System(s) Theory

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Part I

Chapter 1 — Linear DE's with constant coefficients

Overview

1. 0.0 Introduction to ST
2. 1.1 Examples and Classification of DE's
3. 1.2 Homogeneous equation
4. 1.3 First-order homogeneous DE
5. 1.4 Second-order homogeneous DE
6. 1.5 Homogeneous DE
7. 1.6 Inhomogeneous equation – particular solution
8. 1.7 Asymptotic Stability & Time Constants
9. 1.8 Routh-Hurwitz test

- Organization
- What is "(linear) systems theory"
- Chapter 1, § 1.1–§ 1.4:
    - Examples of DE's
    - Homogeneous equation (1st order)
    - Homogeneous equation (2nd order)
    - (Homogeneous equation (arbitrary order))

Organization

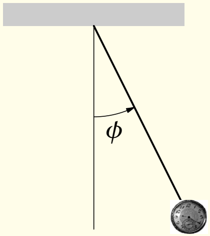
Tugce Akkaya
tutorials

Gjerrit Meinsma
lectures

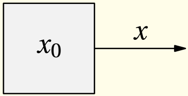
Alexander Wierzba
tutorials

- 4 chapters (we don't do chapter 5)
- 8 Lectures & 8 tutorials
- lecture notes (pdf)
- One standard written test. [4EC; 80% of grade]
- One PROJECT. Starts in week 3. [1EC; 20% grade]

“DE’s” versus “ST”



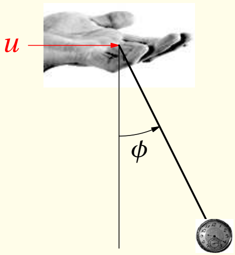
$$\dot{x}(t) = f(x(t), t)$$



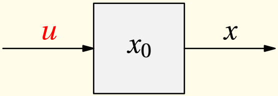
$$x = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix}$$

differential equations

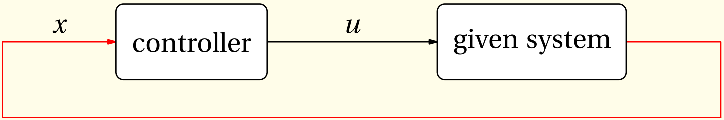
systems theory



$$\dot{x}(t) = f(x(t), u(t), t)$$



Inputs, diagrams, closed loop control, “plotter”



Applications

- "Biological control systems", glucose
- temperature control
- cruise control
- "war" (camera's, targets)
- drones [youtube]
- robots, self-driving cars, platoons
- satellite
- chemical,
- wafer-steppers
- ....

Have a look at the three YOUTUBE clips on CANVAS

1.1 Examples and Classification of DE's

Example (Pandemics — logistic model)

$$\dot{y}(t) = \beta y(t)(1 - y(t))$$

This is a

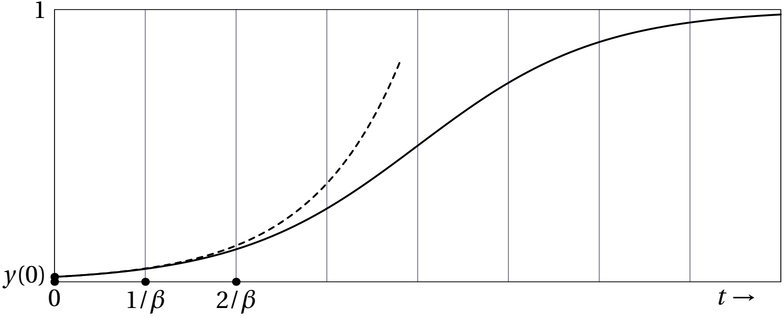
- differential equation (DE)
- first-order DE
- nonlinear DE

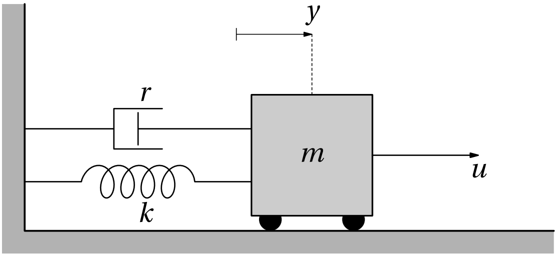
Example (Initial Pandemics — exponential growth)

$$\dot{y}(t) = \beta y(t)$$

This is a linear DE. We solve it soon:

$$y(t) = y(0)\,\mathrm{e}^{\beta t}$$


1
$y(0)$
0
$1/\beta$
$2/\beta$
$t \rightarrow$



Example (Mass-spring-damper system)

$$m\ddot{y}(t) + r\dot{y}(t) + ky(t) = u(t).$$

This is a 2nd-order linear DE with constant coefficients, and with an input $u(t)$

The input $u(t)$ is given; we want to find $y(t)$

We consider linear DE's with constant coefficients:

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = u(t).$$

Today: homogeneous equation:

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = 0.$$

1.2 Homogeneous equation

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = 0.$$

Lemma (Subspace)

The set of all solutions $y : \mathbb{R} \to \mathbb{C}$ of a homogeneous equation is a (linear) subspace of the complex vector space $\{y : \mathbb{R} \to \mathbb{C}\}$.
Hence if $y_1, \ldots, y_k$ are solutions then so is any element of $\operatorname{span}\{y_1, \ldots, y_k\}$

1.3 First-order homogeneous DE

$$
\begin{aligned}
\dot{y}(t) + p_0\, y(t) &= 0 \\
y(t) &= \mathrm{e}^{-p_0 t} \\
y(t) &= {\color{red}c}\, \mathrm{e}^{-p_0 t} \\
y(t) &= {\color{red}c(t)}\, \mathrm{e}^{-p_0 t} \qquad \text{variation of constants} \\
\dot{y}(t) + p_0\, y(t) = 0 &\iff \left(\dot{c}(t)\, \mathrm{e}^{-p_0 t} - p_0\, c(t)\, \mathrm{e}^{-p_0 t}\right) + p_0\, c(t)\, \mathrm{e}^{p_0 t} = 0 \\
&\iff \dot{c}(t)\, \mathrm{e}^{-p_0 t} = 0 \\
&\iff \dot{c}(t) = 0 \\
&\iff c \text{ is constant}
\end{aligned}
$$

Lemma (All solutions of a 1st-order homogeneous DE)

Let $p_0 \in \mathbb{R}$. A differentiable function $y : \mathbb{R} \to \mathbb{C}$ satisfies $\dot{y}(t) + p_0\, y(t) = 0$ for all $t \in \mathbb{R}$ iff there is a $c \in \mathbb{C}$ such that $y(t) = c\, \mathrm{e}^{-p_0 t}$

