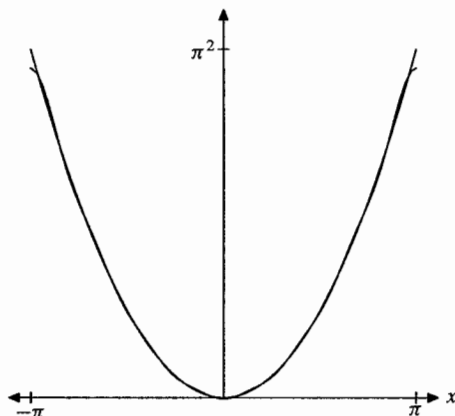
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = 2 \int_0^1 x^2 dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}.$$

Hence, $f(x) = x^2$ has Fourier series

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \\ &= \frac{2/3}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2 \pi^2} \cos n\pi x \\ &= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cos n\pi x}{n^2 \pi^2}. \end{aligned}$$

The partial sum S_7 is displayed on $[-1, 1]$ and $[-2, 2]$.

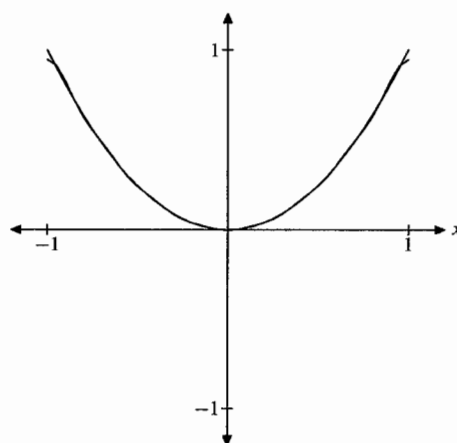


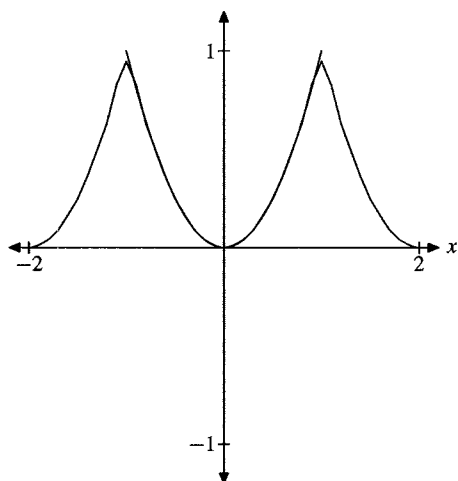
18. Note that $f(x) = x^2$ is even, so the Fourier series will have only cosine terms and

$$a_n = \frac{1}{1} \int_{-1}^1 x^2 \cos n\pi x dx = 2 \int_0^1 x^2 \cos n\pi x dx.$$

Two applications of integration by parts provides

$$\begin{aligned} a_n &= 2 \left[\frac{x^2 \sin n\pi x}{n\pi} + \frac{2x \cos n\pi x}{n^2 \pi^2} - \frac{2 \sin n\pi x}{n^3 \pi^3} \right]_0^1 \\ &= 2 \left[\frac{2 \cos n\pi}{n^2 \pi^2} \right] \\ &= \frac{4}{n^2 \pi^2} (-1)^n, \end{aligned}$$





19. Since $\sin(-x) = -\sin(x)$, we have $|\sin(-x)| = |\sin(x)|$. Hence $f(x) = |\sin x|$ is even.
20. Since $f(-x) = -x + 3(-x)^3 = -(x + 3x^3) = -f(x)$, f is odd.
21. Since $f(-1) = e^{-1} = 1/e$, and $f(1) = e$, we see that f is neither even nor odd.
22. Notice that $f(-1) = 0$, while $f(1) = 2$. Hence f is neither even nor odd.
23. Using the cosine expansion,

$$\begin{aligned} & \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \\ &= \frac{1}{2}[\cos \alpha \cos \beta + \sin \alpha \sin \beta \\ & \quad + \cos \alpha \cos \beta - \sin \alpha \sin \beta] \\ &= \cos \alpha \cos \beta. \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\ &= \frac{1}{2}[\cos \alpha \cos \beta + \sin \alpha \sin \beta \\ & \quad - \cos \alpha \cos \beta + \sin \alpha \sin \beta] \\ &= \sin \alpha \sin \beta. \end{aligned}$$

Finally, the sine expansion delivers

$$\begin{aligned} & \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)] \\ &= \frac{1}{2}[\sin \alpha \cos \beta - \cos \alpha \sin \beta \\ & \quad + \sin \alpha \cos \beta + \cos \alpha \sin \beta] \\ &= \sin \alpha \cos \beta. \end{aligned}$$

24. The function

$$\sin \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx,$$

being the product of an odd and even function, is odd on the interval $[-L, L]$. Thus,

$$\int_{-L}^L \sin \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx = 0$$

for all nonnegative integers p and q .

Secondly, $\cos(p\pi x/L) \cos(q\pi x/L)$ is even on $[-L, L]$, so

$$\begin{aligned} & \int_{-L}^L \cos \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx \\ &= 2 \int_0^L \cos \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx. \end{aligned}$$

Using the product to sum identity,

$$\begin{aligned} & 2 \int_0^L \cos \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx \\ &= 2 \int_0^L \frac{1}{2} \left[\cos \frac{\pi(p-q)x}{L} \right. \\ & \quad \left. + \cos \frac{\pi(p+q)x}{L} \right] dx. \end{aligned}$$

But

$$\begin{aligned} & \int_0^L \cos \frac{\pi(p-q)x}{L} dx \\ &= \frac{L}{\pi(p-q)} \sin \frac{\pi(p-q)x}{L} \Big|_0^L \\ &= \frac{L}{\pi(p-q)} [\sin(p-q)\pi - \sin 0] \\ &= 0, \end{aligned}$$

provided $p \neq q$. Also,

$$\begin{aligned} & \int_0^L \cos \frac{\pi(p+q)x}{L} dx \\ &= \frac{L}{\pi(p+q)} \sin \frac{\pi(p+q)x}{L} \Big|_0^L \\ &= \frac{L}{\pi(p+q)} [\sin(p+q)\pi - 0] \\ &= 0, \end{aligned}$$

provided it is *not* the case that $p = q = 0$. Hence, if it is not the case the $p = q = 0$ and it is not the case that $p = q$,

$$\int_{-L}^L \cos \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx = 0.$$

Consider the case where $p = q = 0$. Then

$$\int_{-L}^L \cos \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx = \int_{-L}^L dx = 2L.$$

Consider the case where $p = q$. Then

$$\begin{aligned} & \int_{-L}^L \cos \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx \\ &= 2 \int_0^L \cos^2 \frac{p\pi x}{L} dx \\ &= 2 \int_0^L \frac{1 + \cos \frac{2p\pi x}{L}}{2} dx \\ &= \left[x + \frac{L}{2p\pi} \sin \frac{2p\pi x}{L} \right]_0^L \\ &= [L + 0] - [0 + 0] \\ &= L. \end{aligned}$$

Next, $\sin(p\pi x/L) \sin(q\pi x/L)$ is even on $[-L, L]$, so

$$\begin{aligned} & \int_{-L}^L \sin \frac{p\pi x}{L} \sin \frac{q\pi x}{L} dx \\ &= 2 \int_0^L \sin \frac{p\pi x}{L} \sin \frac{q\pi x}{L} dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} & 2 \int_0^L \sin \frac{p\pi x}{L} \sin \frac{q\pi x}{L} dx \\ &= 2 \int_0^L \frac{1}{2} \left[\cos \frac{\pi(p-q)x}{L} - \cos \frac{\pi(p+q)x}{L} \right] dx. \end{aligned}$$

But,

$$\begin{aligned} & \int_0^L \cos \frac{\pi(p-q)x}{L} dx \\ &= \frac{L}{\pi(p-q)} \sin \frac{\pi(p-q)x}{L} \Big|_0^L \\ &= \frac{L}{\pi(p-q)} [\sin \pi(p-q) - \sin 0] \\ &= 0, \end{aligned}$$

provided $p \neq q$. Similarly,

$$\int_0^L \cos \frac{\pi(p+q)x}{L} dx = 0,$$

provided it is not the case that $p = q = 0$. Thus,

$$\int_{-L}^L \sin \frac{p\pi x}{L} \sin \frac{q\pi x}{L} dx = 0,$$

provided $p \neq q$ and it is not the case that $p = q = 0$.

In the case that $p = q = 0$,

$$\int_{-L}^L \sin \frac{p\pi x}{L} \sin \frac{q\pi x}{L} dx = \int_{-L}^L 0 dx = 0.$$

In the case that $p = q$,

$$\begin{aligned} & \int_{-L}^L \sin \frac{p\pi x}{L} \sin \frac{q\pi x}{L} dx \\ &= \int_{-L}^L \sin^2 \frac{p\pi x}{L} dx \\ &= \int_{-L}^L \frac{1 - \cos \frac{2p\pi x}{L}}{2} dx \\ &= \frac{1}{2} \left[x - \frac{L}{2p\pi} \sin \frac{2p\pi x}{L} \right]_{-L}^L \\ &= \frac{1}{2} [(L - 0) - (-L - 0)] \\ &= L. \end{aligned}$$

25. We multiply both sides of

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi x + b_k \sin k\pi x$$

by $\sin nx$ and integrate. By the orthogonality relations in Lemma 1.2, all terms will equal zero, save one.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin nx \, dx &= b_k \int_{-\pi}^{\pi} \sin nx \sin nx \, dx \\ \int_{-\pi}^{\pi} f(x) \sin nx \, dx &= b_k \pi \end{aligned}$$

Hence,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

26. If f and g are even, then

$$(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x).$$

Hence, fg is even. Next, if f and g are odd, then

$$\begin{aligned} (fg)(-x) &= f(-x)g(-x) = [-f(x)][-g(x)] \\ &= f(x)g(x) = (fg)(x). \end{aligned}$$

Hence, fg is even. Finally, if f is even and g is odd, then

$$\begin{aligned} (fg)(-x) &= f(-x)g(-x) = f(x)[-g(x)] = \\ &= -f(x)g(x) = -(fg)(x). \end{aligned}$$

Thus, fg is odd.

27. Whether f is even or odd, we can write

$$\int_{-L}^L f(x) \, dx = \int_{-L}^0 f(x) \, dx + \int_0^L f(x) \, dx.$$

However, if we let $u = -x$, the $du = -dx$ and the first integral on the right becomes

$$\int_{-L}^0 f(x) \, dx = \int_L^0 f(-u)(-du).$$

If we reverse the bounds, then

$$= \int_0^L f(-u) \, du.$$

But f is even, so

$$\int_0^L f(u) \, du.$$

But u is just a dummy variable of integration, so

$$\begin{aligned} \int_{-L}^L f(x) \, dx &= \int_{-L}^0 f(x) \, dx + \int_0^L f(x) \, dx \\ &= \int_0^L f(x) \, dx + \int_0^L f(x) \, dx \\ &= 2 \int_0^L f(x) \, dx. \end{aligned}$$

If f is odd, then with $u = -x$ and $du = -dx$,

$$\int_{-L}^0 f(x) \, dx = \int_L^0 f(-u)(-du).$$

Reversing the bounds,

$$= \int_0^L f(-u) \, du.$$

But f is odd, so

$$= - \int_0^L f(u) \, du.$$

But u is just a dummy variable of integration, so

$$\begin{aligned} \int_{-L}^L f(x) \, dx &= \int_{-L}^0 f(x) \, dx + \int_0^L f(x) \, dx \\ &= - \int_0^L f(x) \, dx + \int_0^L f(x) \, dx \\ &= 0. \end{aligned}$$

28. If $f(x)$ is odd, then $f(x) \cos(n\pi x/L)$ is odd and

$$a_k = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{k\pi x}{L} \, dx = 0$$

for $k \geq 0$. Therefore, all cosine terms vanish and

$$f(x) = \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{L},$$

where

$$b_k = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{k\pi x}{L} dx.$$

Further, $f(x) \sin(k\pi x/L)$ is even and

$$b_k = \frac{2}{L} \int_0^L f(x) \sin \frac{k\pi x}{L} dx.$$

29. We start with f being odd on the interval $[-L, L]$. Then the Fourier representation contains only sine terms. Thus,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Now, suppose we have the additional symmetry that f is symmetrical with respect to the line $x = L/2$. That is, $f(L - x) = f(x)$ for all $0 \leq x \leq L$. We can write

$$b_n = \frac{2}{L} \left[\int_0^{L/2} f(x) \sin \frac{n\pi x}{L} dx + \int_{L/2}^L f(x) \sin \frac{n\pi x}{L} dx \right].$$

Now, consider the second integral with the substitution

$$u = L - x \quad \text{and} \quad du = -dx.$$

Then,

$$\begin{aligned} & \int_{L/2}^L f(x) \sin \frac{n\pi x}{L} dx \\ &= \int_{L/2}^0 f(L - u) \sin \frac{n\pi(L - u)}{L} (-du) \\ &= \int_0^{L/2} f(L - u) \sin \left(n\pi - \frac{n\pi u}{L} \right) du. \end{aligned}$$

But $f(L - u) = f(u)$, and the sine expansion provides

$$\begin{aligned} & \int_0^{L/2} f(u) \left[\sin n\pi \cos \frac{n\pi u}{L} - \sin \frac{n\pi u}{L} \cos n\pi \right] du \\ &= -(-1)^n \int_0^{L/2} f(u) \sin \frac{n\pi u}{L} du. \end{aligned}$$

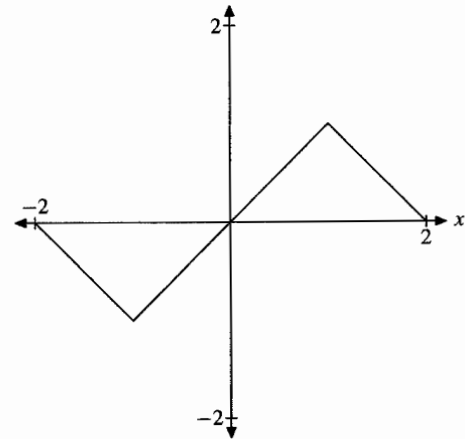
Hence, if n is even,

$$\begin{aligned} b_n &= \frac{2}{L} \left[\int_0^{L/2} f(x) \sin \frac{n\pi x}{L} dx - \int_0^{L/2} f(u) \sin \frac{n\pi u}{L} du \right] \\ &= 0. \end{aligned}$$

As an example, consider

$$f(x) = \begin{cases} -x - 2, & -2 \leq x < -1 \\ x, & -1 \leq x < 1 \\ -x + 2, & 1 \leq x \leq 2 \end{cases}.$$

Note that f is symmetric about $x = 1$ on $[0, 2]$. That is, $f(2 - x) = f(x)$ for $0 \leq x \leq 2$.



30. By the definition of derivative and the periodicity of f we have

$$\begin{aligned} f'(x + T) &= \lim_{h \rightarrow 0} \frac{f(x + T + h) - f(x + T)}{h} \\ &= \frac{f(x + h) - f(x)}{h} \\ &= f'(x). \end{aligned}$$

Therefore f' is periodic with period T .

31. Set $f_{\text{odd}}(x) = [f(x) - f(-x)]/2$ and $f_{\text{even}}(x) = [f(x) + f(-x)]/2$. Then f_{odd} is odd and f_{even} is even, and $f(x) = f_{\text{odd}}(x) + f_{\text{even}}(x)$ for all x .

Section 12.2. Convergence of Fourier Series

1. Since $\sin(x + \pi) = -\sin x$, $|\sin(x + \pi)| = |\sin x|$. Hence f is periodic with period π . For any $T < \pi$, we have $\sin T > 0 = \sin 0$. Hence f cannot be periodic with period T . Therefore the smallest period is π .
2. Since $\cos y$ is periodic with period 2π , we have

$$\begin{aligned} f(x + 2/3) &= \cos(3\pi(x + 2/3)) \\ &= \cos(3\pi x + 2\pi) \\ &= \cos(3\pi x) = f(x). \end{aligned}$$
 Hence f is periodic with period $2/3$.
3. Since f is strictly increasing, $f(x) \neq f(y)$ as long as $x \neq y$. Hence f cannot be periodic.
4. The function $\sin x$ is periodic with period 2π , while $\cos(x/2)$ is periodic with period 4π . Hence f is periodic with period 4π .
5. Since $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$, f cannot be periodic.
6. Since f is strictly increasing, $f(x) \neq f(y)$ as long as $x \neq y$. Hence f cannot be periodic.
7. f_p has one-sided derivatives everywhere, and is continuous everywhere except at odd multiples of π . Therefore, the Fourier series converges to $f_p(x)$ except at the odd multiples of π . At the odd multiples of π the series converges to $[\pi + 0]/2 = \pi/2$.
8. f_p has one-sided derivatives everywhere, and is continuous everywhere, so the Fourier series converges to $f_p(x)$ everywhere.
9. f_p has one-sided derivatives everywhere, and is continuous everywhere except at the odd integers. Therefore, the Fourier series converges to $f_p(x)$ except at the odd integers. At the odd integers, the series converges to $[1 + 0]/2 = 1/2$.
10. f_p has one-sided derivatives everywhere, and is continuous everywhere, so the Fourier series converges to $f_p(x)$ everywhere.
11. f_p has one-sided derivatives everywhere, and is continuous everywhere except at the odd integers. Therefore, the Fourier series converges to $f_p(x)$ except at the odd integers. At the odd integers, the series converges to $[1 - 1]/2 = 0$.
12. f_p has one-sided derivatives everywhere, and is continuous everywhere except at the odd integers. Therefore, the Fourier series converges to $f_p(x)$ except at the odd integers. At the odd integers, the series converges to $[1 + 0]/2 = 1/2$.
13. f_p has one-sided derivatives everywhere, and is continuous everywhere, so the Fourier series converges to $f_p(x)$ everywhere.
14. f_p has one-sided derivatives everywhere, and is continuous everywhere except at the odd multiples of 2. Therefore, the Fourier series converges to $f_p(x)$ except at the odd multiples of 2. At the odd multiples of 2, the series converges to $[2 + 0]/2 = 1$.
15. The function $f(x) = |x|$ is even on the interval $[-\pi, \pi]$. Hence, the Fourier representation contains only cosine terms. Hence,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx,$$
 since $|x| = x$ on $[0, \pi]$. For $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \frac{1}{\pi} x^2 \Big|_0^{\pi} = \pi.$$
 For $n \geq 1$, integration by parts provides

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left\{ \frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right\} \\ &= \frac{2}{\pi n^2} ((-1)^n - 1). \end{aligned}$$
 Thus, if n is even, $a_n = 0$. If n is odd,

$$a_n = -\frac{4}{\pi n^2}.$$
 We can write

$$a_{2n+1} = -\frac{4}{\pi(2n+1)^2},$$

which is valid for $n \geq 1$. Thus, the Fourier representation of f is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \cos((2n+1)x).$$

Using Corollary 2.4, the function $f(x) = |x|$ is continuous at $x = 0$ so the series converges to $f(0) = 0$ at $x = 0$. Thus,

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

16. The function $f(x) = x^2$ is even on the interval $[-\pi, \pi]$ so its Fourier representation has only cosine terms.

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx.$$

With $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2}{3\pi} x^3 \Big|_0^{\pi} = \frac{2\pi^2}{3}.$$

With $n \geq 1$, several applications of integration by parts provides

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{2\pi \cos n\pi}{n^2} \right] \\ &= (-1)^n \frac{4}{n^2}. \end{aligned}$$

Thus, the Fourier representation of $f(x) = x^2$ is

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

Because f is continuous at $x = 0$, Corollary 2.4 implies that the series converges at $x = 0$ to $f(0) = 0$.

$$0 = f(0) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Thus,

$$\begin{aligned} -4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} &= \frac{\pi^2}{3} \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}. \end{aligned}$$

Again, Corollary 2.4 implies that the series converges to $f(\pi)$ at $x = \pi$. Thus,

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi.$$

Since $\cos n\pi = (-1)^n$,

$$\begin{aligned} \frac{2\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}. \end{aligned}$$

17. The function $f(x) = x^4$ is even on the interval $[-\pi, \pi]$, so the Fourier representation has only cosine terms.

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^4 \cos nx \, dx$$

For $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^4 \, dx = \frac{2}{5\pi} x^5 \Big|_0^{\pi} = \frac{2\pi^4}{5}.$$

For $n \geq 1$, several applications of integration by parts provides

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{x^4 \sin nx}{n} + \frac{4x^3 \cos nx}{n^2} - \frac{12x^2 \sin nx}{n^3} \right. \\ &\quad \left. - \frac{24x \cos nx}{n^4} + \frac{24 \sin nx}{n^5} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{4\pi^3 \cos n\pi}{n^2} - \frac{24\pi \cos n\pi}{n^4} \right]. \end{aligned}$$

Since $\cos n\pi = (-1)^n$,

$$a_n = 8(-1)^n \left[\frac{\pi^2}{n^2} - \frac{6}{n^4} \right].$$

Hence, the Fourier representation of $f(x) = x^4$ is

$$f(x) = \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} (-1)^n \left[\frac{\pi^2}{n^2} - \frac{6}{n^4} \right] \cos nx.$$

Because f is continuous at $x = 0$, the Fourier series converges to $f(0)$ at $x = 0$.

$$\begin{aligned} 0 = f(0) &= \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} (-1)^n \left[\frac{\pi^2}{n^2} - \frac{6}{n^4} \right] \\ -\frac{\pi^4}{40} &= \pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - 6 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \end{aligned}$$

In Exercise 16, we found that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Substituting,

$$\begin{aligned} -\frac{\pi^4}{40} &= \pi^2 \left(-\frac{\pi^2}{12} \right) + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \\ 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} &= \frac{\pi^4}{12} - \frac{\pi^4}{40} \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} &= \frac{7\pi^4}{720}. \end{aligned}$$

In similar fashion, $f(x) = x^4$ is continuous at $x = \pi$, so the series converges to $f(\pi)$ at $x = \pi$.

$$\pi^4 = f(\pi) = \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} (-1)^n \left[\frac{\pi^2}{n^2} - \frac{6}{n^4} \right] \cos n\pi$$

Since $\cos n\pi = (-1)^n$, we have $(-1)^n(-1)^n = 1$ and

$$\begin{aligned} \frac{4\pi^4}{5} &= 8 \sum_{n=1}^{\infty} \left[\frac{\pi^2}{n^2} - \frac{6}{n^4} \right] \\ \frac{\pi^4}{10} &= \pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 6 \sum_{n=1}^{\infty} \frac{1}{n^4}. \end{aligned}$$

From Exercise 16,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

so

$$\begin{aligned} \frac{\pi^4}{10} &= \pi^2 \left(\frac{\pi^2}{6} \right) - 6 \sum_{n=1}^{\infty} \frac{1}{n^4} \\ 6 \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{6} - \frac{\pi^4}{10} = \frac{5\pi^4 - 3\pi^4}{30} = \frac{\pi^4}{15} \\ \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}. \end{aligned}$$

18. If

$$f(x) = \begin{cases} 0, & -1 < x \leq -1/2, \\ 1, & -1/2 < x \leq 1/2, \\ 0, & 1/2 < x \leq 1, \end{cases}$$

then f is even on the interval $[-1, 1]$ (except at points of discontinuity) and its Fourier representation contains only cosine terms.

$$\begin{aligned} a_n &= \int_{-1}^1 f(x) \cos n\pi x dx \\ &= \int_{-1/2}^{1/2} \cos n\pi x dx \\ &= \frac{1}{n\pi} \sin n\pi x \Big|_{-1/2}^{1/2} \\ &= \frac{1}{n\pi} \left[\sin \frac{n\pi}{2} - \sin \left(-\frac{n\pi}{2} \right) \right] \\ &= \frac{2}{n\pi} \sin \frac{n\pi}{2}, \end{aligned}$$

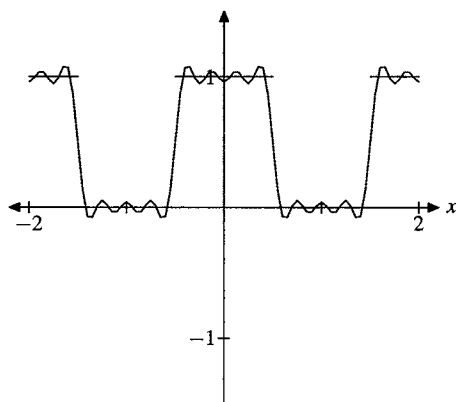
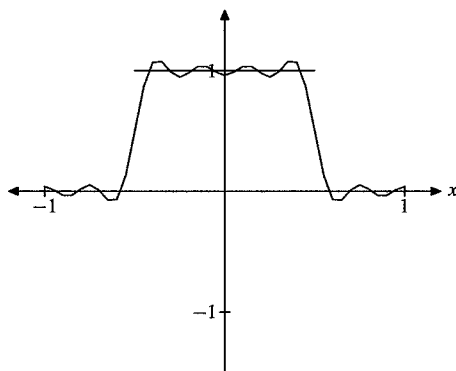
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \int_{-1/2}^{1/2} dx = 1.$$

Thus, the Fourier representation is

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos n\pi x.$$

The plot of S_7 on $[-1, 1]$ and $[-2, 2]$ follows.



Note the slower convergence due to the discontinuities at the half integers.

19. Consider $f(x) = e^{rx}$ on the interval $[-\pi, \pi]$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{rx} dx = \frac{1}{\pi r} e^{rx} \Big|_{-\pi}^{\pi} = \frac{2}{\pi r} \sinh \pi r.$$

For $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{rx} \cos nx dx.$$

Two applications of integration by parts provides

$$\begin{aligned} a_n &= \frac{1}{\pi} \cdot \frac{n^2}{n^2 + r^2} \left[\frac{e^{rx} \sin nx}{n} + \frac{r e^{rx} \cos nx}{n^2} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(n^2 + r^2)} [e^{r\pi}(n \sin n\pi + r \cos n\pi)]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(n^2 + r^2)} [e^{r\pi} r \cos n\pi - e^{-r\pi} r \cos n\pi] \\ &= \frac{r(-1)^n}{\pi(n^2 + r^2)} [e^{r\pi} - e^{-r\pi}] \\ &= \frac{2r(-1)^n}{\pi(n^2 + r^2)} \sinh \pi r. \end{aligned}$$

Next,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{rx} \sin nx dx.$$

Again, two applications of integration by parts provides the answer.

$$\begin{aligned} b_n &= \frac{1}{\pi} \cdot \frac{n^2}{n^2 + r^2} \left[-\frac{e^{rx} \cos nx}{n} + \frac{r e^{rx} \sin nx}{n^2} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(n^2 + r^2)} [e^{r\pi}(-n \cos n\pi + r \sin n\pi)]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(n^2 + r^2)} [e^{r\pi}(-n \cos n\pi) - e^{-r\pi}(-n \cos n\pi)] \\ &= \frac{n(-1)^{n+1}}{\pi(n^2 + r^2)} [e^{r\pi} - e^{-r\pi}] \\ &= \frac{2n(-1)^{n+1}}{\pi(n^2 + r^2)} \sinh \pi r. \end{aligned}$$

Thus, the Fourier representation is

$$\begin{aligned} e^{rx} &= \frac{\sinh \pi r}{\pi r} + \sum_{n=1}^{\infty} \frac{2r(-1)^n \sinh \pi r}{\pi(n^2 + r^2)} \cos nx \\ &\quad - \sum_{n=1}^{\infty} \frac{2n(-1)^n \sinh \pi r}{\pi(n^2 + r^2)} \sin nx. \end{aligned}$$

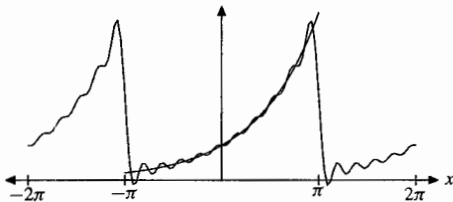
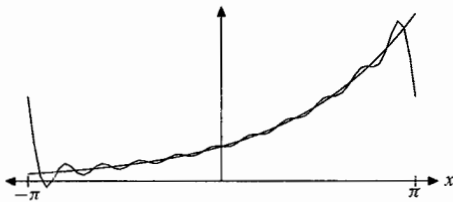
In the case that $r = 1/2$,

$$f(x) = \frac{2}{\pi} \sinh \frac{\pi}{2} + \frac{\sinh(\pi/2)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1/4} \cos nx \\ - \frac{2 \sinh(\pi/2)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1/4} \sin nx.$$

This is perhaps better written as

$$f(x) = \frac{1}{\pi} \sinh \frac{\pi}{2} \left[2 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1/4} \cos nx \right. \\ \left. - 2 \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1/4} \sin nx \right].$$

The following plots use $N = 10$ terms.



20. In Exercise 19, we found that on the interval $[-\pi, \pi]$,

$$e^{rx} = \frac{\sinh \pi r}{\pi r} + \sum_{n=1}^{\infty} \frac{2r(-1)^n \sinh \pi r}{\pi(n^2 + r^2)} \cos nx \\ - \sum_{n=1}^{\infty} \frac{2n(-1)^n \sinh \pi r}{\pi(n^2 + r^2)} \sin nx.$$

Thus, with $r = 1$,

$$e^x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \cos nx \\ - \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \sin nx.$$

Similarly, with $r = -1$,

$$e^{-x} = \frac{\sinh(-\pi)}{-\pi} - \frac{2 \sinh(-\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \cos nx \\ - \frac{2 \sinh(-\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \sin nx.$$

However, $\sinh x$ is an odd function, since

$$\sinh(-\pi) = \frac{e^{-\pi} - e^{\pi}}{2} \\ = -\frac{e^{\pi} - e^{-\pi}}{2} = -\sinh(\pi).$$

Thus,

$$e^{-x} = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \cos nx \\ + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \sin nx.$$

Subtracting,

$$e^x - e^{-x} = -\frac{4 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \sin nx.$$

Thus,

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \\ = \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 + 1} \sin nx.$$

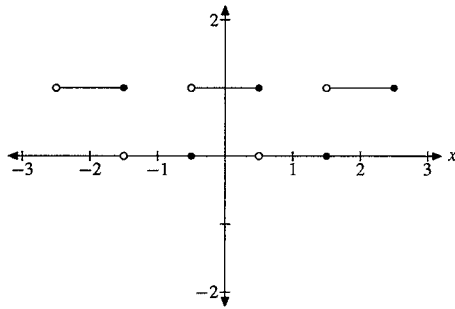
Note that the answer contains only sine terms, which is consistent with the fact that $f(x) = \sinh x$ is odd on the interval $[-\pi, \pi]$. In similar fashion,

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \\ = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \cos nx.$$

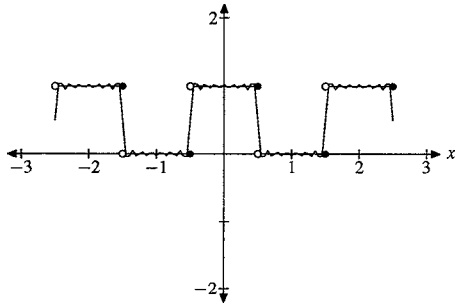
21. Consider

$$f(x) = \begin{cases} 0, & -1 < x \leq -1/2, \\ 1, & -1/2 < x \leq 1/2, \\ 0, & 1/2 < x \leq 1. \end{cases}$$

The plot of $f_p(x)$ is shown in blue.



The Fourier series will converge to $f_p(x)$ whenever f_p is continuous at x . At each point of discontinuity we have a limit from one side equalling 1, while the limit coming in from the other side is 0. Thus, the Fourier series should converge to $1/2$ at these points, as is evident in the following image depicting $S_{13}(x)$.



22. Using the limit quotient definition of the derivative

and the periodicity of f we have

$$\begin{aligned} f''(x+T) &= \lim_{h \rightarrow 0} \frac{f(x+T+h) - f(x+T)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= f'(x). \end{aligned}$$

Hence f' is periodic with period T .

23. First notice that

$$\int_0^T f(x) dx = \int_0^a f(x) dx + \int_a^T f(x) dx.$$

For the first integral, we use the periodicity of f and the substitution $x = u + T$ to get

$$\int_0^a f(x) dx = \int_T^{a+T} f(u+T) du = \int_T^{a+T} f(u) du.$$

Since u is a dummy variable in the last integral, we can replace it by x . Then we get

$$\begin{aligned} \int_0^T f(x) dx &= \int_T^{a+T} f(x) dx + \int_a^T f(x) dx \\ &= \int_a^{a+T} f(x) dx. \end{aligned}$$

Using this with a replaced by b , we get the final result

$$\int_b^{b+T} f(x) dx = \int_0^T f(x) dx = \int_a^{a+T} f(x) dx$$

If f is periodic with period L , then $F(x) = f(x) \cos(n\pi x/L)$ is also periodic with period L . Hence, for any c ,

$$a_n = \frac{1}{L} \int_{-L}^L F(x) dx = \frac{1}{L} \int_c^{c+2L} F(x) dx.$$

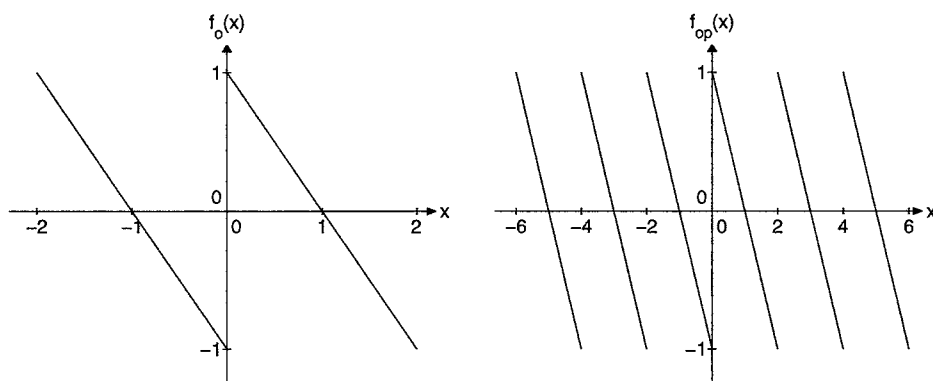
The same argument works for b_n .

Section 12.3. Fourier Cosine and Sine Series

1. If $f(x) = 1 - x$ on $[0, 2]$, then

$$f_o(x) = \begin{cases} -(1+x), & -2 \leq x < 0, \\ 0, & x = 0, \\ 1-x, & 0 < x \leq 2. \end{cases}$$

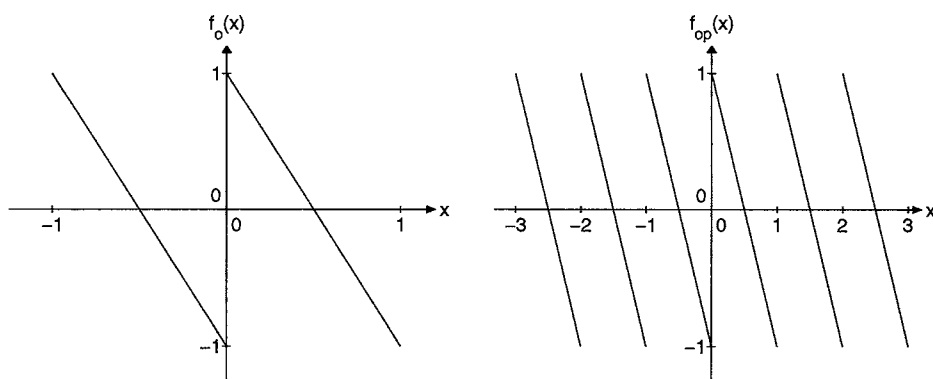
The graphs of the odd extension f_o on $[-2, 2]$ and the odd periodic extension f_{op} on $[-6, 6]$ follow.



2. If $f(x) = 1 - 2x$ on $[0, 1]$, then

$$f_o(x) = \begin{cases} -(1+2x), & -1 \leq x < 0, \\ 0, & x = 0, \\ 1-2x, & 0 < x \leq 1. \end{cases}$$

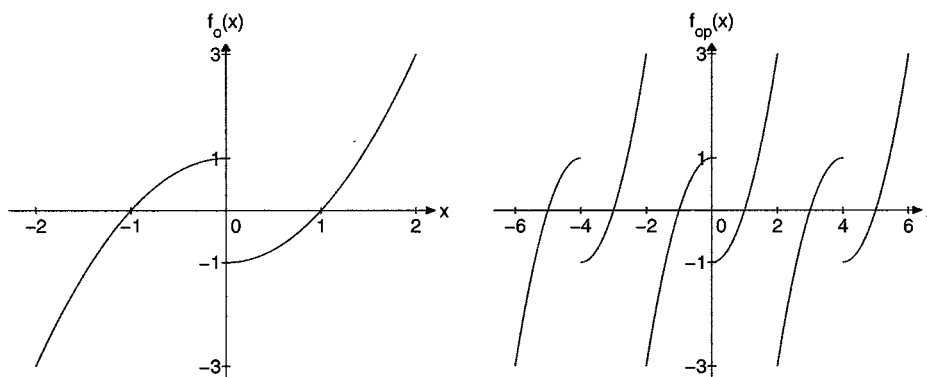
The graphs of the odd extension f_o on $[-1, 1]$ and the odd periodic extension f_{op} on $[-3, 3]$ follow.



3. If $f(x) = x^2 - 1$ on $[0, 2]$, then

$$f_o(x) = \begin{cases} -(x^2 - 1), & -2 \leq x < 0, \\ 0, & x = 0, \\ x^2 - 1, & 0 < x \leq 2. \end{cases}$$

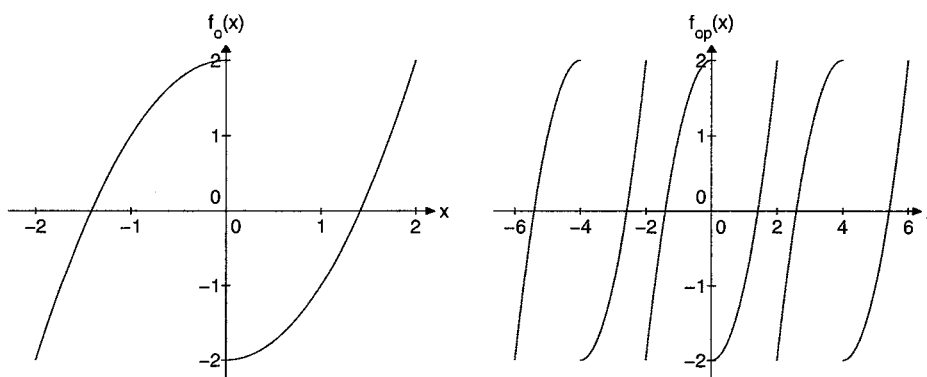
The graphs of the odd extension f_o on $[-2, 2]$ and the odd periodic extension f_{op} on $[-6, 6]$ follow.



4. If $f(x) = x^2 - 2$ on $[0, 2]$, then

$$f_o(x) = \begin{cases} -(x^2 - 2), & -2 \leq x < 0, \\ 0, & x = 0, \\ x^2 - 2, & 0 < x \leq 2. \end{cases}$$

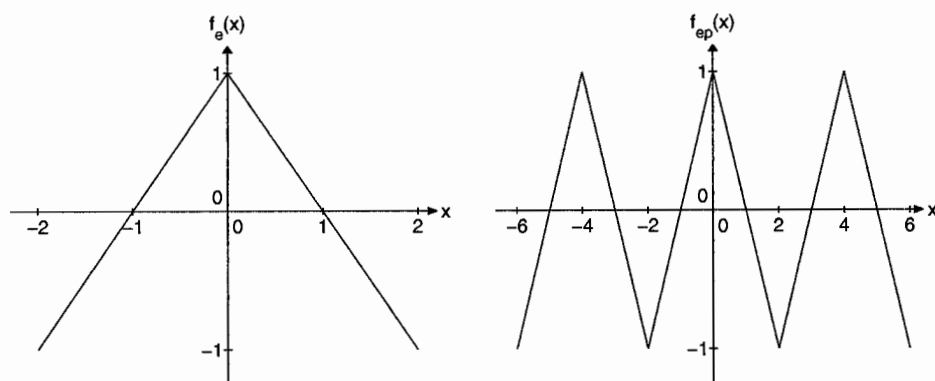
The graphs of the odd extension f_o on $[-2, 2]$ and the odd periodic extension f_{op} on $[-6, 6]$ follow.



5. If $f(x) = 1 - x$ on $[0, 2]$, then

$$f_e(x) = \begin{cases} 1 + x, & -2 \leq x < 0, \\ 1 - x, & 0 \leq x \leq 2. \end{cases}$$

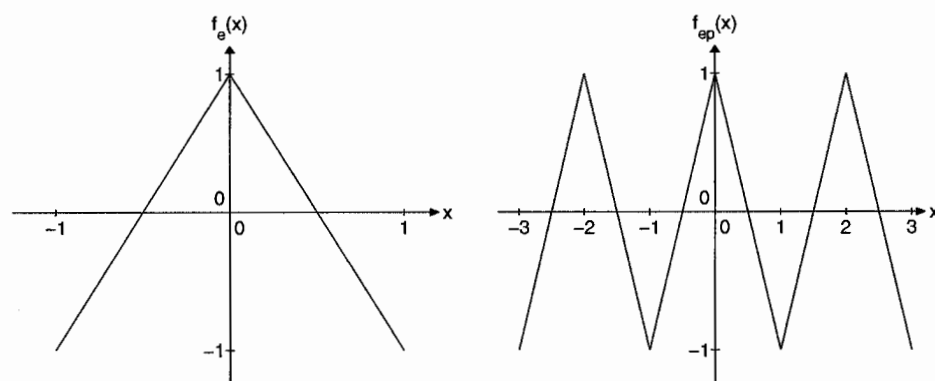
The graphs of the even extension f_e on $[-2, 2]$ and the even periodic extension f_{ep} on $[-6, 6]$ follow.



6. If $f(x) = 1 - 2x$ on $[0, 1]$, then

$$f_e(x) = \begin{cases} 1 + 2x, & -1 \leq x < 0, \\ 1 - 2x, & 0 \leq x \leq 1. \end{cases}$$

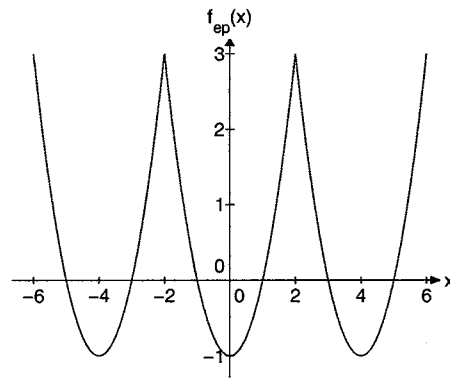
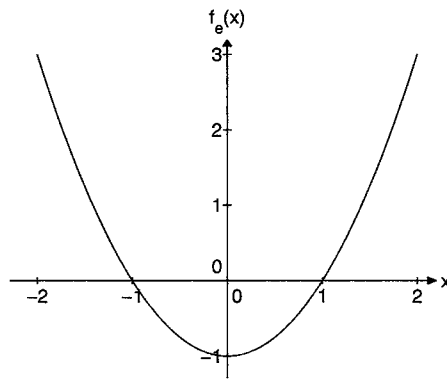
The graphs of the even extension f_e on $[-1, 1]$ and the even periodic extension f_{ep} on $[-3, 3]$ follow.



7. If $f(x) = x^2 - 1$ on $[0, 2]$, then

$$f_e(x) = \begin{cases} x^2 - 1, & -2 \leq x < 0, \\ x^2 - 1, & 0 \leq x \leq 2. \end{cases}$$

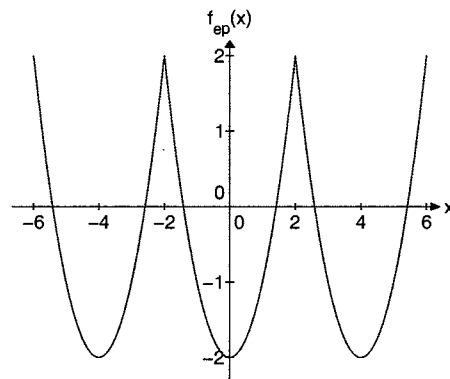
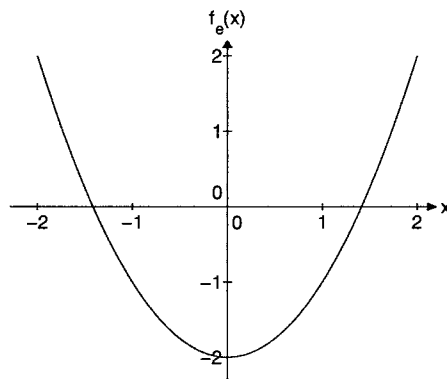
The graphs of the even extension f_e on $[-2, 2]$ and the even periodic extension f_{ep} on $[-6, 6]$ follow.



8. If $f(x) = x^2 - 2$ on $[0, 2]$, then

$$f_e(x) = \begin{cases} x^2 - 2, & -2 \leq x < 0, \\ x^2 - 2, & 0 \leq x \leq 2. \end{cases}$$

The graphs of the even extension f_e on $[-2, 2]$ and the even periodic extension f_{ep} on $[-6, 6]$ follow.



9. The function $f(x) = x$ on $[0, \pi]$ has even extension

$$f_e(x) = \begin{cases} -x, & -\pi \leq x < 0, \\ x, & 0 \leq x \leq \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right] \\ &= \frac{2}{\pi n^2} [(-1)^n - 1], \end{aligned}$$

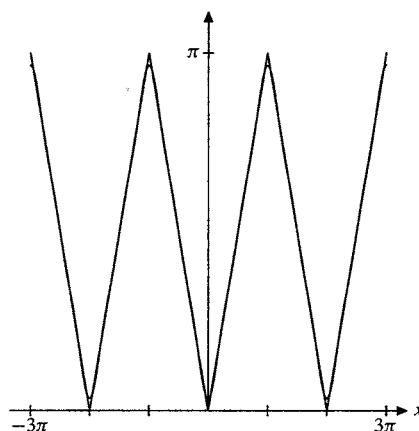
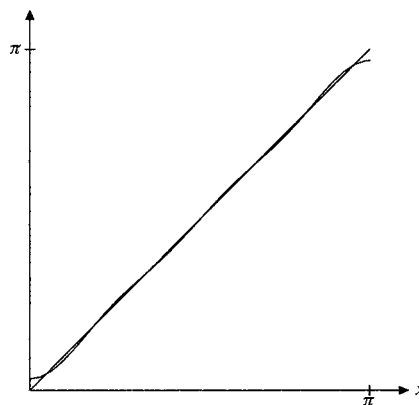
provided $n \neq 0$. In the case $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \frac{2}{\pi} \left[\frac{1}{2} x^2 \right]_0^{\pi} = \pi.$$

Thus, the Fourier series expansion is

and $[-3\pi, 3\pi]$.

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [(-1)^n - 1] \cos nx. \end{aligned}$$



The partial sum S_6 is shown on the interval $[0, \pi]$

10. The function $f(x) = \sin x$ has even extension

$$f_e(x) = \begin{cases} -\sin x, & -\pi \leq x < 0, \\ \sin x, & 0 \leq x \leq \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. Hence, the Fourier series expansion has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi \frac{1}{2} [\sin(1-n)x + \sin(1+n)x] dx \\
 &= \frac{1}{\pi} \int_0^\pi [\sin(n+1)x - \sin(n-1)x] dx \\
 &= \frac{1}{\pi} \left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^\pi \\
 &= \frac{1}{\pi} \left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \left(\frac{-1}{n+1} + \frac{1}{n-1} \right) \right] \\
 &= \frac{1}{\pi} \left[\frac{-n(-1)^{n+1} + (-1)^{n+1} + n(-1)^{n-1} + (-1)^{n-1} + n-1 - n-1}{(n+1)(n-1)} \right] \\
 &= \frac{1}{\pi} \left[\frac{n(-1)^{n-1}[1 - (-1)^2] + (-1)^{n-1}[1 + (-1)^2] - 2}{n^2 - 1} \right] \\
 &= \frac{1}{\pi} \left[\frac{2(-1)^{n-1} - 2}{n^2 - 1} \right] \\
 &= \frac{2}{\pi} \left[\frac{(-1)^{n-1} - 1}{n^2 - 1} \right],
 \end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$a_1 = \frac{2}{\pi} \int_0^\pi \sin x \cos x dx = \frac{1}{\pi} \int_0^\pi \sin 2x dx = 0.$$

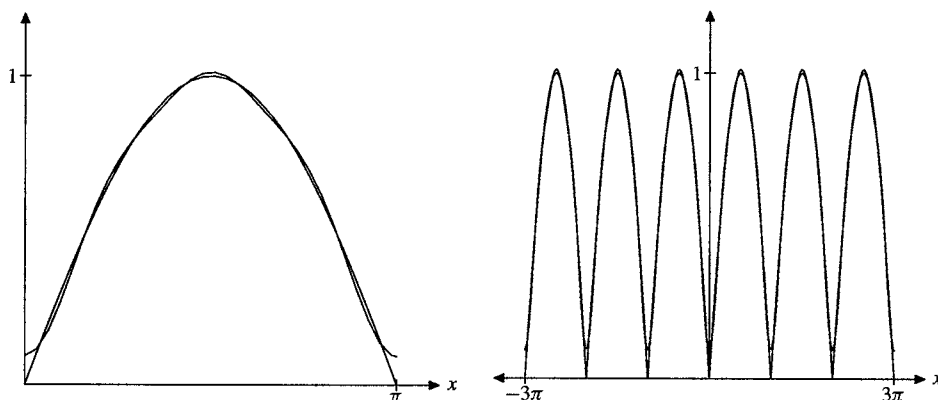
Note that

$$a_0 = \frac{2}{\pi} \left[\frac{(-1)^{-1} - 1}{0^2 - 1} \right] = \frac{2}{\pi} \left[\frac{-2}{-1} \right] = \frac{4}{\pi}.$$

Hence, the Fourier series expansion for f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi} \left[\frac{(-1)^{n-1} - 1}{n^2 - 1} \right] \cos nx.$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



11. As $f(x) = \cos x$ is already a cosine series, any analysis we do should agree with this fact. The function has even extension

$$\begin{aligned} f_e(x) &= \begin{cases} \cos(-x), & -\pi \leq x < 0, \\ \cos x, & 0 \leq x \leq \pi, \end{cases} \\ &= \begin{cases} \cos x, & -\pi \leq x < 0, \\ \cos x, & 0 \leq x \leq \pi, \end{cases} \end{aligned}$$

on the interval $[-\pi, \pi]$. Hence, the Fourier series expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \cos x \cos nx \, dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} [\cos(1-n)x + \cos(1+n)x] \, dx \\ &= \frac{1}{\pi} \left[\frac{\sin(n-1)x}{n-1} + \frac{\sin(n+1)x}{n+1} \right]_0^{\pi} \\ &= 0, \end{aligned}$$

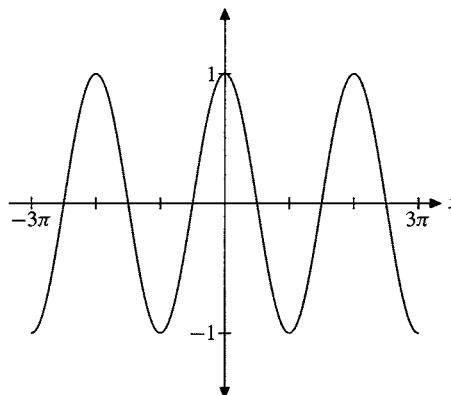
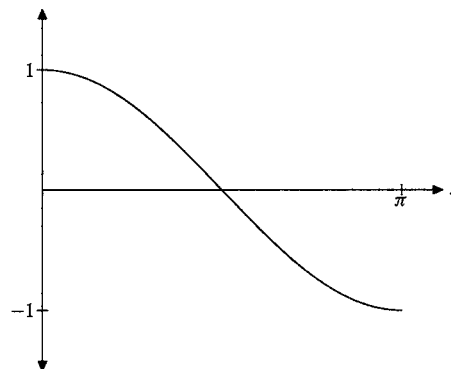
provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned} a_1 &= \frac{2}{\pi} \int_0^{\pi} \cos^2 x \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \frac{1 + \cos 2x}{2} \, dx \\ &= \frac{1}{\pi} \left[x + \frac{1}{2} \sin 2x \right]_0^{\pi} \\ &= \frac{1}{\pi} [\pi] \\ &= 1. \end{aligned}$$

Hence, the Fourier series expansion is

$$f(x) = \cos x.$$

No surprises here! The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



12. The function $f(x) = 1$, being already even on $[0, \pi]$, has even extension $f_e(x) = 1$. Moreover, the function is already a Fourier series with $a_0 = 1$ and $a_n = b_n = 0, n \geq 1$. So any analysis should agree with this fact. Thus, $f_e(x)$, being even on $[-\pi, \pi]$ will only contain cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \cos nx \, dx \\ &= \frac{2}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi} \\ &= 0, \end{aligned}$$

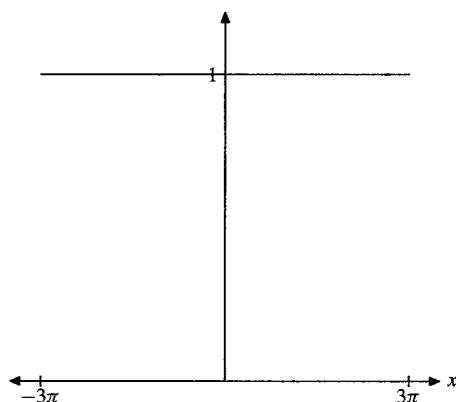
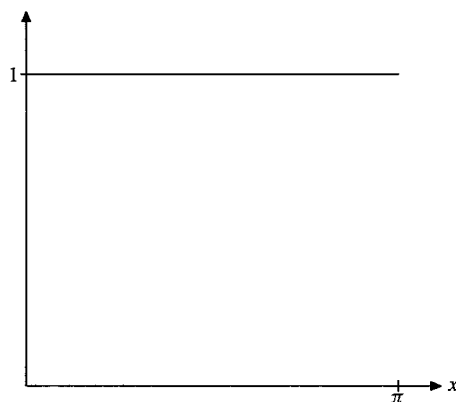
provided $n \neq 0$. For the case $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} dx = 2.$$

Hence, the Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = 1.$$

No surprises here! The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



13. The function $f(x) = \pi - x$ on $[0, \pi]$ has even extension

$$f_e(x) = \begin{cases} \pi + x, & -\pi \leq x < 0, \\ \pi - x, & 0 \leq x \leq \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier series expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx \, dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{(\pi - x) \sin nx}{n} - \frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[-\frac{\cos n\pi}{n^2} + \frac{1}{n^2} \right] \\ &= \frac{2}{\pi n^2} [1 - (-1)^n], \end{aligned}$$

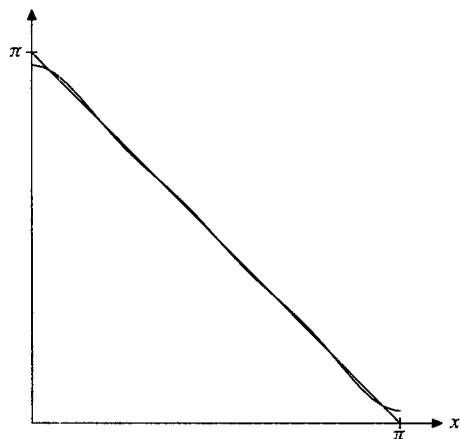
provided $n \neq 0$. In the case that $n = 0$,

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx \\ &= \frac{2}{\pi} \left[\pi x - \frac{1}{2} x^2 \right]_0^{\pi} \\ &= \pi. \end{aligned}$$

Thus, the Fourier series expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [1 - (-1)^n] \cos nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^\pi \\ &= \frac{2}{\pi} \left[\frac{2\pi \cos n\pi}{n^2} \right] \\ &= (-1)^n \frac{4}{n^2}, \end{aligned}$$

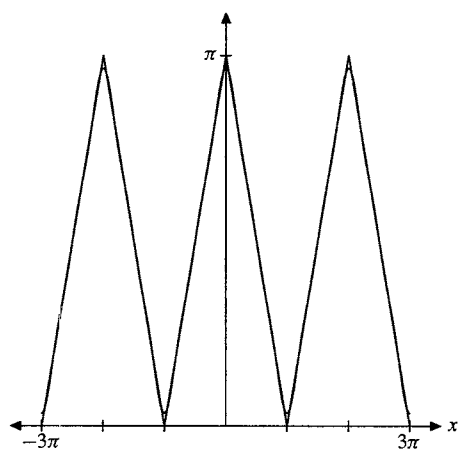
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^\pi x^2 dx = \frac{2}{\pi} \left[\frac{1}{3} x^3 \right]_0^\pi = \frac{2\pi^2}{3}.$$

Hence, the Fourier expansion is

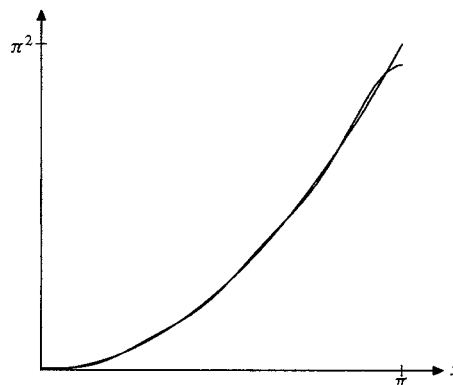
$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx. \end{aligned}$$

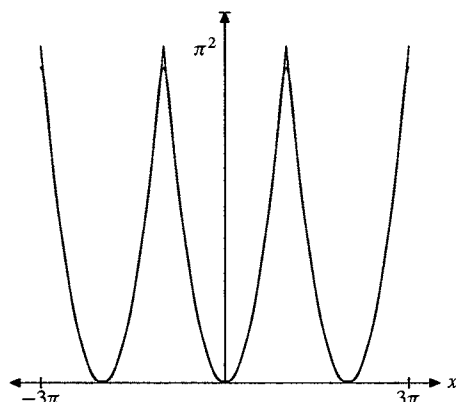
The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



14. The function $f(x) = x^2$ on $[0, \pi]$ has even extension $f_e(x) = x^2$ on $[-\pi, \pi]$. The Fourier expansion will have cosine terms only and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx dx \\ &= \frac{2}{\pi} \int_0^\pi x^2 \cos nx dx \end{aligned}$$





15. The function $f(x) = x^3$ on $[0, \pi]$ has even extension

$$f_e(x) = \begin{cases} -x^3, & -\pi \leq x < 0, \\ x^3, & 0 \leq x \leq \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^3 \cos nx \, dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{x^3 \sin nx}{n} + \frac{3x^2 \cos nx}{n^2} \right. \\ &\quad \left. - \frac{6x \sin nx}{n^3} - \frac{6 \cos nx}{n^4} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{3\pi^2 \cos n\pi}{n^2} - \frac{6 \cos n\pi}{n^4} + \frac{6}{n^4} \right] \\ &= \frac{6\pi(-1)^n}{n^2} - \frac{12}{\pi n^4} ((-1)^n - 1), \end{aligned}$$

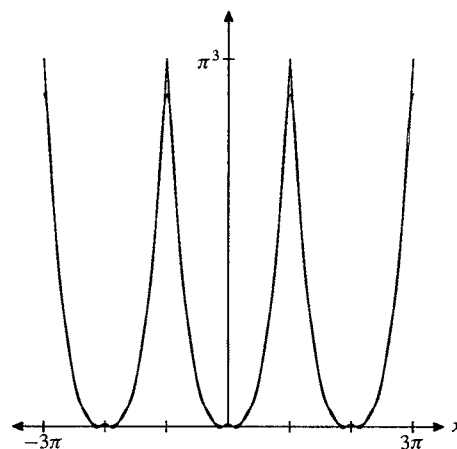
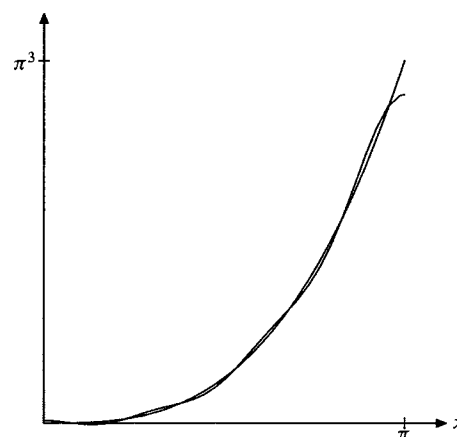
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^3 \, dx = \frac{2}{\pi} \left[\frac{1}{4} x^4 \right]_0^{\pi} = \frac{\pi^3}{2}.$$

Hence, the Fourier series expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi^3}{4} + \sum_{n=1}^{\infty} \left[\frac{6\pi(-1)^n}{n^2} - \frac{12}{\pi n^4} ((-1)^n - 1) \right] \cos nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



16. The function $f(x) = x^4$ on $[0, \pi]$ has even extension $f_e(x) = x^4$ on $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^4 \cos nx \, dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{x^4 \sin nx}{n} + \frac{4x^3 \cos nx}{n^2} - \frac{12x^2 \sin nx}{n^3} \right. \\ &\quad \left. - \frac{24x \cos nx}{n^4} + \frac{24 \sin nx}{n^5} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{4\pi^3 \cos n\pi}{n^2} - \frac{24\pi \cos n\pi}{n^4} \right] \\ &= 8(-1)^n \left[\frac{\pi^2}{n^2} - \frac{6}{n^4} \right] \\ &= 8(-1)^n \frac{\pi^2 n^2 - 6}{n^4}, \end{aligned}$$

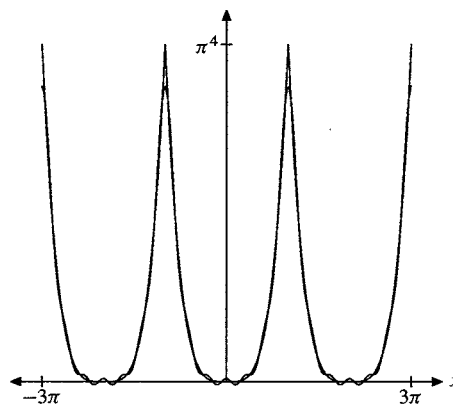
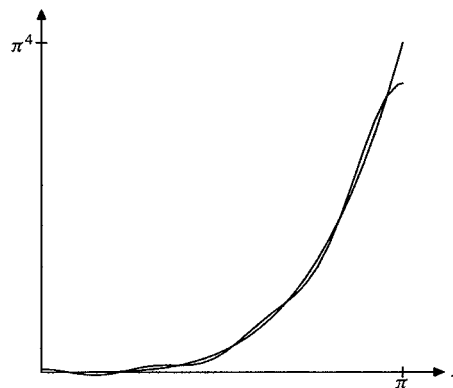
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^4 \, dx = \frac{2}{\pi} \left[\frac{1}{5} x^5 \right]_0^{\pi} = \frac{2\pi^4}{5}.$$

Thus, the Fourier series expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi^4}{5} + \sum_{n=1}^{\infty} 8(-1)^n \frac{\pi^2 n^2 - 6}{n^4} \cos nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



17. The function

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi/2, \\ 0, & \pi/2 \leq x \leq \pi, \end{cases}$$

has even extension

$$f_e(x) = \begin{cases} 0, & -\pi \leq x < -\pi/2, \\ 1, & -\pi/2 \leq x < \pi/2, \\ 0, & \pi/2 \leq x < \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f_e(x) \cos nx \, dx. \end{aligned}$$

However, $f_e(x) = 0$ on $[\pi/2, \pi]$, so this last integral becomes

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi/2} \cos nx \, dx = \frac{2}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi/2} \\ &= \frac{2}{n\pi} \sin \frac{n\pi}{2}, \end{aligned}$$

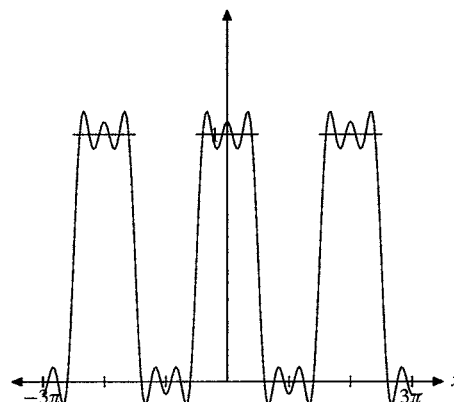
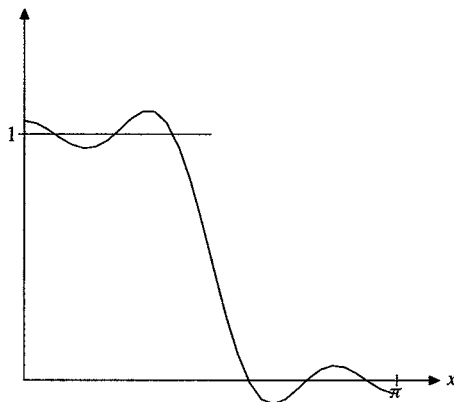
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} dx = 1.$$

Thus, the Fourier series expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



18. The function

$$f(x) = \begin{cases} x, & 0 \leq x < \pi/2, \\ \pi/2, & \pi/2 \leq x \leq \pi, \end{cases}$$

has even extension

$$f_e(x) = \begin{cases} \pi/2, & -\pi \leq x < -\pi/2, \\ -x, & -\pi/2 \leq x < 0, \\ x, & 0 \leq x < \pi/2, \\ \pi/2, & \pi/2 \leq x \leq \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} x \cos nx \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} \frac{\pi}{2} \cos nx \, dx \end{aligned}$$

Integrating by parts, the first integral is

$$\begin{aligned} & \frac{2}{\pi} \int_0^{\pi/2} x \cos nx \, dx \\ &= \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi/2} \\ &= \frac{2}{\pi} \left[\frac{\frac{\pi}{2} \sin \frac{n\pi}{2}}{n} + \frac{\cos \frac{n\pi}{2}}{n^2} - \frac{1}{n^2} \right] \\ &= \frac{\sin \frac{n\pi}{2}}{n} + \frac{2 \cos \frac{n\pi}{2}}{\pi n^2} - \frac{2}{\pi n^2}. \end{aligned}$$

The second integral is

$$\begin{aligned} \frac{2}{\pi} \int_{\pi/2}^{\pi} \frac{\pi}{2} \cos nx \, dx &= \int_{\pi/2}^{\pi} \cos nx \, dx \\ &= \frac{1}{n} \sin nx \Big|_{\pi/2}^{\pi} \\ &= -\frac{1}{n} \sin \frac{n\pi}{2} \end{aligned}$$

Adding these last two results,

$$\begin{aligned} a_n &= \frac{2}{\pi n^2} \cos \frac{n\pi}{2} - \frac{2}{\pi n^2} \\ &= \frac{2}{\pi n^2} \left[\cos \frac{n\pi}{2} - 1 \right], \end{aligned}$$

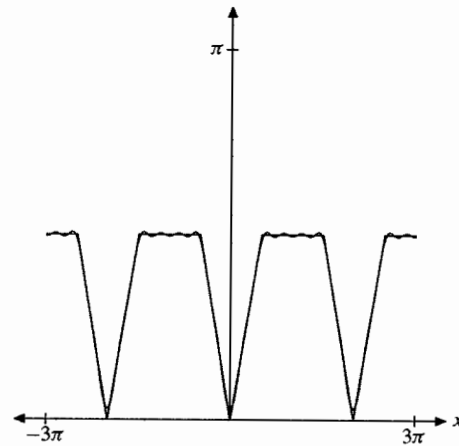
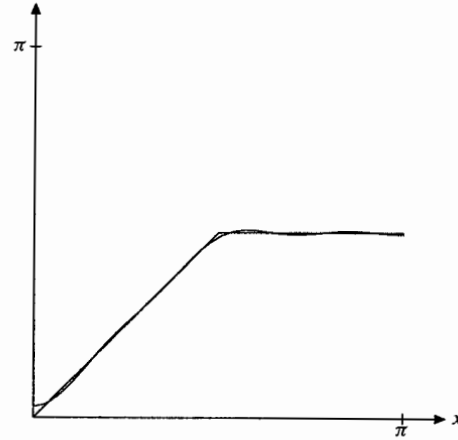
provided $n \neq 0$. In the case that $n = 0$,

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi/2} x \, dx + \int_{\pi/2}^{\pi} dx \\ &= \frac{2}{\pi} \left[\frac{1}{2} x^2 \right]_0^{\pi/2} + \left(\pi - \frac{\pi}{2} \right) \\ &= \frac{3\pi}{4}. \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{3\pi}{8} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \left[\cos \frac{n\pi}{2} - 1 \right] \cos nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



19. The function $f(x) = x \cos x$ on $[0, \pi]$ has even extension

$$f_e(x) = \begin{cases} -x \cos x, & -\pi \leq x < 0, \\ x \cos x, & 0 \leq x < \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos x \cos nx \, dx.$$

Using a product to sum identity,

$$a_n = \frac{2}{\pi} \int_0^{\pi} \frac{x}{2} [\cos(1-n)x + \cos(1+n)x] \, dx = \frac{1}{\pi} \int_0^{\pi} x \cos(n-1)x \, dx + \frac{1}{\pi} \int_0^{\pi} x \cos(n+1)x \, dx.$$

Integrating by parts, the first integral becomes

$$\begin{aligned} \frac{1}{\pi} \int_0^{\pi} x \cos(n-1)x \, dx &= \frac{1}{\pi} \left[\frac{x \sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[\frac{\cos(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2} \right] \\ &= \frac{1}{\pi(n-1)^2} [(-1)^{n-1} - 1], \end{aligned}$$

provided $n \neq 1$. Similarly, the second integral is

$$\frac{1}{\pi} \int_0^{\pi} x \cos(n+1)x \, dx = \frac{1}{\pi(n+1)^2} [(-1)^{n+1} - 1].$$

Adding,

$$\begin{aligned} a_n &= \frac{1}{\pi(n-1)^2} [(-1)^{n-1} - 1] + \frac{1}{\pi(n+1)^2} [(-1)^{n+1} - 1] \\ &= \frac{(n+1)^2 ((-1)^{n-1} - 1) + (n-1)^2 ((-1)^{n+1} - 1)}{\pi(n-1)^2(n+1)^2} \\ &= \frac{((-1)^{n-1} - 1)((n+1)^2 + (n-1)^2)}{\pi(n-1)^2(n+1)^2} \\ &= \frac{2((-1)^{n-1} - 1)(n^2 + 1)}{\pi(n-1)^2(n+1)^2}, \end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \cos^2 x \, dx.$$

Using the half angle identity,

$$\begin{aligned}
 a_1 &= \frac{2}{\pi} \int_0^{\pi} x \left(\frac{1 + \cos 2x}{2} \right) dx \\
 &= \frac{1}{\pi} \int_0^{\pi} (x + x \cos 2x) dx \\
 &= \frac{1}{\pi} \left[\frac{1}{2} x^2 + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\pi} \\
 &= \frac{1}{\pi} \left[\frac{\pi^2}{2} \right] \\
 &= \frac{\pi}{2}.
 \end{aligned}$$

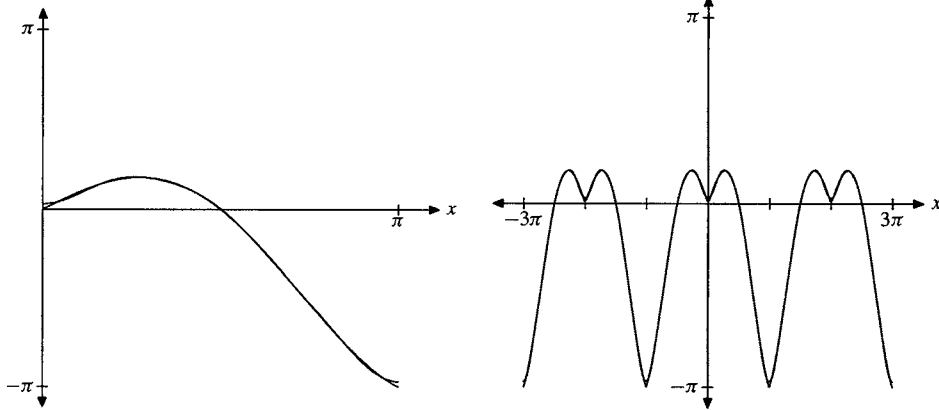
Also, note that with computation above,

$$a_0 = \frac{2(-1 - 1)(1)}{\pi \cdot 1 \cdot 1} = -\frac{4}{\pi}.$$

Hence, the Fourier expansion is

$$\begin{aligned}
 f(x) &= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx \\
 &= -\frac{2}{\pi} + \frac{\pi}{2} \cos x + \sum_{n=2}^{\infty} \frac{2((-1)^{n-1} - 1)(n^2 + 1)}{\pi(n-1)^2(n+1)^2} \cos nx.
 \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



20. The function $f(x) = x \sin x$ on $[0, \pi]$ has even extension $f_e(x) = x \sin x$ on the interval $[-\pi, \pi]$. Hence, the Fourier expansion of $f_e(x)$ has only co-

sine terms and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx.$$

Using a product to sum identity,

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi \frac{x}{2} [\sin(1-n)x + \sin(1+n)x] dx \\ &= \frac{1}{\pi} \int_0^\pi x \sin(n+1)x dx \\ &\quad - \frac{1}{\pi} \int_0^\pi x \sin(n-1)x dx. \end{aligned}$$

Using integration by parts, the first integral becomes

$$\begin{aligned} &\frac{1}{\pi} \int_0^\pi x \sin(n+1)x dx \\ &= \frac{1}{\pi} \left[\frac{-x \cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} \right]_0^\pi \\ &= \frac{1}{\pi} \left[\frac{-\pi \cos(n+1)\pi}{n+1} \right] \\ &= \frac{(-1)^{n+2}}{n+1}. \end{aligned}$$

Similarly, the second integral is

$$\frac{1}{\pi} \int_0^\pi x \sin(n-1)x dx = \frac{(-1)^n}{n-1}.$$

Subtracting,

$$\begin{aligned} a_n &= \frac{(-1)^{n+2}}{n+1} - \frac{(-1)^n}{n-1} \\ &= \frac{(-1)^{n+2}(n-1) - (-1)^n(n+1)}{(n+1)(n-1)} \\ &= \frac{(-1)^n[n-1-n-1]}{(n+1)(n-1)} \\ &= \frac{(-1)^n(-2)}{n^2-1} \\ &= \frac{2(-1)^{n+1}}{n^2-1}, \end{aligned}$$

provided $n \neq 1$. Note that $a_0 = 2$. In the event that $n = 1$,

$$\begin{aligned} a_1 &= \frac{2}{\pi} \int_0^\pi x \sin x \cos x dx \\ &= \frac{1}{\pi} \int_0^\pi x \sin 2x dx. \end{aligned}$$

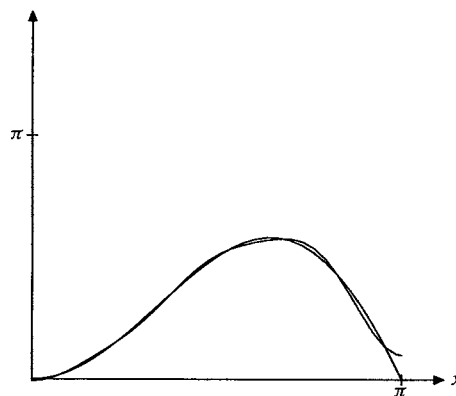
Integrating by parts,

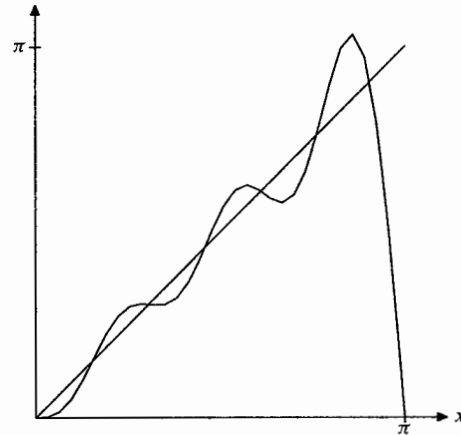
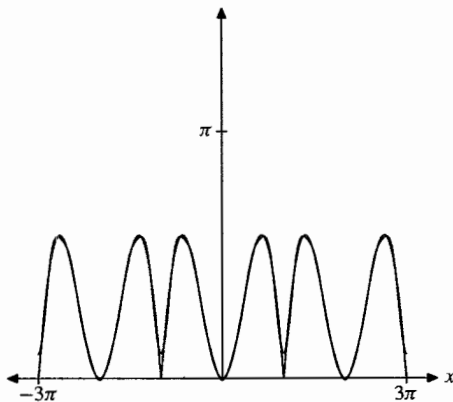
$$\begin{aligned} a_1 &= \frac{1}{\pi} \left[\frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^\pi \\ &= \frac{1}{\pi} \left[\frac{-\pi}{2} \right] \\ &= -\frac{1}{2}. \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= 1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.





21. The function $f(x) = x$ on $[0, \pi]$ is already odd on the interval $[-\pi, \pi]$, so it is the same as its odd extension. Thus the Fourier expansion has only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx. \end{aligned}$$

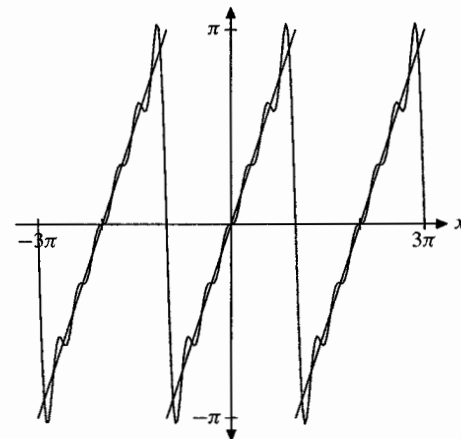
Integrating by parts,

$$\begin{aligned} b_n &= \frac{2}{\pi} \left[\frac{-x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{-\pi \cos n\pi}{n} \right] \\ &= 2 \left[\frac{-(-1)^n}{n} \right] \\ &= (-1)^{n+1} \frac{2}{n}, \end{aligned}$$

for $n \geq 1$. Thus, the Fourier series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx.$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



22. The function $f(x) = \sin x$ on $[0, \pi]$ is already odd on the interval $[-\pi, \pi]$, so it is the same as its odd extension. Thus, the Fourier expansion has only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin x \sin nx \, dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi \frac{1}{2} [\cos(x - nx) - \cos(x + nx)] dx \\ &= \frac{1}{\pi} \int_0^\pi [\cos(n-1)x - \cos(n+1)x] dx \\ &= \frac{1}{\pi} \left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right]_0^\pi \\ &= 0, \end{aligned}$$

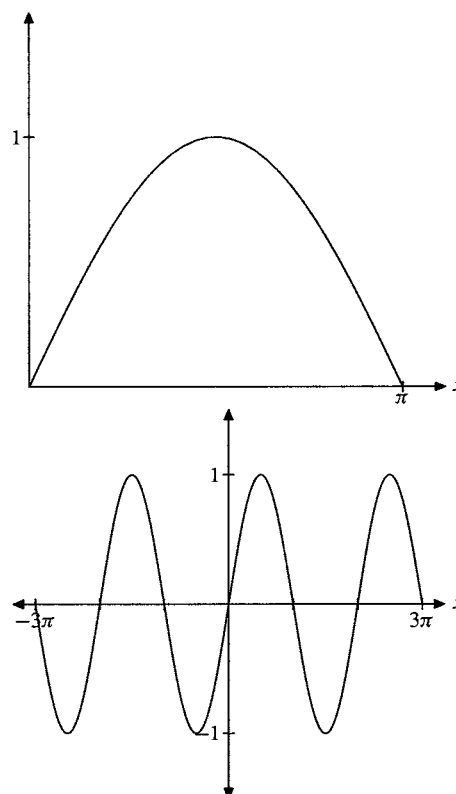
provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned} b_1 &= \frac{2}{\pi} \int_0^\pi \sin^2 x dx \\ &= \frac{2}{\pi} \int_0^\pi \frac{1 - \cos 2x}{2} dx \\ &= \frac{1}{\pi} \left[x - \frac{1}{2} \sin 2x \right]_0^\pi \\ &= \frac{1}{\pi} [\pi] \\ &= 1. \end{aligned}$$

Thus the Fourier expansion for $f(x) = \sin x$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sin x,$$

as it should be. The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



23. An odd extension for $f(x) = \cos x$ on the interval $[-\pi, \pi]$ is

$$f_o(x) = \begin{cases} -\cos x, & -\pi \leq x < 0, \\ \cos x, & 0 \leq x \leq \pi. \end{cases}$$

The Fourier expansion of $f_o(x)$ contains only sine terms and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_o(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx dx.$$

Using a product to sum identity,

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \int_0^\pi \frac{1}{2} [\sin(n-1)x + \sin(n+1)x] dx \\
 &= \frac{1}{\pi} \left[-\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1} \right]_0^\pi \\
 &= -\frac{1}{\pi} \left\{ \frac{(-1)^{n-1}}{n-1} + \frac{(-1)^{n+1}}{n+1} - \frac{1}{n-1} - \frac{1}{n+1} \right\} \\
 &= -\frac{1}{\pi} \left\{ \frac{(-1)^{n-1}(n+1) + (-1)^{n+1}(n-1) - (n+1) - (n-1)}{n^2 - 1} \right\}.
 \end{aligned}$$

Because $(-1)^{n+1} = (-1)^{n-1}$,

$$b_n = -\frac{1}{\pi} \left\{ \frac{(-1)^{n-1}[(n+1) + (n-1)] - 2n}{n^2 - 1} \right\} = \frac{2n}{\pi} \cdot \frac{(-1)^n + 1}{n^2 - 1},$$

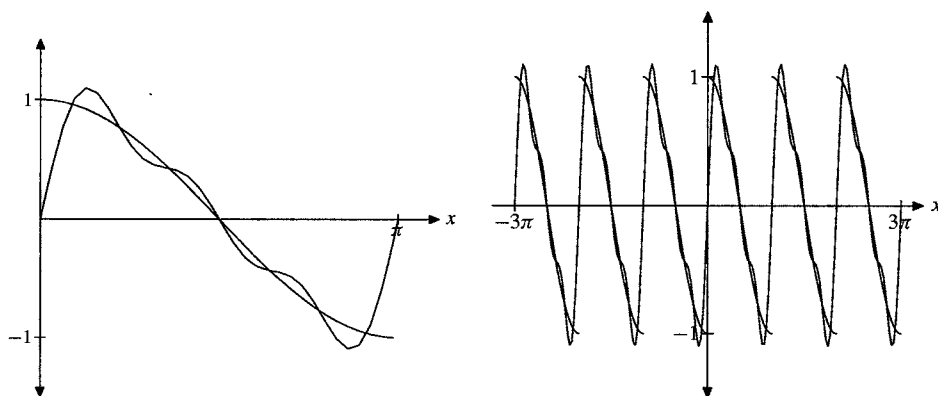
provided $n \neq 1$. In the case that $n = 1$,

$$b_1 = \frac{2}{\pi} \int_0^\pi \cos x \sin x dx = \frac{1}{\pi} \int_0^\pi \sin 2x dx = \frac{1}{\pi} \left[-\frac{1}{2} \cos 2x \right]_0^\pi = 0.$$

Thus, the Fourier expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=2}^{\infty} \frac{2((-1)^n + 1)n}{\pi(n^2 - 1)} \sin nx.$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



24. An odd extension for $f(x) = 1$ on the interval $[-\pi, \pi]$ is

$$f_o = \begin{cases} -1, & -\pi \leq x < 0, \\ 1, & 0 \leq x \leq \pi. \end{cases}$$

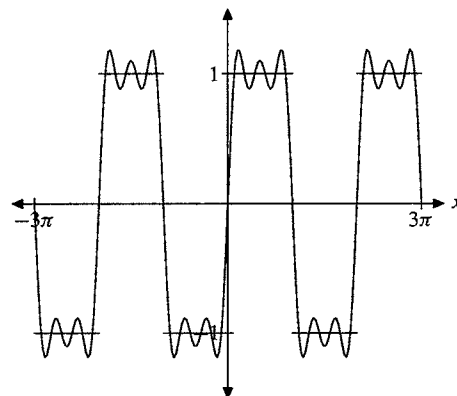
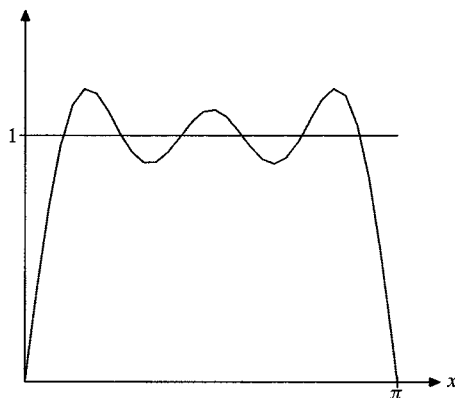
The Fourier expansion of $f_o(x)$ contains only sine terms with

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_o(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx \\ &= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \\ &= -\frac{2}{\pi n} [\cos n\pi - 1] \\ &= -\frac{2}{\pi n} [(-1)^n - 1] \\ &= \frac{2}{\pi n} [(-1)^{n+1} + 1]. \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \sum_{n=1}^{\infty} \frac{2((-1)^{n+1} + 1)}{\pi n} \sin nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



25. An odd extension of $f(x) = \pi - x$ on the interval $[-\pi, \pi]$ is

$$f_o(x) = \begin{cases} -\pi - x, & -\pi \leq x < 0, \\ \pi - x, & 0 \leq x \leq \pi. \end{cases}$$

The Fourier expansion contains only sine terms with

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_o(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin nx \, dx. \end{aligned}$$

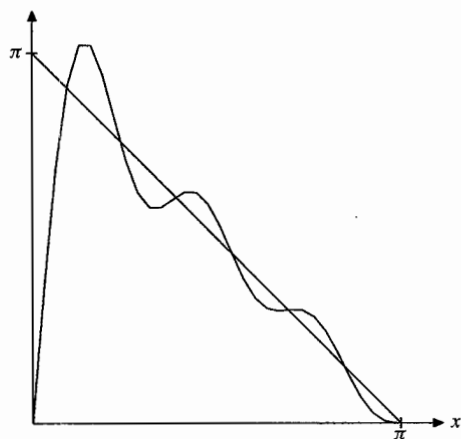
Integrating by parts,

$$\begin{aligned} b_n &= \frac{2}{\pi} \left[\frac{-(\pi - x) \cos nx}{n} - \frac{\sin nx}{n^2} \right]_0^{\pi} \\ &= -\frac{2}{\pi n^2} [n(\pi - x) \cos nx + \sin nx]_0^{\pi} \\ &= -\frac{2}{\pi n^2} (-n\pi) \\ &= \frac{2}{n}. \end{aligned}$$

Hence, the Fourier expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx.$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



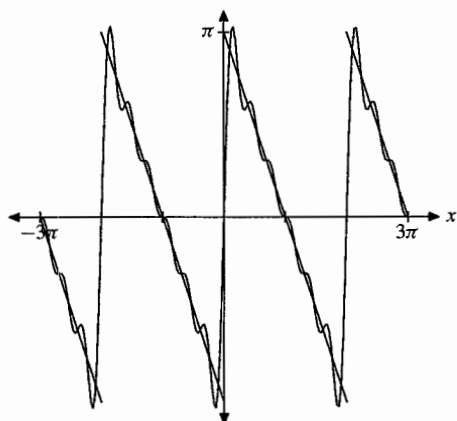
Integrating by parts,

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \left[-\frac{x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3} \right]_0^\pi \\
 &= \frac{2}{\pi} \left[\left(-\frac{\pi^2 \cos n\pi}{n} + \frac{2 \cos n\pi}{n^3} \right) - \frac{2}{n^3} \right] \\
 &= \frac{2}{\pi n^3} (-\pi^2 n^2 (-1)^n + 2(-1)^n - 2) \\
 &= \frac{2((-1)^n (2 - \pi^2 n^2) - 2)}{\pi n^3}.
 \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\
 &= \sum_{n=1}^{\infty} \frac{2((-1)^n (2 - \pi^2 n^2) - 2)}{\pi n^3} \sin nx.
 \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.

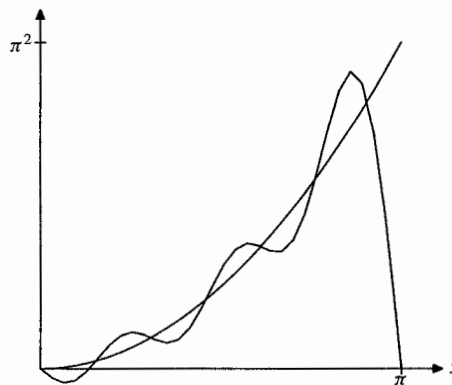


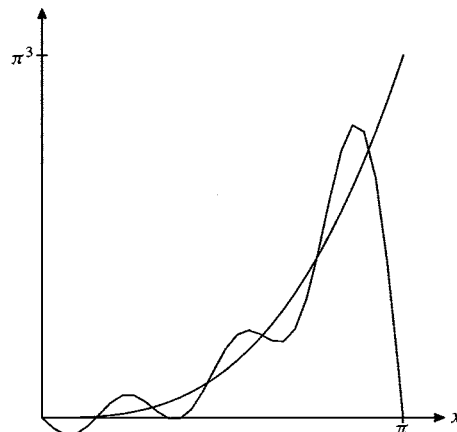
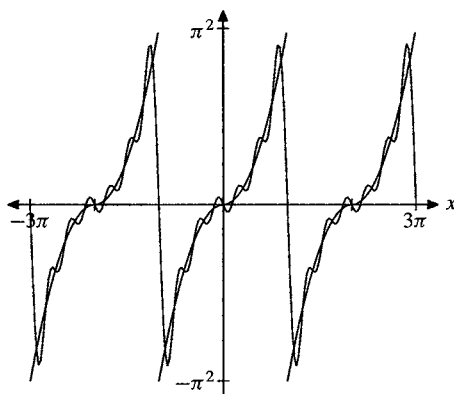
26. An odd extension of $f(x) = x^2$ on the interval $[-\pi, \pi]$ is

$$f_o(x) = \begin{cases} -x^2, & -\pi \leq x < 0, \\ x^2, & 0 \leq x \leq \pi. \end{cases}$$

The Fourier expansion contains only sine terms and

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_o(x) \sin nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi} x^2 \sin nx \, dx.
 \end{aligned}$$





27. The function $f(x) = x^3$ is already odd on the interval $[-\pi, \pi]$, so its odd extension is itself and its Fourier expansion contains only sine terms.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx.$$

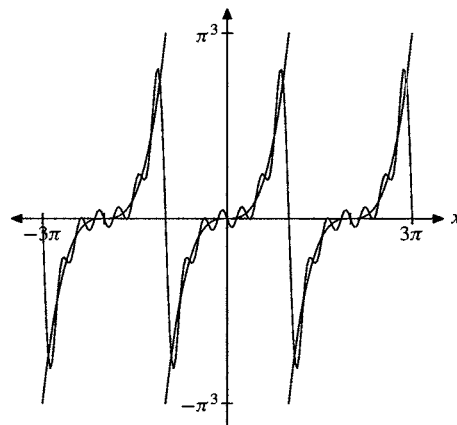
Integrating by parts,

$$\begin{aligned} b_n &= \frac{2}{\pi} \left[-\frac{x^3 \cos nx}{n} + \frac{3x^2 \sin nx}{n^2} \right. \\ &\quad \left. + \frac{6x \cos nx}{n^3} - \frac{6 \sin nx}{n^4} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[-\frac{\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^3} \right] \\ &= \frac{2}{\pi} \left[\frac{-\pi^3 n^2 (-1)^n + 6\pi (-1)^n}{n^3} \right] \\ &= \frac{2(-1)^n (6 - n^2 \pi^2)}{n^3} \end{aligned}$$

Thus, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \sum_{n=1}^{\infty} \frac{2(-1)^n (6 - n^2 \pi^2)}{n^3} \sin nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



28. The function $f(x) = x^4$ has odd extension

$$f_o(x) = \begin{cases} -x^4, & -\pi \leq x < 0, \\ x^4, & 0 \leq x \leq \pi. \end{cases}$$

The Fourier expansion contains only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^4 \sin nx \, dx. \end{aligned}$$

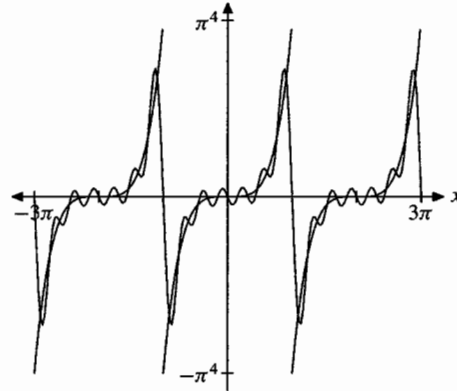
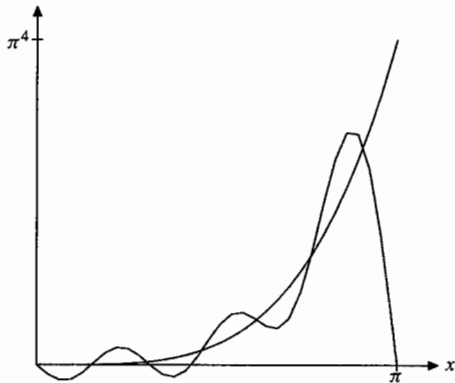
Integrating by parts,

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \left[-\frac{x^4 \cos nx}{n} + \frac{4x^3 \sin nx}{n^2} \right. \\
 &\quad \left. + \frac{12x^2 \cos nx}{n^3} - \frac{24x \sin nx}{n^4} \right. \\
 &\quad \left. - \frac{24 \cos nx}{n^5} \right]_0^\pi \\
 &= \frac{2}{\pi} \left[-\frac{\pi^4 \cos n\pi}{n} + \frac{12\pi^2 \cos n\pi}{n^3} \right. \\
 &\quad \left. - \frac{24 \cos n\pi}{n^5} + \frac{24}{n^5} \right] \\
 &= \frac{2}{\pi n^5} [(-1)^{n+1}(\pi^4 n^4 - 12\pi^2 n^2 + 24) \\
 &\quad + 24]
 \end{aligned}$$

Thus, the Fourier series expansion is

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\
 &= \sum_{n=1}^{\infty} \frac{2}{\pi n^5} [(-1)^{n+1}(\pi^4 n^4 - 12\pi^2 n^2 + 24) \\
 &\quad + 24] \sin nx.
 \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



29. The function

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi/2, \\ 0, & \pi/2 \leq x \leq \pi, \end{cases}$$

has odd extension

$$f_o(x) = \begin{cases} 0, & -\pi \leq x < -\pi/2, \\ -1, & -\pi/2 \leq x < 0, \\ 1, & 0 \leq x < \pi/2, \\ 0, & \pi/2 \leq x \leq \pi, \end{cases}$$

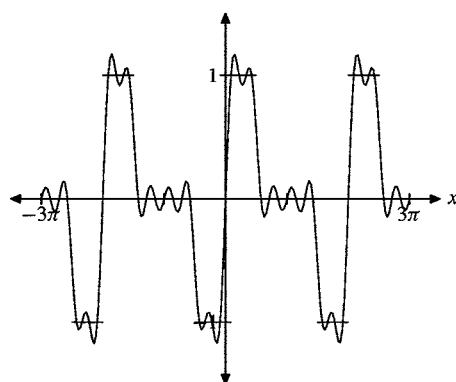
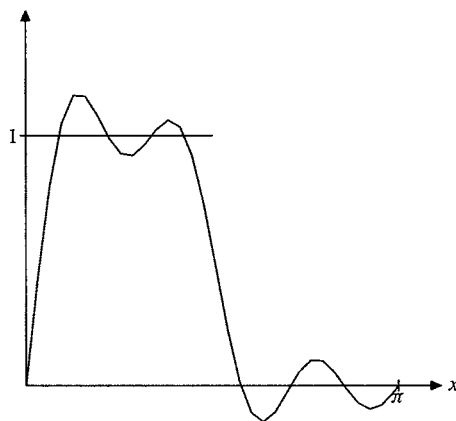
on the interval $[-\pi, \pi]$. The Fourier expansion of $f_o(x)$ has only sine terms with

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_o(x) \sin nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \sin nx \, dx \\
 &= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi/2} \\
 &= -\frac{2}{\pi n} \left[\cos \frac{n\pi}{2} - 1 \right].
 \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\
 &= \sum_{n=1}^{\infty} \frac{2}{\pi n} \left[1 - \cos \frac{n\pi}{2} \right] \sin nx.
 \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



30. The function

$$f(x) = \begin{cases} x, & 0 \leq x < \pi/2, \\ \pi/2, & \pi/2 \leq x \leq \pi, \end{cases}$$

has odd extension

$$f_o(x) = \begin{cases} -\pi/2, & -\pi \leq x < -\pi/2, \\ x, & -\pi/2 \leq x < 0, \\ x, & 0 \leq x < \pi/2, \\ \pi/2, & \pi/2 \leq x \leq \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. Hence, the Fourier expansion has only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_o(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \left\{ \int_0^{\pi/2} x \sin nx \, dx \right. \\ &\quad \left. + \int_{\pi/2}^{\pi} \frac{\pi}{2} \sin nx \, dx \right\}. \end{aligned}$$

The first integral is found by integrating by parts.

$$\begin{aligned} \int_0^{\pi/2} x \sin nx \, dx &= \left[\frac{-x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi/2} \\ &= -\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \end{aligned}$$

The second integral is

$$\begin{aligned} \frac{\pi}{2} \int_{\pi/2}^{\pi} \sin nx \, dx &= \frac{\pi}{2} \left[-\frac{\cos nx}{n} \right]_{\pi/2}^{\pi} \\ &= -\frac{\pi}{2n} \left[\cos n\pi - \cos \frac{n\pi}{2} \right]. \end{aligned}$$

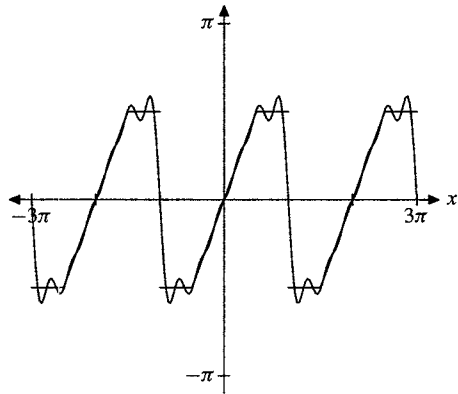
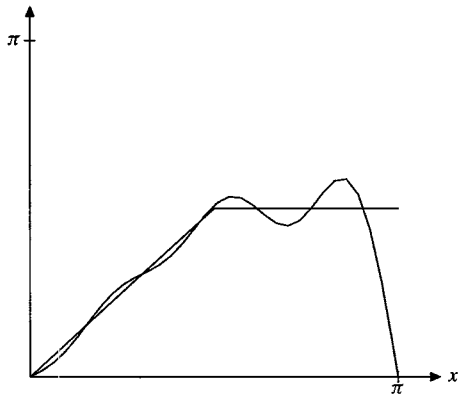
Hence,

$$\begin{aligned} b_n &= \frac{2}{\pi} \left\{ -\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right. \\ &\quad \left. - \frac{\pi}{2n} \left(\cos n\pi - \cos \frac{n\pi}{2} \right) \right\} \\ &= \frac{2}{\pi} \left\{ \frac{1}{n^2} \sin \frac{n\pi}{2} - \frac{\pi}{2n} \cos n\pi \right\}. \end{aligned}$$

Thus, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \sin \frac{n\pi}{2} - \frac{\pi}{2n} \cos n\pi \right) \sin nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



31. The function $f(x) = x \cos x$ is already odd on the interval $[-\pi, \pi]$, so the Fourier expansion contains only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin nx \, dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \left\{ \frac{1}{2} [\sin(n-1)x + \sin(n+1)x] \right\} dx \\ &= \frac{1}{\pi} \left\{ \int_0^{\pi} x \sin(n-1)x \, dx \right. \\ &\quad \left. + \int_0^{\pi} x \sin(n+1)x \, dx \right\}. \end{aligned}$$

Integrating by parts, the first integral is

$$\begin{aligned} &\int_0^{\pi} x \sin(n-1)x \, dx \\ &= \left[-\frac{x \cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{\pi} \\ &= \frac{-\pi \cos(n-1)\pi}{n-1}. \end{aligned}$$

Integrating by parts, the second integral is

$$\begin{aligned} &\int_0^{\pi} x \sin(n+1)x \, dx \\ &= \left[-\frac{x \cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} \right]_0^{\pi} \\ &= \frac{-\pi \cos(n+1)\pi}{n+1}. \end{aligned}$$

Hence,

$$\begin{aligned} b_n &= \frac{1}{\pi} \left\{ \frac{-\pi \cos(n-1)\pi}{n-1} \right. \\ &\quad \left. - \frac{\pi \cos(n+1)\pi}{n+1} \right\} \\ &= \frac{-(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1} \\ &= \frac{(-1)^n}{n-1} + \frac{(-1)^{n+2}}{n+1} \\ &= (-1)^n \frac{2n}{n^2-1}, \end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned} b_1 &= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin x \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx. \end{aligned}$$

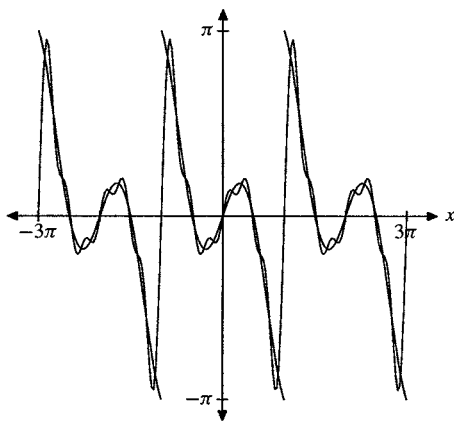
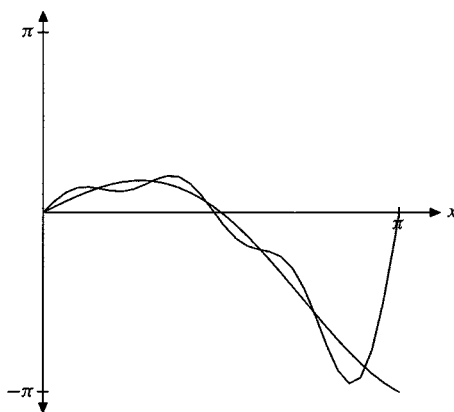
Integrating by parts,

$$\begin{aligned} b_1 &= \frac{1}{\pi} \left[\frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi} \\ &= -\frac{1}{2}. \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= -\frac{1}{2} \sin x + \sum_{n=2}^{\infty} \frac{(-1)^n 2n}{n^2 - 1} \sin nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



32. The function $f(x) = x \sin x$ has odd extension

$$f_o(x) = \begin{cases} -x \sin x, & -\pi \leq x < 0, \\ x \sin x, & 0 \leq x \leq \pi. \end{cases}$$

Thus, the Fourier expansion has only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_o(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \sin x \sin nx \, dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \left[\frac{1}{2} (\cos(x - nx) - \cos(x + nx)) \right] dx \\ &= \frac{1}{\pi} \left[\int_0^{\pi} x \cos(n-1)x \, dx - \int_0^{\pi} x \cos(n+1)x \, dx \right]. \end{aligned}$$

Integrating by parts, the first integral is

$$\begin{aligned} &\int_0^{\pi} x \cos(n-1)x \, dx \\ &= \left[\frac{x \sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2} \right]_0^{\pi} \\ &= \frac{\cos(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2} \\ &= \frac{(-1)^{n-1}}{(n-1)^2} - \frac{1}{(n-1)^2}. \end{aligned}$$

Similarly, the second integral is

$$\begin{aligned} &\int_0^{\pi} x \cos(n+1)x \, dx \\ &= \left[\frac{x \sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2} \right]_0^{\pi} \\ &= \frac{\cos(n+1)\pi}{(n+1)^2} - \frac{1}{(n+1)^2} \\ &= \frac{(-1)^{n+1}}{(n+1)^2} - \frac{1}{(n+1)^2}. \end{aligned}$$

Hence,

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \left[\frac{(-1)^{n-1}}{(n-1)^2} - \frac{1}{(n-1)^2} \right. \\
 &\quad \left. - \frac{(-1)^{n+1}}{(n+1)^2} + \frac{1}{(n+1)^2} \right] \\
 &= \frac{1}{\pi} \left[\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} \right] ((-1)^n + 1) \\
 &= \frac{1}{\pi} \left[\frac{(n^2 - 2n + 1) - (n^2 + 2n + 1)}{(n+1)^2(n-1)^2} \right] \\
 &\quad \times ((-1)^n + 1) \\
 &= \frac{4n((-1)^{n+1} - 1)}{\pi(n+1)^2(n-1)^2},
 \end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned}
 b_1 &= \frac{2}{\pi} \int_0^\pi x \sin^2 x \, dx \\
 &= \frac{1}{\pi} \int_0^\pi (x - x \cos 2x) \, dx.
 \end{aligned}$$

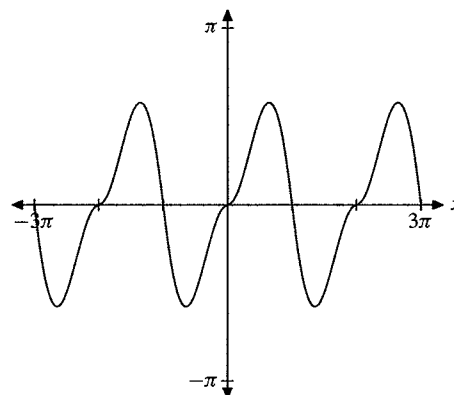
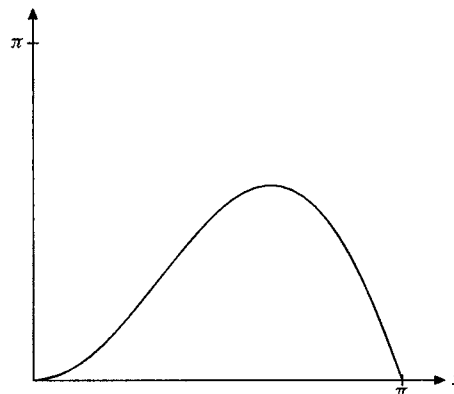
Integrating by parts,

$$\begin{aligned}
 b_1 &= \frac{1}{\pi} \left[\frac{x^2}{2} - \left(\frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right) \right]_0^\pi \\
 &= \frac{1}{\pi} \left[\frac{\pi^2}{2} \right] \\
 &= \frac{\pi}{2}.
 \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\
 &= \frac{\pi}{2} \sin x + \sum_{n=2}^{\infty} \frac{4n((-1)^{n+1} - 1)}{\pi(n+1)^2(n-1)^2} \sin nx.
 \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



33. Use the trig identity $\cos \alpha \cos \beta = 1/2[\cos(\alpha - \beta) +$

$\cos(\alpha + \beta)]$ to write (if $n \neq p$),

$$\begin{aligned} & \int_0^L \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} dx \\ &= \frac{1}{2} \int_0^L \left[\cos \frac{(n-p)\pi x}{L} + \cos \frac{(n+p)\pi x}{L} \right] dx \\ &= \frac{1}{2} \left[\frac{L}{(n-p)\pi} \sin \frac{(n-p)\pi x}{L} \right. \\ & \quad \left. + \frac{L}{(n+p)\pi} \sin \frac{(n+p)\pi x}{L} \right]_0^L \\ &= \frac{1}{2} \left[\frac{L}{(n-p)\pi} \sin(n-p)\pi \right. \\ & \quad \left. + \frac{L}{(n+p)\pi} \sin(n+p)\pi \right] \end{aligned}$$

Because $n-p$ and $n+p$ are integers, $\sin(n-p)\pi = 0$ and $\sin(n+p)\pi = 0$. Thus, if $n \neq p$, then

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} dx = 0.$$

34. Using a product to sum trig identity,

$$\begin{aligned} & \int_0^L \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L} dx \\ &= \int_0^L \frac{1}{2} \left[\cos \left(\frac{n\pi x}{L} - \frac{p\pi x}{L} \right) \right. \\ & \quad \left. - \cos \left(\frac{n\pi x}{L} + \frac{p\pi x}{L} \right) \right] dx \\ &= \frac{1}{2} \int_0^L \left[\cos \frac{(n-p)\pi x}{L} - \cos \frac{(n+p)\pi x}{L} \right] dx \\ &= \frac{1}{2} \left[\frac{L}{(n-p)\pi} \sin \frac{(n-p)\pi x}{L} \right. \\ & \quad \left. - \frac{L}{(n+p)\pi} \sin \frac{(n+p)\pi x}{L} \right]_0^L \\ &= 0, \end{aligned}$$

provided n and p are integers and $n \neq p$.

35. Use the trig identity $\sin \alpha \cos \beta = 1/2[\sin(\alpha - \beta) +$

$\sin(\alpha + \beta)]$ to write

$$\begin{aligned} & \int_0^1 \cos(2n\pi x) \sin(2k\pi x) dx \\ &= \frac{1}{2} \int_0^1 [\sin 2(k-n)\pi x + \sin 2(k+n)\pi x] dx \\ &= \frac{1}{2} \left[-\frac{1}{2(k-n)\pi} \cos 2(k-n)\pi x \right. \\ & \quad \left. - \frac{1}{2(k+n)\pi} \cos 2(k+n)\pi x \right]_0^1 \\ &= \frac{1}{2} \left[-\frac{1}{2(k-n)\pi} (\cos 2(k-n)\pi - 1) \right. \\ & \quad \left. - \frac{1}{2(k+n)\pi} (\cos 2(k+n)\pi - 1) \right] \end{aligned}$$

If k and n are different integers, $\cos 2(k-n)\pi = 1$ and $\cos 2(k+n)\pi = 1$. Thus, if $k \neq n$,

$$\int_0^1 \cos(2n\pi x) \sin(2k\pi x) dx = 0.$$

Now, if $k = n$,

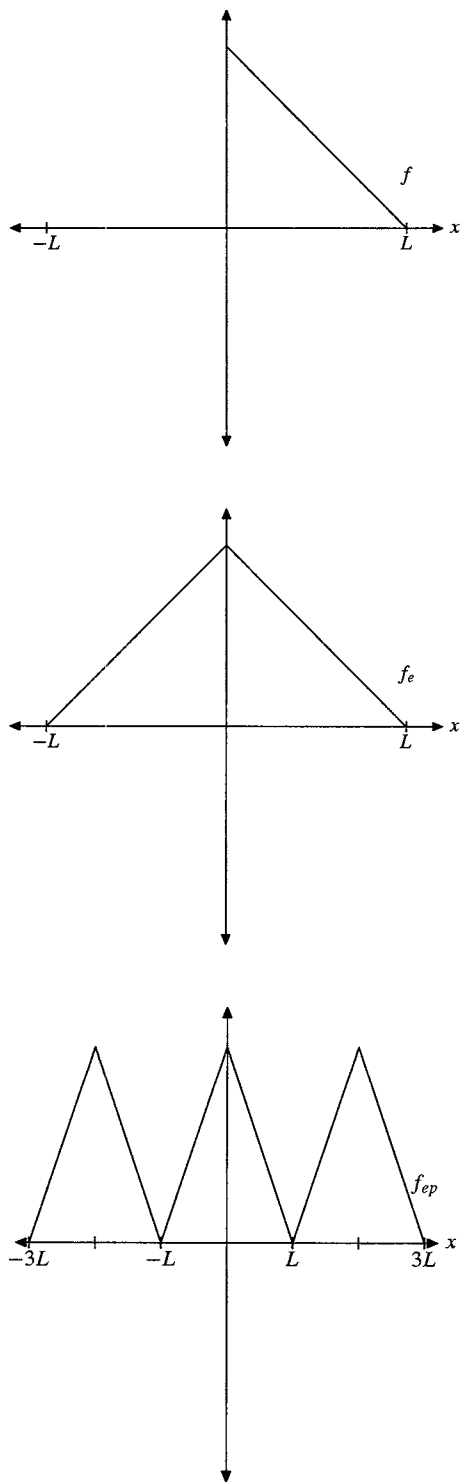
$$\begin{aligned} & \int_0^1 \cos(2n\pi x) \sin(2k\pi x) dx \\ &= \int_0^1 \cos(2n\pi x) \sin(2n\pi x) dx \\ &= \frac{1}{2} \int_0^1 \sin(4n\pi x) dx \\ &= \frac{1}{2} \cdot \frac{-1}{4n\pi} \cos(4n\pi x) \Big|_0^1 \\ &= 0. \end{aligned}$$

36. The point here is that starting with a continuous function f on $[0, L]$, then its even extension

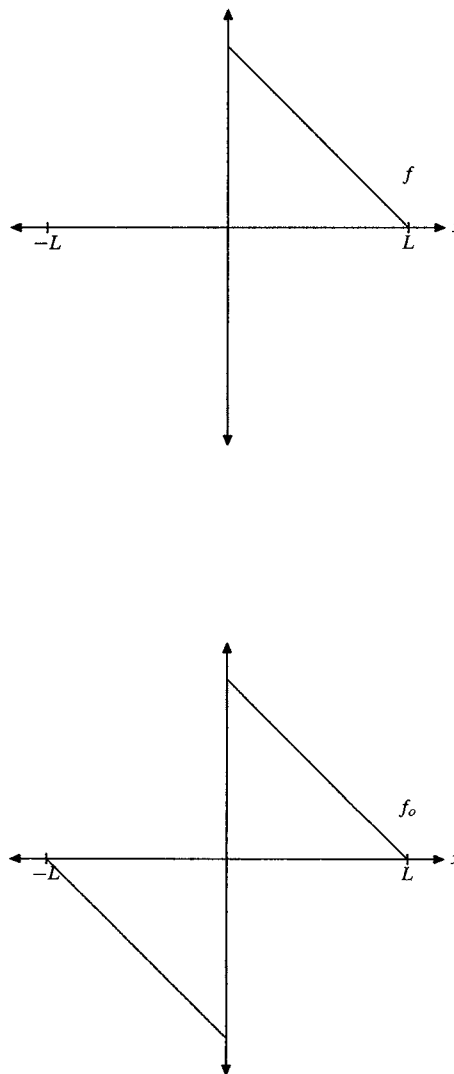
$$f_e(x) = \begin{cases} f(-x), & -L \leq x < 0 \\ f(x), & 0 \leq x \leq L, \end{cases}$$

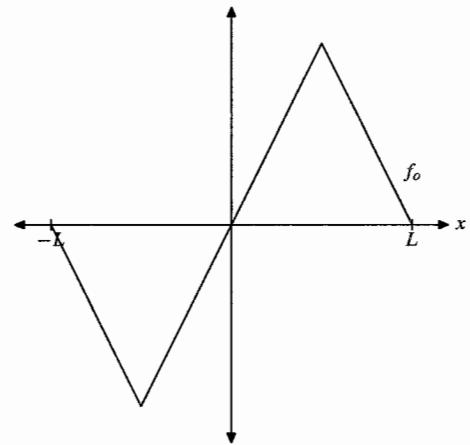
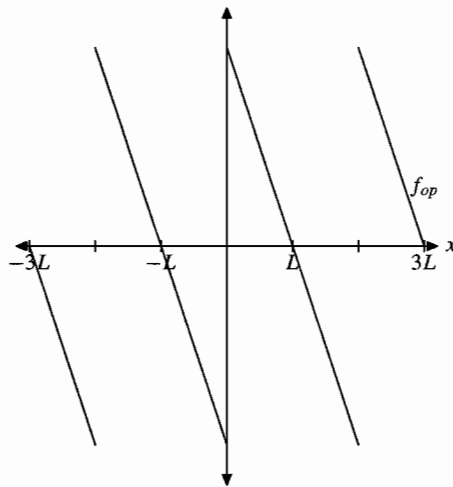
will hook up when extended periodically. For example start with f and its even extension, as shown in

the first two images.

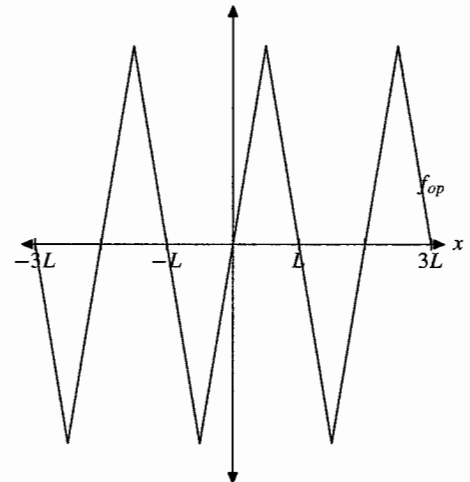
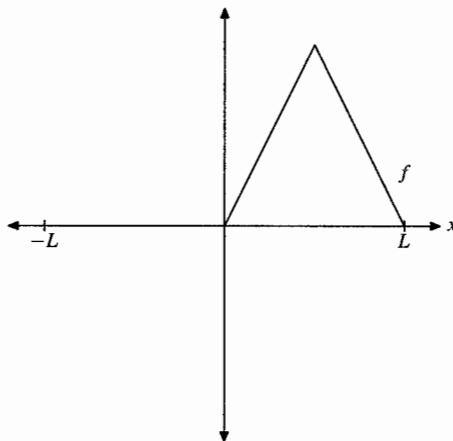


Note that when f_e is extended periodically, the result is continuous everywhere, as shown in the third image above. However this is not the case for an odd extension. Consider, for example, this graph of f and its odd extension f_o , shown in the first and second images that follow.





When f_o is extended periodically, note that it is not continuous at even multiples of L , as shown in the third image above. One thing we can do is require that $f(0) = f(L) = 0$. Then the periodic extension is continuous everywhere, as shown in the images that follow.



Section 12.4. The Complex Form of a Fourier Series

1. If f is real, we know that

$$\overline{\alpha_n} = \alpha_{-n}, \quad \text{for all } n.$$

However, if

$$\alpha_n = \frac{1}{L} \int_{-L}^L f(x) e^{-in\pi x/L} dx,$$

then

$$\alpha_{-n} = \frac{1}{L} \int_{-L}^L f(x) e^{in\pi x/L} dx.$$

Since f is even, $f(-x) = f(x)$ and

$$\alpha_{-n} = \frac{1}{L} \int_{-L}^L f(-x) e^{in\pi x/L} dx.$$

Now, let

$$u = -x \quad \text{and} \quad du = -dx,$$

and write

$$\begin{aligned} \alpha_{-n} &= \frac{1}{L} \int_L^{-L} f(u) e^{-in\pi u/L} (-du) \\ &= \frac{1}{L} \int_{-L}^L f(u) e^{-in\pi u/L} du. \end{aligned}$$

But this is precisely the definition for α_n (ignore the dummy variable u). Thus,

$$\overline{\alpha_n} = \alpha_{-n} = \alpha_n.$$

Because $\overline{\alpha_n} = \alpha_n$, it must be the case that α_n is real.

Next, suppose that f is real and odd. Again, because f is real, we know that

$$\overline{\alpha_n} = \alpha_{-n}, \quad \text{for all } n.$$

The coefficient α_n is purely imaginary if and only if we can show that $\overline{\alpha_n} = -\alpha_n$ for all n . However,

$$\alpha_n = \frac{1}{L} \int_{-L}^L f(x) e^{-in\pi x/L} dx,$$

so again,

$$\alpha_{-n} = \frac{1}{L} \int_{-L}^L f(x) e^{in\pi x/L} dx.$$

Because f is odd, $f(-x) = -f(x)$ for all $x \in [-L, L]$. We can write

$$\alpha_{-n} = \frac{1}{L} \int_{-L}^L -f(-x) e^{in\pi x/L} dx$$

and again make the change of variable

$$u = -x \quad \text{and} \quad du = -dx.$$

Thus,

$$\begin{aligned} \alpha_{-n} &= \frac{1}{L} \int_L^{-L} -f(u) e^{-in\pi u/L} (-du) \\ &= -\frac{1}{L} \int_{-L}^L f(u) e^{-in\pi u/L} du \\ &= -\alpha_n, \end{aligned}$$

the last statement being true because u is just a dummy variable. Thus,

$$\overline{\alpha_n} = \alpha_{-n} = -\alpha_n,$$

and thus α_n is purely imaginary for all n .

2. With $L = \pi$, the n th coefficient becomes

$$\begin{aligned} \alpha_n &= \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx \end{aligned}$$

Integrating by parts,

$$\begin{aligned}
 \alpha_n &= \frac{1}{2\pi} \left[-\frac{x e^{-inx}}{in} + \frac{e^{-inx}}{n^2} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{2\pi} \left[e^{-inx} \left(\frac{1}{n^2} - \frac{x}{in} \right) \right]_{-\pi}^{\pi} \\
 &= \frac{1}{2\pi} \left[e^{-in\pi} \left(\frac{1}{n^2} - \frac{\pi}{in} \right) \right. \\
 &\quad \left. - e^{in\pi} \left(\frac{1}{n^2} + \frac{\pi}{in} \right) \right] \\
 &= \frac{1}{2\pi} \left[(-1)^n \left(\frac{1}{n^2} - \frac{\pi}{in} \right) \right. \\
 &\quad \left. - (-1)^n \left(\frac{1}{n^2} + \frac{\pi}{in} \right) \right] \\
 &= \frac{1}{2\pi} \left[(-1)^n \left(-\frac{2\pi}{in} \right) \right] \\
 &= \frac{(-1)^{n+1}}{in},
 \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0.$$

Thus,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} = \sum_{n \neq 0} \frac{(-1)^{n+1}}{in} e^{inx}.$$

3. With $L = \pi$, the n th coefficient becomes

$$\begin{aligned}
 \alpha_n &= \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} \, dx \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{-inx} \, dx \\
 &= \frac{1}{2\pi} \int_{-\pi}^0 (-x) e^{-inx} \, dx \\
 &\quad + \frac{1}{2\pi} \int_0^{\pi} x e^{-inx} \, dx.
 \end{aligned}$$

Integrating by parts, the first integral becomes

$$\begin{aligned}
 &\frac{1}{2\pi} \int_{-\pi}^0 (-x) e^{-inx} \, dx \\
 &= \frac{1}{2\pi} \left[\frac{x e^{-inx}}{in} - \frac{e^{-inx}}{n^2} \right]_{-\pi}^0 \\
 &= \frac{1}{2\pi} \left[e^{-inx} \left(\frac{x}{in} - \frac{1}{n^2} \right) \right]_{-\pi}^0 \\
 &= \frac{1}{2\pi} \left[-\frac{1}{n^2} - e^{in\pi} \left(\frac{-\pi}{in} - \frac{1}{n^2} \right) \right] \\
 &= \frac{1}{2\pi} \left[-\frac{1}{n^2} - (-1)^n \left(-\frac{\pi}{in} - \frac{1}{n^2} \right) \right] \\
 &= \begin{cases} \frac{1}{2\pi} \left(\frac{\pi}{in} \right), & n \text{ even,} \\ \frac{1}{2\pi} \left(-\frac{2}{n^2} - \frac{\pi}{in} \right), & n \text{ odd} \end{cases} \\
 &= \begin{cases} \frac{1}{2in}, & n \text{ even} \\ -\frac{1}{\pi n^2} - \frac{1}{2in}, & n \text{ odd} \end{cases}
 \end{aligned}$$

The second integral is

$$\begin{aligned}
 &\frac{1}{2\pi} \int_0^{\pi} x e^{-inx} \, dx \\
 &= \frac{1}{2\pi} \left[\frac{-x e^{-inx}}{in} + \frac{e^{-inx}}{n^2} \right]_0^{\pi} \\
 &= \frac{1}{2\pi} \left[e^{-inx} \left(\frac{1}{n^2} - \frac{x}{in} \right) \right]_0^{\pi} \\
 &= \frac{1}{2\pi} \left[e^{-in\pi} \left(\frac{1}{n^2} - \frac{\pi}{in} \right) - \frac{1}{n^2} \right] \\
 &= \frac{1}{2\pi} \left[(-1)^n \left(\frac{1}{n^2} - \frac{\pi}{in} \right) - \frac{1}{n^2} \right] \\
 &= \begin{cases} \frac{1}{2\pi} \left(-\frac{\pi}{in} \right), & n \text{ even,} \\ \frac{1}{2\pi} \left(\frac{\pi}{in} - \frac{2}{n^2} \right), & n \text{ odd} \end{cases} \\
 &= \begin{cases} -\frac{1}{2in}, & n \text{ even,} \\ \frac{1}{2in} - \frac{1}{\pi n^2}, & n \text{ odd} \end{cases}
 \end{aligned}$$

Thus,

$$\alpha_n = \begin{cases} 0, & n \text{ even,} \\ -\frac{2}{\pi n^2}, & n \text{ odd,} \end{cases}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{1}{2\pi} (\pi^2) = \frac{\pi}{2}.$$

Hence,

$$\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} \\ &= \frac{\pi}{2} + \sum_{n \text{ odd}} \left(-\frac{2}{\pi n^2} \right) e^{inx} \\ &= \frac{\pi}{2} - \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} e^{inx}. \end{aligned}$$

4. With $L = \pi$, the n th coefficient becomes

$$\begin{aligned} \alpha_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^0 (-1) e^{-inx} dx \\ &\quad + \frac{1}{2\pi} \int_0^{\pi} (1) e^{-inx} dx \end{aligned}$$

The first integral is

$$\begin{aligned} -\frac{1}{2\pi} \int_{-\pi}^0 e^{-inx} dx &= -\frac{1}{2\pi} \frac{e^{-inx}}{-in} \Big|_{-\pi}^0 \\ &= \frac{e^{-inx}}{2\pi in} \Big|_{-\pi}^0 \\ &= \frac{1 - e^{in\pi}}{2\pi in}. \end{aligned}$$

The second integral is

$$\begin{aligned} \frac{1}{2\pi} \int_0^{\pi} e^{-inx} dx &= \frac{1}{2\pi} \frac{e^{-inx}}{-in} \Big|_0^{\pi} \\ &= -\frac{e^{-inx}}{2\pi in} \Big|_0^{\pi} \\ &= -\frac{e^{-in\pi} - 1}{2\pi in}. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_n &= \frac{1 - (-1)^n}{2\pi in} - \frac{(-1)^n - 1}{2\pi in} \\ &= \frac{2 - 2(-1)^n}{2\pi in} = \frac{1 - (-1)^n}{\pi in}, \\ &= \begin{cases} 0, & n \text{ even,} \\ 2/(\pi in), & n \text{ odd,} \end{cases} \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = 0,$$

because f is odd on $[-\pi, \pi]$. Consequently,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} = \sum_{n \text{ odd}} \frac{2}{\pi in} e^{inx}.$$

5. With $L = \pi$, the n th coefficient becomes

$$\begin{aligned} \alpha_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_0^{\pi} e^{-inx} dx \\ &= \frac{1}{2\pi} \left[\frac{e^{-inx}}{-in} \right]_0^{\pi} \\ &= -\frac{1}{2\pi in} [e^{-in\pi} - 1] \\ &= \frac{1 - (-1)^n}{2\pi in} \\ &= \begin{cases} 0, & n \text{ even,} \\ \frac{1}{\pi in}, & n \text{ odd,} \end{cases} \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_0^{\pi} dx = \frac{1}{2}.$$

Thus,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} = \frac{1}{2} + \sum_{n \text{ odd}} \frac{1}{\pi in} e^{inx}.$$

6. With $L = \pi$, the n th coefficient becomes

$$\alpha_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx.$$

Integrating by parts,

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi} \left[\frac{-x^2 e^{-inx}}{in} + \frac{2x e^{-inx}}{n^2} + \frac{2e^{-inx}}{in^3} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[e^{-inx} \left(-\frac{x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3} \right) \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[(-1)^n \left(-\frac{\pi^2}{in} + \frac{2\pi}{n^2} + \frac{2}{in^3} \right) \right. \\ &\quad \left. - (-1)^n \left(-\frac{\pi^2}{in} - \frac{2\pi}{n^2} + \frac{2}{in^3} \right) \right] \\ &= \frac{1}{2\pi} \left[(-1)^n \left(\frac{4\pi}{n^2} \right) \right] \\ &= (-1)^n \frac{2}{n^2},\end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}.$$

Hence,

$$\begin{aligned}f(x) &\sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} \\ &= \frac{\pi^2}{3} + \sum_{n \neq 0} (-1)^n \frac{2}{n^2} e^{inx}.\end{aligned}$$

7. With $L = \pi$, the n th coefficient becomes

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{bx} e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(b-in)x} dx \\ &= \frac{1}{2\pi} \frac{e^{(b-in)x}}{b-in} \Big|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi(b-in)} [e^{(b-in)\pi} - e^{(b-in)(-\pi)}] \\ &= \frac{1}{2\pi(b-in)} [e^{b\pi} e^{-in\pi} - e^{-b\pi} e^{in\pi}] \\ &= \frac{(-1)^n}{\pi(b-in)} \left[\frac{e^{b\pi} - e^{-b\pi}}{2} \right] \\ &= \frac{(-1)^n \sinh b\pi}{\pi(b-in)},\end{aligned}$$

provided, of course, that $b \neq in$. Hence,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} = \frac{\sinh b\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{b-in} e^{inx}$$

8. With $L = \pi$, the n th coefficient becomes

$$\alpha_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^3 e^{-inx} dx.$$

Several integrations by parts provides

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi} \left[\frac{ix^3 e^{-inx}}{n} + \frac{3x^2 e^{-inx}}{n^2} \right. \\ &\quad \left. - \frac{6xe^{-inx}}{n^3} - \frac{6e^{-inx}}{n^4} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi n^4} [e^{-inx} (in^3 x^3 + 3n^2 x^2 - 6inx - 6)]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi n^4} \left[(-1)^n (in^3 \pi^3 + 3n^2 \pi^2 - 6in\pi - 6) \right. \\ &\quad \left. - (-1)^n (-in^3 \pi^3 + 3n^2 \pi^2 + 6in\pi - 6) \right] \\ &= \frac{(-1)^n}{2\pi n^4} [2in^3 \pi^3 - 12in\pi] \\ &= \frac{(-1)^n i}{n^3} (n^2 \pi^2 - 6),\end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^3 dx = 0,$$

since x^3 is odd on $[-\pi, \pi]$. Hence,

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} = \sum_{n \neq 0} \frac{(-1)^n i (n^2 \pi^2 - 6)}{n^3} e^{inx}.$$

9. If $f(x) = \pi - x$ on the interval $[-\pi, \pi]$, then

$$\alpha_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - x) e^{-inx} dx.$$

Integration by parts provides

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi} \left[-\frac{(\pi-x)e^{-inx}}{in} - \frac{e^{-inx}}{n^2} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[-e^{-inx} \left(\frac{\pi-x}{in} + \frac{1}{n^2} \right) \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[-(-1)^n \cdot \frac{1}{n^2} + (-1)^n \left(\frac{2\pi}{in} + \frac{1}{n^2} \right) \right] \\ &= (-1)^{n+1} \frac{i}{n},\end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\begin{aligned}\alpha_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi-x) dx \\ &= \frac{1}{2\pi} \left[\pi x - \frac{1}{2} x^2 \right]_{-\pi}^{\pi} \\ &= \pi.\end{aligned}$$

Hence,

$$f(x) \sim \pi + i \sum_{n \neq 0} \frac{(-1)^{n+1}}{n} e^{inx}.$$

10. With $L = \pi$,

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\cos x| e^{-inx} dx \\ &= -\frac{1}{2\pi} \int_{-\pi}^{-\pi/2} (\cos x) e^{-inx} dx \\ &\quad + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (\cos x) e^{-inx} dx \\ &\quad - \frac{1}{2\pi} \int_{\pi/2}^{\pi} (\cos x) e^{-inx} dx.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}& -\frac{1}{2\pi} \int_{-\pi}^{-\pi/2} (\cos x) e^{-inx} dx \\ &= \frac{-1}{2\pi(1-n^2)} \left[e^{-inx} (\sin x - in \cos x) \right]_{-\pi}^{-\pi/2} \\ &= \frac{1}{2\pi(1-n^2)} [i^n + (-1)^n in]\end{aligned}$$

Similarly,

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (\cos x) e^{-inx} dx = \frac{(-1)^n i^n + i^n}{2\pi(1-n^2)},$$

and

$$-\frac{1}{2\pi} \int_{\pi/2}^{\pi} (\cos x) e^{-inx} dx = \frac{(-1)^n i^n - (-1)^n in}{2\pi(1-n^2)}.$$

Adding,

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi(1-n^2)} [2i^n + 2(-1)^n i^n] \\ &= \frac{1 + (-1)^n}{\pi(1-n^2)} i^n,\end{aligned}$$

providing $n \neq \pm 1$. In the case that $n = -1$,

$$\begin{aligned}\alpha_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\cos x| e^{ix} dx \\ &= -\frac{1}{2\pi} \int_{-\pi}^{-\pi/2} (\cos x) e^{ix} dx \\ &\quad + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (\cos x) e^{ix} dx \\ &\quad - \frac{1}{2\pi} \int_{\pi/2}^{\pi} (\cos x) e^{ix} dx.\end{aligned}$$

Note that

$$\begin{aligned}& \int (\cos x) e^{ix} dx \\ &= \int \cos x (\cos x + i \sin x) dx \\ &= \int \left[\frac{1 + \cos 2x}{2} + i \frac{1}{2} \sin 2x \right] dx \\ &= \frac{1}{2} \left[x + \frac{1}{2} \sin 2x \right] - \frac{1}{4} i \cos 2x.\end{aligned}$$

Thus,

$$\begin{aligned}& -\frac{1}{2\pi} \int_{-\pi}^{-\pi/2} (\cos x) e^{ix} dx \\ &= -\frac{1}{2\pi} \left[\frac{2x + \sin 2x}{4} - \frac{i \cos 2x}{4} \right]_{-\pi}^{-\pi/2} \\ &= -\frac{1}{2\pi} \left\{ \frac{\pi}{4} + \frac{2i}{4} \right\} \\ &= -\frac{\pi + 2i}{8\pi}.\end{aligned}$$

Similarly,

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (\cos x) e^{ix} dx = \frac{1}{4},$$

and

$$-\frac{1}{2\pi} \int_{\pi/2}^{\pi} (\cos x) e^{ix} dx = -\frac{\pi - 2i}{8\pi}.$$

Adding,

$$\begin{aligned} \alpha_{-1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\cos x| e^{ix} dx \\ &= -\frac{\pi + 2i}{8\pi} + \frac{2\pi}{8\pi} - \frac{\pi - 2i}{8\pi} \\ &= 0. \end{aligned}$$

Similarly,

$$\alpha_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\cos x| e^{ix} dx = 0.$$

Hence,

$$f(x) = \sum_{n \neq \pm 1, -1} \frac{1 + (-1)^n}{\pi(1 - n^2)} i^n e^{inx}.$$

11. With $L = \pi$, the n th coefficient becomes

$$\begin{aligned} \alpha_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin x| e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^0 (-\sin x) e^{-inx} dx \\ &\quad + \frac{1}{2\pi} \int_0^{\pi} (\sin x) e^{-inx} dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} &-\frac{1}{2\pi} \int_{-\pi}^0 (\sin x) e^{-inx} dx \\ &= -\frac{1}{2\pi} \cdot \frac{n^2}{n^2 - 1} \left[\frac{-e^{-inx} \sin x}{in} \right. \\ &\quad \left. + \frac{e^{-inx} \cos x}{n^2} \right]_{-\pi}^0 \\ &= \frac{-n^2}{2\pi(n^2 - 1)} \left[e^{-inx} \left(\frac{\cos x}{n^2} - \frac{\sin x}{in} \right) \right]_{-\pi}^0 \\ &= \frac{-n^2}{2\pi(n^2 - 1)} \left[\frac{1}{n^2} - e^{in\pi} \left(\frac{-1}{n^2} \right) \right] \\ &= \frac{-(-1)^n - 1}{2\pi(n^2 - 1)}. \end{aligned}$$

Similarly,

$$\frac{1}{2\pi} \int_0^{\pi} (\sin x) e^{-inx} dx = \frac{-1 - (-1)^n}{2\pi(n^2 - 1)}.$$

Adding,

$$\begin{aligned} \alpha_n &= \frac{-2 - 2(-1)^n}{2\pi(n^2 - 1)} = \frac{-1 - (-1)^n}{\pi(1 - n^2)} \\ &= \begin{cases} 2/(\pi(n^2 - 1)), & n \text{ even,} \\ 0, & n \text{ odd,} \end{cases} \end{aligned}$$

providing $n \neq \pm 1, 0$. In the case that $n = -1$,

$$\begin{aligned} \alpha_{-1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin x| e^{ix} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^0 (-\sin x) e^{ix} dx \\ &\quad + \frac{1}{2\pi} \int_0^{\pi} (\sin x) e^{ix} dx. \end{aligned}$$

Note that

$$\begin{aligned} &\int e^{ix} \sin x dx \\ &= \int (\cos x + i \sin x) \sin x dx \\ &= \int \left(\frac{1}{2} \sin 2x + i \left(\frac{1 - \cos 2x}{2} \right) \right) dx \\ &= -\frac{1}{4} \cos 2x + i \left(\frac{1}{2} x - \frac{1}{4} \sin 2x \right). \end{aligned}$$

Thus

$$\begin{aligned}
 & -\frac{1}{2\pi} \int_{-\pi}^0 e^{ix} \sin x dx \\
 &= -\frac{1}{2\pi} \left[-\frac{1}{4} \cos 2x + i \left(\frac{1}{2} x - \frac{1}{4} \sin 2x \right) \right]_{-\pi}^0 \\
 &= -\frac{1}{2\pi} \left[-\frac{1}{4} - \left(-\frac{1}{4} - \frac{\pi}{2} i \right) \right] \\
 &= -\frac{1}{4} i.
 \end{aligned}$$

Similarly,

$$\frac{1}{2\pi} \int_0^{\pi} e^{ix} \sin x dx = \frac{1}{4} i.$$

Thus, $\alpha_1 = 0$. In a similar manner, $\alpha_{-1} = 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin x| dx = \frac{1}{\pi} \int_0^{\pi} \sin x dx = \frac{2}{\pi}.$$

Thus,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} = \frac{2}{\pi} + \sum_{\substack{n \text{ even} \\ n \neq 0}} \frac{2}{\pi(n^2 - 1)} e^{inx}$$

12. Suppose that p and q are integers, $p \neq q$. Then

$$\begin{aligned}
 \int_{-\pi}^{\pi} e^{ipx} \overline{e^{iqx}} dx &= \int_{-\pi}^{\pi} e^{ipx} e^{-iqx} dx \\
 &= \int_{-\pi}^{\pi} e^{i(p-q)x} dx \\
 &= \frac{1}{i(p-q)} e^{i(p-q)x} \Big|_{-\pi}^{\pi} \\
 &= \frac{1}{i(p-q)} [e^{i(p-q)\pi} - e^{-i(p-q)\pi}].
 \end{aligned}$$

Now, the key idea is the fact that $e^{i\pi} = e^{-i\pi} = -1$. Thus,

$$\int_{-\pi}^{\pi} e^{ipx} \overline{e^{iqx}} dx = \frac{1}{i(p-q)} [(-1)^{p-q} - (-1)^{p-q}] = 0.$$

Thus, e^{ipx} and e^{iqx} are orthogonal, provided $p \neq q$.

13. Begin with the assumption that

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}, \quad -\pi \leq x \leq \pi.$$

Multiply both sides by $\overline{e^{ikx}} = e^{-ikx}$ and integrate.

$$\begin{aligned}
 f(x) e^{-ikx} &= \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} e^{-ikx} \\
 \int_{-\pi}^{\pi} f(x) e^{-ikx} dx &= \sum_{n=-\infty}^{\infty} \alpha_n \int_{-\pi}^{\pi} e^{inx} e^{-ikx} dx.
 \end{aligned}$$

In Exercise 12, we showed that e^{inx} and e^{ikx} are orthogonal, provided n and k are integers, $n \neq k$. Thus, the right side simplifies to

$$\int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \alpha_k \int_{-\pi}^{\pi} e^{ikx} e^{-ikx} dx = \alpha_k (2\pi)$$

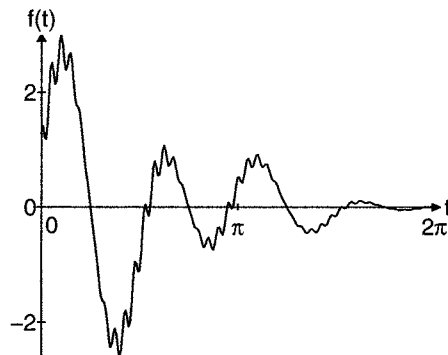
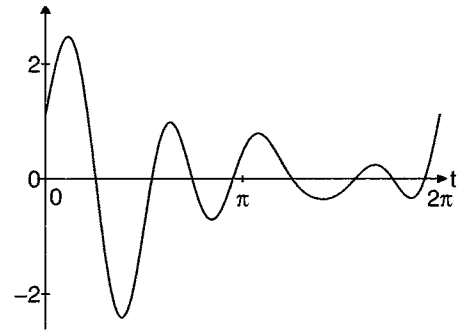
Therefore,

$$\alpha_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

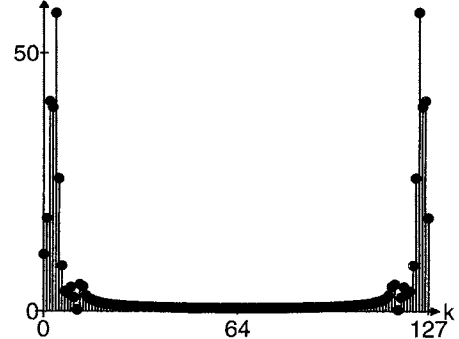
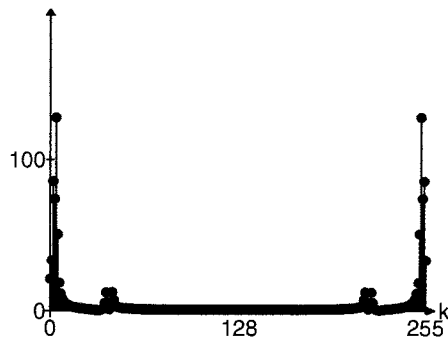
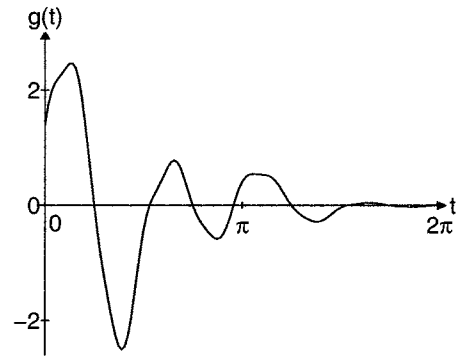
Now, k is a dummy variable, so just substitute n for k .

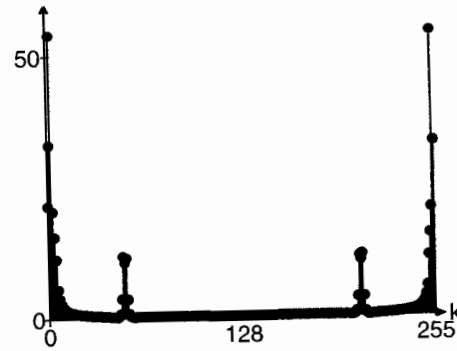
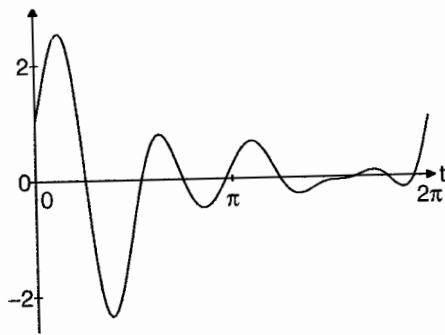
Section 12.5. The Discrete Fourier Transform and the FFT

1. The function f is plotted on the left below. Because of the terms involving $\cos 2x$ and $\sin 4x$, we would expect the coefficients for $n = 4$ to be the largest, followed by $n = 2$. The middle figure plots the discrete Fourier transform with $N = 256$, and it verifies the conjecture. The third figure is a plot of the partial sum of the Fourier series of order $n = 6$. The partial sum keeps the main features of f , but misses the high frequency vibrations in f .



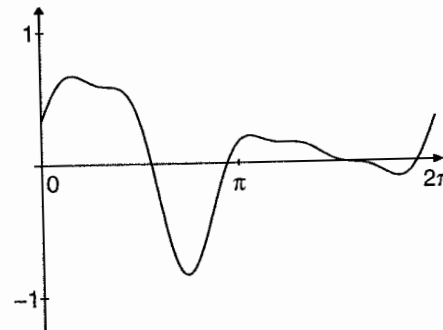
2. The function g is plotted on the left below. The middle figure plots the discrete Fourier transform with $N = 128$. The third figure is a plot of the Fourier series with the coefficients of small magnitude filtered out. There were 13 nonzero coefficients.





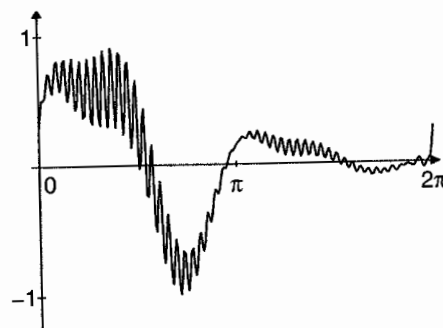
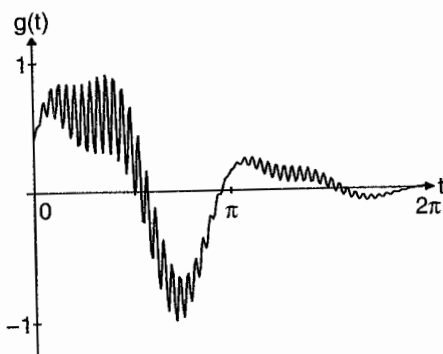
3. By 5.5, $\hat{y}_m = \sum_{j=0}^{N-1} y_j e^{(-2\pi i m j/N)} = \sum_{j=0}^{N-1} y_j \bar{w}^{jm}$. Hence, using the facts that $\overline{y_j} = y_j$ and $w = \bar{w}^{-1}$, we have

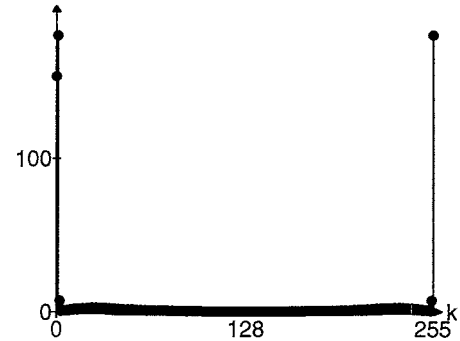
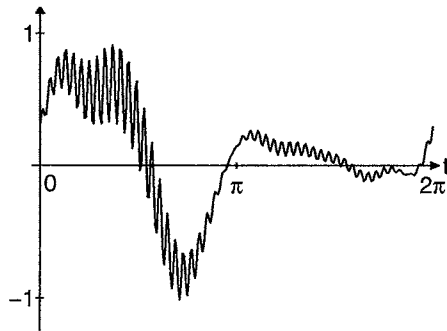
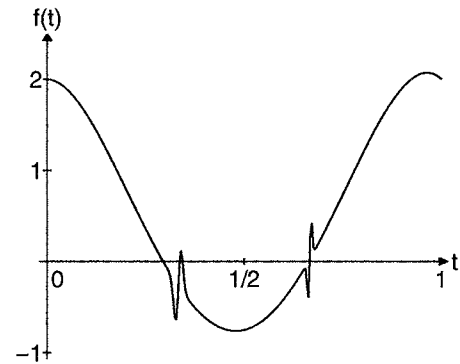
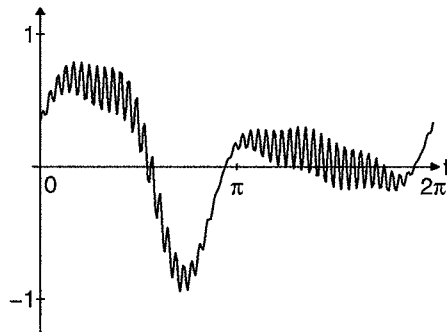
$$\begin{aligned}\overline{\hat{y}_m} &= \overline{\sum_{j=0}^{N-1} y_j \bar{w}^{jm}} = \sum_{j=0}^{N-1} \overline{y_j \bar{w}^{jm}} = \sum_{j=0}^{N-1} \overline{y_j} w^{jm} = \sum_{j=0}^{N-1} y_j w^{jm} \\ &= \hat{y}_{-m} = \hat{y}_{N-m}.\end{aligned}$$



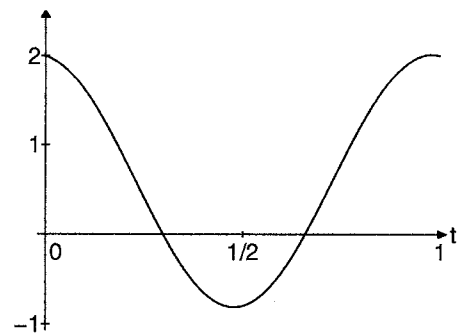
4. The results are shown in the following three figures. The first shows the graph of f . The second shows the graph of the FFT, which shows that the low frequencies dominate except for a few large coefficients that account for the high frequency noise in the graph of f . The third figure shows the result of using only the low frequency components.

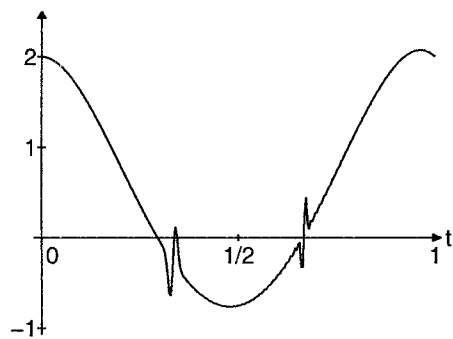
5. The graph of f and the FFT are shown in the previous exercise. The three figures shown below show the results of filtering with different tolerances. The first used $\text{tol} = 0.01$, and needed 71 nonzero coefficients. The second used $\text{tol} = 0.2$, and needed 15 nonzero coefficients. The second used $\text{tol} = 0.05$, and needed 27 nonzero coefficients.





6. The results are shown in the following four figures. The first shows the graph of f showing a relatively smooth function with the exception of two spikes. The second is the graph of the FFT, which shows that the low frequencies dominate. The third figure shows the result of using only the low frequency components. In fact we use $m = 2$, so there are only 3 nonzero coefficients. The spikes are filtered out. In the fourth figure we zero out the coefficients with small magnitude, and we wanted to see the spikes. This required the use of a very small tolerance. For the figure we used $\text{tol} = 0.0025$, and the result was 179 nonzero coefficients.





Chapter 13. Partial Differential Equations

Section 13.1. Derivation of the Heat Equation

1. The temperature satisfies Dirichlet conditions as given in (1.16). With the data as given the temperature must satisfy

$$\begin{aligned} u_t(x, t) &= ku_{xx}, \quad \text{for } 0 \leq x \leq L \text{ and } t > 0, \\ u(0, t) &= 5, \quad \text{and} \quad u(L, t) = 25, \quad \text{for } t > 0, \\ u(x, 0) &= 15, \quad \text{for } 0 \leq x \leq L. \end{aligned}$$

2. This follows immediately from the maximum principle.
3. By the equation just before (1.3),

$$\frac{dQ}{dt} = S \int_a^b \frac{\partial}{\partial t} [c\rho u] dx.$$

By (1.7)

$$\frac{dQ}{dt} = S \int_a^b \frac{\partial}{\partial x} \left(C \frac{\partial u}{\partial x} \right) dx.$$

Setting these equal, and using the argument leading to (1.9), we get the result.

4. Since the rod is insulated at the endpoints, we have $u_x(0, t) = u_x(L, t) = 0$. Then (1.7) says that $Q_t = 0$, so Q is unchanging.
5. Since α and β are constants, the linearity of differentiation implies that $w_t = \alpha u_t + \beta v_t$ and $w_{xx} = \alpha u_{xx} + \beta v_{xx}$. The result follows immediately.
6. Let $w(x, t) = u(x, t) - v(x, t)$. By the linearity of the heat equation, w is a solution. Since u and v satisfy the same initial and boundary conditions, we

have $w(0, t) = 0$ and $w(L, t) = 0$ for $t \geq 0$, and $w(x, 0) = 0$ for $0 \leq x \leq L$. By the maximum principle, $w(x, t) = 0$ for $t \geq 0$ and $0 \leq x \leq L$. Therefore, $u(x, t) = v(x, t)$ for $t \geq 0$ and $0 \leq x \leq L$.

7. From Table 1 we see that the thermal diffusivity of aluminum is $k = 0.86$. The boundary conditions are Dirichlet conditions so modifying (1.16) we get

$$\begin{aligned} u_t(x, t) &= 0.86u_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0, \\ u(0, t) &= 20, \quad \text{and} \quad u(L, t) = 35, \quad \text{for } t > 0, \\ u(x, 0) &= 15, \quad \text{for } 0 \leq x \leq L. \end{aligned}$$

8. From Table 1 we see that the thermal diffusivity of gold is $k = 1.18$. We have a Dirichlet condition at the left-hand endpoint and a Neumann condition at the right-hand endpoint, so the initial/boundary value problem is

$$\begin{aligned} u_t(x, t) &= 1.18u_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0, \\ u(0, t) &= 20, \quad \text{and} \quad u_x(L, t) = 0, \quad \text{for } t > 0, \\ u(x, 0) &= 15, \quad \text{for } 0 \leq x \leq L. \end{aligned}$$

9. From Table 1 we see that the thermal diffusivity of silver is $k = 1.71$. We now have a Robin condition at the left-hand endpoint.

$$\begin{aligned} u_t(x, t) &= 1.71u_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0, \\ u_x(0, t) &= 0.0013(u(0, t) - 15), \quad \text{and} \\ u(L, t) &= 35, \quad \text{for } t > 0, \\ u(x, 0) &= 15, \quad \text{for } 0 \leq x \leq L. \end{aligned}$$

Section 13.2. Separation of Variables for the Heat Equation

1. The thermal diffusivity of gold is $k = 1.18 \text{ cm}^2/\text{sec}$. We will let the unit of length be centimeters, so $L = 50$. The boundary conditions are $u(0, t) = 0$ and $u(50, t) = 0$, so the steady-state temperature is 0. Hence the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin(n\pi x/L),$$

where the coefficients are

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L 100 \sin(n\pi x/L) dx \\ &= \frac{200}{n\pi} [1 - \cos(n\pi)] \\ &= \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Thus the solution is given by

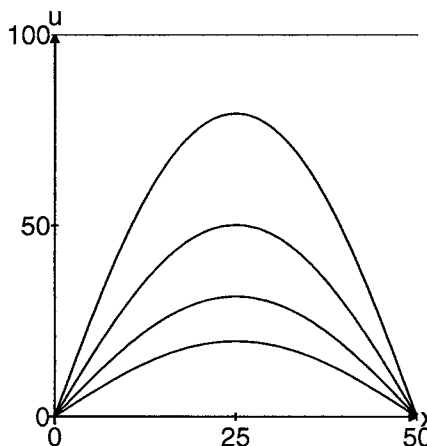
$$u(x, t) = \sum_{p=0}^{\infty} \frac{400}{(2p+1)\pi} e^{-1.18 \times (2p+1)^2 \pi^2 t/2500} \times \sin\left(\frac{n\pi x}{50}\right).$$

For $t = 100$, the term for $p = 1$ is bounded by 0.65. Hence one term of the series will suffice to estimate the temperature within 1° . Using just this one term, we solve

$$\frac{400}{\pi} e^{-1.18 \times \pi^2 t/2500} = 10$$

for $t = 546$ sec. Hence we see that it will take about 546 sec for the temperature to drop below 10°C . The

temperature at 100 second intervals is plotted below.



2. The thermal diffusivity of aluminum is $k = 0.84 \text{ cm}^2/\text{sec}$. For $t = 100$, the term for $p = 1$ is bounded by 2.14, while that for $p = 2$ is bounded by 0.006. Hence two terms will suffice to compute the temperature for $t = 100$ sec. Looking at the term for $p = 0$, we solve

$$\frac{400}{\pi} e^{-0.84 \times \pi^2 t/2500} = 10$$

for $t = 767$ sec. For such a time, all of the other terms are extremely small, so we see that it will take about 767 sec for the temperature to drop below 10°C .

The thermal diffusivity of silver is $k = 1.7 \text{ cm}^2/\text{sec}$. For $t = 100$, the term for $p = 1$ is bounded by 0.1. Hence one term will suffice to compute the temperature for $t = 100$ sec. Looking at the term for $p = 0$, we solve

$$\frac{400}{\pi} e^{-1.7 \times \pi^2 t/2500} = 10$$

for $t = 379$ sec. For such a time, all of the other terms are extremely small, so we see that it will take about 379 sec for the temperature to drop below 10°C .

The thermal diffusivity of PVC is $k = 0.0004 \text{ cm}^2/\text{sec}$. For $t = 1 \text{ day} = 86,400 \text{ sec}$, the term for $p = 2$ is bounded by 0.1.5, while that for $p = 3$ is bounded by 0.07. Hence three terms are needed to compute the temperature for $t = 1 \text{ day}$. Looking at the the term for $p = 0$, we solve

$$\frac{400}{\pi} e^{-0.0004 \times \pi^2 t / 2500} = 10$$

for $t = 1.611 \times 10^6 \text{ sec} = 18.6 \text{ days}$. For such a time, all of the other terms are extremely small, so we see that it will take about 18, 5 days for the temperature to drop below 10°C .

3. (a) For $0 \leq x \leq 10$, the steady-state temperature is given by $f(x) = 20 - x$.
 (b) Let $u(x, t)$ be the temperature. It must solve the initial/boundary value problem

$$\begin{aligned} u_t - k u_{xx} &= 0, \quad \text{for } 0 < x < 10 \text{ and } t > 0, \\ u(0, t) &= 20 \quad \text{and} \quad u(10, t) = -10, \\ u(x, 0) &= 20 - x. \end{aligned}$$

This solution will tend to a new steady-state temperature $u_{ss}(x)$ with $u_{ss}(0) = 20$ and $u_{ss}(10) = -10$. Hence $u_{ss}(x) = 20 - 3x$. The difference $v(x, t) = u(x, t) - u_{ss}(x)$ must solve

$$\begin{aligned} v_t - k v_{xx} &= 0, \quad \text{for } 0 < x < 10 \text{ and } t > 0, \\ v(0, t) &= 0 \quad \text{and} \quad v(10, t) = 0, \\ v(x, 0) &= (20 - x) - (20 - 3x) = 2x. \end{aligned}$$

The solution v is given by

$$v(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/100} \sin \frac{n\pi x}{10},$$

where $k = 0.0057 \text{ cm}^2/\text{sec}$ is the thermal diffusivity of brick, and

$$\begin{aligned} b_n &= \frac{2}{10} \int_0^{10} 2x \sin \left(\frac{n\pi x}{10} \right) dx \\ &= (-1)^{n+1} \frac{40}{n\pi}. \end{aligned}$$

Hence the solution is

$$\begin{aligned} u(x, t) &= (20 - 3x) \\ &+ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{40}{n\pi} e^{-0.0057 \times n^2 \pi^2 t / 100} \\ &\times \sin \frac{n\pi x}{10}. \end{aligned}$$

4. The temperature $u(x, t) = u_{ss}(x) + v(x, t)$, where u_{ss} is the steady-state temperature and v is the transient temperature. The steady state temperature satisfies $u_{ss}(0) = 20$ and $u_{ss}(10) = 420$, so $u_{ss}(x) = 20 + 40x$. The transient temperature satisfies

$$\begin{aligned} v_t - k v_{xx} &= 0, \quad \text{for } 0 < x < 10 \text{ and } t > 0, \\ v(0, t) &= 0 \quad \text{and} \quad v(10, t) = 0, \\ v(x, 0) &= u(x, 0) - u_{ss}(x) = -40x. \end{aligned}$$

The solution is given by

$$v(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/100},$$

where $k = 5 \times 10^{-5} \text{ cm}^2/\text{sec}$, and

$$\begin{aligned} b_n &= \frac{2}{10} \int_0^{10} (-40x) \sin \left(\frac{n\pi x}{10} \right) dx \\ &= (-1)^n \frac{800}{n\pi}. \end{aligned}$$

Hence the solution is

$$\begin{aligned} u(x, t) &= (20 + 40x) \\ &+ \sum_{n=1}^{\infty} (-1)^n \frac{800}{n\pi} e^{-0.00005 \times n^2 \pi^2 t / 100}. \end{aligned}$$

5. Since $T_0 = T_L = 0$, the steady-state temperature is 0° . The solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-4n^2\pi^2 t} \sin(n\pi x),$$

where the coefficients are computed from

$$b_n = 2 \int_0^1 x(1-x) \sin(n\pi x) dx.$$

Computing this integral we see that $b_n = 0$ if n is even, and $b_{2n+1} = 4/((2n+1)^3\pi^3)$. Thus the solution is

$$u(x, t) = \frac{4}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} e^{-4(2n+1)^2\pi^2 t} \times \sin((2n+1)\pi x).$$

6. Since $T_0 = T_L = 0$, the steady-state temperature is 0° . The Fourier series for the initial temperature is given, so we can immediately write down the formula for the temperature

$$u(x, t) = e^{-8t} \sin 2x - e^{-32t} \sin 4x.$$

7. Since $T_0 = T_L = 0$, the steady-state temperature is 0° . The solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx),$$

where the coefficients are

$$b_n = \frac{2}{\pi} \int_0^\pi \sin^2 x \sin(nx) dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -8/[\pi n(n^2 - 4)] & \text{if } n \text{ is odd.} \end{cases}$$

8. Since $T_0 = 0$ and $T_L = 2$, the steady-state temperature is $u_s(x) = 2x$. It remains to find $v(x, t) = u(x, t) - u_s(x)$, which solves the initial/boundary value problem

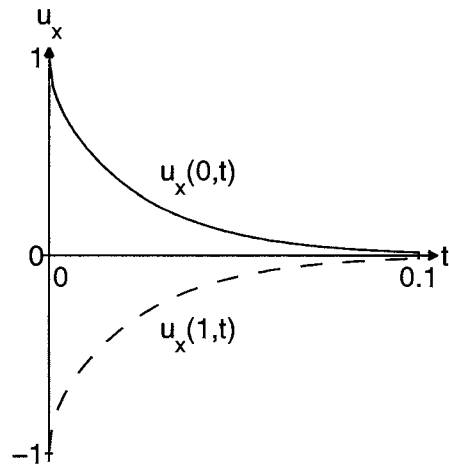
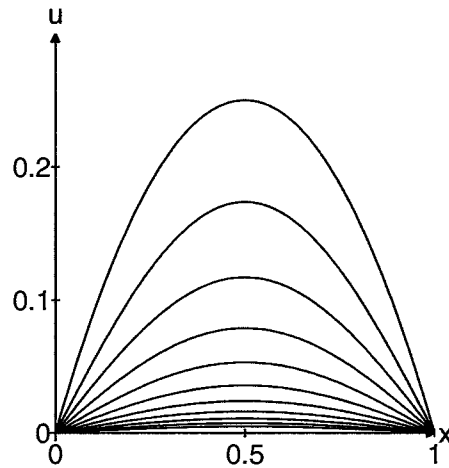
$$\begin{aligned} v_t(x, t) &= v_{xx}(x, t), \quad \text{for } t > 0 \text{ and } 0 \leq x \leq L, \\ v(0, t) &= 0 \quad \text{and} \quad v(L, t) = 0, \quad \text{for } t > 0, \\ v(x, 0) &= -x, \quad \text{for } 0 \leq x \leq L. \end{aligned}$$

This problem is almost the same as the one in Example 2.17. The solution is

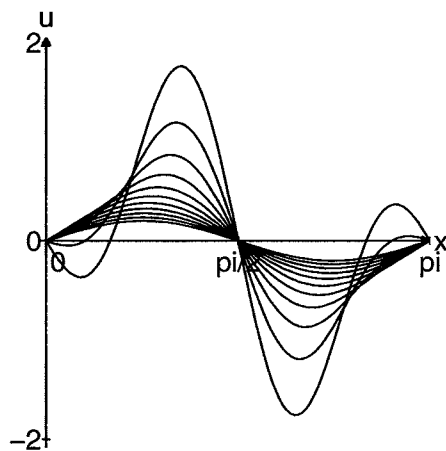
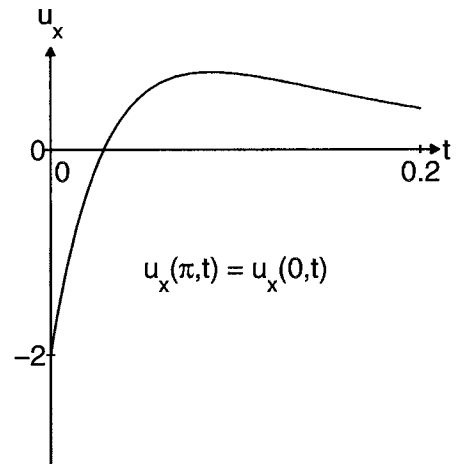
$$v(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2\pi^2 t} \sin n\pi x.$$

The complete solution is $u(x, t) = u_s(x) + v(x, t)$.

9. This solution is plotted in the first figure for $t = 0$ and then for values of t at intervals of 0.01. The temperature profiles show a steady decrease in temperature as it approaches steady-state. The partial u_x at the endpoints is plotted in the second figure. Notice that $u_x(0, t) > 0$ and $u_x(1, t) < 0$, reflecting the fact that heat is flowing out of the rod at each endpoint, leading to the decrease in the temperature in the rod.

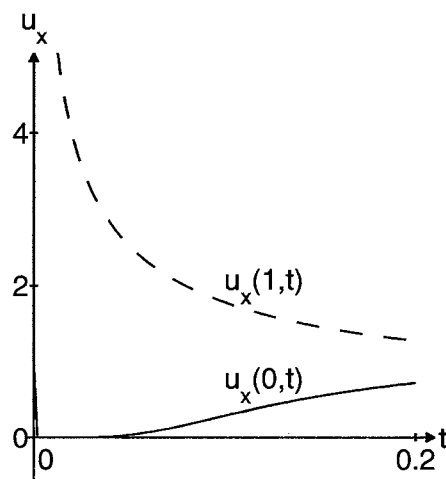
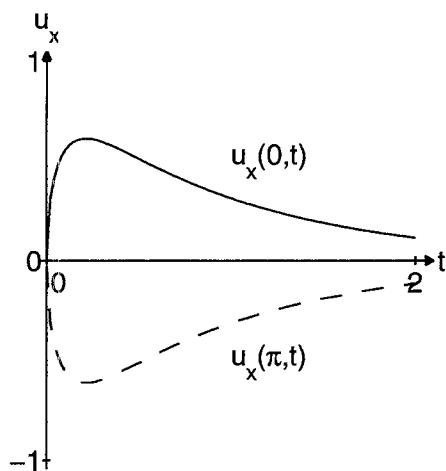
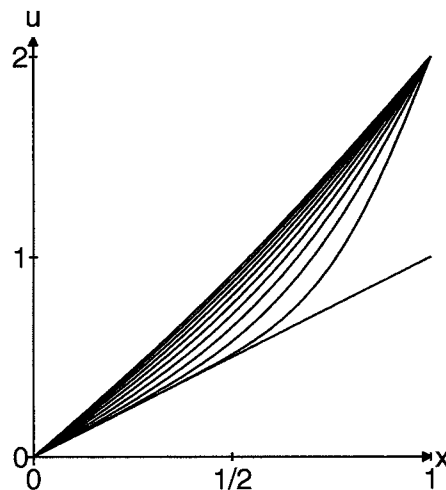
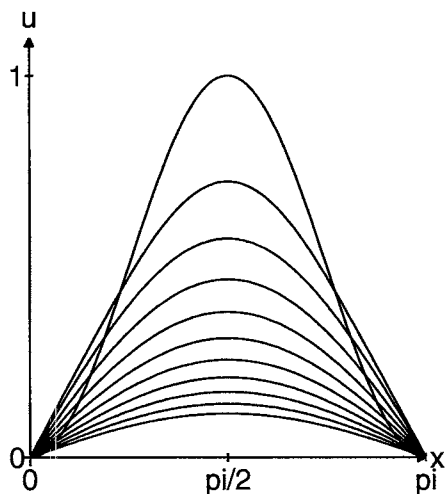


10. The solution is plotted in the first figure for $t = 0$ and then for values of t at intervals of 0.02. The temperature profiles show a steady decrease in the total heat in the rod as time increases. The temperature decreases from hot spots and increases from cold spots as it approaches steady-state. The partial u_x at the endpoints is plotted in the second figure. Notice that $u_x(0, t) = u_x(\pi, t)$, for all t . As a result, heat flows into the rod at one endpoint as fast as it leaves at the other. The total amount of heat in the rod is constant. Nevertheless the temperature is approaching 0 at each point.



11. This solution is plotted in the first figure for $t = 0$ and then for values of t at intervals of 0.2. The temperature profiles show a steady decrease in temperature as it approaches steady-state. The partial u_x at the endpoints is plotted in the second figure. Notice that $u_x(0, t) > 0$ and $u_x(\pi, t) < 0$, reflecting the fact that heat is flowing out of the rod at each endpoint, leading to the decrease in the temperature in the rod. However, notice that there is an increase in the temperature near the endpoints for a short period of time. This is because heat flows faster from the center of the rod into these areas than it flows out through the endpoints.

$x = 0$ is smaller than that at $x = 1$, heat flows from $x = 1$ to $x = 0$.



12. This solution is plotted in the first figure for $t = 0$ and then for values of t at intervals of 0.02. The temperature profiles show a steady increase in temperature as it approaches steady-state. The partial u_x at the endpoints is plotted in the second figure. Notice that $u_x(0, t) > 0$ and $u_x(1, t) > 0$, reflecting the fact that heat is flowing out of the rod at $x = 0$, but into the rod at $x = 1$. Since the temperature at

13. The solution is

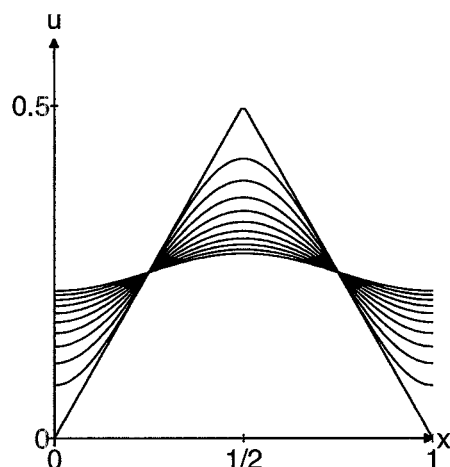
$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 t} \cos(n\pi x),$$

where the coefficients a_n are the Fourier cosine coefficients of f . After computing the coefficients, we

get the series

$$u(x, t) = \frac{1}{4} - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(4n+2)^2} e^{-(4n+2)^2 \pi^2 t} \times \cos((4n+2)\pi x).$$

The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.005. The temperature approaches the steady-state temperature of $1/4$, which is the average temperature in the rod initially.

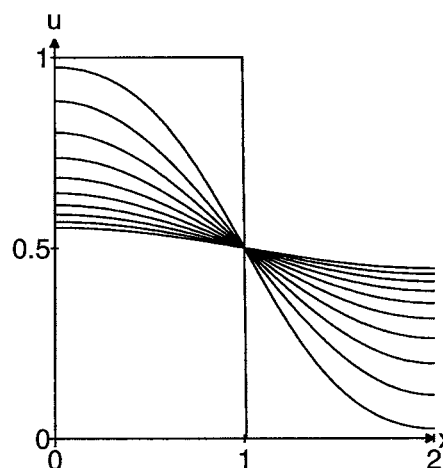


14. The solution is

$$u(x, t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n} e^{-(2n+1)^2 \pi^2 t/4} \times \cos\left(\frac{(2n+1)\pi x}{2}\right).$$

The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.1. The temperature approaches the steady-state temperature of $1/2$, which

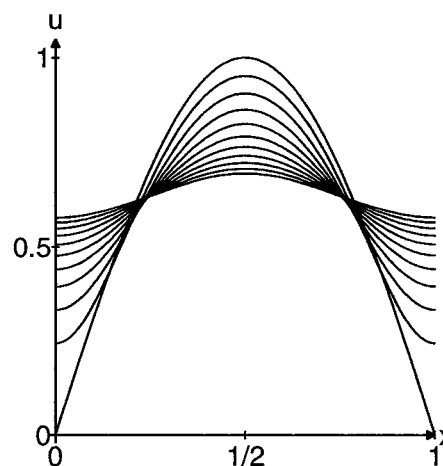
is the average temperature in the rod initially.



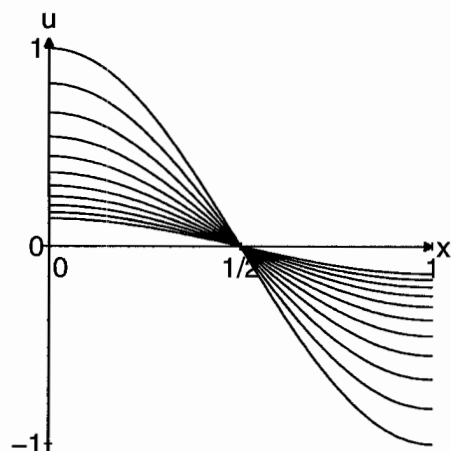
15. The solution is

$$u(x, t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} e^{-4n^2 \pi^2 t} \cos(2n\pi x).$$

The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.1. The temperature approaches the steady-state temperature of $2/\pi$, which is the average temperature in the rod initially.



16. The solution is $u(x, t) = e^{-\pi^2 t} \cos(\pi x)$. The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.02. The temperature approaches the steady-state temperature of 0, which is the average temperature in the rod initially.



17. The solution is

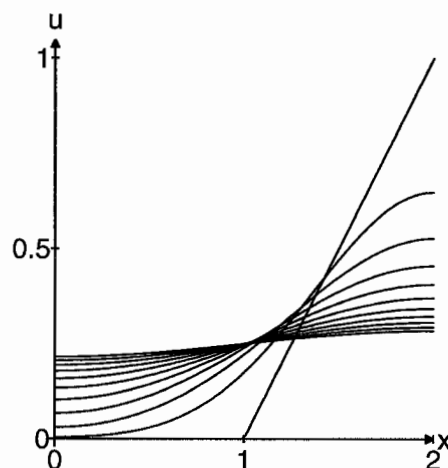
$$u(x, t) = \frac{1}{4} + \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 t/4} \cos\left(\frac{n\pi x}{2}\right),$$

where for $n \geq 1$,

$$a_n = \frac{4}{n^2 \pi^2} \times \begin{cases} 0, & \text{if } n = 4k, \\ -1, & \text{if } n = 4k \pm 1, \\ 2, & \text{if } n = 4k + 2. \end{cases}$$

The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.1. The temperature approaches the steady-state temperature of $1/4$, which

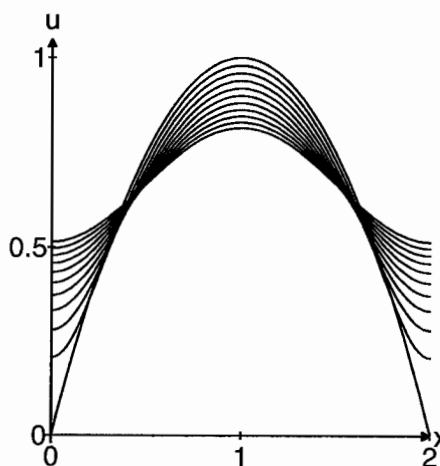
is the average temperature in the rod initially.



18. The solution is

$$u(x, t) = \frac{2}{3} - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{4n^2} e^{-n^2 \pi^2 t} \cos(n\pi x).$$

The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.01. The temperature approaches the steady-state temperature of $2/3$, which is the average temperature in the rod initially.



19. Since v does not depend on t , the partial differential equation reduces to $-kv'' = p/c\rho$, which is equivalent to what is to be shown. If $u(x, t) = u_h(x, t) + v(x)$, then

$$\frac{\partial u}{\partial t} = \frac{\partial u_h}{\partial t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u_h}{\partial x^2} + v'' = \frac{\partial^2 u_h}{\partial x^2} - \frac{P}{k}.$$

Hence

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u_h}{\partial t} - k \left(\frac{\partial^2 u_h}{\partial x^2} - \frac{P}{k} \right) = P.$$

In addition, $u(x, 0) = u_h(x, 0) + v(x) = f(x)$. Finally, $u(0, t) = u_h(0, t) + v(0) = A$ and $u(L, t) = u_h(L, t) + v(L) = B$.

20. Since $v'' = -P(x)/k = -6x$, we have $v(x) = -x^2 + \alpha x + \beta$, where α and β are constants of integration. The boundary conditions say that $0 = \beta$ and $1 = -1 + \alpha + \beta$. Hence $\beta = 0$ and $\alpha = 2$, so $v(x) = 2x - x^3$.

We have to solve the initial/boundary value problem for the heat equation with initial value $u_h(x, 0) = f(x) - v(x) = \sin \pi x - 2x + x^3$. The Fourier sine series for v is

$$v(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x,$$

where

$$\begin{aligned} a_n &= 2 \int_0^1 v(x) \sin n\pi x \, dx \\ &= 2 \int_0^1 (2x - x^3) \sin n\pi x \, dx. \end{aligned}$$

After integrating by parts three times, we find that

$$a_n = (-1)^{n+1} \left[\frac{2}{n\pi} + \frac{12}{n^3\pi^3} \right].$$

Therefore

$$\begin{aligned} u_h(x, t) &= e^{-\pi^2 t} \sin \pi^2 x \\ &+ \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi^2 t} \left[\frac{2}{n\pi} + \frac{12}{n^3 \pi^3} \right] \sin n\pi x. \end{aligned}$$

Finally,

$$\begin{aligned} u(x, t) &= u_h(x, t) + v(x) \\ &= e^{-\pi^2 t} \sin \pi^2 x \\ &+ \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi^2 t} \left[\frac{2}{n\pi} + \frac{12}{n^3 \pi^3} \right] \sin n\pi x \\ &+ 2x - x^3. \end{aligned}$$

21. Since $v'' = -P(x)/k = -e^{-x}$, we have $v(x) = -e^{-x} + \alpha x + \beta$, where α and β are constants of integration. The boundary conditions say that $1 = \beta$ and $-1/e = -1/e + \alpha + \beta$. Hence $\beta = 1$ and $\alpha = -1$, so $v(x) = 1 - x - e^{-x}$.

We have to solve the initial/boundary value problem for the heat equation with initial value $u_h(x, 0) = f(x) - v(x) = \sin \pi x + e^{-x} - 1 + x$. The Fourier sine series for v is

$$v(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x,$$

where

$$\begin{aligned} a_n &= 2 \int_0^1 v(x) \sin n\pi x \, dx \\ &= 2 \int_0^1 (1 - x - e^{-x}) \sin n\pi x \, dx \\ &= \frac{2}{n\pi} - \frac{2n\pi[1 - (-1)^n/e]}{1 + n^2\pi^2}. \end{aligned}$$

Therefore

$$u_h(x, t) = e^{-\pi^2 t} \sin \pi x - \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 t} \sin n\pi x.$$

Finally,

$$\begin{aligned} u(x, t) &= u_h(x, t) + v(x) \\ &= 1 - x - e^{-x} + e^{-\pi^2 t} \sin \pi x \\ &- \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 t} \sin n\pi x. \end{aligned}$$

22. Suppose that $\lambda < 0$, and write $\lambda = -\mu^2$, where $\mu > 0$. The differential equation in (2.27) becomes $X'' - \mu^2 X = 0$, which has general solution $X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$. Since $X'(x) = \mu[C_1 e^{\mu x} - C_2 e^{-\mu x}]$, the first boundary condition says that $0 = X'(0) = C_1 - C_2$. Hence $C_2 = C_1$ and

$X(x) = C_1(e^{\mu x} + e^{-\mu x})$. The second boundary condition then says that $0 = X'(L) = C_1(e^{\mu L} - e^{-\mu L})$. For all $L > 0$, $e^{\mu L} > 0 > e^{-\mu L}$, so we conclude that $C_1 = 0$. Hence the only solution to the boundary value problem is $X(x) = 0$, so $\lambda = -\mu^2$ is not an eigenvalue.

Section 13.3. The Wave Equation

1. The Fourier sine series for f is $\sum_{n=1}^{\infty} a_n \sin n\pi x$, where

$$\begin{aligned} a_n &= 2 \int_0^1 f(x) \sin n\pi x \, dx \\ &= \frac{1}{2} \int_0^1 x(1-x) \sin n\pi x \, dx \\ &= \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{2}{n^3 \pi^3}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

From (3.5), we conclude that

$$\sum_{k=0}^{\infty} \frac{2 \sin((2k+1)\pi x) \cos((2k+1)\pi t)}{(2k+1)^3 \pi^3}.$$

2. The Fourier sine series for f is $\sum_{n=1}^{\infty} a_n \sin n\pi x$, where

$$a_n = \frac{1}{5} \int_0^{10} f(x) \sin \frac{n\pi x}{10} \, dx.$$

Since $f(10-x) = f(x)$, and $\sin(n\pi - \alpha) = -\cos n\pi \sin \alpha$, it follows that

$$\begin{aligned} &\int_0^{10} f(x) \sin \frac{n\pi x}{10} \, dx \\ &= \int_0^5 f(x) \sin \frac{n\pi x}{10} \, dx \\ &\quad + \int_5^{10} f(x) \sin \frac{n\pi x}{10} \, dx \\ &= [1 - (-1)^n] \int_0^5 f(x) \sin \frac{n\pi x}{10} \, dx. \end{aligned}$$

Hence $a_n = 0$ if n is even, and

$$\begin{aligned} a_{2k+1} &= \frac{1}{50} \int_0^5 x \sin \frac{(2k+1)\pi x}{10} \, dx \\ &= (-1)^k \frac{4}{(2k+1)^2 \pi^2}. \end{aligned}$$

From (3.5), we conclude that

$$u(x, t) = 4 \sum_{k=0}^{\infty} \frac{(-1)^k \sin((2k+1)\pi x) \cos((2k+1)\pi t)}{(2k+1)^2 \pi^2}.$$

3. The Fourier sine series for g is $\sum_{n=1}^{\infty} b_n \sin n\pi x$, where

$$\begin{aligned} b_n &= 2 \int_0^1 \sin n\pi x \, dx \\ &= \frac{2}{n\pi} [1 - \cos n\pi] \\ &= \begin{cases} 0 & n \text{ even} \\ 4/n\pi & n \text{ odd.} \end{cases} \end{aligned}$$

From (3.5) and (3.7), we conclude that

$$u(x, t) = 4 \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi x \cdot \sin(2n+1)\pi t}{(2n+1)^2 \pi^2}.$$

4. The Fourier sine series for g is $\sum_{n=1}^{\infty} b_n \sin n\pi x$,

where

$$\begin{aligned} b_n &= 2 \int_0^1 g(x) \sin n\pi x \, dx \\ &= - \int_0^{1/2} \sin n\pi x \, dx + \int_{1/2}^1 \sin n\pi x \, dx \\ &= \frac{1}{n\pi} \left[\cos \frac{n\pi}{2} - 1 - \cos n\pi + \cos \frac{n\pi}{2} \right]. \end{aligned}$$

Thus $b_n = 0$ if n is odd, $b_{4n} = 0$ and $b_{4n+2} = -4/n\pi$. From (3.5) and (3.7), we conclude that

$$\begin{aligned} u(x, t) &= -4 \sum_{k=0}^{\infty} \frac{\sin(4k+2)\pi x \cdot \sin(4k+2)\pi t}{(4n+2)^2 \pi^2}. \end{aligned}$$

5. The Fourier sine series for g is $\sum_{n=1}^{\infty} b_n \sin n\pi x$, where

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^3 g(x) \sin \frac{n\pi x}{3} \, dx \\ &= \frac{2}{3} \int_1^2 \sin \frac{n\pi x}{3} \, dx \\ &= \frac{2[\cos(n\pi/3) - \cos(2n\pi/3)]}{n\pi}. \end{aligned}$$

From (3.5) and (3.7), we conclude that

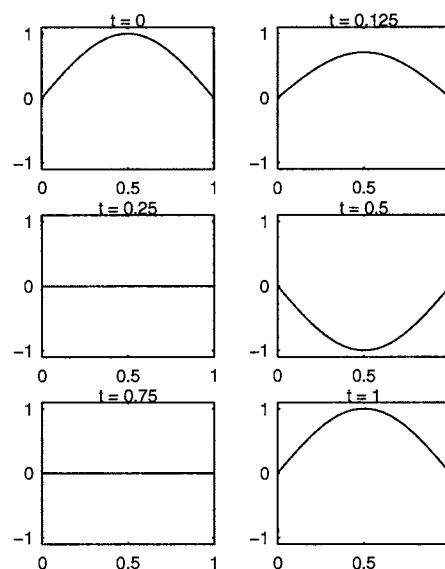
$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/3) \sin(n\pi t/3),$$

where

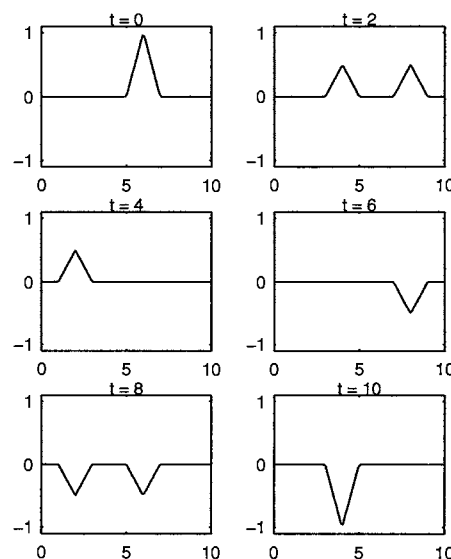
$$B_n = \frac{3b_n}{n\pi} = \frac{6[\cos(n\pi/3) - \cos(2n\pi/3)]}{n^2 \pi^2}.$$

6. Let u_1 be the solution with all of the same data, except that $g = 0$, and let u_2 be the solution with all the same data except that $f = 0$. We found u_1 in Exercise 1 and we found $-u_2$ in Exercise 3. By the linearity of the wave equation, $u = u_1 + u_2$.
7. The odd periodic extension of $f(x) = \sin \pi x$ is $f_{op}(x) = \sin \pi x$. Hence the solution is

$$\begin{aligned} u(x, t) &= [f_{op}(x + ct) + f_{op}(x - ct)]/2 \\ &= [\sin \pi(c + 2t) + \sin \pi(x - 2t)]/2 \\ &= \sin \pi x \cos 2\pi t. \end{aligned}$$



8. The solution is $u(x, t) = [f_{op}(x + ct) + f_{op}(x - ct)]/2$. It is plotted in the figure.



9. The Fourier sine series converges to $f_{op}(x)$ for all x , so

$$f_{op}(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x, \quad \text{for all } x.$$

Using the trigonometric identity, we get

$$\begin{aligned} u_1(x, t) &= \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n\pi t \\ &= \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin n\pi(x+t) \\ &\quad + \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin n\pi(x-t) \\ &= \frac{1}{2} [f_{op}(x+t) + f_{op}(x-t)] \\ &= u_2(x, t). \end{aligned}$$

10. We look for product solutions of the form $u(x, t) = X(x)T(t)$. Substituting into the differential equation, it becomes $[T'' + T' + T]X = X''T$. Dividing by $u = XT$, we get

$$\frac{T'' + T' + T}{T} = \frac{X''}{X}.$$

Since the left-hand side is a function of t and the right-hand side is a function of x , there must be a constant λ such that both are equal to $-\lambda$. Hence we have

$$T'' + T' + T = -\lambda T \quad \text{and} \quad -X'' = \lambda X.$$

Since u satisfies the boundary conditions, we also have $X(0) = X(1) = 0$. The Sturm Liouville problem for X has solutions

$$\lambda_n = n^2\pi^2 \quad \text{with} \quad X_n(x) = \sin n\pi x.$$

The equation for T becomes

$$T'' + T' + (n^2\pi^2 + 1)T = 0.$$

This is a second order equation with constant coefficients. If we set

$$\omega_n = \sqrt{n^2\pi^2 + 3/4},$$

the equation has the fundamental set of solutions

$$e^{-t/2} \sin \omega_n t \quad \text{and} \quad e^{-t/2} \cos \omega_n t.$$

Hence we have the general solution

$$u(x, t) = e^{-t/2} \sum_{n=1}^{\infty} [a_n \cos \omega_n t + b_n \sin \omega_n t] \times \sin n\pi x.$$

The initial conditions become

$$\begin{aligned} f(x) &= u(x, 0) = \sum_{n=1}^{\infty} a_n \sin n\pi x, \\ 0 &= u_t(0, x) = \sum_{n=1}^{\infty} \left[\omega_n b_n - \frac{a_n}{2} \right] \sin n\pi x. \end{aligned}$$

The first series is the Fourier sine series for f , so

$$a_n = 2 \int_0^1 f(x) \sin n\pi x \, dx.$$

From the second series we see that $b_n = a_n/(2\omega_n)$. hence the solution is

$$u(x, t) = e^{-t/2} \sum_{n=1}^{\infty} a_n \left[\cos \omega_n t + \frac{1}{2\omega_n} \sin \omega_n t \right] \times \sin n\pi x.$$

11. We start with the d'Alembert solution

$$u(x, t) = F(x+ct) + G(x-ct).$$

The initial conditions are

$$\begin{aligned} 0 &= F(x) + G(x) \\ g(x) &= c[F'(x) - G'(x)]. \end{aligned}$$

By the first equation, $G = -F$, and then by the second equation

$$F'(x) = \frac{1}{2c} g(x) \quad \text{for } 0 \leq x \leq L.$$

The first boundary condition is

$$0 = u(0, t) = F(ct) - F(-ct),$$

from which we conclude that F is an even function. The second boundary condition is

$$0 = u(L, t) = F(L + ct) - F(L - ct).$$

If we set $ct = y + L$, and remember that F is even, this equation implies that

$$F(y + 2L) = F(-y) = F(y).$$

Hence F is even and $2L$ -periodic. Therefore, the derivative F' is odd and periodic. Since $F'(x) = g(x)/2c$ for $0 \leq x \leq L$, we set $F'(x) = g_{op}(x)$ for all x . Then by the fundamental theorem of calculus,

$$\begin{aligned} u(x, t) &= F(x + ct) - F(x - ct) \\ &= \int_{x-ct}^{x+ct} F'(s) ds \\ &= \frac{1}{2c} \int_{x-ct}^{x+ct} g_{op}(s) ds. \end{aligned}$$

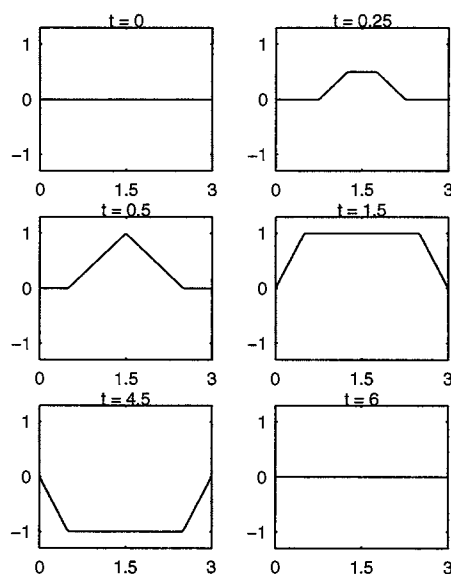
12. When we integrate the function g , we get

$$\begin{aligned} F(x) &= \int_0^x g(x) dx \\ &= \begin{cases} 0, & \text{for } 0 \leq x \leq 1, \\ 2x - 2, & \text{for } 1 < x \leq 2, \\ 2, & \text{for } 2 < x \leq 3. \end{cases} \end{aligned}$$

The function F must be even and periodic with period 6. Then

$$u(x, t) = \frac{1}{2} [F(x + t) - F(x - t)].$$

The solution is plotted below.



13. Let u_1 be the solution with the same data, but with $g = 0$. According to (3.18)

$$u_1(x, t) = \frac{1}{2} [f_{op}(x + ct) - f_{op}(x - ct)].$$

Let u_2 be the solution with the same data, but with $f = 0$. According to Exercise 11

$$u_2(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g_{op}(s) ds.$$

Since the wave equation is linear, the function $u = u_1 + u_2$ is a solution, and satisfies the needed boundary and initial conditions. Hence

$$\begin{aligned} u(x, t) &= \frac{1}{2} [f_{op}(x + ct) - f_{op}(x - ct)] \\ &\quad + \frac{1}{2c} \int_{x-ct}^{x+ct} g_{op}(s) ds. \end{aligned}$$

14. (a) If we substitute $v(x)$ into (3.20) we get $0 = c^2 v_{xx} - g$. This is an ordinary differential equation for v , and its general solution is $v(x) = gx^2/(2c^2) + Ax + B$. From the boundary conditions we get $0 = v(0) = B$, and then $0 =$

$v(L) = gL^2/(2c^2) - AL$, or $A = -gL/(2c^2)$.
Hence

$$v(x) = \frac{g}{2c^2}L(L-x).$$

The function $w = u - v$ satisfies $w_{tt} = u_{tt} = c^2u_{xx} - g$, and $w_{xx} = u_{xx} - v_{xx} = u_{xx} - g/c^2$. Hence $w_{tt} = c^2w_{xx}$, so w satisfies the wave equation.

- (b) The function $w(x, t) = u(x, t) - v(x)$ satisfies the wave equation, the Dirichlet boundary conditions, and the initial conditions $w(x, 0) = u(x, 0) - v(x) = -v(x) = -gL(L-x)/(2c^2)$ and $w_t(x, 0) = u_t(x, 0) = 0$. Hence

$$w(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi t}{L}\right),$$

where

$$a_n = -\frac{2}{L} \int_0^L \frac{g}{2c^2}L(L-x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \begin{cases} 0 & \text{if } n \text{ is even,} \\ -4gL^2/(c^2\pi^3n^3) & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$w(x, t) = -\frac{4gL^2}{c^2\pi^3} \sum_{n \text{ odd}} \frac{1}{n^3} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi t}{L}\right),$$

and our solution is

$$u(x, t) = v(x) + w(x, t).$$

15. Proceeding according to the hint, we get

$$E'(t) = \frac{1}{2} \int_0^L [\rho u_t^2 + T u_x^2]_t dx$$

$$= \int_0^L [\rho u_t u_{tt} + T u_x u_{xt}] dx$$

$$= T \int_0^L [u_t u_{xx} + u_x u_{xt}] dx,$$

since $u_{tt} = c^2 u_{xx} = T u_{xx}/\rho$. It then follows that

$$E'(t) = T \int_0^L [u_t u_x]_x dx$$

$$= 0 \quad \text{since } u_t(0, t) = 0 = u_t(L, t).$$

For the last step we notice that $u_t(0, t) = [u(0, t)]_t = 0$, and a similar calculation for the other endpoint.

16. By fingering the string precisely in the middle, you are halving the length of the string. As a result the fundamental frequency $\omega_1 = c\pi/L$ is doubled.

17. (a) When we substitute $u(x, t) = X(x)T(t)$ into equation (3.22), and separate variables, we get

$$\frac{T'' + 2kT'}{c^2T} = \frac{X''}{X}.$$

Since the left-hand side is a function of t and the right-hand side is a function of x , both must be equal to a constant, which we designate as $-\lambda$. Thus X must satisfy $X'' = -\lambda X$, with $X(0) = X(L) = 0$. This is the same boundary value problem as discussed in the text. Hence the solutions are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \sin \frac{n\pi x}{L},$$

for $n = 1, 2, \dots$

The equation for T becomes

$$T'' + 2kT' + [n^2\pi^2c^2/L^2]T = 0.$$

Since $k < \pi c/L$, the functions $e^{-kt} \sin(\mu_n)$ and $e^{-kt} \cos(\mu_n)$, where

$$\mu_n = \sqrt{(n^2\pi^2c^2/L^2) - k^2},$$

form a fundamental set of solutions. Hence we have product solutions

$$u_n(x, t) = e^{-kt} \cos(\mu_n t) \sin\left(\frac{n\pi x}{L}\right) \quad \text{and}$$

$$v_n(x, t) = e^{-kt} \sin(\mu_n t) \sin\left(\frac{n\pi x}{L}\right)$$

for $n = 1, 2, \dots$

- (b) The general solution to equation (3.22) and the boundary conditions is the infinite series

$$u(x, t) = e^{-kt} \sum_{n=1}^{\infty} [a_n u_n(x, t) + b_n v_n(x, t)].$$

To satisfy the initial conditions we must have

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and}$$

$$g(x) = u_t(x, 0) = -\sum_{n=1}^{\infty} \mu_n b_n \sin\left(\frac{n\pi x}{L}\right).$$

Hence we must have

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{and}$$

$$b_n = -\frac{2}{L\mu_n} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

————— × —————

Section 13.4. Laplace's Equation

1. Using Figure 1, we have $f(x) = 10$, and $g = h = k = 0$. According to (4.12), (4.13), and (4.14), the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh[n\pi(y-2)] \sin(n\pi x),$$

where

$$b_n = -\frac{2}{\sinh(2n\pi)} \int_0^1 10 \sin(n\pi x) dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -40/n\pi \sinh(2n\pi) & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$u(x, y) = \sum_{k=0}^{\infty} \frac{40}{(2k+1)\pi} \frac{\sinh[(2k+1)\pi(2-y)]}{\sinh[(4k+2)\pi]} \sin(n\pi x).$$

2. Using Figure 1, we have $f(x) = 100x$, $h(x) = 20$, and $g = k = 0$. According to (4.12), (4.13), and (4.14), the solution is

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{10}\right) \sin\left(\frac{n\pi x}{10}\right) + \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi(y-25)}{10}\right) \sin\left(\frac{n\pi x}{10}\right),$$

where

$$a_n = \frac{2}{10 \sinh(2.5n\pi)} \int_0^{10} 20 \sin\left(\frac{n\pi x}{10}\right) dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 80/[n\pi \sinh(2.5n\pi)] & \text{if } n \text{ is odd.} \end{cases}$$

and

$$b_n = \frac{-2}{10 \sinh(2.5n\pi)} \int_0^{10} 100x \sin\left(\frac{n\pi x}{10}\right) dx = (-1)^n \frac{200}{n\pi \sinh(2.5n\pi)}.$$

Hence,

$$u(x, y) = 80 \sum_{k=0}^{\infty} \frac{1}{(2k+1)\pi} \frac{\sinh((2k+1)\pi y/10)}{\sinh(2.5(2k+1)\pi)} \sin\left(\frac{(2k+1)\pi x}{10}\right)$$

$$+ 200 \sum_{n=1}^{\infty} (-1)^n \frac{\sinh(n\pi(y-25)/10)}{\sinh(2.5n\pi)} \sin\left(\frac{(2k+1)\pi x}{10}\right).$$

3. The derivation is just like that in the text. In outline, we use separation of variables to find the product solutions

$$\sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \quad \text{and} \quad \sinh \frac{n\pi(x-a)}{b} \sin \frac{n\pi y}{b}.$$

The general solution therefore has the form

$$u(x, y) = \sum_{n=1}^{\infty} \left[a_n \sinh \frac{n\pi x}{b} + b_n \sinh \frac{n\pi(x-a)}{b} \right] \sin \frac{n\pi y}{b},$$

The constants are determined from the boundary conditions. At $x = 0$, we have

$$g(y) = u(0, y) = \sum_{n=1}^{\infty} -b_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b},$$

and at $x = a$,

$$k(y) = u(a, y) = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b}.$$

These are the Fourier sine series for g and k , so the result follows.

4. For $j = 1, 2, 3, 4$, let u_j be the temperature where the j th side is kept at 100° , while the other three sides are at 0° . By symmetry, all four temperatures are equal at the center. On the other hand, the sum $u_1 + u_2 + u_3 + u_4$ is a solution to Laplace's equation and is equal to 100° on the entire boundary. Hence at the center, $100 = u_1 + u_2 + u_3 + u_4 = 4u_1$. Hence the temperature at the center is 25° .
5. Since $f = 0$ and $a = b = 1$, we see from (4.12) that the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi y) \sin(n\pi x),$$

where

$$a_n = \frac{2}{\sinh(n\pi)} \int_0^1 h(x) \sin(n\pi x) dx.$$

We find that $a_n = 0$ if n is even, and

$$a_{2n+1} = \frac{4}{\sinh((2n+1)\pi)} \cdot \frac{(-1)^n}{(2n+1)^2\pi^2}.$$

Hence the temperature is

$$u(x, y) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n \sinh((2n+1)\pi y) \sin((2n+1)\pi x)}{(2n+1)^2\pi^2 \sinh((2n+1)\pi)}.$$

6. Notice that the odd periodic extension of h is the square wave which we considered in Example 2.8 in Section 12.2. We found that its Fourier sine series is

$$h(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin((2n+1)\pi x).$$

Therefore the temperature is

$$u(x, y) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sinh((2n+1)\pi y) \cdot \sin((2n+1)\pi x)}{(2n+1) \sinh((2n+1)\pi)}.$$

7. The function $f(x) = \sin(2\pi x)$ is expressed in terms of its Fourier sine series, so we can immediately write down the temperature:

$$u(x, y) = \frac{-\sinh(2\pi(y-1)) \sin(2\pi x)}{\sinh(2\pi)}.$$

8. Since $h = 0$ and $a = 1$, we see from (4.12) that the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi(y-2)) \sin(n\pi x),$$

where

$$b_n = \frac{-2}{\sinh(2n\pi)} \int_0^1 f(x) \sin(n\pi x) dx.$$

We have $f(x) = \sin^2(\pi x) = [1 - \cos(2\pi x)]/2$. Using the identity $\cos(a) \sin(b) = [\sin(b+a) + \sin(b-a)]/2$, we have

$$\begin{aligned} \int_0^1 f(x) \sin(n\pi x) dx &= \int_0^1 \sin^2(\pi x) \sin(n\pi x) dx \\ &= \int_0^1 \left[\frac{1}{2} \sin(n\pi x) - \frac{1}{4} \sin((n+2)\pi x) - \frac{1}{4} \sin((n-2)\pi x) \right] dx \\ &= \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{-4}{\pi n(n^2-4)}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Therefore,

$$u(x, y) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{\sinh(n\pi(y-2)) \sin(n\pi x)}{(2n+1)[(2n+1)^2-4] \sinh(2n\pi)}.$$

9. We see from (4.12) that the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} [a_n \sinh(n\pi y) + b_n \sinh(n\pi(y-2))] \sin(n\pi x),$$

where

$$\begin{aligned} a_n = b_n &= \frac{2}{\sinh(2n\pi)} \int_0^1 \sin(n\pi x) dx \\ &= \frac{2}{\sinh(2n\pi)} \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{2}{n\pi}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Therefore the temperature is

$$u(x, y) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sinh((2n+1)\pi y) + \sinh((2n+1)\pi(y-2))}{(2n+1) \sinh((4n+2)\pi)} \sin((2n+1)\pi x).$$

10. Let u_1 be the temperature with all of the data the same, except that $f = 0$, and let u_2 be the temperature with all of the data the same, except that $h = 0$. From Exercise 5 we see that

$$u_1(x, y) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n \sinh((2n+1)\pi y) \sin((2n+1)\pi x)}{(2n+1)^2 \pi^2 \sinh((2n+1)\pi)}.$$

From Exercise 7 we see that

$$u_2(x, y) = \frac{-\sinh(2\pi(y-1)) \sin(2\pi x)}{\sinh(2\pi)}.$$

By the linearity of the problem, $u = u_1 + u_2$.

11. (a) Using (4.12), we see that the temperature is given by

$$u(x, y) = - \sum_{n=0}^{\infty} B_n \frac{\sinh(n\pi(y-L))}{\sinh(n\pi L)} \sin(n\pi x).$$

- (b) In part (a), the dependence on L comes from the factor

$$\frac{-\sinh(n\pi(y-L))}{\sinh(n\pi L)} = \frac{e^{n\pi(L-y)} - e^{n\pi(y-L)}}{e^{n\pi L} - e^{-n\pi L}}.$$

If we factor $e^{n\pi L}$ from both numerator and denominator, we get

$$\frac{e^{-n\pi y} - e^{n\pi(y-2L)}}{1 - e^{-2n\pi L}}.$$

When we let L increase to ∞ , this converges to $e^{-n\pi y}$. Hence the solution over the finite rectangle converges to

$$u(x, y) = \sum_{n=1}^{\infty} B_n e^{-n\pi y} \sin(n\pi x).$$

12. Since Laplace's equation is linear, the function $w(x, y) = u(x, y) - v(x, y)$ satisfies Laplace's equation. By assumption, $w(x, y) = 0$ at every point $(x, y) \in \partial D$. Since, by the maximum principle, w must achieve both its maximum and minimum on ∂D , we must have $w(x, y) = 0$ at every point $(x, y) \in D$. Therefore, $u(x, y) = v(x, y)$ at every point $(x, y) \in D$.

Section 13.5. Laplace's Equation on a Disk

1. The computations of u_{xx} and u_{yy} proceed just as they do in polar coordinates, since the equations for x and y are the same. Since the coordinate z is unchanged, u_{zz} is also unchanged. Hence

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

2. Differentiating $r^2 = x^2 + y^2 + z^2$ with respect to x , y , and z , we get

$$r_x = \frac{x}{r} = \cos \theta \sin \phi, \quad r_y = \frac{y}{r} = \sin \theta \sin \phi, \quad \text{and} \quad r_z = \frac{z}{r} = \cos \phi.$$

From $\tan \theta = y/x$, we see that $1 + \tan^2 \theta = (x^2 + y^2)/x^2$. Notice also that $x^2 + y^2 = r^2 \sin^2 \phi$. Differentiating $\tan \theta = y/x$ with respect to x , and using these formulas, we get $(1 + \tan^2 \theta)\theta_x = -y/x^2$. Thus $\theta_x = -y/(x^2 + y^2)$. Doing the same for y , we see that

$$\theta_x = \frac{-y}{x^2 + y^2} = \frac{-\sin \theta}{r \sin \phi}, \quad \theta_y = \frac{x}{x^2 + y^2} = \frac{\cos \theta}{r \sin \phi}, \quad \text{and} \quad \theta_z = 0.$$

Since $\tan \phi = \sqrt{x^2 + y^2}/z$, $1 + \tan^2 \phi = r^2/z^2$. Differentiating $\tan^2 \phi = (x^2 + y^2)/z^2$ we then compute that

$$\phi_x = \frac{\cos \theta \cos \phi}{r}, \quad \phi_y = \frac{\sin \theta \cos \phi}{r}, \quad \text{and} \quad \phi_z = \frac{-\sin \phi}{r}.$$

By the chain rule,

$$u_x = u_r r_x + u_\theta \theta_x + u_\phi \phi_x = u_r \cdot \cos \theta \sin \phi + u_\theta \cdot \frac{-\sin \theta}{r \sin \phi} + u_\phi \cdot \frac{\cos \theta \cos \phi}{r}.$$

Differentiating once again we get

$$\begin{aligned} u_{xx} = & u_{rr} \cdot \cos^2 \theta \sin^2 \phi + \left(\frac{u_\theta}{r} \right)_r \cdot \frac{-\sin \theta \cos \phi}{r} + \left(\frac{u_\phi}{r} \right)_r \cdot \frac{\cos^2 \theta \sin \phi \cos \phi}{r} \\ & + (u_r \cos \theta)_\theta \cdot \frac{-\sin \theta}{r} + (u_\theta \sin \theta)_\theta \cdot \frac{\sin \theta}{r^2 \sin^2 \phi} + (u_\phi \cos \theta)_\theta \cdot \frac{-\sin \theta \cos \phi}{r^2 \sin \phi} \\ & + (u_r \sin \phi)_\phi \cdot \frac{\cos^2 \theta \cos \phi}{r} + \left(\frac{u_\theta}{-\sin \phi} \right)_\phi \cdot \frac{\sin \theta \cos \theta \cos \phi}{r^2} + (u_\phi \cos \phi)_\phi \cdot \frac{\cos^2 \theta \cos \phi}{r^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} u_{yy} = & u_{rr} \cdot \sin^2 \theta \sin^2 \phi + \left(\frac{u_\theta}{r} \right)_r \cdot \frac{\sin \theta \cos \phi}{r} + \left(\frac{u_\phi}{r} \right)_r \cdot \frac{\sin^2 \theta \sin \phi \cos \phi}{r} \\ & + (u_r \sin \theta)_\theta \cdot \frac{\cos \theta}{r} + (u_\theta \cos \theta)_\theta \cdot \frac{\cos \theta}{r \sin^2 \phi} + (u_\phi \sin \theta)_\theta \cdot \frac{\cos \theta \cos \phi}{r^2 \sin \phi} \\ & + (u_r \sin \phi)_\phi \cdot \frac{\sin^2 \theta \cos \phi}{r} + \left(\frac{u_\theta}{\sin \phi} \right)_\phi \cdot \frac{\sin \theta \cos \theta \cos \phi}{r^2} + (u_\phi \cos \phi)_\phi \cdot \frac{\sin^2 \theta \cos \phi}{r^2}. \end{aligned}$$

Finally,

$$u_{zz} = u_{rr} \cdot \cos^2 \phi + \left(\frac{u_\phi}{r}\right)_r \cdot (-\sin \phi \cos \phi) + (u_r \cos \phi)_\phi \cdot \frac{-\sin \phi}{r} + (u_\phi \sin \phi)_\phi \cdot \frac{\sin \phi}{r^2}.$$

The last term is simpler because $\theta_z = 0$.

Since $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ we have to combine these terms. This is made easier if you systematically combine terms by type. A term is of type (a, b) if it contains a factor of $(u_a \cdot *)_b$, where $*$ stands for an arbitrary expression. Of course, a and b stand for r, θ , or ϕ . For example, the (r, r) term is

$$u_{rr} \cdot (\cos^2 \theta \sin^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \phi) = u_{rr}.$$

The (θ, r) term is

$$\left(\frac{u_\theta}{r}\right)_r \cdot \frac{-\sin \theta \cos \phi}{r} + \left(\frac{u_\theta}{r}\right)_r \cdot \frac{\sin \theta \cos \phi}{r} = 0.$$

The (r, θ) term is

$$(u_r \cos \theta)_\theta \cdot \frac{-\sin \theta}{r} + (u_r \sin \theta)_\theta \cdot \frac{\cos \theta}{r} = \frac{u_r}{r}.$$

Proceeding in this manner through all 9 terms we get

$$\nabla^2 u = u_{rr} + \frac{2u_r}{r} + \frac{u_{\theta\theta}}{r^2 \sin \phi} + \frac{u_\phi \cos \phi}{r^2 \sin \phi} + \frac{u_{\phi\phi}}{r^2}.$$

The first two terms and the last two terms can be combined to get

$$\nabla^2 u = \frac{1}{r^2} (r^2 u_r)_r + \frac{u_{\theta\theta}}{r^2 \sin \phi} + \frac{1}{r^2 \sin \phi} (u_\phi \cdot \sin \phi)_\phi.$$

- The function $u(x, y) = 1 + y$ is a solution to Laplace's equation, and is equal to $f(x, y)$ on the surface of the can. Hence the temperature is $u(x, y) = 1 + y$.
- The effect of the rays of the sun should be proportional to the normal component of the ray. This is $\sin \phi$ times the full force of the ray. A little geometry shows that $\phi = \theta$, the polar angle. Hence assuming the boundary temperature is

$$f(\theta) = \begin{cases} \sin \theta, & \text{for } 0 \leq \theta \leq \pi, \\ 0, & \text{for } \pi < \theta \leq 2\pi. \end{cases}$$

The Fourier series for f is

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos n\theta + B_n \sin n\theta],$$

where $A_0 = (1/\pi) \int_0^\pi \sin \theta \, d\theta = 2/\pi$, and

$$A_n = \frac{1}{\pi} \int_0^\pi \sin \theta \cos n\theta \, d\theta = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ -2, & \text{if } n \text{ is even,} \end{cases}$$

and

$$B_n = \frac{1}{\pi} \int_0^\pi \sin \theta \sin n\theta \, d\theta = \begin{cases} 1/2, & \text{for } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$u(r, \theta) = \frac{1}{\pi} + \frac{r \sin \theta}{2} - 2 \sum_{n=1}^{\infty} \frac{r^{2n} \cos 2n\theta}{\pi(4n^2 - 1)}.$$

5. Since the Fourier series for f is $f(\theta) = \sin^2 \theta = [1 - \cos 2\theta]/2$, the temperature is $u(r, \theta) = [1 - r^2 \cos 2\theta]/2$.
6. Since the Fourier series for f is $f(\theta) = \cos^2 \theta = [1 + \cos 2\theta]/2$, the temperature is $u(r, \theta) = [1 + r^2 \cos 2\theta]/2$.
7. The Fourier series for f is

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos n\theta + B_n \sin n\theta].$$

First, we have

$$A_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta) d\theta = \frac{1}{\pi} \int_0^{2\pi} \theta(2\pi - \theta) d\theta = \frac{4\pi^2}{3},$$

Next, for $n \geq 1$, we integrate by parts twice to get

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta = \frac{1}{\pi} \int_0^{2\pi} \pi(2\pi - \theta) \cos n\theta d\theta = \frac{-4}{n^2}.$$

Finally, since the periodic extension of f is even, the sine coefficients are $B_n = 0$. Thus the temperature is

$$u(r, \theta) = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{r^n \cos n\theta}{n^2}.$$

8. The Fourier series for f is $f(\theta) = \sin \theta \cos \theta = (1/2) \sin 2\theta$, so the temperature is $u(r, \theta) = (r/2) \sin 2\theta$.
9. Let $u(r)$ denote the temperature at distance r from the center of the circles. Then by (5.2), $\nabla^2 u = u_{rr} + u_r/r = 0$. Thus u satisfies the Euler equation $ru_{rr} + u_r = 0$, and has the form $u(r) = A + B \ln r$. Using the boundary conditions to solve for the coefficients A and B we find that

$$u(r) = T_1 + (T_2 - T_1) \frac{\ln(r/a)}{\ln(b/a)}.$$

10. As in Exercise 9, we know that $u(r) = A + B \ln r$. Using the boundary conditions we see that $B = 0$ and $A = T$. Thus $u(r) = T$, and the temperature is constant throughout the plate.
11. Since the origin is not in the ring-shaped plate, we cannot eliminate the second solution to Euler's equation in (5.9). Hence the general series solution has the form

$$u(r, \theta) = \frac{A_0 + C_0 \ln r}{2} + \sum_{n=1}^{\infty} [(A_n r^n + C_n r^{-n}) \cos n\theta + (B_n r^n + D_n r^{-n}) \sin n\theta]$$

Since $u(a, \theta) = 0$, all of the coefficients of the trigonometric functions must equal 0 at $r = a$. Using this to evaluate A_n and B_n we get

$$u(r, \theta) = \frac{C_0 \ln(r/a)}{2} + \sum_{n=1}^{\infty} [C_n(r^{-n} - a^{-2n}r^n) \cos n\theta + D_n(r^{-n} - a^{-2n}r^n) \sin n\theta]$$

We can then use $u(b, \theta) = f(\theta)$ to compute the upper case coefficients, since these reduce to the Fourier coefficients of f . We get

$$\begin{aligned} C_0 \ln(b/a) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \\ C_n(b^{-n} - a^{-2n}b^n) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \\ D_n(b^{-n} - a^{-2n}b^n) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta \end{aligned}$$

12. Separation of variables in polar coordinates proceeds as before, but instead of periodic boundary conditions, we now have that $u(r, 0) = u(r, \pi) = 0$. Hence the Sturm-Liouville problem is

$$T'' + \lambda T = 0 \quad \text{with} \quad T(0) = T(\pi) = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = n^2 \quad \text{and} \quad T_n(\theta) = \sin n\theta \quad \text{for } n \geq 1.$$

Hence the product solutions are of the form $r^n \sin n\theta$, and the general series solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta.$$

At $r = a$ we have $u(a, \theta) = f(\theta)$ so we get the Fourier sine series for f . Hence the coefficients are given by

$$B_n = \frac{2}{\pi a^n} \int_0^{\pi} f(\theta) \sin n\theta d\theta.$$

13. Separation of variables in polar coordinates proceeds as before, but we now have that $u(r, 0) = u(r, \theta_0) = 0$. Hence the Sturm-Liouville problem is

$$T'' + \lambda T = 0 \quad \text{with} \quad T(0) = T(\theta_0) = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = n^2/\theta_0^2 \quad \text{and} \quad T_n(\theta) = \sin(n\theta/\theta_0) \quad \text{for } n \geq 1.$$

Hence the product solutions are of the form $r^{n\pi/\theta_0} \sin(n\theta/\theta_0)$, and the general series solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{n\pi/\theta_0} \sin(n\theta/\theta_0).$$

At $r = a$ we have $u(a, \theta) = f(\theta)$ so we get the Fourier sine series for f . Hence the coefficients are given by

$$B_n = \frac{2}{\theta_0 a^n} \int_0^{\theta_0} f(\theta) \sin(n\theta/\theta_0) d\theta.$$

Section 13.6. Sturm-Liouville Problems

1. The operator in (b) is $L\phi = x\phi'' + \phi' = (x\phi)'$. The operator in (e) is $L\phi = \sin x\phi'' + \cos x\phi' = (\sin x\phi)'$. The operator in (f) is $L\phi = (1-x^2)\phi'' - 2x\phi' = [(1-x^2)\phi]'$. These are therefore formally self-adjoint. The others are not.
2. This problem was solved in Section 2. According to (2.28), the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \\ \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

for $n = 0, 1, 2, 3, \dots$

3. We use Proposition 6.24 to conclude that all of the eigenvalues are positive. Setting $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A \cos \omega x + B \sin \omega x$. By the first boundary condition, $0 = \phi'(0) = \omega B$. Hence $B = 0$. Then, by the second boundary condition, $0 = \phi(1) = A \cos \omega$. Hence $\omega = \pi/2 + n\pi$ for a nonnegative integer n . We conclude that the eigenvalues and eigenfunctions are

$$\lambda_n = (\pi/2 + n\pi)^2 \quad \text{and} \\ \phi_n(x) = \cos(\pi/2 + n\pi)x$$

for $n = 0, 1, 2, 3, \dots$

4. We use Proposition 6.24 to conclude that all of the eigenvalues are positive. Setting $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A \cos \omega x + B \sin \omega x$. By the first boundary condition, $0 = \phi'(0) = \omega B$. Hence $B = 0$. Then, by the second boundary condition, $0 = \phi'(1) + \phi(1) = A[-\omega \sin \omega + \cos \omega]$. Hence, we are looking for solutions to the equation $\tan \omega = 1/\omega$. For every $n \geq 1$, there is a solution ω_n satisfying $(n-1)\pi < \omega_n < (n-1)\pi + \pi/2$. We conclude that the eigenvalues and eigenfunctions are

$$\lambda_n = \omega_n^2 \quad \text{and} \\ \phi_n(x) = \cos \omega_n x$$

for $n = 1, 2, 3, \dots$

5. We use Proposition 6.24 to conclude that all of the eigenvalues are positive. Setting $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A \cos \omega x + B \sin \omega x$. By the first boundary condition, $0 = \phi'(0) - \phi(0) = \omega B - A$. Hence $A = \omega B$, so the solution becomes $\phi(x) = B(\omega \cos \omega x + \sin \omega x)$. Then, by the second boundary condition, $0 = \phi(1) = B[\omega \cos \omega + \sin \omega]$. Hence, we are looking for solutions to the equation $\tan \omega = -1/\omega$. For every $n \geq 1$, there is a solution ω_n satisfying $n\pi - \pi/2 < \omega_n < n\pi$. We conclude that the eigenvalues and eigenfunctions are

$$\lambda_n = \omega_n^2 \quad \text{and} \\ \phi_n(x) = \omega_n \cos \omega_n x + \sin \omega_n x,$$

for $n = 1, 2, 3, \dots$

6. We need to find functions p, q , and μ such that

$$\mu[\phi'' + 4\phi' + \lambda\phi] = (p\phi')' + \lambda q\phi \\ = p\phi'' + p'\phi' + \lambda q\phi.$$

Equating the coefficients of the derivatives of ϕ , we see that $p = \mu$, $p' = 4\mu$, and $q = \mu$. From the first two equations we get the equation $p' = 4p$, which has solution $p(x) = e^{4x}$. Since $q = \mu = p$, we see that the equation becomes $-(e^{4x}\phi')' = \lambda e^{4x}\phi$.

7. We need to find functions p, q , and μ such that

$$\mu[2x\phi'' + \lambda\phi] = (p\phi')' + \lambda q\phi \\ = p\phi'' + p'\phi' + \lambda q\phi.$$

Equating the coefficients of the derivatives of ϕ , we see that $p = 2x\mu$, $p' = 0$, and $q = \mu$. From the second equation we see that p is a constant, so we set $p = 1$. Then $q = \mu = 1/2x$, so the equation becomes $-\phi'' = \lambda(1/2x)\phi$.

8. We need to find functions p, q , and μ such that

$$\mu[x(x-1)\phi'' + 2x\phi' + \lambda\phi] \\ = (p\phi')' + \lambda q\phi \\ = p\phi'' + p'\phi' + \lambda q\phi.$$

Equating the coefficients of the derivatives of ϕ , we see that $p = x(x-1)\mu$, $p' = 2x\mu$, and

$q = \mu$. From the first two equations we get the equation $p' = (2/(x-1))p$, which has solution $p(x) = (x-1)^2$. Hence $\mu = (x-1)/x$, and $q = \mu = (x-1)/x$, so the equation becomes $-(x-1)^2\phi' = \lambda[(x-1)/x]\phi$.

9. We need to find functions p , q , and μ such that

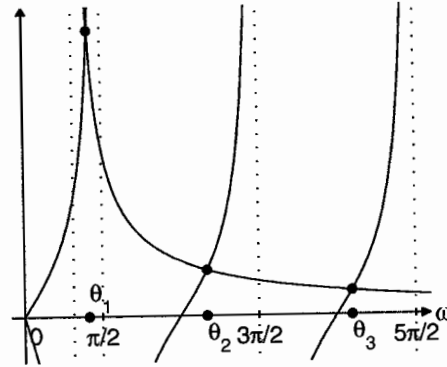
$$\begin{aligned}\mu[x^2\phi'' - 2x\phi' + \lambda\phi] &= (p\phi')' + \lambda q\phi \\ &= p\phi'' + p'\phi' + \lambda q\phi.\end{aligned}$$

Equating the coefficients of the derivatives of ϕ , we see that $p = x^2\mu$, $p' = -2x\mu$, and $q = \mu$. From the first two equations we get the equation $p' = -2x^{-1}p$, which has solution $p(x) = x^{-2}$. Hence $\mu = x^{-4}$, and $q = \mu = x^{-4}$, so the equation becomes $-(x^{-2}\phi')' = \lambda x^{-4}\phi$.

10. Notice that the boundary conditions imply that $\phi(0)\phi'(0) = \phi(0)^2$, and $\phi(1)\phi'(1) = -\phi(1)^2$. Hence $\phi\phi'|_0^1 = -[\phi(0)^2 + \phi(1)^2] \leq 0$. Therefore Proposition 6.24 implies that the eigenvalues are nonnegative. Furthermore, if $\lambda = 0$ is an eigenvalue, then the eigenfunction ϕ is a constant. Since $\phi' = 0$, the boundary conditions imply that $\phi(0) = \phi(1) = 0$, so $\phi = 0$. Thus all of the eigenvalues are positive. Setting $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A \cos \omega x + B \sin \omega x$. By the first boundary condition, $0 = \phi'(0) - \phi(0) = \omega B - A$. Hence $A = \omega B$, so the solution becomes $\phi(x) = B(\omega \cos \omega x + \sin \omega x)$. Then, by the second boundary condition, $0 = \phi'(1) + \phi(1) = B[(1 - \omega^2) \sin \omega + 2\omega \cos \omega]$. Hence, we are looking for solutions to the equation $\tan \omega = 2\omega/(\omega^2 - 1)$. The functions $\tan \omega$ and $2\omega/(\omega^2 - 1)$ are plotted in the accompanying figure. The points of intersection determine the solutions ω_n . Notice that for $0 < \omega < 1$, $\tan \omega > 0$, while $2\omega/(\omega^2 - 1) < 0$. Hence the first root satisfies $1 < \omega_1 < \pi/2$. For $n \geq 2$ we have $(n-1)\pi < \omega_n < (n-1)\pi + \pi/2$. We conclude that the eigenvalues and eigenfunctions are

$$\begin{aligned}\lambda_n &= \omega_n^2 \quad \text{and} \\ \phi_n(x) &= \omega_n \cos \omega_n x + \sin \omega_n x,\end{aligned}$$

for $n = 1, 2, 3, \dots$



11. (a) If $\phi \neq 0$ satisfies the boundary conditions we have $p\phi\phi'|_0^1 = \phi(1)\phi'(1) = a\phi(1)^2 \geq 0$.
- (b) First notice that when $\lambda = 0$, the equation is $\phi'' = 0$, which has solution $\phi(x) = Ax + B$. The first boundary condition implies that $B = \phi(0) = 0$, so $\phi(x) = Ax$. The second condition then implies that $A(1-a) = 0$. We get a nontrivial solution when $A \neq 0$, and therefore only when $a = 1$. Thus 0 is an eigenvalue only when $a = 1$. Next set $\lambda = -v^2$ where $v > 0$. The differential equation becomes $\phi'' = v^2\phi$, and has solution $\phi(x) = Ae^{vx} + Be^{-vx}$. The first boundary condition is $0 = \phi(0) = A + B$. Hence $B = -A$, and $\phi(x) = A(e^{vx} - e^{-vx}) = 2A \sinh vx$. The second boundary condition can be written as $\phi'(1) = a\phi(1)$, which becomes $2vA \cosh v = aA \sinh v$. This can be written as $v = a \tanh v$. To examine solutions, we graph the two functions v and $a \tanh v$. Since $\tanh'(0) = 1$, we see that the slope of $a \tanh v$ at $v = 0$ is a . If $a < 1$, the graph of $a \tanh v$ lies below the graph of v , so there is no solution, and therefore no negative eigenvalue. On the other hand if $a > 1$, the two graphs intersect precisely once, and the value of v for which this occurs is the lone negative eigenvalue.

12. Using the definition of L , we have

$$\begin{aligned}
 \int_a^b Lf \cdot g \, dx &= \int_a^b [-(pf')' + qf](x)g(x) \, dx \\
 &= - \int_a^b (pf')' g \, dx \\
 &\quad + \int_a^b qfg \, dx \\
 &= -pf'g \Big|_a^b + \int_a^b f'(pg') \, dx \\
 &\quad + \int_a^b qfg \, dx \\
 &= -pf'g \Big|_a^b + pf'g' \Big|_a^b \\
 &\quad - \int_a^b f(pg')' \, dx + \int_a^b qfg \, dx \\
 &= \int_a^b f \cdot Lg \, dx + p(fg' - f'g) \Big|_a^b.
 \end{aligned}$$

13. (a) Proposition 6.12 says that if L is a formally self-adjoint operator, then

$$\int_a^b Lf \cdot g \, dx = \int_a^b f \cdot Lg \, dx + p(fg' - f'g) \Big|_a^b,$$

for any two functions f and g which have two continuous derivatives. If both f and g vanish at the endpoints, then $p(fg' - f'g) \Big|_a^b = 0$, so the result follows.

- (b) Direct computation shows that

$$\begin{aligned}
 \int_0^1 Lf \cdot g \, dx &= - \int_0^1 [x^2 + x^3 - 2x^4] \, dx \\
 &= \frac{-13}{30},
 \end{aligned}$$

while

$$\begin{aligned}
 \int_a^b f \cdot Lg \, dx &= \int_0^1 [2x - 6x^2 + x^3 + 3x^4] \, dx \\
 &= \frac{-3}{20}.
 \end{aligned}$$

14. The differential equation to be solved is

$$-\phi'' = \lambda\phi \quad \text{or} \quad \phi'' + \lambda\phi = 0.$$

By Proposition 6.24 there are no negative eigenvalues. For $\lambda = 0$, the only possible eigenfunction is a constant function. Checking shows that constants are eigenfunctions. Hence we set $\lambda_0 = 0$ and it is simplest to take $c_0(x) = 1$.

If $\lambda > 0$, we can set $\lambda = \omega^2$, where $\omega > 0$. Then the differential equation is $\phi'' + \omega^2\phi = 0$. This equation has general solution $\phi(x) = C_1 \cos \omega x + C_2 \sin \omega x$. Using the fact that $\sin$ is odd and $\cos$ is even, the periodic boundary conditions yield

$$0 = B_1\phi = \phi(\pi) - \phi(-\pi) = 2C_2 \sin \omega\pi$$

$$0 = B_2\phi = \phi'(\pi) - \phi'(-\pi) = 2C_1 \sin \omega\pi$$

If $\sin \omega\pi \neq 0$ we have $C_1 = C_2 = 0$. The only solution is $\phi = 0$, so $\lambda = \omega^2$ is not an eigenvalue. However, if $\sin \omega\pi = 0$, the constants C_1 and C_2 are both arbitrary, and any solution to the differential equation is an eigenfunction. Of course $\sin \omega\pi = 0$ if and only if ω is an integer, and since $\omega > 0$ it must be a positive integer. Thus for each positive integer we get the eigenvalue $\lambda = n^2$, and for each such eigenvalue there are two linearly independent eigenfunctions $\cos nx$ and $\sin nx$. We will set $\lambda_n = n^2$, with corresponding eigenfunctions $c_n(x) = \cos nx$ and $s_n(x) = \sin nx$.

15. Substituting $u(x, t) = X(x)T(t)$ into the differential equation, we get $XT_t = k[X_{xx} - qX]T$. Dividing by kXT , we get

$$\frac{T_t}{kT} = \frac{X_{xx} - qX}{X}.$$

Since the left-hand side is a function of t and the right-hand side is a function of x , both must be constant. Therefore, there is a constant λ such that

$$T' = \lambda T \quad \text{and} \quad -X'' + qX = \lambda X.$$

At $x = 0$ we have $0 = u(0, t) = X(x)T(t)$. Hence $X(0) = 0$. At $x = L$ the boundary condition is $0 = u_t(L, t) = X'(L)T(t)$, so $X'(L) = 0$.

16. Suppose that the boundary condition at $x = a$ is unmixed, so that

$$B_1\phi = \alpha_1\phi'(a) + \beta_1\phi(a) = 0.$$

The eigenfunctions ϕ_1 and ϕ_2 both satisfy this boundary condition. Consequently α_1 and β_1 satisfy the system of equations

$$\begin{pmatrix} \phi_1'(a) & \phi_1(a) \\ \phi_2'(a) & \phi_2(a) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $(\alpha_1, \beta_1)^T$ is a nonzero vector, the matrix must have determinant equal to 0. Hence $\phi_1'(a)\phi_2(a) -$

$\phi_1(a)\phi_2'(a) = 0$. Thus the Wronskian of ϕ_1 and ϕ_2 , defined by $W = \phi_1\phi_2' - \phi_1'\phi_2$ satisfies $W(a) = 0$. Since both ϕ_1 and ϕ_2 are solutions to the differential equation (6.18), Proposition 1.25 of section 4.1 implies that $W(x) = 0$ for all x . According to Proposition 1.26 of section 4.1, this implies that the functions ϕ_1 and ϕ_2 are linearly dependent. Therefore two linearly independent eigenfunctions cannot exist.

————— × —————

Section 13.7. Orthogonality and Generalized Fourier Series

1. The eigenfunctions are $\phi_n(x) = \sin((2n+1)\pi x/2)$. We compute

$$\begin{aligned} (\phi_n, \phi_n) &= \int_0^1 \sin^2 \frac{(2n+1)\pi x}{2} dx = \frac{1}{2}, \quad \text{and} \\ (f, \phi_n) &= \int_0^1 \sin \frac{(2n+1)\pi x}{2} dx = \frac{2}{(2n+1)\pi}. \end{aligned}$$

Consequently, the coefficients are

$$c_n = \frac{4}{(2n+1)\pi},$$

and the generalized Fourier series is

$$1 = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin \frac{(2n+1)\pi x}{2}.$$

2. From Exercise 1 we know that $(\phi_n, \phi_n) = 1/2$. We compute that

$$\begin{aligned} (f, \phi_n) &= \int_0^1 \sin \pi x \cdot \sin \frac{(2n+1)\pi x}{2} dx \\ &= \frac{1}{2} \int_0^1 \cos \frac{(2n-1)\pi x}{2} \\ &\quad - \frac{1}{2} \int_0^1 \cos \frac{(2n+3)\pi x}{2} dx \\ &= (-1)^{n+1} \left[\frac{1}{\pi(2n-1)} - \frac{1}{\pi(2n+3)} \right] \\ &= (-1)^{n+1} \frac{4}{\pi(2n-1)(2n+3)}. \end{aligned}$$

The generalized Fourier series is

$$\begin{aligned} f(x) &= \sin \pi x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 8}{\pi(2n-1)(2n+3)} \sin \frac{(2n+1)\pi x}{2} \end{aligned}$$

3. From Exercise 1 we know that $(\phi_n, \phi_n) = 1/2$. Using integration by parts, we compute that

$$\begin{aligned} (f, \phi_n) &= \int_0^1 (1-x) \cdot \sin \frac{(2n+1)\pi x}{2} dx \\ &= -\frac{2(1-x)}{(2n+1)\pi} \cos \frac{(2n+1)\pi x}{2} \Big|_0^1 \\ &\quad - \frac{2}{(2n+1)\pi} \int_0^1 \cos \frac{(2n+1)\pi x}{2} dx \\ &= \frac{2}{(2n+1)\pi} + (-1)^{n+1} \frac{4}{(2n+1)^2 \pi^2}. \end{aligned}$$

The generalized Fourier series is

$$\begin{aligned} f(x) &= 1-x \\ &= \sum_{n=0}^{\infty} \left[\frac{4}{(2n+1)\pi} + (-1)^{n+1} \frac{8}{(2n+1)^2 \pi^2} \right] \\ &\quad \times \sin \frac{(2n+1)\pi x}{2}. \end{aligned}$$

4. From Exercise 1 we know that $(\phi_n, \phi_n) = 1/2$. Since

$\sin^2 x = [1 - \cos 2x]/2$, we have

$$\begin{aligned}(f, \phi_n) &= \int_0^1 \sin^2 \pi x \cdot \sin \frac{(2n+1)\pi x}{2} dx \\&= \frac{1}{2} \int_0^1 \sin \frac{(2n+1)\pi x}{2} dx \\&\quad - \frac{1}{2} \int_0^1 \cos 2\pi x \cdot \sin \frac{(2n+1)\pi x}{2} dx\end{aligned}$$

For the first integral we have

$$\int_0^1 \sin \frac{(2n+1)\pi x}{2} dx = \frac{2}{(2n+1)\pi}.$$

For the second we use the identity $\sin \alpha \cdot \cos \beta = [\sin(\alpha + \beta) + \sin(\alpha - \beta)]/2$. The second integral becomes

$$\begin{aligned}&\int_0^1 \cos 2\pi x \cdot \sin \frac{(2n+1)\pi x}{2} dx \\&= \frac{1}{2} \int_0^1 \sin \frac{(2n+5)\pi x}{2} dx \\&\quad + \frac{1}{2} \int_0^1 \sin \frac{(2n-3)\pi x}{2} dx \\&= \frac{1}{2} \cdot \frac{2}{(2n+5)\pi} + \frac{1}{2} \cdot \frac{2}{(2n-3)\pi} \\&= \frac{4n+2}{\pi(2n-3)(2n+5)}.\end{aligned}$$

Thus

$$(f, \phi_n) = \frac{-16}{\pi(2n-3)(2n+1)(2n+5)},$$

and

$$\begin{aligned}\sin^2 \pi x &= \sum_{n=1}^{\infty} \frac{-32}{\pi(2n-3)(2n+1)(2n+5)} \\&\quad \times \sin \frac{(2n+1)\pi x}{2}.\end{aligned}$$

5. The eigenfunctions are $\phi_n(x) = \sin(\theta_n x)$, where θ_n is the n th root of the equation $\tan \theta = -\theta$. This equation can also be written as $\sin \theta = -\theta \cos \theta$.

We compute

$$\begin{aligned}(\phi_n, \phi_n) &= \int_0^1 \sin^2 \theta_n x dx \\&= \frac{1}{2} \int_0^1 [1 - \cos 2\theta_n x] dx \\&= \frac{1}{2} - \frac{1}{4\theta_n} \sin 2\theta_n \\&= \frac{1}{2} \left[1 - \frac{1}{\theta_n} \sin \theta_n \cos \theta_n \right] \\&= \frac{1}{2} [1 + \cos^2 \theta_n].\end{aligned}$$

We also compute

$$(f, \phi_n) = \int_0^1 \sin \theta_n x dx = \frac{1 - \cos \theta_n}{\theta_n}.$$

Thus the coefficients are

$$c_n = \frac{2(1 - \cos \theta_n)}{\theta_n(1 + \cos^2 \theta_n)}.$$

We have

$$1 = f(x) = \sum_{n=1}^{\infty} \frac{2(1 - \cos \theta_n)}{\theta_n(1 + \cos^2 \theta_n)} \sin(\theta_n x).$$

6. We compute

$$\begin{aligned}(f, \phi_n) &= \int_0^1 \sin \pi x \cdot \sin \theta_n x dx \\&= \frac{1}{2} \int_0^1 [\cos(\theta_n - \pi)x - \cos(\theta_n + \pi)x] dx \\&= \frac{\pi \sin \theta_n}{\pi^2 - \theta_n^2}.\end{aligned}$$

Therefore the coefficients are

$$c_n = \frac{2\pi \sin \theta_n}{(\pi^2 - \theta_n^2)(1 + \cos^2 \theta_n)},$$

and

$$\sin \pi x = \sum_{n=1}^{\infty} \frac{2\pi \sin \theta_n \cdot \sin \theta_n x}{(\pi^2 - \theta_n^2)(1 + \cos^2 \theta_n)}.$$

7. Integrating by parts, we get

$$\begin{aligned}(f, \phi_n) &= \int_0^1 (1-x) \sin \theta_n x \, dx \\ &= -\frac{1}{\theta_n} (1-x) \cos \theta_n x \Big|_0^1 \\ &\quad + \frac{1}{\theta_n} \int_0^1 \cos \theta_n x \, dx \\ &= \frac{\theta_n - \sin \theta_n}{\theta_n^2}.\end{aligned}$$

Using the fact that $\sin \theta_n = -\theta_n \cos \theta_n$, we see that the coefficients are

$$c_n = \frac{2(\theta_n - \sin \theta_n)}{\theta_n^2(1 + \cos^2 \theta_n)} = \frac{2(1 + \cos \theta_n)}{\theta_n(1 + \cos^2 \theta_n)},$$

and

$$1-x = \sum_{n=1}^{\infty} \frac{2(1 + \cos \theta_n)}{\theta_n(1 + \cos^2 \theta_n)} \sin \theta_n x.$$

8. Since $\sin^2 \alpha = [1 - \cos 2\alpha]/2$, and $\cos \alpha \cdot \sin \beta =$

$$\frac{\sin(\beta + \alpha) + \sin(\beta - \alpha)}{2},$$

$$[\sin(\beta + \alpha) + \sin(\beta - \alpha)] / 2,$$

$$\begin{aligned}(f, \phi_n) &= \frac{1}{2} \int_0^1 \sin^2 \pi x \cdot \sin \theta_n x \, dx \\ &= \frac{1}{2} \int_0^1 \sin \theta_n x \, dx \\ &\quad - \frac{1}{4} \int_0^1 \sin(\theta_n + 2\pi)x \, dx \\ &\quad - \frac{1}{4} \int_0^1 \sin(\theta_n - 2\pi)x \, dx \\ &= \frac{1 - \cos \theta_n}{2} \left[\frac{1}{\theta_n} - \frac{1}{2(\theta_n + 2\pi)} \right. \\ &\quad \left. - \frac{1}{2(\theta_n - 2\pi)} \right] \\ &= \frac{2\pi^2(1 - \cos \theta_n)}{\theta_n(4\pi^2 - \theta_n^2)}.\end{aligned}$$

Hence

$$\sin^2 \pi x = \sum_{n=1}^{\infty} \frac{4\pi^2(1 - \cos \theta_n)}{\theta_n(4\pi^2 - \theta_n^2)(1 + \cos^2 \theta_n)} \sin \theta_n x.$$

9. The solution in general is given in 7.22. For $f(x) = 1$ we computed the generalized Fourier coefficients in Exercise 1. The solution is

$$\begin{aligned}u(t, x) &= \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} e^{-(2n+1)^2 \pi^2 t/4} \\ &\quad \times \sin \frac{(2n+1)\pi x}{2}.\end{aligned}$$

10. The solution in general is given in 7.22. For $f(x) = \sin \pi s$ we computed the generalized Fourier coefficients in Exercise 2. The solution is

$$u(t, x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{8}{\pi(2n-1)(2n+3)} e^{-(2n+1)^2 \pi^2 t/4} \sin \frac{(2n+1)\pi x}{2}.$$

11. The solution in general is given in 7.22. For $f(x) = 1-x$ we computed the generalized Fourier coefficients in Exercise 3. The solution is

$$u(t, x) = \sum_{n=0}^{\infty} \left[\frac{4}{(2n+1)\pi} + (-1)^{n+1} \frac{8}{(2n+1)^2 \pi^2} \right] e^{-(2n+1)^2 \pi^2 t/4} \sin \frac{(2n+1)\pi x}{2}.$$

12. The solution in general is given in 7.22. For $f(x) = \sin^2 \pi x$ we computed the generalized Fourier coefficients in Exercise 4. The solution is

$$u(t, x) = \sum_{n=0}^{\infty} \frac{-32}{\pi(2n-3)(2n+1)(2n+5)} e^{-(2n+1)^2 \pi^2 t/4} \sin \frac{(2n+1)\pi x}{2}.$$

13. The solution to the initial/boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin \frac{\theta_n x}{L},$$

where $\lambda_n = \theta_n^2$. For $f(x) = 1$ we computed the generalized Fourier coefficients in Exercise 5. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} \frac{2(1 - \cos \theta_n)}{\theta_n(1 + \cos^2 \theta_n)} e^{-\theta_n^2 t} \sin \theta_n x.$$

14. The solution to the initial/boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin \frac{\theta_n x}{L},$$

where $\lambda_n = \theta_n^2$. For $f(x) = \sin \pi x$ we computed the generalized Fourier coefficients in Exercise 6. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} \frac{2\pi \sin \theta_n}{(\pi^2 - \theta_n^2)(1 + \cos^2 \theta_n)} e^{-\theta_n^2 t} \sin \theta_n x.$$

15. The solution to the initial/boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin \frac{\theta_n x}{L},$$

where $\lambda_n = \theta_n^2$. For $f(x) = 1 - x$ we computed the generalized Fourier coefficients in Exercise 7. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} \frac{2(1 + \cos \theta_n)}{\theta_n(1 + \cos^2 \theta_n)} e^{-\theta_n^2 t} \sin \theta_n x.$$

16. The solution to the initial/boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin \frac{\theta_n x}{L},$$

where $\lambda_n = \theta_n^2$. For $f(x) = \sin^2 \pi x$ we computed the generalized Fourier coefficients in Exercise 8. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} \frac{4\pi^2(1 - \cos \theta_n)}{\theta_n(4\pi^2 - \theta_n^2)(1 + \cos^2 \theta_n)} e^{-\theta_n^2 t} \sin \theta_n x.$$

17. We look for product solutions of the form $u(x, y) = X(x)Y(y)$ which satisfy the boundary conditions in the variable x . Separating the variables in the differential equation

$$\nabla^2 u = X''Y + XY'' = 0,$$

we get the ordinary differential equations

$$-X'' = \lambda X \quad \text{and} \quad Y'' - \lambda Y = 0,$$

where λ is a constant. Invoking the boundary conditions, we see that X must be a solution to the Sturm Liouville problem

$$-X'' = \lambda X \quad \text{with} \quad X(0) = X'(1) = 0.$$

This is the Sturm Liouville problem in Example 6.26. The solutions are

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4} \quad \text{and} \quad \phi_n(x) = \sin \frac{(2n+1)\pi x}{2} \quad \text{for } n = 0, 1, 2, \dots$$

In analogy with equation 4.11, for each positive integer n , we get two product solutions

$$\begin{aligned} u_n(x, y) &= \sinh \left(\frac{(2n+1)\pi y}{2} \right) \sin \left(\frac{(2n+1)\pi x}{2} \right) \quad \text{and} \\ v_n(x, y) &= \sinh \left(\frac{(2n+1)\pi(y-1)}{2} \right) \sin \left(\frac{(2n+1)\pi x}{2} \right) \end{aligned}$$

Thus we get the general solution

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} a_n u_n(x, y) + \sum_{n=1}^{\infty} b_n v_n(x, y) \\ &= \sum_{n=1}^{\infty} a_n \sinh \left(\frac{(2n+1)\pi y}{2} \right) \sin \left(\frac{(2n+1)\pi x}{2} \right) \\ &\quad + \sum_{n=1}^{\infty} b_n \sinh \left(\frac{(2n+1)\pi(y-1)}{2} \right) \sin \left(\frac{(2n+1)\pi x}{2} \right), \end{aligned}$$

For the boundary condition at $y = 0$ we have the generalized Fourier series

$$T_1 = u(x, 0) = - \sum_{n=1}^{\infty} b_n \sinh \left(\frac{(2n+1)\pi}{2} \right) \sin \left(\frac{(2n+1)\pi x}{2} \right).$$

The coefficients are given by

$$-b_n \sinh \left(\frac{(2n+1)\pi}{2} \right) = \frac{(T_1, X_n)}{(X_n, X_n)} = \frac{4T_1}{(2n+1)\pi},$$

or

$$b_n = \frac{-4T_1}{(2n+1)\pi \sinh((2n+1)\pi/2)}.$$

Similarly

$$a_n = \frac{4T_2}{(2n+1)\pi \sinh((2n+1)\pi/2)}.$$

Hence the temperature is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4}{(2n+1)\pi \sinh((2n+1)\pi/2)} \sin\left(\frac{(2n+1)\pi x}{2}\right) \cdot \left[T_2 \sinh\left(\frac{(2n+1)\pi y}{2}\right) - T_1 \sinh\left(\frac{(2n+1)\pi(y-1)}{2}\right) \right].$$

————— × —————

Section 13.8. Temperatures in a Ball—Legendre Polynomials

1. We have the boundary temperature $f(z) = T$. If a is the radius of the sphere, we have $f(as) = T = T P_0(s)$. Hence $u(r, \phi) = T P_0(\cos \phi) = T$.
2. We have the boundary temperature $f(z) = 1 - z$. Since $f(s) = 1 - s = P_0(s) - P_1(s)$, the temperature in the sphere is $u(r, \phi) = P_0(\cos \phi) + r P_1(\cos \phi) = 1 - r \cos \phi$. Since $z = r \cos \phi$, the formula for the temperature in cartesian coordinates is $u(x, y, z) = 1 - z$.
3. We have $f(s) = s^3 = [2P_3(s) + 3P_1(s)]/5$. Therefore the temperature is

$$u(r, \phi) = \frac{2}{5}r^3 P_3(\cos \phi) + \frac{3}{5}r P_1(\cos \phi) = \frac{1}{5}r^3 [5 \cos^3 \phi - 3 \cos \phi] + \frac{3}{5}r \cos \phi.$$

When put into cartesian coordinates, this becomes

$$u(x, y, z) = z^3 + \frac{3}{5}z(1 - x^2 - y^2 - z^2).$$

4. We have $f(s) = s^4 = [8P_4(s) + 20P_2(s) + 7P_0(s)]/35$. Therefore the temperature is

$$\begin{aligned} u(r, \phi) &= \frac{1}{35} [8r^4 P_4(\cos \phi) + 20r^2 P_2(\cos \phi) + 7] \\ &= \frac{1}{35} [r^4 (35 \cos^4 \phi - 30 \cos^2 \phi + 3) + 10r^2 (3 \cos^2 \phi - 1) + 7]. \end{aligned}$$

When put into cartesian coordinates, this becomes

$$u(x, y, z) = z^4 + \frac{1}{35} [30z^2(1 - x^2 - y^2 - z^2) - 7(x^2 - y^2 - z^2) + 7].$$

5. We have

$$f(s) = \begin{cases} 10, & \text{for } s > 0, \text{ and} \\ 0, & \text{for } s \leq 0. \end{cases}$$

The Legendre coefficients for f are given by

$$c_n = \frac{2n+1}{2} \int_0^1 P_n(s) ds.$$

Evaluating explicitly we get

$$\begin{aligned} c_0 &= \frac{1}{2} \int_0^1 10 ds = 5, & c_1 &= \frac{3}{2} \int_0^1 10s ds = \frac{15}{2}, \\ c_2 &= \frac{5}{2} \int_0^1 10 \frac{3s^2 - 1}{2} ds = 0, & c_3 &= \frac{7}{2} \int_0^1 10 \frac{5s^3 - 3s}{2} ds = -\frac{35}{8}. \end{aligned}$$

Hence the temperature is given approximately by

$$\begin{aligned} u(r, \phi) &= \frac{1}{2} + \frac{3}{4} r \cos \phi - \frac{7}{16} r^3 \frac{5 \cos^3 \phi - 3 \cos \phi}{2} + \dots \\ &= \frac{1}{2} + \frac{3}{4} z - \frac{35}{32} z^3 + \frac{21}{32} z(x^2 + y^2 + z^2) + \dots \end{aligned}$$

6. According to (8.9), the solution is $u(r, \phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos \phi)$, where the coefficients are given by (8.11). Using the hint and the fact that $P_n(1) = 1$, we get

$$\begin{aligned} n \int_0^1 P_n(s) ds &= \int_0^1 s P'_n(s) ds - \int_0^1 P'_{n-1}(s) ds \\ &= 1 - \int_0^1 P_n(s) ds - [1 - P_{n-1}(0)]. \end{aligned}$$

Collecting the terms involving the integral of P_n on the left, we see that

$$(n+1) \int_0^1 P_n(s) ds = P_{n-1}(0).$$

We get

$$\begin{aligned} \int_0^1 P_{2n}(s) ds &= 0 \quad \text{for } n \geq 1, \text{ and} \\ \int_0^1 P_{2n+1}(s) ds &= (-1)^n \frac{(2n)!}{2^{2n+2}(n+1)(n!)^2} \quad \text{for } n \geq 0. \end{aligned}$$

Then we have $c_0 = 10/2 = 5$, $c_{2n} = 0$ for $n \geq 1$, and

$$c_{2n+1} = (-1)^n \frac{10 \cdot (4n+3)(2n)!}{2^{2n+2}(n+1)(n!)^2} \quad \text{for } n \geq 0.$$

Therefore

$$u(r, \phi) = 5 + 10 \sum_{n=0}^{\infty} (-1)^n \frac{(4n+3)(2n)!}{2^{2n+2}(n+1)(n!)^2} r^{2n+1} P_{2n+1}(\cos \phi).$$

7. The Legendre coefficients for f are given by

$$c_n = \frac{2n+1}{2} \int_0^1 s P_n(s) ds.$$

Evaluating explicitly we get

$$\begin{aligned} c_0 &= \frac{1}{2} \int_0^1 s ds = \frac{1}{4}, & c_1 &= \frac{3}{2} \int_0^1 s \cdot s ds = \frac{1}{2}, \\ c_2 &= \frac{5}{2} \int_0^1 s \cdot \frac{3s^2-1}{2} ds = \frac{5}{16}, & c_3 &= \frac{7}{2} \int_0^1 s \cdot \frac{5s^3-3s}{2} ds = 0. \end{aligned}$$

8. According to (8.9), the solution is $u(r, \phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos \phi)$, where the coefficients are given by (8.11). Using the hint and the fact that $P_n(1) = 1$, we get

$$\begin{aligned} n \int_0^1 s P_n(s) ds &= \int_0^1 s^2 P'_n(s) ds - \int_0^1 s P'_{n-1}(s) ds \\ &= 1 - 2 \int_0^1 s P_n(s) ds - 1 + \int_0^1 P_{n-1}(s) ds. \end{aligned}$$

Collecting the terms involving the integral of P_n on the left, we see that

$$(n+2) \int_0^1 P_n(s) ds = \int_0^1 P_{n-1}(s) ds.$$

This last integral was computed in Exercise 6, and we get

$$\int_0^1 P_n(s) ds = \frac{1}{n(n+2)} P_{n-2}(0) \quad \text{for } n \geq 2.$$

We get

$$\begin{aligned} \int_0^1 s P_{2n+1}(s) ds &= 0 \quad \text{for } n \geq 1, \text{ and} \\ \int_0^1 s P_{2n}(s) ds &= (-1)^{n-1} \frac{(2n-2)!}{2^{2n} n(n+1)[(n-1)!]^2} \quad \text{for } n \geq 1. \end{aligned}$$

Then we have $c_0 = 1/4$, $c_1 = 1/2$, $c_{2n+1} = 0$ for $n \geq 1$, and

$$c_{2n} = (-1)^{n-1} \frac{(4n+1)(2n-2)!}{2^{2n+1} n(n+1)[(n-1)!]^2} \quad \text{for } n \geq 1.$$

Therefore

$$u(r, \phi) = \frac{1}{4} + \frac{1}{2} r P_1(\cos \phi) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(4n+1)(2n-2)!}{2^{2n+1} n(n+1)[(n-1)!]^2} r^{2n} P_{2n}(\cos \phi).$$

9. Consider the temperature v of the ball with the bottom half at 10° and the top half at 0° . Then $u + v = 100$ on the boundary, so $u + v = 100$ throughout the ball. In addition, by symmetry, $u = v$ at the center of the ball. Therefore $u = 5^\circ$ at the center.

10. Since the boundary temperature is independent of the angle variables, we can assume that the temperature is a function of the radius only. Hence $u = u(r)$. Laplace's equation for such a function is

$$0 = \nabla^2 u = \frac{1}{r^2}(r^2 u')' = 0.$$

We conclude that $u(r) = Ar^{-1} + B$, for suitable constants A and B . The boundary conditions reduce to $A + B = 0$ and $A/2 + B = 10$. Solving, we find that

$$u(r) = 20 \left(1 - \frac{1}{r} \right).$$

————— × —————

Section 13.9. The Heat and Wave Equations in Higher Dimension

1. If $u(t, x, y) = T(t)\phi(x, y)$, then the equation $u_t = \nabla^2 u$ becomes $T_t\phi = T\nabla^2\phi$. Separating variables we get

$$\frac{T_t}{T} = \frac{\nabla^2\phi}{\phi}.$$

Since the left-hand side depends only on t , while the right-hand side depends on (x, y) , we conclude that both are equal to the same constant, which we write as $-\lambda$. Both differential equations follow. Since $0 = u(t, x, 0) = T(t)\phi(x, 0)$, we conclude that $\phi(x, 0) = 0$. The other boundary conditions follow in the same way.

2. We look for product solutions of the form $\phi(x, y) = X(x)Y(y)$. The differential equation becomes $-X_{xx}Y - XY_{yy} = \lambda XY$. Dividing by XY we get

$$-\frac{X_{xx}}{X} - \frac{Y_{yy}}{Y} = \lambda.$$

Each summand must be constant, so we conclude that there are constants μ and ν such that

$$-X_{xx} = \mu X \quad \text{and} \quad -Y_{yy} = \nu Y, \quad \text{with} \quad \mu + \nu = \lambda.$$

The boundary conditions for ϕ translate into

$$X(0) = X'(1) = 0 \quad \text{and} \quad Y(0) = Y(1) = 0.$$

Thus we have two Sturm Liouville problems. The problem for Y we solved in Section 2. The solutions are

$$\nu_q = q^2\pi^2 \quad \text{with} \quad Y_q(y) = \sin q\pi y, \quad \text{for } q \geq 1.$$

The problem for X we solved in Section 6. The solutions are

$$\mu_p = \frac{(2p+1)^2\pi^2}{4} \quad \text{with} \quad X_p(x) = \sin\left(\frac{(2p+1)\pi x}{2}\right), \quad \text{for } p \geq 0.$$

Thus the eigenvalues are $\lambda_{p,q} = \mu_p + \nu_q$, and the eigenfunctions are $\phi_{p,q}(x, y) = X_p(x)Y_q(y)$.

3.

$$\begin{aligned} & \int_D \phi_{p,q}(x, y) \phi_{p',q'}(x, y) dx dy \\ &= \int_0^1 \sin\left(\frac{(2p+1)\pi x}{2}\right) \sin\left(\frac{(2p'+1)\pi x}{2}\right) dx \int_0^1 \sin(q\pi y) \sin(q'\pi y) dy \end{aligned}$$

The first integral is equal to 0 if $p' \neq p$ and equal to $1/2$ if $p' = p$. The second integral is equal to 0 if $q' \neq q$ and equal to $1/2$ if $q' = q$. The result follows from this.

4. With $\lambda = \lambda_{p,q}$, the equation for T in Exercise 1 has solution $T_{p,q}(t) = e^{-\lambda_{p,q}t}$. Hence we have the product solutions $u_{p,q}(t, x, y) = T_{p,q}(t)\phi_{p,q}(x, y)$ to the heat equation that also satisfy the Dirichlet boundary conditions. Hence, the series

$$u(t, x, y) = \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{p,q} e^{-\lambda_{p,q}t} \phi_{p,q}(x, y),$$

is a solution, provided that it converges.

The initial condition is

$$f(x, y) = u(0, x, y) = \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{p,q} \phi_{p,q}(x, y).$$

If this is true, then multiplying f by $\phi_{p',q'}$ and integrating over D , we get, using the orthogonality relations in Exercise 3,

$$\begin{aligned} \int_D f \phi_{p',q'} dx dy &= \int_D \phi_{p',q'} \left(\sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{p,q} \phi_{p,q} \right) dx dy \\ &= \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{p,q} \int_D \phi_{p',q'} \phi_{p,q} dx dy \\ &= c_{p',q'} \cdot \frac{1}{4}. \end{aligned}$$

The result follows.

5. It is necessary to start by finding a steady-state temperature u_s that solves the boundary value problem

$$\begin{aligned} \nabla^2 u_s &= 0, \quad \text{in } D, \\ u_s(x, 0) &= u_s(x, 1) = T_1, \quad \text{and} \quad u_s(0, y) = (u_s)_x(1, y) = 0. \end{aligned}$$

Then the function $v(t, x, y) = u(t, x, y) - u_s(x, y)$ must be found to solve the initial/boundary value problem

$$\begin{aligned} v_t &= \nabla^2 v, \\ v(t, x, 0) &= v(t, x, 1) = T_1, \quad \text{and} \quad v(t, 0, y) = v_x(t, 1, y) = 0 \\ v(0, x, y) &= f(x, y) - u_s(x, y), \end{aligned}$$

which is a variant of the problem in these exercises. Then the solution is $u(t, x, y) = u_s(x, y) + v(t, x, y)$

6. In this case the eigenvalue problem for the Laplacian is modified to $-\nabla^2 \phi = \lambda \phi$ with the boundary conditions

$$\phi(x, 0) = \phi(x, 1) = 0, \quad \text{and} \quad \phi_x(0, y) = \phi_x(1, y) = 0.$$

When we separate variables on this problem we get the two Sturm Liouville problems

$$\begin{aligned} -X'' &= \mu X & \text{with } X'(0) &= X'(1) = 0, & \text{and} \\ -Y'' &= \nu Y & \text{with } Y(0) &= Y(1) = 0. \end{aligned}$$

We solved both of these in Section 2, getting

$$\begin{aligned} \mu_p &= p^2\pi^2 & \text{with } X_p(x) &= \cos p\pi x, & \text{for } p \geq 0, \text{ and} \\ \nu_q &= q^2\pi^2 & \text{with } Y_q(y) &= \sin q\pi y, & \text{for } q \geq 1. \end{aligned}$$

Thus the solutions to the eigenvalue problem are

$$\lambda_{p,q} = (p^2 + q^2)\pi^2 \quad \text{with} \quad \phi_{p,q}(x, y) = \cos(p\pi x) \sin(q\pi y), \quad \text{for } p \geq 0 \text{ and } q \geq 1.$$

Everything else is the same.

7. (a) We check the four cases. Remember that $0 < x < \pi$ and $0 < y < \pi$.

$$0 < x < \min\{y, \pi - x, \pi - y\} \Rightarrow \begin{cases} -\pi < x - y < 0 & \text{so } F(x - y) = \pi - y + x \\ 0 < x < \pi/2 \\ 0 < x + y < \pi & \text{so } F(x + y) = \pi - x - y \end{cases}$$

Therefore, $F(x - y) - F(x + y) = 2x$.

$$0 < y < \min\{x, \pi - x, \pi - y\} \Rightarrow \begin{cases} 0 < x - y < \pi & \text{so } F(x - y) = \pi - x + y \\ 0 < x + y < \pi & \text{so } F(x + y) = \pi - x - y \\ 0 < y < \pi/2 \end{cases}$$

Therefore, $F(x - y) - F(x + y) = 2y$.

$$0 < \pi - x < \min\{x, y, \pi - y\} \Rightarrow \begin{cases} x > \pi/2 \\ x + y > \pi \\ -\pi < x - y < 0 & \text{so } F(x - y) = \pi - y + x \end{cases}$$

The second inequality and the periodicity of F implies that $F(x + y) = F(x + y - 2\pi) = \pi - (2\pi - x - y) = x + y - \pi$. Therefore, $F(x - y) - F(x + y) = 2(\pi - x)$.

$$0 < \pi - y < \min\{x, y, \pi - x\} \Rightarrow \begin{cases} x + y > \pi \\ y > \pi/2 \\ -\pi < x - y < 0 & \text{so } F(x - y) = \pi - y + x \end{cases}$$

The first inequality and the periodicity of F implies that $F(x + y) = F(x + y - 2\pi) = \pi - (2\pi - x - y) = x + y - \pi$. Therefore, $F(x - y) - F(x + y) = 2(\pi - y)$.

- (b) F is an even function, so its Fourier series is in the cosine series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz,$$

where the coefficients are $a_0 = \pi$, and

$$a_n = \frac{2}{\pi} \int_0^\pi (\pi - z) \cos nz \, dz = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 4/(n^2\pi), & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$F(z) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)z}{(2n+1)^2}.$$

(c) The addition formula for the cosine is $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. Hence

$$F(x-y) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x \cdot \cos(2n+1)y + \sin(2n+1)x \cdot \sin(2n+1)y}{(2n+1)^2}, \quad \text{and}$$

$$F(x+y) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x \cdot \cos(2n+1)y - \sin(2n+1)x \cdot \sin(2n+1)y}{(2n+1)^2}.$$

Hence

$$\begin{aligned} f(x, y) &= [F(x-y) - F(x+y)]/2 \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n+1)x \cdot \sin(2n+1)y}{(2n+1)^2}. \end{aligned}$$

(d) Notice that only the diagonal terms $p = q = 2n + 1$ occur. The eigenvalues are $\lambda_{2n+1, 2n+1} = 2(2n+1)^2$. Hence,

$$u(t, x, y) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n+1)x \cdot \sin(2n+1)y \cdot \cos \sqrt{2}c(2n+1)t}{(2n+1)^2}.$$

The time frequencies are $(2n+1) \cdot \sqrt{2}ct$. Since they are all integer multiples of the fundamental frequency $\sqrt{2}ct$, the vibration is periodic in time with period $2\pi/[\sqrt{2}ct]$.

8. By computation with the chain rule, we see that

$$\begin{aligned} u_{xx}(t, x, y) &= u_{yy}(t, x, y) \\ &= \frac{1}{4} [F''(x-y+\sqrt{2}ct) + F''(x-y-\sqrt{2}ct) \\ &\quad - F''(x+y+\sqrt{2}ct) - F''(x+y-\sqrt{2}ct)], \end{aligned}$$

while

$$\begin{aligned} u_{tt}(t, x, y) &= 2c^2 \cdot \frac{1}{4} [F''(x-y+\sqrt{2}ct) + F''(x-y-\sqrt{2}ct) \\ &\quad - F''(x+y+\sqrt{2}ct) - F''(x+y-\sqrt{2}ct)]. \end{aligned}$$

Hence $u_{tt} = c^2[u_{xx} + u_{yy}]$. Setting $t = 0$, we see that

$$\begin{aligned} u(0, x, y) &= \frac{1}{4} [F(x-y) + F(x-y) - F(x+y) - F(x+y)] \\ &= f(x, y). \end{aligned}$$

Differentiating u with respect to t , and evaluating at $t = 0$, we get

$$\begin{aligned} u_t(0, x, y) &= \frac{\sqrt{2}c}{4}[F'(x-y) - F'(x-y) - F'(x+y) + F'(x+y)] \\ &= 0. \end{aligned}$$

For the boundary conditions, first setting $x = 0$,

$$\begin{aligned} u(t, 0, y) &= \frac{1}{4}[F(-y + \sqrt{2}ct) + F(-y - \sqrt{2}ct) - F(y + \sqrt{2}ct) - F(y - \sqrt{2}ct)] \\ &= 0, \end{aligned}$$

since F is an even function. Next setting $x = \pi$,

$$\begin{aligned} u(t, \pi, y) &= \frac{1}{4}[F(\pi - y + \sqrt{2}ct) + F(\pi - y - \sqrt{2}ct) \\ &\quad - F(\pi + y + \sqrt{2}ct) - F(\pi + y - \sqrt{2}ct)] \\ &= \frac{1}{4}[F(\pi - y + \sqrt{2}ct) + F(\pi - y - \sqrt{2}ct) \\ &\quad - F(-\pi + y + \sqrt{2}ct) - F(-\pi + y - \sqrt{2}ct)] \\ &= 0, \end{aligned}$$

since F is 2π -periodic and even. The other boundary conditions are verified in a similar way.

—————×—————

Section 13.10. Domains with Circular Symmetry—Bessel Functions

- Using polar coordinates, we have

$$\begin{aligned} \int_D \phi_{n,k} \phi_{n',k'} dx dy &= \int_D \cos n\theta \cdot J_n(\alpha_{n,k}r/a) \cdot \cos n'\theta \cdot J_{n'}(\alpha_{n',k'}r/a) dx dy \\ &= \int_0^{2\pi} \cos n\theta \cdot \cos n'\theta d\theta \cdot \int_0^a J_n(\alpha_{n,k}r/a) \cdot J_{n'}(\alpha_{n',k'}r/a) r dr. \end{aligned}$$

If $n \neq n'$, the first integral is equal to 0. If $k \neq k'$, the second integral is equal to 0. If $n = n'$, the first integral is equal to π , and if $k = k'$ the second is equal to $a^2 J_{n+1}^2(\alpha_{n,k})/2$, so the first orthogonality relation is verified. Similarly,

$$\begin{aligned} \int_D \psi_{n,k} \psi_{n',k'} dx dy &= \int_D \sin n\theta \cdot J_n(\alpha_{n,k}r/a) \cdot \sin n'\theta \cdot J_{n'}(\alpha_{n',k'}r/a) dx dy \\ &= \int_0^{2\pi} \sin n\theta \cdot \sin n'\theta d\theta \cdot \int_0^a J_n(\alpha_{n,k}r/a) \cdot J_{n'}(\alpha_{n',k'}r/a) r dr. \end{aligned}$$

Again, if $n \neq n'$, the first integral is equal to 0, and if $k \neq k'$, the second integral is equal to 0. If $n = n'$, the first integral is equal to π , and if $k = k'$ the second is equal to $a^2 J_{n+1}^2(\alpha_{n,k})/2$, so the second orthogonality relation is verified. Finally

$$\begin{aligned} \int_D \phi_{n,k} \psi_{n',k'} dx dy &= \int_D \cos n\theta \cdot J_n(\alpha_{n,k}r/a) \cdot \sin n'\theta \cdot J_{n'}(\alpha_{n',k'}r/a) dx dy \\ &= \int_0^{2\pi} \cos n\theta \cdot \sin n'\theta d\theta \cdot \int_0^a J_n(\alpha_{n,k}r/a) \cdot J_{n'}(\alpha_{n',k'}r/a) r dr. \end{aligned}$$

The first integral is equal to 0, so we are through.

2. If we multiply the series for f_0 by $\phi_{p,q}$ and integrate term by term, we get

$$\begin{aligned} \int_D f_0(x, y) \phi_{p,q}(x, y) dx dy &= \sum_{k=1}^{\infty} a_{0,k} \int_0^a \int_0^{2\pi} J_0\left(\frac{\alpha_{0,k}r}{a}\right) \cdot \cos p\theta \cdot J_p(\alpha_{p,q}r/a) d\theta r dr \\ &\quad + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_0^a \int_0^{2\pi} J_n\left(\frac{\alpha_{n,k}r}{a}\right) [a_{n,k} \cos n\theta + b_{n,k} \sin n\theta] \cdot \cos p\theta \cdot J_p(\alpha_{p,q}r/a) d\theta r dr. \end{aligned}$$

By the orthogonality relations, each summand is equal to 0, except when $n = p$ and $k = q$, and we get

$$\int_D f_0(x, y) \phi_{p,q}(x, y) dx dy = a_{p,q} \frac{\pi a^2 J_{p+1}^2(\alpha_{p,q})}{2}.$$

Thus

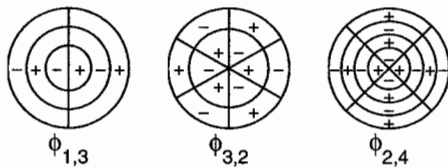
$$a_{p,q} = \frac{2}{\pi a^2 J_{p+1}^2(\alpha_{p,q})} \int_D f_0(x, y) \phi_{p,q}(x, y) dx dy.$$

The argument for $b_{p,q}$ is similar. Differentiating u with respect to t , and evaluating at $t = 0$, we get

$$f_1(r, \theta) = \frac{c\alpha_{n,k}}{a} \left\{ \sum_{k=1}^{\infty} c_{0,k} J_0\left(\frac{\alpha_{0,k}r}{a}\right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k}r}{a}\right) [c_{n,k} \cos n\theta + d_{n,k} \sin n\theta] \right\}.$$

Then the same argument as before yields the formulas for $c_{p,q}$ and $d_{p,q}$.

- 3.



4. The coefficients as given in (10.15) reduce to

$$\begin{aligned} a_{n,k} &= \frac{2}{\pi a^2 J_{n+1}(\alpha_{n,k})} \int_0^a \int_0^{2\pi} f(r) J_n\left(\frac{\alpha_{n,k} r}{a}\right) \cos n\theta r dr d\theta \\ &= \int_0^a f(r) J_n\left(\frac{\alpha_{n,k} r}{a}\right) r dr \cdot \int_0^{2\pi} \cos n\theta d\theta. \end{aligned}$$

The last integral is equal to 0 if $n \geq 1$, so $a_{n,k} = 0$ for $n \geq 1$. Similarly, $b_{n,k} = 0$ for $n \geq 1$. Thus the series for u reduces to

$$u(t, r, \theta) = \sum_{k=1}^{\infty} J_0\left(\frac{\alpha_{0,k} r}{a}\right) \left[a_{0,k} \cos \frac{c\alpha_{0,k} t}{a} + b_{0,k} \sin \frac{c\alpha_{0,k} t}{a} \right].$$

5. (a) The Laplacian of u in polar coordinates is $\nabla^2 u = u_{rr} + u_r/r + u_{\theta\theta}/r^2$. Since u does not depend on θ , this reduces to $\nabla^2 u = u_{rr} + u_r/r$.
 (b) If we substitute $u(t, r) = T(t)R(r)$ into the heat equation, we get $RT_t = [R_{rr} + R_r/r]T$. When we separate variables, we get the two equations

$$T' = -\lambda k T \quad \text{and} \quad -[R_{rr} + \frac{1}{r}R_r] = \lambda R.$$

The first equation has solution $T(t) = e^{-\lambda k t}$. If we multiply the second equation by r it becomes $-(rR')' = \lambda rR$. Thus we are left with the Sturm Liouville problem in (10.7) with $n = 0$. The solutions are given in (10.8):

$$\lambda_p = \frac{\alpha_{0,p}^2}{a^2} \quad \text{and} \quad R_p(r) = J_0(\alpha_{0,p}r/a), \quad \text{for } p = 1, 2, \dots$$

Hence the product solutions are

$$e^{-k\alpha_{0,p}^2 t/a^2} J_0(\alpha_{0,p}r/a), \quad \text{for } p = 1, 2, \dots$$

- (c) The solution is

$$u(t, r) = \sum_{p=1}^{\infty} c_p e^{-k\alpha_{0,p}^2 t/a^2} J_0(\alpha_{0,p}r/a),$$

where

$$f(r) = u(0, r) = \sum_{p=1}^{\infty} c_p J_0(\alpha_{0,p}r/a)$$

is the Bessel series for f . The coefficients are computed in (10.10):

$$c_p = \frac{2}{a^2 J_1^2(\alpha_{0,p})} \int_0^a f(r) J_0(\alpha_{0,p}r/a) r dr.$$

6. We follow the procedure in text, and look for product solutions of the form $u(t, x, y) = T(t)\phi(x, y)$. When we substitute into the heat equation and separate variables, we see that there is a constant λ such that $T' = -\lambda k T$,

and ϕ is a solution to the eigenvalue problem in (10.1). The solutions are in (10.11). Since $T(t) = e^{-k\lambda t}$, where λ is the appropriate eigenvalue, we see that the solution is

$$u(t, r, \theta) = \sum_{p=1}^{\infty} a_{0,p} e^{-k\alpha_{0,p}^2 t/a^2} J_0\left(\frac{\alpha_{0,p} r}{a}\right) + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} [a_{n,p} \cos n\theta + b_{n,p} \sin n\theta] e^{-k\alpha_{n,p}^2 t/a^2} J_n\left(\frac{\alpha_{n,p} r}{a}\right).$$

The coefficients are found by using the initial condition:

$$f(r, \theta) = u(0, r, \theta) = \sum_{p=1}^{\infty} a_{0,p} J_0\left(\frac{\alpha_{0,p} r}{a}\right) + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} [a_{n,p} \cos n\theta + b_{n,p} \sin n\theta] J_n\left(\frac{\alpha_{n,p} r}{a}\right).$$

Using the orthogonality relations in the usual way, we get

$$a_{n,p} = \frac{2}{\pi a^2 J_{n+1}^2(\alpha_{n,p})} \int_0^a \int_0^{2\pi} f(r, \theta) J_n\left(\frac{\alpha_{n,p} r}{a}\right) \cos n\theta r dr d\theta, \quad \text{and} \\ b_{n,p} = \frac{2}{\pi a^2 J_{n+1}^2(\alpha_{n,p})} \int_0^a \int_0^{2\pi} f(r, \theta) J_n\left(\frac{\alpha_{n,p} r}{a}\right) \sin n\theta r dr d\theta.$$

7. (a) We look for product solutions $u(t, x, y) = T(t)\phi(x, y)$. When we substitute into the heat equation and separate variables, we see that there is a constant λ such that $T' = -k\lambda T$, and $-\nabla^2 \phi = \lambda \phi$. Since

$$\frac{\partial u}{\partial \mathbf{n}}(t, x, y) T(t) = T(t) \frac{\partial \phi}{\partial \mathbf{n}}(x, y),$$

we conclude that ϕ must satisfy a Neumann condition. Hence the eigenvalue problem is

$$-\nabla^2 \phi(x, y) = \lambda \phi(x, y), \quad \text{for } (x, y) \in D, \text{ and} \\ \frac{\partial \phi}{\partial \mathbf{n}}(x, y) = 0, \quad \text{for } (x, y) \in \partial D.$$

- (b) Since we know the Laplacian in polar coordinates, it is only necessary to understand what $\partial \phi / \partial \mathbf{n}$ is in polar coordinates. Since the normal direction to the boundary at a point on the circular boundary of the disk points in the direction of increasing r , we see that $\partial \phi / \partial \mathbf{n} = \phi_r$. Hence the problem becomes

$$-\left[\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta}\right](r, \theta) = \lambda \phi(r, \theta), \quad \text{for } r < 1, \\ \phi_r(a, \theta) = 0 \quad \text{for } 0 \leq \theta \leq 2\pi.$$

- (c) We look for product solutions $\phi(r, \theta) = R(r)U(\theta)$. Substituting and separating variables, we get the equations in [refcdm3](#). The Sturm Liouville problem for U is the same and has the solutions in (10.4). The Sturm Liouville problem for R , however, is different. It becomes

$$-(rR')' + \frac{n^2}{r}R = \lambda rR \quad \text{for } 0 < r < 1, \\ R \text{ and } R' \text{ are continuous at } r = 0, \\ R'(a) = 0.$$

Proposition 6.24 implies that all of the eigenvalues are nonnegative. If $\lambda = 0$ is an eigenvalue, the corresponding eigenfunction must be a constant. Notice that this occurs when $n = 0$. Thus $\lambda_{0,0} = 0$ as an eigenvalue, with eigenfunction $R_0 = 1$. In other cases the eigenvalues are positive, so we set $\lambda = \nu^2$. As before, the differential equation is transformed into Bessel's equation and has solutions $J_n(\nu r)$ which are bounded at $r = 0$. To satisfy the second boundary condition, we must have $J'_n(\nu) = 0$. Let $\beta_{n,k}$ be the sequence of zeros of J'_n . Thus the eigenvalues and eigenfunctions are

$$\lambda_k = \beta_{n,k}^2 \quad \text{and} \quad R_k(r) = J_n(\beta_{n,k}r/a) \quad \text{for } k = 1, 2, \dots,$$

for $k \geq 1$, with the addition of $\lambda_{0,0} = 0$ and $R_0 = 1$ when $n = 0$. Consequently, the eigenvalues for the eigenvalue problem on the disk are $\lambda_{0,0} = 0$ and $\lambda_{n,k} = \beta_{n,k}^2$ for $n \geq 0$ and $k \geq 1$.

8. (a) We have $\nabla^2 u = \nabla^2 \phi \cdot Z + \phi \cdot Z_{zz} = 0$. Dividing by ϕZ , we get

$$\frac{\nabla^2 \phi}{\phi} = -\frac{Z''}{Z} = -\lambda,$$

where λ is a constant, since the two sides of the first equality depend on different variables. Hence

$$-\nabla^2 \phi = \lambda \phi, \quad \text{and} \quad Z'' = \lambda Z.$$

- (b) λ and ϕ must solve the eigenvalue problem for the disk in (10.2). The solutions are in (10.11). If we set $\omega_{n,k} = \sqrt{\lambda_{n,k}}$, then the equation for Z becomes $Z + \omega_{n,k}^2 Z = 0$. As we found in Section 4, a convenient fundamental set of solutions is

$$\sinh(\omega_{n,k}z) = \frac{e^{\omega_{n,k}z} - e^{-\omega_{n,k}z}}{2} \quad \text{and} \quad \sinh(\omega_{n,k}(z-L)) = \frac{e^{\omega_{n,k}(z-L)} - e^{-\omega_{n,k}(z-L)}}{2}.$$

Therefore the product solutions in the cylinder are:

$$\begin{aligned} & \sinh(\omega_{n,k}z) \cdot \phi_{n,k}(r, \theta), \quad \sinh(\omega_{n,k}z) \cdot \psi_{n,k}(r, \theta), \\ & \sinh(\omega_{n,k}(z-L)) \cdot \phi_{n,k}(r, \theta), \quad \text{and} \quad \sinh(\omega_{n,k}(z-L)) \cdot \psi_{n,k}(r, \theta), \end{aligned}$$

- (c) By linearity, any function of the form

$$\begin{aligned} u(r, \theta, z) = & \sum_{k=1}^{\infty} J_0\left(\frac{\alpha_{0,k}r}{a}\right) [a_{0,k} \sinh(\omega_{0,k}z) + c_{0,k} \sinh(\omega_{0,k}(z-L))] \\ & + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k}r}{a}\right) [a_{n,k} \cos n\theta + b_{n,k} \sin n\theta] \sinh(\omega_{n,k}z) \\ & + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k}r}{a}\right) [c_{n,k} \cos n\theta + d_{n,k} \sin n\theta] \sinh(\omega_{n,k}(z-L)) \end{aligned}$$

is formally a solution to Laplace's equation on the cylinder that vanishes on the curved portion of the boundary. To satisfy the boundary conditions on the flat portions, we first set

$$\begin{aligned} 0 = u(r, \theta, L) = & \sum_{k=1}^{\infty} a_{0,k} J_0\left(\frac{\alpha_{0,k}r}{a}\right) \sinh(\omega_{0,k}L) \\ & + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k}r}{a}\right) [a_{n,k} \cos n\theta + b_{n,k} \sin n\theta] \sinh(\omega_{n,k}L) \end{aligned}$$

Thus $a_{n,k} = b_{n,k} = 0$. At the other end we have

$$\begin{aligned} f(r, \theta) = u(r, \theta, 0) = & - \sum_{k=1}^{\infty} c_{0,k} J_0\left(\frac{\alpha_{0,k} r}{a}\right) \sinh(\omega_{n,k} L) \\ & - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k} r}{a}\right) [c_{n,k} \cos n\theta + d_{n,k} \sin n\theta] \sinh(\omega_{n,k} L) \end{aligned}$$

From this expansion for f we can evaluate the coefficients using the orthogonality relations in (10.12). We get

$$\begin{aligned} c_{n,k} &= \frac{-2}{\pi a^2 J_{n+1}(\alpha_{n,k}) \sinh(\omega_{n,k} L)} \int_0^a \int_0^{2\pi} f(r, \theta) J_n\left(\frac{\alpha_{n,k} r}{a}\right) \cos n\theta r dr d\theta, \quad \text{and} \\ d_{n,k} &= \frac{-2}{\pi a^2 J_{n+1}^2(\alpha_{n,k}) \sinh(\omega_{n,k} L)} \int_0^a \int_0^{2\pi} f(r, \theta) J_n\left(\frac{\alpha_{n,k} r}{a}\right) \sin n\theta r dr d\theta. \end{aligned}$$

9. (a) We have $-\nabla^2 u = -\nabla^2 A \cdot B - A \cdot B_{zz} = \lambda AB$. If we divide by $u = AB$, this becomes

$$\frac{-\nabla^2 A}{A} + \frac{-B''}{B} = \lambda.$$

Since each of the summands depend on different variables, they must both be constant. Hence there exist constants μ and ν such that $\mu + \nu = \lambda$ for which

$$-\nabla^2 A = \mu A \quad \text{and} \quad -B'' = \nu B.$$

Since $u = AB$ vanishes on the boundary, we must have

$$A(a, \theta) = 0 \quad \text{for } (r, \theta) \in \partial D, \quad \text{and} \quad B(0) = B(L) = 0.$$

- (b) The eigenvalue problem for A is the same as that in (10.1). The solutions are in (10.11):

$$\begin{aligned} \mu_{0,k} &= \frac{\alpha_{0,k}^2}{a^2} \quad \text{with} \quad \phi_{0,k}(r, \theta) = J_0(\alpha_{0,k} r/a) \\ &\quad \text{for } n = 0 \text{ and } k = 1, 2, 3, \dots \\ \mu_{n,k} &= \frac{\alpha_{n,k}^2}{a^2} \quad \text{with} \quad \begin{cases} \phi_{n,k}(r, \theta) = \cos n\theta \cdot J_n(\alpha_{n,k} r/a) \\ \psi_{n,k}(r, \theta) = \sin n\theta \cdot J_n(\alpha_{n,k} r/a) \end{cases} \quad \text{and} \\ &\quad \text{for } n = 1, 2, 3, \dots \text{ and } k = 1, 2, 3, \dots \end{aligned}$$

We are well familiar with the Sturm Liouville problem for B . The solutions are

$$\nu_l = \frac{l^2 \pi^2}{L^2} \quad \text{with} \quad Z_l(z) = \sin \frac{l\pi z}{L}.$$

Consequently, the eigenvalues for the cylinder are

$$\lambda_{n,k,l} = \mu_{n,k} + \nu_l = \frac{\alpha_{n,k}^2}{a^2} + \frac{l^2 \pi^2}{L^2},$$

for $n = 0, 1, 2, \dots$, $k = 1, 2, \dots$, and $l = 1, 2, \dots$. The associated eigenfunctions are

$$u_{0,k,l} = \phi_{0,k} \cdot Z_l = J_0(\alpha_{0,k}r/a) \cdot \sin \frac{l\pi z}{L},$$

for $n = 0$, and

$$u_{n,k,l} = \phi_{n,k} \cdot Z_l = \cos n\theta \cdot J_n(\alpha_{n,k}r/a) \cdot \sin \frac{l\pi z}{L},$$

$$v_{n,k,l} = \psi_{n,k} \cdot Z_l = \sin n\theta \cdot J_n(\alpha_{n,k}r/a) \cdot \sin \frac{l\pi z}{L},$$

for $n \geq 1$.