Appendix F

Answers to some exercises

F.1 Chapter 1

1.4 Existence: since f is continuous we are guaranteed that $\|f\|_1$ exists

positive homogeneous: for every scalar α and function f we have $\|\alpha f\|_1 = \int_a^b |\alpha f(t)|\,\mathrm{d}t = |\alpha| \int_a^b |f(t)|\,\mathrm{d}t = |\alpha| \|f\|_1$.

triangle inequality: for every two functions f, g we have $\|f+g\|_1 = \int_a^b |f(t) + g(t)|\,\mathrm{d}t \leq \int_a^b |f(t)| + |g(t)|\,\mathrm{d}t = \|f\|_1 + \|g\|_1$.

positive definite: if f is not the zero function, then, $f(t_0) \neq 0$ for some $t_0 \in [a, b]$. Then because of continuity $|f(t)| > |f(t_0)|/2$ for all t in some small enough neighborhood of t_0. Then $\|f\|_1 > 0$.

1.5 Yes because:

Existence: $\|x\|_a + \|x\|_b$ is well defined because both $\|x\|_a$ and $\|x\|_b$ are defined (because they are norms).

Positive homogeneous: $\|\alpha x\| := \|\alpha x\|_a + \|\alpha x\|_b = |\alpha| \|x\|_a + |\alpha| \|x\|_b = |\alpha| (\|x\|_a + \|x\|_b) = |\alpha| \|x\|$. So it is positive homogeneous.

Triangle inequality: $\|x+y\| := \|x+y\|_a + \|x+y\|_b \leq (\|x\|_a + \|y\|_a) + (\|x\|_b + \|y\|_b) = (\|x\|_a + \|x\|_b) + (\|y\|_a + \|y\|_b) = \|x\| + \|y\|$. So triangle inequality holds.

Positive definite: if $x \neq 0$ then $\|x\| := \|x\|_a + \|x\|_b > 0$ because $\|x\|_a > 0$ (since $\|\cdot\|_a$ is a norm.) So all axioms of norm hold.

1.8 (By the way: this assumes that the cartesian product is equipped with the natural vector addition and scalar product: $(x_a, x_b) + (y_a, y_b) = (x_a + y_a, x_b + y_b)$ and $\alpha(x_a, x_b) = (\alpha x_a, \alpha x_b)$.)

Existence: since $\|\cdot\|_A$ and $\|\cdot\|_B$ are norms, they are well defined for all its elememts. Then $\max(\|x_a\|_A, \|x_b\|_B)$ is also well defined (finite).

Positive homogeneous: $\|\alpha(x_a, x_b)\| = \|(\alpha x_a, \alpha x_b)\| = \max(\|\alpha x_a\|_A, \|\alpha x_b\|_B)$ which by the norm properties of $\|\cdot\|_A, \|\cdot\|_B$ equals $|\alpha| \max(\|x_a\|_A, \|x_b\|_B)$. So it is positive homogeneous.

Triangle inequality: $\|(x_a, x_b) + (y_a, y_b)\| = \|(x_a + y_a, x_b + y_b)\| = \max(\|x_a + y_a\|_A, \|x_b + y_b\|_B)$. Case 1: *suppose that* $\|x_a + y_a\|_A \geq \|x_b + y_b\|_B$. Then the above gives $\|(x_a, x_b) + (y_a, y_b)\| = \|x_a + y_a\|_A \leq \|x_a\|_A + \|y_a\|_A \leq \max(\|x_a\|_A, \|x_b\|_B) + \max(\|x_b\|_B, \|y_b\|_B) = \|(x_a, x_b)\| + \|(y_a, y_b)\|$. This is the triangle inequality. The other case (so $\|x_a + y_a\|_A \leq \|x_b + y_b\|_B$) results in the same triangle inequality.

Positive definiteness: if (x_a, x_b) is nonzero then at least one of the two is nonzero, so then $\|(x_a, x_b)\| = \max(\|x_a\|_A, \|x_b\|_B) > 0$.

1.9 yes...

1.10 On $\mathbb{R}$ we take the absolute value as norm (Exercise 1.6 explains that this is, essentially, without loss of generality). Suppose f_n is Cauchy. Then $\forall \epsilon > 0$ there is an $N > 0$ such that $\|f_n - f_m\| < \epsilon$ for all $n > N$. Then $a_n := \|f_n\|$ by the reverse triangle inequality satisfies $|a_n - a_m| = |\|f_n\| - \|f_m\|| < \|f_n - f_m\| < \epsilon$ for all $n, m > N$. Hence a_n is a Cauchy sequence (in $\mathbb{R}$ with absolute value as norm).

1.11 Take an N. For all $n, m > N$ we have that $f_n(t) - f_m(t) = 0$ for all $t > 1/N$. Hence for all $n, m > N$ we have

$$\|f_n - f_m\|_2^2 \leq \int_0^{1/N} |f_n(t) - f_m(t)|^2 \, \mathrm{d}t$$

All functions $f_n(t), f_m(t)$ are nonnegative and bounded from above by $\frac{1}{t^{1/5}} = t^{-0.2}$ so we have for all $n, m > N$ that

$$\|f_n - f_m\|_2^2 \leq \int_0^{1/N} (2t^{-0.2})^2 \, \mathrm{d}t = \left[4\frac{t^{0.6}}{0.6}\right]_0^{1/N} = \frac{4}{0.6}\frac{1}{N^{0.6}}$$

This converges to zero as $N \to \infty$ so f_n is a Cauchy sequence.

1.12 Suppose f_n is Cauchy. Take an $\epsilon > 0$ and let N be such that $\|f_n - f_m\| < \epsilon$ for all $n > N$. This implies that $\|f_n\| \leq \|f_m\| + \epsilon$ for all $n, m > N$. For fixed such m and ϵ this shows that $|f_n|$ for all $n > N$ is bounded by $M := \|f_m\| + \epsilon$. Then f_n for all n is bounded by

$$\|f_n\| \leq \max(\|f_1\|, \|f_2\|, \ldots, \|f_N\|, M) < \infty$$

The upper bound does not depend on n, implying that f_n is bounded.

1.13 $\|A(x)\|_2 = \|\left[\begin{smallmatrix} 2x_1 \\ 3x_2 \end{smallmatrix}\right]\|_2 = \sqrt{4|x_1|^2 + 9|x_2|^2} \leq \sqrt{9|x_1|^2 + 9|x_2|^2} = 3\|x\|_2$. So $\|A\| \leq 3$. It is in fact equal to 3 because for $x = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ have have $\|x\|_2 = 1$ and $\|A(x)\|_2 = \|\left[\begin{smallmatrix} 0 \\ 3 \end{smallmatrix}\right]\|_2 = 3$ (so $\|A(x)\|_2 / \|x\|_2 = 3$ for this x)

1.14 If v_i is an eigenvector with eigenvalue λ_i then

$$\frac{\|A(v_i)\|}{\|v_i\|} = \frac{\|\lambda_i v_i\|}{\|v_i\|} = |\lambda_i|.$$

Since the eigenvectors are only a subset of $\mathbb{V}$ we have

$$\|A\| := \sup_{v \in \mathbb{V}} \frac{\|A(v)\|}{\|v\|} \geq \sup_{v = v_i} \frac{\|A(v)\|}{\|v\|} = \max_i |\lambda_i|.$$

1.15 (a) $\|K(u)\| = \|(0, u_1, u_2, \ldots)\|_2 = \sqrt{0^2 + u_1^2 + u_2^2 + \cdots} = \|u\|$ so $\|K\| = 1$

(b) $\|L(u)\| = \|(u_2, u_3, \ldots)\|_2 = \sqrt{u_2^2 + u_3^2 + \cdots} \leq \sqrt{u_1^2 + u_2^2 + u_3^2 + \cdots} = \|u\|$ so $\|L\| \leq 1$.

Now let $u = (0, 1, 0, 0, \ldots)$. Then $L(u) = (1, 0, 0, \ldots)$ so $\|L(u)\|_2 = \|u\|_2 = 1$ in this case, showing that $\|L\| \geq 1$. So we have $\|L\| = 1$

(c) $\|M(u)\|_2 = \sqrt{\sum_{k=1}^{\infty} (2 - 1/k)^2 u_k^2} \leq \sqrt{\sum_{k=1}^{\infty} 2^2 u_k^2} = 2\|u\|$. So $\|M\| \leq 2$.

Now consider the "sequence of sequences" $e_n := (0 \ldots, 0, 0, 1, 0, 0, \ldots)$ (meaning e_n has a 1 at its nth position and all other entries are zero). Then $M(e_n) = (2 - 1/n)e_n$ implying that $\|M\| \geq \|M(e_n)\| / \|e_n\| = (2 - 1/n)$. In the limit as $n \to \infty$ this lower bound for $\|M\|$ converges to 2. So $\|M\| = 2$.

1.16 -

1.17 Exploit the triangle inequalitty of norm: $\|cA(f)\|_2 = \sqrt{\int_{-\infty}^{\infty} |f(t) + f(-t)|^2 \, \mathrm{d}t} = \|f(t) + f(-t)\|_2 \leq \|f(t)\|_2 + \|f(-t)\|_2 = 2\|f\|_2$. So $\|A\| \leq 2$. For, say,

$$f(t) = \begin{cases} 1 & \text{if } t \in [0, 5] \\ 0 & \text{elsewhere} \end{cases}$$

we have $\|f\|_2 = 5$ and $\|A(f)\|_2 = \sqrt{\int_{-5}^{5} 1^2 \, \mathrm{d}t} = 10$. So $\|A(f)\| / \|f\|_2 = 2$ here. Combined with $\|A\| \leq 2$ shows that $\|A\| = 2$.

1.18

1.20

1.22 (a) Suppose f and g are two limits of a convergent f_n. Since convergent, for every $\epsilon > 0$ there are N_f, N_g such that $\|f - f_n\| < \epsilon \forall n > N_f$ and $\|g - f_n\| < \epsilon \forall n > N_g$. Then $\|f - g\| = \|(f - f_n) - (g - f_n)\| \leq \|f - f_n\| + \|g - f_n\| < 2\epsilon$ for all $n > \max(N_f, N_g)$. Since ϵ is arbitrary we must have $\|f - g\| = 0$. By the third axiom norm this means $f = g$.

(b) Let $f = \lim_{n \to \infty} f_n$ as an element of $\mathbb{X}$. If $f \in \mathbb{Y}$ then clearly f_n converges in $\mathbb{Y}$. If $f \notin \mathbb{Y}$ then f_n cannot converge in $\mathbb{Y}$ because if it would then this limit (call it y) would mean that f_n has two different limits (f and y) in $\mathbb{X}$. Not possible.

(c) Since $\mathscr{L}^1([-1, 1]; \mathbb{R})$ (with its 1-norm) is Banach, and f_n is a Cauchy sequence in the 1-norm, it follows that $f := \lim_{n \to \infty} f_n$ is $\mathscr{L}^1([-1, 1]; \mathbb{R})$ and is unique. Clearly this is $f(t) = 0$ for $t < 0$ and $f(t) = 1$ for all $t > 1$. This function is (essentially) not continuous. Apply the previous part (using that $\mathscr{C}([-1, 1]; \mathbb{R}) \subset \mathscr{L}^1([-1, 1]; \mathbb{R})$).

1.23 -