Example (Newton's law of cooling)

$$\dot{y}(t) = -k y(t), \qquad \text{for some heat transfer coef } k > 0$$

$$\dot{y}(t) + k y(t) = 0$$

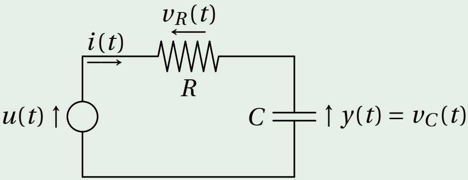
$$y(t) = c\,\mathrm{e}^{-kt}$$

$$y(t) = y(0)\,\mathrm{e}^{-kt} \qquad \text{initial condition } y(0) \text{ determines } y(t)$$

Example (RC network)

Let $\alpha = 1/(RC)$

$$\dot{y}(t) + \alpha y(t) = \alpha u(t),$$



$$\dot{y}(t) + \alpha y(t) = 0$$

$$y(t) = c\,\mathrm{e}^{-\alpha t} = y(0)\,\mathrm{e}^{-t/(RC)}$$

1.4 Second-order homogeneous DE

$$
\begin{aligned}
\ddot{y}(t) + p_1\,\dot{y}(t) + p_0\,y(t) &= 0 \\
y(t) &= \mathrm{e}^{\lambda t} \\
(\lambda^2 + p_1\lambda + p_0)\,\mathrm{e}^{\lambda t} &= 0 \\
\lambda^2 + p_1\lambda + p_0 &= 0 && \text{characteristic equation} \\
y(t) &= c_1\,\mathrm{e}^{\lambda_1 t} + c_2\,\mathrm{e}^{\lambda_2 t} && \text{there might be more:}
\end{aligned}
$$

Lemma (All complex solutions of a 2nd-order DE)

Let $p_0, p_1 \in \mathbb{R}$, and let $\lambda_{1,2} \in \mathbb{C}$ be the characteristic roots. A twice differentiable function $y : \mathbb{R} \to \mathbb{C}$ satisfies

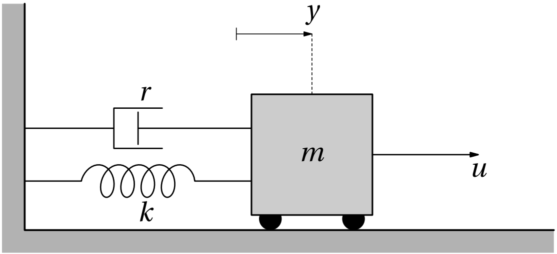
$$
\ddot{y}(t) + p_1\,\dot{y}(t) + p_0\,y(t) = 0 \quad \Longleftrightarrow \quad y(t) = \begin{cases} c_1\,\mathrm{e}^{\lambda_1 t} + c_2\,\mathrm{e}^{\lambda_2 t} & \text{if } \lambda_1 \neq \lambda_2 \\ (c_1 + c_2\,t)\,\mathrm{e}^{\lambda_1 t} & \text{if } \lambda_1 = \lambda_2 \end{cases}
$$

Example (Polynomial solution)

$$
\begin{aligned}
\ddot{y}(t) &= 0 \\
\lambda^2 &= 0 \\
y(t) &= (c_1 + c_2\,t)\,\mathrm{e}^{0\,t} \\
&= c_1 + c_2\,t
\end{aligned}
$$

Example (Complex characteristic roots)

$$
\begin{aligned}
\ddot{y}(t) + 4y(t) &= 0 \\
\lambda^2 + 4 &= 0 \\
\lambda_{1,2} &= \pm \mathrm{i}2 \\
\mathbb{Y} &= \operatorname{span}\{\mathrm{e}^{\mathrm{i}2t}, \mathrm{e}^{-\mathrm{i}2t}\} \\
&= \operatorname{span}\{\cos(2t), \sin(2t)\} \\
\text{all complex solutions:} \quad y(t) &= \hat{c}_1 \cos(2t) + \hat{c}_2 \sin(t) && \hat{c}_1, \hat{c}_2 \in \mathbb{C} \\
\text{all real solutions:} \quad y(t) &= c_1 \cos(2t) + c_2 \sin(t) && c_1, c_2 \in \mathbb{R} \\
&= A\cos(2t + \phi) && A, \phi \in \mathbb{R}
\end{aligned}
$$



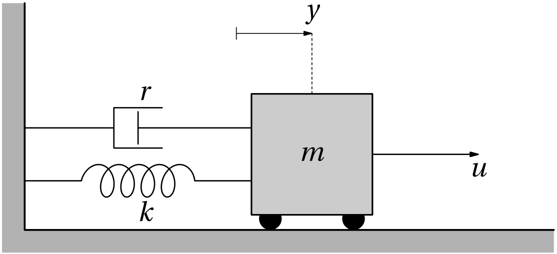
Example (Car connected to a wall — no damping)

$$m\ddot{y}(t) + r\dot{y}(t) + ky(t) = u(t) = 0$$

$$m\lambda^2 + r\lambda + k = 0$$

if $r = 0$: $\qquad \lambda_1 = \mathrm{i}\sqrt{k/m}, \qquad \lambda_2 = -\mathrm{i}\sqrt{k/m}$

$$y(t) = c_1 \cos(\sqrt{k/m}\,t) + c_2 \sin(\sqrt{k/m}\,t)$$



Example (Car connected to a wall — with damping)

Next suppose there is damping: we take $m = 1$, $r = 1$, $k = 9.25$:

$$m\ddot{y}(t) + r\dot{y}(t) + ky(t) = u(t) := 0$$

$$\lambda^2 + \lambda + 9.25 = 0$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{-36}}{2} = -0.5 \pm \mathrm{i}3$$

$$y(t) = \mathrm{e}^{-0.5t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

1.5 Homogeneous DE

$$y(t) = \mathrm{e}^{\lambda t}$$

$$\begin{aligned}
& 0 = y^{(n)}(t) + p_{n-1} y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) \\
\Longleftrightarrow\ & 0 = \lambda^n \mathrm{e}^{\lambda t} + p_{n-1}\lambda^{n-1} \mathrm{e}^{\lambda t} + \cdots + p_1 \lambda \mathrm{e}^{\lambda t} + p_0 \mathrm{e}^{\lambda t} \\
\Longleftrightarrow\ & 0 = \left(\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0\right) \mathrm{e}^{\lambda t} \\
\Longleftrightarrow\ & 0 = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 \qquad \text{characteristic equation} \\
& 0 = \prod_{i=1}^{n} (\lambda - \lambda_i)
\end{aligned}$$

$$y(t) = \sum_{i=1}^{n} c_i \mathrm{e}^{\lambda_i t} \qquad \text{but there might be more:}$$

Theorem (General solution of homogeneous equation)

Factorize the characteristic polynomial as

$$(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

with all λ_i distinct ($\lambda_i \neq \lambda_j\ \forall i \neq j$). Then $y(t)$ solves the homogeneous DE iff

$$y(t)=\sum_{i=1}^{k}\left(c_{i,0}+c_{i,1}\,t+\cdots+c_{i,m_i-1}\,t^{m_i-1}\right)\mathrm{e}^{\lambda_i t}$$

for some $c_{i,j}\in\mathbb{C}$

m_i is the multiplicity of root λ_i

Example

$$
\begin{aligned}
y^{(n)}(t) &= 0 \\
\lambda^n &= 0 \\
y(t) &= \left(c_{1,0} + c_{1,1}\,t + \cdots + c_{1,n-1}\,t^{n-1}\right) \mathrm{e}^{0t} \\
&= c_{1,0} + c_{1,1}\,t + \cdots + c_{1,n-1}\,t^{n-1}
\end{aligned}
$$

A polynomial of degree $n-1$ or less

Example

$$y^{(3)}(t) - 4y^{(2)}(t) + 5y^{(1)}(t) - 2y(t) = 0$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$(\lambda - 1)^2(\lambda - 2) = 0$$

$$y(t) = (c_1 + c_2 t)\,\mathrm{e}^{t} + c_3\,\mathrm{e}^{2t}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

1.6 Inhomogeneous equation – particular solution

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_0 y(t) = u(t)$$

Lemma (Particular + homogeneous = all)

Let $u : \mathbb{R} \to \mathbb{C}$. Suppose $y_{\text{part}} : \mathbb{R} \to \mathbb{C}$ is a particular solution of the DE. Then $y(t)$ satisfies inhomogeneous DE iff

$$y(t) = y_{\text{part}}(t) + y_{\text{hom}}(t)$$

for some solution with y_{hom} of the corresponding homogeneous eqn.

right-hand side $u(t)$	possible particular solution $y_{\text{part}}(t)$
polynomial of degree k	polynomial of degree k
$d\,\mathrm{e}^{\alpha t}$	$c\,\mathrm{e}^{\alpha t}$
$d_1 \cos(\beta t) + d_2 \sin(\beta t)$	$a \cos(\beta t) + b \sin(\beta t)$
$\mathrm{e}^{\alpha t}(d_1 \cos(\beta t) + d_2 \sin(\beta t)$	$\mathrm{e}^{\alpha t}(a \cos(\beta t) + b \sin(\beta t))$

Example (Newton's law of cooling — constant medium temperature)

$$\dot{y}(t) = -k(y(t) - m), \qquad k > 0,$$

$$\dot{y}(t) + ky(t) = km$$

$$y_{\text{part}}(t) = a?$$

$$ka = km, \qquad \Longrightarrow a = m$$

$$y(t) = m + c\,\mathrm{e}^{-kt}, \qquad c \in \mathbb{R}$$

Example (Newton's law of cooling — increasing medium temperature)

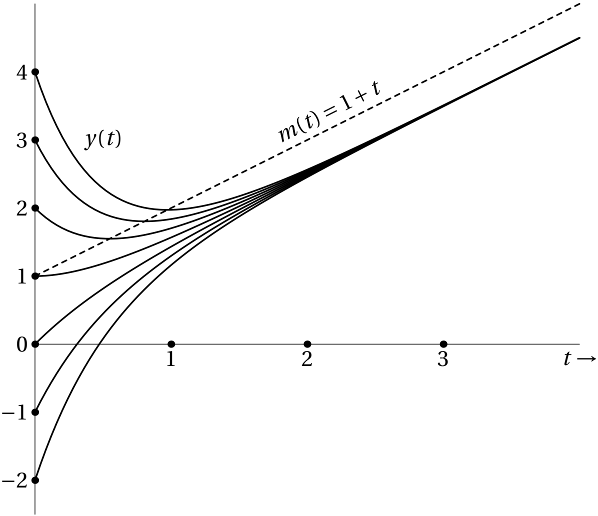
$$\dot{y}(t) = -k(y(t) - (1+t)), \qquad k > 0$$

$$\dot{y}(t) + k\,y(t) = k(1+t)$$

$$y_{\text{part}}(t) = a + bt$$

$$b + k(a+bt) = k(1+t) \qquad \Longrightarrow b = 1 \text{ and } a = 1 - 1/k$$

$$y(t) = 1 - 1/k + t + c\,\mathrm{e}^{-kt}, \qquad c \in \mathbb{R}$$



For $m(t) = 1 + t$ and $k = 2$.

Example (harmonic)

$\ddot{y}(t) - 4\dot{y}(t) + 4y(t) = 5\cos(t)$

Example (exponential & initial conditions)

$\ddot{y}(t) - 5\dot{y}(t) + 6\dot{y}(t) = \mathrm{e}^{2t}$ and $y(0) = 1, y(0) = 2.$

1.7 Asymptotic Stability & Time Constants

Definition (Asymptotic stability)

A DE

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = u(t), \qquad (p_i \in \mathbb{R})$$

is asymptotically stable if $\lim_{t\to\infty} y(t) = 0$ for all homogeneous solutions y

Lemma (Asymptotic stability)

A DE is asymptotically stable iff all its characteristic roots have negative real part: $\mathrm{re}(\lambda_i) < 0$

Definition (Asymptotically stable polynomial)

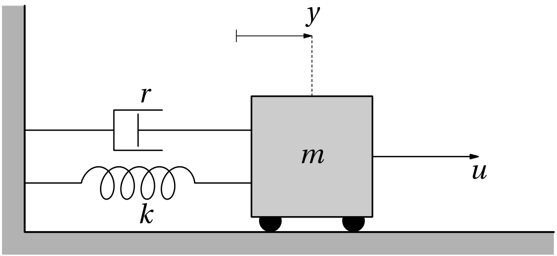
A polynomial is said to be asymptotically stable if all its zeros have negative real part

A constant polynomial $p(\lambda) = p_0$ is as.stable iff $p_0 \neq 0$

Lemma (Asymptotic Stability of DE's of order ≤ 3)

Let $p_0, p_1, p_2 \in \mathbb{R}$. Then

$$\begin{aligned}
\lambda + p_0 \text{ is asympt. stable} &\iff p_0 > 0 \\
\lambda^2 + p_1\lambda + p_0 \text{ is asympt. stable} &\iff p_1, p_0 > 0 \\
\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 \text{ is asympt. stable} &\iff p_2, p_1, p_0 > 0 \ \& \ p_2 p_1 > p_0
\end{aligned}$$



Example (Mechanical system)

$$m\ddot{y}(t) + r\dot{y}(t) + ky(t) = u(t)$$

$$1\ddot{y}(t) + (r/m)\dot{y}(t) + (k/m)y(t) = (1/m)u(t)$$

As.stable iff both $r > 0$ and $k > 0$.........

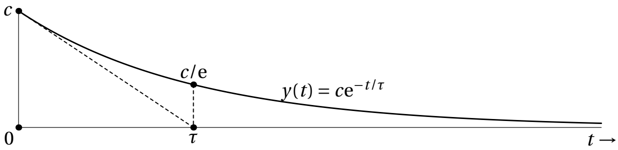
Time constant

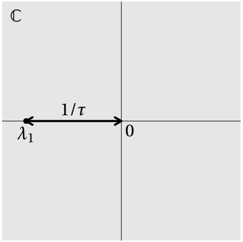
If as.stable then homogeneous converges to zero, but how fast?:
Consider 1st-order as.stable DE:

$$\dot{y}(t) + p_0 y(t) = 0, \qquad p_0 > 0$$

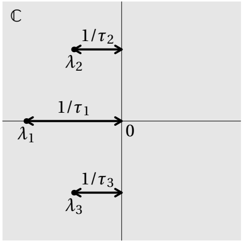
$$\tau := 1/p_0 \qquad \text{time constant}$$

$$y(t) = c\,\mathrm{e}^{-t/\tau}$$





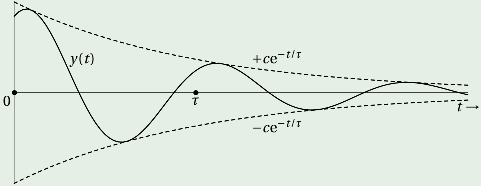
(a)



(b)

Example (2nd-order DE (mass-spring problem))

$$\lambda_{1,2} = -1/2 \pm i3 \quad \Longrightarrow \quad \tau = 1/\operatorname{re}(-\lambda_{1,2}) = 2$$



1.8 Routh-Hurwitz test

There are formulas for zeros of polynomials of degree ≤ 4:

0. $a = 0$ funny
1. $a\lambda + b = 0$ trivial
2. $a\lambda^2 + b\lambda + c = 0$ easy
3. $a\lambda^3 + b\lambda^2 + c\lambda + d = 0$ (Cardano, 16th century)
4. $a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e = 0$ (Ferrari, 16th century)

Theorem (Abel(1826) & Ruffini(1799) – Abel's impossibility thm)

There is no algebraic expression for zeros of arbitrary degree 5 polynomials

- Algebraic expression: $+, -, *, /$ and rational powers
- Awesome theorem, but also bad news for us:
- Finding zeros requires "infinitely many calculations"
- But a finite test for as.stability might still exist! It does exist:

Definition (Routh table)

The Routh table of a degree n polynomial

$$p_n\lambda^n + p_{n-1}\lambda^{n-1} + p_{n-2}\lambda^{n-2} + \cdots + p_1\lambda + p_0, \qquad p_n \neq 0$$

has $n+1$ rows:

$$\begin{array}{ccccc} p_n & p_{n-2} & p_{n-4} & p_{n-6} & \cdots \\ p_{n-1} & p_{n-3} & p_{n-5} & p_{n-7} & \cdots \\ b_0 & b_2 & b_4 & \cdots & \\ b_1 & b_3 & b_5 & \cdots & \\ \cdots & & & & \end{array}$$

Row 3 follows from rows 1 and 2 as

$$\begin{bmatrix} b_0 & b_2 \cdots \end{bmatrix} = \begin{bmatrix} p_{n-2} & p_{n-4} & \cdots \end{bmatrix} - \tfrac{p_n}{p_{n-1}} \begin{bmatrix} p_{n-3} & p_{n-5} & \cdots \end{bmatrix}.$$

Every row $i \geq 3$ likewise follows from rows $i-2$ and $i-1$.

Theorem (Routh-Hurwitz test – a finite test!)

As.stable $\iff$ *all elements in 1st column of Routh table have same sign*

Example

$$5\lambda^4 + 4\lambda^3 + 3\lambda^2 + 2\lambda + 1$$

Routh table has $n + 1 = 5$ rows:

$$\begin{array}{ccc} 5 & 3 & 1 \\ 4 & 2 & 0 \\ 1/2 & 1 & 0 \\ -6 & 0 & 0 \\ 1 & 0 & 0 \end{array}$$

Sign change in first column.
Hence not asymptotically stable.

Example

$$5\lambda^5 + 5\lambda^4 + 4\lambda^3 + 3\lambda^2 + 2\lambda + 1$$

Routh table has $n+1 = 6$ rows:

$$\begin{array}{ccc} 5 & 4 & 2 \\ 5 & 3 & 1 \\ 1 & 1 & 0 \\ -2 & 1 & 0 \\ 3/2 & 0 & 0 \\ 1 & 0 & 0 \end{array}$$

not as.stable. See below: 3 zeros $\mathrm{re}(\lambda) < 0$ and 2 zeros $\mathrm{re}(\lambda) > 0$

The proof of the Routh-Hurwitz test in fact shows:

- # of sign changes in 1st column = # of zeros $\mathrm{re}(\lambda_i) > 0$
- # of "no sign changes" = # number of zeros $\mathrm{re}(\lambda_i) < 0$

(Counting sign changes as you move down entire 1st column)

Example (Imaginary zeros)

$$\lambda^2 + 1$$

Routh table has $n+1=3$ rows:

$$\begin{array}{cc} 1 & 1 \\ 0 & 0 \\ ?? & ?? \end{array}$$

Procedure terminates due to division by zero.
Then not as.stable

Proof of RH-test shows: test terminates if there are imaginary zeros

Example

$$6\lambda^4 + 3\lambda^3 + c\lambda^2 + 2\lambda + 1$$

Routh table has $n+1=5$ rows:

$$\begin{array}{ccc} 6 & c & 1 \\ 3 & 2 & \\ c-4 & 1 & \\ 2-3/(c-4) & & \\ 1 & & \end{array}$$

First column entries have same sign iff $c>4$ and $2>3/(c-4)$.
Hence as.stable iff $c>5.5$.

For $c=5.5$ it has imaginary zeros: $\pm \mathrm{i}\sqrt{2/3}$ (and $-\frac{1}{4} \pm \mathrm{i}\frac{1}{4}\sqrt{3}$)

Part II

Chapter 2 — State Representations

Overview

10 2.1 Introduction

11 2.2 Solution of State Equations

12 2.2.1 The entries of e^{At}

13 2.3 Stability of Equilibria

14 2.4 A final note on Higher-order DE's

2.1 Introduction

An *nth* order DE (with input)

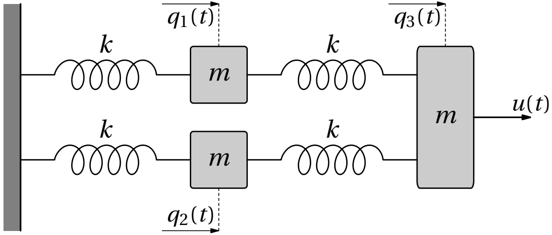
$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = u(t)$$

can also we represented as a system of 1st-order DE's:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} x(t).$$

This is an example of a state representation



Example ($n = 6$)

$$m\ddot{q}_1 = -kq_1 + k(q_3 - q_1),$$
$$m\ddot{q}_2 = -kq_2 + k(q_3 - q_2),$$
$$m\ddot{q}_3 = -k(q_3 - q_1) - k(q_3 - q_2) \ + \ u$$

Let $x = (q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_2)$, then....

Might choose $y := q_1$ as output, or $y := q_3$, or ...

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_{n_u}(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_{n_y}(t) \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{state equation}$$

$$y(t) = Cx(t) + Du(t) \qquad \text{output equation}$$

2.2 Solution of State Equations

For $n = 1$ the state equation is easy (use $x(t) = z(t)\,\mathrm{e}^{at}$):

$$\begin{aligned}
\dot{x}(t) = ax(t) + bu(t) &\iff \dot{z}(t)\,\mathrm{e}^{at} + az(t)\,\mathrm{e}^{at} = az(t)\,\mathrm{e}^{at} + bu(t) \\
&\iff \dot{z}(t)\,\mathrm{e}^{at} = bu(t) \\
&\iff \dot{z}(t) = \mathrm{e}^{-at}\,bu(t) \\
&\iff z(t) = z_0 + \int_0^t \mathrm{e}^{-a\tau}\,bu(\tau)\,\mathrm{d}\tau \qquad (z_0 \in \mathbb{C}) \\
&\iff x(t) = \mathrm{e}^{at}\,z_0 + \int_0^t \mathrm{e}^{a(t-\tau)}\,bu(\tau)\,\mathrm{d}\tau
\end{aligned}$$

(we always assume that $u(t)$ is “piecewise continuous”)

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^k = 1 + a + \frac{1}{2!} a^2 + \frac{1}{3!} a^3 + \cdots$$

Definition (Matrix exponential)

The matrix exponential e^A of a matrix $A \in \mathbb{R}^{n \times n}$ is defined as

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots.$$

It is an $n \times n$ matrix

Lemma (Matrix exponential)

Let $A, F \in \mathbb{R}^{n\times n}$. *Then:*

1. $\mathrm{e}^0 = I$ *for the zero matrix* $0 \in \mathbb{R}^{n\times n}$.
2. *If* $AF = FA$, *then* $\mathrm{e}^A \mathrm{e}^F = \mathrm{e}^{A+F}$.
3. e^A *is invertible and* $(\mathrm{e}^A)^{-1} = \mathrm{e}^{-A}$.
4. *Let* $t \in \mathbb{R}$; *then* $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{At} = A\mathrm{e}^{At} = \mathrm{e}^{At}A$.

Variations of constants: $x(t) = \mathrm{e}^{At} z(t)$:

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t)\\
&\qquad \iff A\mathrm{e}^{At} z(t) + \mathrm{e}^{At} \dot{z}(t) = A\mathrm{e}^{At} z(t) + Bu(t)\\
&\qquad \iff \mathrm{e}^{At} \dot{z}(t) = Bu(t)\\
&\qquad \iff \dot{z}(t) = \mathrm{e}^{-At} Bu(t)\\
&\qquad \iff z(t) = z(t_0) + \int_{t_0}^{t} \mathrm{e}^{-A\tau} Bu(\tau)\,\mathrm{d}\tau \qquad \text{recall } x(t) = \mathrm{e}^{At} z(t):\\
&\qquad \iff \boxed{x(t) = \mathrm{e}^{At} \mathrm{e}^{-At_0} x(t_0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau}\\
&\qquad \implies y(t) = C\mathrm{e}^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} C\mathrm{e}^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau + Du(t)
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= C\,\mathrm{e}^{A(t-t_0)}\,x(t_0) + \int_{t_0}^{t} C\,\mathrm{e}^{A(t-\tau)}\,Bu(\tau)\,\mathrm{d}\tau + Du(t) \\
&= \mathscr{H}\Big(x(t_0), u(\tau)\Big|_{\tau\in[t_0,t]}\Big)
\end{aligned}
$$

"the only information we need from the past, $t < t_0$, in order to continue into the future is the present $x(t_0)$"

- x is called the state of the system
- $x(0)$ is called the initial state
- since $x(t) \in \mathbb{R}^n$ the system is finite dimensional

Homogeneous:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

$$x(t) = e^{At}\, x_0$$

Easy but useful:

Lemma (Fundamental set of solutions)

If all columns x of $X : \mathbb{R} \to \mathbb{R}^{n\times n}$ satisfy $\dot{x}(t) = Ax(t)$, and $X(0)$ invertible, then

$$e^{At} = X(t)X^{-1}(0)$$

Example

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Then....

2.2.1 The entries of e^{At}

If

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix},$$

then

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}.$$

If only all A were diagonal: eigendecomposition to the rescue:

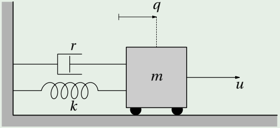
$$
\begin{aligned}
\mathrm{e}^{TMT^{-1}} &= I + TMT^{-1} + \frac{1}{2!}(TMT^{-1})(TMT^{-1}) + \cdots \\
&= I + TMT^{-1} + \frac{1}{2!}TM^2T^{-1} + \cdots \\
&= T\left(I + M + \frac{1}{2!}M^2 + \cdots\right)T^{-1} \\
&= T\,\mathrm{e}^{M}\,T^{-1}
\end{aligned}
$$

$$
\begin{aligned}
A &= T\Lambda T^{-1} \qquad \text{eigendecomposition} \\
AT &= T\Lambda \\
AT_k &= T_k\lambda_k \\
\det(\lambda I - A) &= 0
\end{aligned}
$$

$$
\mathrm{e}^{At} = T\,\mathrm{e}^{\Lambda t}\,T^{-1}
$$

Example (Matrix exponential for a mass-damper system)

$$\begin{bmatrix}\dot q\\ \ddot q\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k=0/m & -r/m\end{bmatrix}\begin{bmatrix}q\\ \dot q\end{bmatrix} + \begin{bmatrix}0\\ 1/m\end{bmatrix}u$$



$$\chi_A(\lambda) = \det\left(\begin{bmatrix}\lambda & -1\\ 0 & \lambda+\frac{r}{m}\end{bmatrix}\right) = \lambda(\lambda+\tfrac{r}{m}): \qquad \lambda_1 = 0,\ \lambda_2 = -r/m$$

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix}, \qquad v_2 = \begin{bmatrix}1\\ -\frac{r}{m}\end{bmatrix}$$

$$\mathrm{e}^{At} = T\,\mathrm{e}^{\Lambda t}\,T^{-1} = \begin{bmatrix}1 & 1\\ 0 & -\frac{r}{m}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \mathrm{e}^{-\frac{r}{m}t}\end{bmatrix}\begin{bmatrix}1 & \frac{m}{r}\\ 0 & -\frac{m}{r}\end{bmatrix} = \begin{bmatrix}1 & \frac{m}{r}(1-\mathrm{e}^{-\frac{r}{m}t})\\ 0 & \mathrm{e}^{-\frac{r}{m}t}\end{bmatrix}$$

$$\begin{bmatrix}q(t)\\ \dot q(t)\end{bmatrix} = \begin{bmatrix}q(0) + \frac{m}{r}(1-\mathrm{e}^{-\frac{r}{m}t})\dot q(0)\\ \mathrm{e}^{-\frac{r}{m}t}\,\dot q(0)\end{bmatrix}$$

Example (Matrix exponential for a mass-spring system)

no damping: $r = 0$. Then

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}$$

$$\chi_A(\lambda) = \lambda^2 + \frac{k}{m}$$

$$\lambda_1 = \mathrm{i}\omega, \qquad \lambda_2 = -\mathrm{i}\omega \qquad \text{for } \omega := \sqrt{k/m}$$

$$v_1 = \begin{bmatrix} -\mathrm{i} \\ \omega \end{bmatrix}, \quad v_2 = \begin{bmatrix} \mathrm{i} \\ \omega \end{bmatrix}$$

$$\mathrm{e}^{At} = \begin{bmatrix} -\mathrm{i} & \mathrm{i} \\ \omega & \omega \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\omega t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\omega t} \end{bmatrix} \begin{bmatrix} -\mathrm{i} & \mathrm{i} \\ \omega & \omega \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}}{2} & \dfrac{\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}}{2\mathrm{i}\omega} \\ -\omega \dfrac{\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}}{2\mathrm{i}} & \dfrac{\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}}{2} \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \frac{\sin(\omega t)}{\omega} \\ -\omega \sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

Jordan normal form (for the record)

$$A = T\,J\,T^{-1}$$

$$J = \begin{bmatrix} J_1 & 0 & \cdots \\ 0 & J_2 & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix} \qquad J_k = \begin{bmatrix} \lambda_k & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

$$e^{At} = T \begin{bmatrix} e^{J_1 t} & 0 & \cdots \\ 0 & e^{J_2 t} & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix} T^{-1} \qquad e^{J_k t} = e^{\lambda_k t} \begin{bmatrix} 1 & t & \frac{1}{2!}t^2 & \cdots & \frac{1}{(m-1)!}t^{m-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{2!}t^2 \\ \vdots & \ddots & \ddots & \ddots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

2.3 Stability of Equilibria

Definition (Equilibrium point)

An $\bar{x} \in \mathbb{R}^n$ is an equilibrium (point) of $\dot{x}(t) = Ax(t)$ if $x(t) := \bar{x}$ is a constant solution of the differential equation.

$$0 = A\bar{x}.$$

Definition (Asymptotic stability)

A state equation $\dot{x}(t) = Ax(t) + Bu(t)$ is asymptotically stable if for $u = 0$ all solutions $x(t)$ converge to 0 as $t \to \infty$.

Lemma (Asymptotic stability)

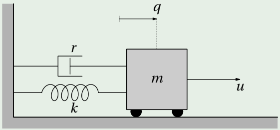
Consider the system $\dot{x}(t) = Ax(t)$ *with equilibrium point* $\bar{x} = 0$.
The following four statements are equivalent:

1. *the system is asymptotically stable.*
2. $\lim_{t\to\infty} x(t) = 0$ *for all* $x(0) \in \mathbb{R}^n$.
3. *All eigenvalues of* A *have negative real part.*
4. *The characteristic polynomial* $\det(\lambda I - A)$ *is an asymptotically stable polynomial.*

Example (Mass-spring-damper)

Clearly $m > 0$:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{r}{m} \end{bmatrix}$$



$$\chi_A(\lambda) = \det(\lambda I - A)$$

$$= \lambda(\lambda + \tfrac{r}{m}) + \tfrac{k}{m}$$

$$= \lambda^2 + \tfrac{r}{m}\lambda + \tfrac{k}{m}$$

asymptotically stable $\iff$ $r, k > 0$

2.4 A final note onf Higher-order DE's

DE

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = u(t)$$

for

$$x := \begin{bmatrix} y & y^{(1)} & \dots & y^{(n-2)} & y^{(n-1)} \end{bmatrix}^{\mathrm{T}}$$

becomes state rep:

$$\begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-1)} \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} x$$

This does work for arbitrary DE's (with derivatives of u):

$$\begin{aligned} & y^{(n)}(t) + p_{n-1} y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) \\ = \quad & q_n u^{(n)}(t) + q_{n-1} u^{(n-1)}(t) + \cdots + q_1 u^{(1)}(t) + q_0 u(t) \end{aligned}$$

Equivalent state rep still exists:

Example

$$\ddot{y} + 5\dot{y} + 6y = 7\dot{u} + 8u$$

$$\ddot{y} = -5\dot{y} + 7\dot{u} - 6y + 8u$$

$$y = \int \left[-5y + 7u + \int [-6y + 8u] \right]$$

$$y = \underbrace{\int \left[-5y + 7u + \underbrace{\int [-6y + 8u]}_{x_1} \right]}_{x_2}$$

Example

$$y = \underbrace{\int \left[-5y + 7u + \underbrace{\int \left[-6y + 8u \right]}_{x_1} \right]}_{x_2}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

General case:

$$y^{(3)} = q_3 u^{(3)} + [q_2 u^{(2)} - p_2 y^{(2)}] + [q_1 u^{(1)} - p_1 y^{(1)}] + [q_0 u - p_0 y]$$

$$y = q_3 u + \underbrace{\int \Big[q_2 u - p_2 y + \underbrace{\int \big[q_1 u - p_1 y + \underbrace{\int [q_0 u - p_0 y]}_{x_1} \big]}_{x_2} \Big]}_{x_3}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} q_0 & -p_0 \\ q_1 & -p_1 \\ q_2 & -p_2 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + q_3 u$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} q_0 & -p_0 \\ q_1 & -p_1 \\ q_2 & -p_2 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix},$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + q_3 u$$

Replace y with $x_3 + q_3 u$ to obtain state repr:

$$\dot{x} = \begin{bmatrix} 0 & 0 & -p_0 \\ 1 & 0 & -p_1 \\ 0 & 1 & -p_2 \end{bmatrix} x + \begin{bmatrix} q_0 - p_0 q_3 \\ q_1 - p_1 q_3 \\ q_2 - p_2 q_3 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + q_3 u$$

This is called observer canonical form

Example

DE

$$\dot{y} + 0y = \dot{u} + 0u$$

Then

$$\dot{y} = \dot{u} + 0y + 0u$$

$$y = u + \underbrace{\int 0}_{x}$$

Observer canonical form:

$$\dot{x} = 0$$

$$y = x + u$$

Hence y and u differ by a constant. Agreed.

Notice: u need not be differentiable now!

Part III

Chapter 3
Controllability & Observability

Overview

15 3.1 Reachability

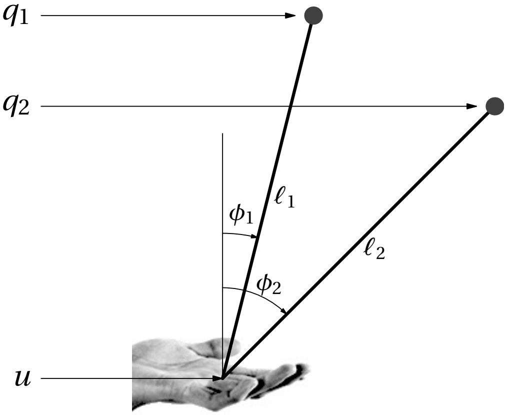
16 3.2 Controllability

17 3.3 Kalman Controllability Decomposition & Hautustest

18 3.4 Observability

19 3.5 Canonical Forms

Cliffhanger: can you "control both pendula?"



For linear systems this is "solvable" with linear algebra:

§ 3.1: Reachability

Definition (Reachability)

$\dot{x} = Ax + Bu$ is reachable if for every $x_1 \in \mathbb{R}^n$ and

$$x(0) = 0$$

there is a $t_1 > 0$ and $u : [0, t_1] \to \mathbb{R}^{n_u}$ such that

$$x(t_1) = x_1$$

- It does not fix t_1
- we say "the pair (A, B) is reachable"
- $x(t) = \int_0^t \mathrm{e}^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau$

Example

Not reachable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Is the next one reachable?:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u?$$

For instance can we steer the state to

$$\begin{bmatrix} x_1(t_1) \\ x_2(t_1) \end{bmatrix} = \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix}?$$

linear algebra:

Example (orthogonal to $x(t)$)

In this unreachable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad x(0) = 0$$

all solutions

$$\begin{bmatrix} x_1(t) = 0 \\ x_2(t) \end{bmatrix}$$

are orthogonal to $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2(t) \end{bmatrix} = 0$$

This generalizes:

Recall: $x(t) = \mathrm{e}^{At}\,x(0) + \int_0^t \mathrm{e}^{A(t-\tau)}\,Bu(\tau)\,\mathrm{d}\tau$

Definition (Reachable subspace)

$$\mathbb{X}(t_1) := \left\{ \int_0^{t_1} \mathrm{e}^{A(t_1-\tau)}\,Bu(\tau)\,\mathrm{d}\tau \,\middle|\, u : [0,t_1] \to \mathbb{R}^{n_u} \right\}$$

$\mathbb{X}(t_1)$ is a subspace of $\mathbb{R}^n$. For now fix $t_1 > 0$:

Lemma

Let $t_1 > 0$ and $\eta \in \mathbb{R}^n$. The following statements are equivalent (TFSAE):

1. $\eta \perp \mathbb{X}(t_1)$
2. $\eta^{\mathrm{T}}\,\mathrm{e}^{At}\,B = 0$ *for all* $t \in [0, t_1]$
3. $\eta^{\mathrm{T}} A^k B = 0$ *for all* $k = 0, 1, \ldots$
4. $\eta^{\mathrm{T}} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$

Here n is the # of state components: $x(t) \in \mathbb{R}^n$

- $\eta^{\mathrm{T}}\, \mathbb{X}(t_1) = 0 \quad \iff \quad \eta^{\mathrm{T}} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$
- $\mathbb{X}(t_1)$ and $\operatorname{im} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ same orthogonal complement
- Since $\mathbb{X}(t_1)$ is a subspace (of finite dimensional $\mathbb{R}^n$):
- $\mathbb{X}(t_1) = \operatorname{im}(\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix})$
- so reachable subspace $\mathbb{X}(t_1)$ does not depend on t_1 (only > 0)
- Define controllability matrix

$$\mathscr{C} := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times (n\, n_u)}$$

- conclusion:

$$\begin{aligned} \text{reachable} &\iff \mathbb{X}(t_1) = \mathbb{R}^n \\ &\iff \operatorname{im}(\mathscr{C}) = \mathbb{R}^n \\ &\iff \mathscr{C} \text{ full row rank (rank } n) \end{aligned}$$

Example ($n = 2$)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \implies \mathscr{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$\mathscr{C}$ not full row rank, so not reachable, and "$\mathbb{X} = \mathrm{im}(\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 \\ \mathbb{R} \end{smallmatrix}\right]$"

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \implies \mathscr{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$\mathscr{C}$ has full row rank, so reachable, $\mathbb{X} = \mathbb{R}^2$ (i.e. "$\mathbb{X} = \left[\begin{smallmatrix} \mathbb{R} \\ \mathbb{R} \end{smallmatrix}\right]$")

Theorem

The following statements are equivalent.

1. *the pair (A, B) is reachable*
2. $\mathrm{im}(\mathscr{C}) = \mathbb{R}^n$
3. *$\mathscr{C}$ has full row rank (rank n)*
4. *(if $\mathscr{C}$ square: $\mathscr{C}$ is invertible)*
5. *controllability gramian* $P(t) = \int_0^t \mathrm{e}^{A\tau} BB^\mathrm{T} \mathrm{e}^{A^\mathrm{T}\tau}\,\mathrm{d}\tau$ *invertible* $\forall\, t > 0$
6. *$P(t)$ is invertible for some $t > 0$.*

Then $x(t_1) = x_1$ if we apply

$$u_*(t) := B^\mathrm{T} \mathrm{e}^{A^\mathrm{T}(t_1 - t)} P^{-1}(t_1) x_1,$$

and, given t_1, it has the smallest possible norm:

$$\|u_*\|^2 := \int_0^{t_1} u_*^\mathrm{T}(t) u_*(t)\,\mathrm{d}t = x_1^\mathrm{T} P^{-1}(t_1) x_1 \leq \|u\|^2$$

for all u that achieve $x(t_1) = x_1$

§ 3.2: Controllability

Definition (Controllability)

$\dot{x} = Ax + Bu$ is controllable if for every $x_0, x_1 \in \mathbb{R}^n$ with

$$x(0) = x_0$$

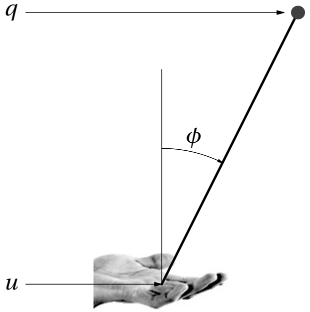
a $t_1 > 0$ exists and a $u : [0, t_1] \to \mathbb{R}^{n_u}$ such that

$$x(t_1) = x_1$$

controllable $\Longrightarrow$ reachable		trivial.
reachable $\Longrightarrow$ controllable		also true, because then:

$$x(t_1) = \mathrm{e}^{At_1}\, x_0 + \underbrace{\int_0^{t_1} \mathrm{e}^{A(t-\tau)}\, Bu(\tau)\, \mathrm{d}\tau}_{\in \mathbb{X}(t_1)=\mathbb{R}^n}$$

$$x(t_1) = \mathrm{e}^{At_1}\, x_0 + (x_1 - \mathrm{e}^{At_1}\, x_0) = x_1$$



Example (juggler)

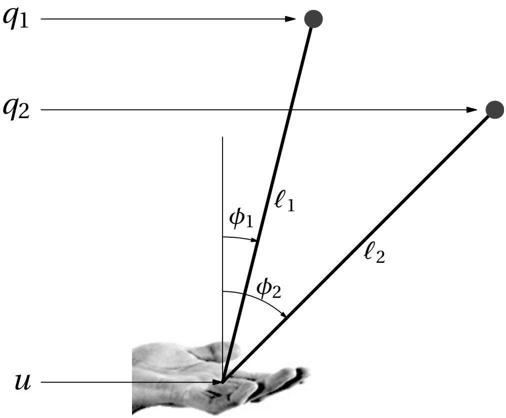
$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} u.$$

Hence

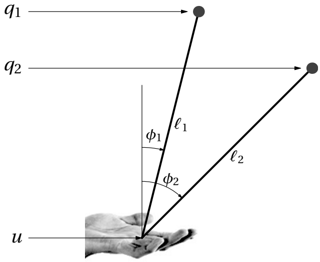
$$\mathscr{C} = \begin{bmatrix} 0 & -\frac{g}{\ell} \\ -\frac{g}{\ell} & 0 \end{bmatrix}.$$

Invertible, so controllable

So can achieve “any” $(q(t_1), v(t_1))$



When is this controllable?



Example (Juggler)

$$\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot v_1 \\ \dot v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{g}{\ell_1} & 0 & 0 & 0 \\ 0 & \frac{g}{\ell_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{g}{\ell_1} \\ -\frac{g}{\ell_2} \end{bmatrix} u$$

For $\alpha := -\frac{g}{\ell_1}, \beta := -\frac{g}{\ell_2}$:

$$\mathscr{C} = \begin{bmatrix} 0 & \alpha & 0 & -\alpha^2 \\ 0 & \beta & 0 & -\beta^2 \\ \alpha & 0 & -\alpha^2 & 0 \\ \beta & 0 & -\beta^2 & 0 \end{bmatrix}.$$

Then $\det(\mathscr{C}) = [\alpha\beta(\beta-\alpha)]^2$?

Controllable iff $\ell_1 \neq \ell_2$!

- Recap: (one slide)
- Recap: Kalman & Hautus
- Observability
- Canonical forms

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$y(t) = C\mathrm{e}^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} C\mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t)$$

$$\mathscr{C} := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$
$$\mathbb{X} = \mathrm{im}(\mathscr{C})$$

controllable $\iff \mathscr{C}$ has full row rank

$\iff \begin{bmatrix} sI - A & B \end{bmatrix}$ full row rank for all $s \in \mathbb{C}$

$$\dot{x} = Ax + Bu \implies z := T^{-1}x \implies \begin{bmatrix} \dot{z}_{\mathrm{c}} \\ \dot{z}_{\mathrm{uc}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_{\mathrm{c}} \\ z_{\mathrm{uc}} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

§ 3.3 Kalman Controllability Decomposition & Hautustest

If state z (not x) satisfies

$$\begin{bmatrix} \dot{z}_{\mathrm{c}} \\ \dot{z}_{\mathrm{uc}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_{\mathrm{c}} \\ z_{\mathrm{uc}} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

then

$$\mathscr{C}_z = \begin{bmatrix} B_1 & A_{11}B_1 & \cdots & A_{11}^{n-1}B_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and the reachable subspace $\mathrm{im}(\mathscr{C}_z)$ is (part of)

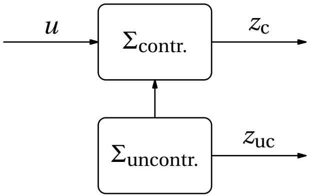
$$\begin{bmatrix} \mathbb{R}^q \\ 0 \end{bmatrix}$$

Lemma (Kalman Controllability Decomposition)

Suppose first q columns of an invertible $T \in \mathbb{R}^{n\times n}$ span $\operatorname{im}(\mathscr{C}_x)$.
Then $z := T^{-1}x$ gives

$$\begin{bmatrix} \dot{z}_{\mathrm{c}} \\ \dot{z}_{\mathrm{uc}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_{\mathrm{c}} \\ z_{\mathrm{uc}} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

with (A_{11}, B_1) controllable, and $\mathscr{C}_z = T^{-1}\mathscr{C}_x$, and $\operatorname{im}(\mathscr{C}_z) = \left[\begin{smallmatrix} \mathbb{R}^q \\ 0 \end{smallmatrix}\right]$



Example

$$\dot{x} = \begin{bmatrix} 4 & 3 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u \qquad \mathscr{C}_x = \begin{bmatrix} 0 & 2 & 4 \\ 1 & -1 & -1 \\ -1 & -1 & -3 \end{bmatrix}$$

the reachable subspace has dimension 2

$$T = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} 0 & 1 & .5 \\ 1 & 2 & .5 \\ 0 & 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad \mathscr{C}_z = T^{-1}\mathscr{C}_x = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

It splits (the eigenvalues of) the A-matrix

Hautustest

Theorem (Hautustest — PBH-test)

$\dot{x} = Ax + Bu$ is controllable iff

$$\begin{bmatrix} sI - A & B \end{bmatrix}$$

has full row rank for all $s \in \mathbb{C}$.

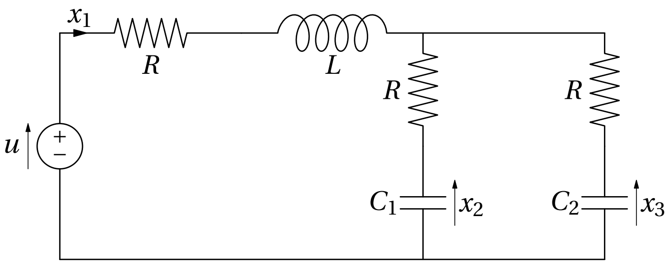
Equivalent: full row rank for all eigenvalues s of A

Example

$$\dot{x} = \begin{bmatrix} 4 & 3 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u$$

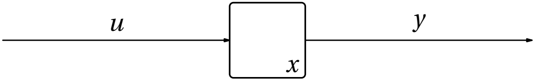
$$\begin{aligned} \begin{bmatrix} sI - A & B \end{bmatrix} &= \begin{bmatrix} s-4 & -3 & -1 & 0 \\ 1 & s+1 & 0 & 1 \\ 2 & 1 & s & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} s-4 & -3 & -1 & 0 \\ 3 & s+2 & s & 0 \\ 2 & 1 & s & \boxed{-1} \end{bmatrix} \\ &\sim \begin{bmatrix} s-4 & -3 & \boxed{-1} & 0 \\ s^2-4s+3 & -2s+2 & 0 & 0 \\ 2 & 1 & s & \boxed{-1} \end{bmatrix} \end{aligned}$$

Row in the middle is zero for $s = 1$. