F.2 Chapter 2

2.3 $= \lim_{t\to 0} \frac{\cos(2t)+\mathrm{i}\sin(2t)-1}{t} = \lim_{t\to\ 0} \frac{\cos(2t)-1}{t} + \mathrm{i}\times\lim_{t\to 0}\frac{\sin 2t}{t} = 0+\mathrm{i}2 = 2\mathrm{i}$. (The latter two limits follow from "real" l'Hôpital. (One may also directly apply "complex l'Hôpital")

2.5 The limit is zero because $\lim_{t\to\infty} |\frac{\mathrm{e}^{\mathrm{i}t}}{t} - 0| = \lim_{t\to\infty}\frac{1}{t} = 0$.

2.6 $f(t) = A\cos(\omega t+\phi)$. Amplitude: $A = 110\sqrt{5}$, angular frequency: $\omega = \dfrac{\pi}{25}$, initial phase: $\phi = -\arctan(2)$.

2.7

$$\begin{aligned}\sin(\alpha)\cos(\beta) &= \frac{\mathrm{e}^{\mathrm{i}\alpha}-\mathrm{e}^{-\mathrm{i}\alpha}}{2\mathrm{i}}\,\frac{\mathrm{e}^{\mathrm{i}\beta}+\mathrm{e}^{-\mathrm{i}\beta}}{2}\\ &= \frac{\mathrm{e}^{\mathrm{i}(\alpha+\beta)}+\mathrm{e}^{\mathrm{i}(\alpha-\beta)}+\mathrm{e}^{\mathrm{i}(-\alpha+\beta)}+\mathrm{e}^{\mathrm{i}(-\alpha-\beta)}}{4\mathrm{i}}\\ &= \frac{\mathrm{e}^{\mathrm{i}(\alpha+\beta)}-\mathrm{e}^{-\mathrm{i}(\alpha+\beta)}}{4\mathrm{i}} + \frac{\mathrm{e}^{\mathrm{i}(\alpha-\beta)}-\mathrm{e}^{-\mathrm{i}(\alpha-\beta)}}{4\mathrm{i}}\\ &= \frac{1}{2}\sin(\alpha+\beta)+\frac{1}{2}\sin(\alpha-\beta).\end{aligned}$$

2.10 If $q = 2$ then there is no smallest period. If q is even (and $q \neq 2$) then the smallest period is π. If q is odd then the smallest period is 2π.

2.15 (a) (Intuitively this makes perfect sense: the power is "the average of the squared function" and this does not change if we shift the function. Now math:) Use that for periodic signals we have

$$P_f = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2\,\mathrm{d}t$$

The the power of the shifted signal $f(t-t_0)$ (which has the same period)

$$P_{f(t-t_0)} = \frac{1}{T}\int_{-T/2}^{T/2} |f(t-t_0)|^2\,\mathrm{d}t$$

do substitution $\tau = t - t_0$:

$$= \frac{1}{T}\int_{-T/2-t_0}^{T/2+t_0} |f(\tau)|^2\,\mathrm{d}\tau = \frac{1}{T}\int_{-T/2}^{T/2} |f(\tau)|^2\,\mathrm{d}\tau = P_f.$$

Here we used Lemma 2.4.3

(b) (Intuievly $f(t)$ and $f(2t)$ have the same power because the graph of $f(2t)$ is that of $f(t)$ squeezed with a factor 2, so the average over "all time" of $|f(2t)|^2$ and $|f(t)|^2$ are probably the same.) Actually I will derive a relation that is also valid for NON-periodic functions.

$$P_{f(2t)} = \lim_{M\to\infty}\frac{1}{M}\int_{-M/2}^{M/2} |f(2t)|^2\,\mathrm{d}t$$

do substitution $\tau = 2t$:

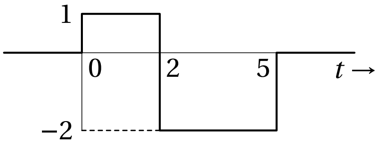
$$= \lim_{M\to\infty}\frac{1}{M}\int_{-M}^{M} |f(\tau)|^2\,\mathrm{d}\frac{\tau}{2} = \lim_{M\to\infty}\frac{1}{2M}\int_{-M}^{M} |f(\tau)|^2\,\mathrm{d}\tau$$

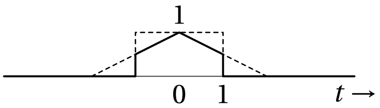
do substitution $N = 2M$:

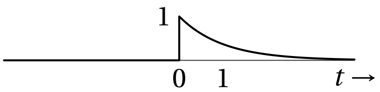
$$= \lim_{N\to\infty} \frac{1}{N} \int_{-N/2}^{N/2} |f(\tau)|^2 \,\mathrm{d}\tau = P_f$$

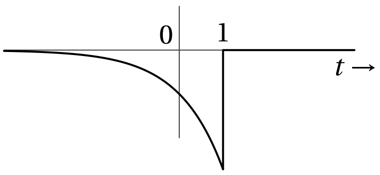
2.16 (a) $4a$

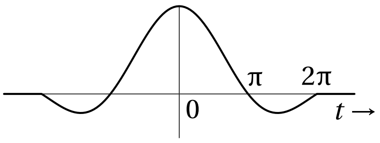
(b) $7/6$

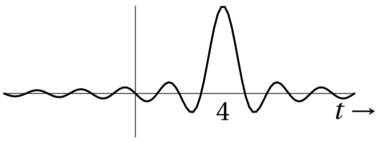
2.19 (a)


(b)


(c)


(d)


(e)


(f)


2.20 (a) 0,

(b) $\delta(t)$,

(c) 0,

(d) $35\delta(t-5)$,

(e) $15\delta(t+5)$,

(f) $\mathbb{1}(t-5)$,

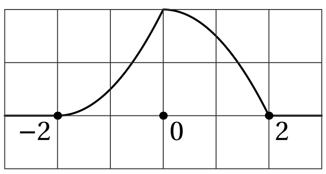
(g) $\frac{1}{5}\mathrm{e}^{5t+5}$

2.26 (a) -

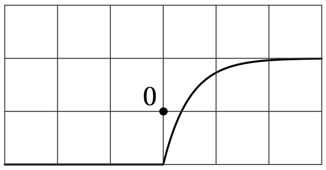
(b) $\frac{1}{a}\mathrm{e}^{a(t-1)}$

(c) $\begin{cases} 0 & \text{if } t < -1 \\ t+1 & \text{if } t \in [-1,1] \\ 2 & \text{if } t > 1 \end{cases}$ (Check: the derivative of this function is $\mathrm{rect}_2(t)$.)

(d) This is a nasty one (sorry): $\begin{cases} 0 & \text{if } t < -2 \\ (t+2)^2/2 & \text{if } t \in [-2,0] \\ 2 - t^2/2 & \text{if } t \in [0,2] \\ 0 & \text{if } t > 2 \end{cases}$



2.27 (a) $-1 + 2(1-\mathrm{e}^{-t})\,\mathbb{1}(t) = \begin{cases} -1 & \text{if } t < 0 \\ 1-2\mathrm{e}^{-t} & \text{if } t > 0 \end{cases}$



(b) Nasty: the answer is $g(t+1/2)$ where $g(t) = \int_{-\infty}^{t} \mathrm{e}^{-|\tau|}\,\mathrm{d}\tau = \mathrm{e}^{t}\,\mathbb{1}(-t) + (2-\mathrm{e}^{-t})\,\mathbb{1}(t)$

(c) $t\,\mathbb{1}(t)$

(d) $(t-1)\,\mathbb{1}(t-1)$

(e) $(t-2)\,\mathbb{1}(t-2)$

(f) $(-\frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{t})\,\mathbb{1}(t)$

(g) If $a \neq b$, then $(\frac{1}{a-b}\mathrm{e}^{at} - \frac{1}{a-b}\mathrm{e}^{bt})\,\mathbb{1}(t)$. If $a = b$, then $t\mathrm{e}^{at}\,\mathbb{1}(t)$

2.28 (a) -

(b) $\left[-\frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)\right]\mathbb{1}(t)$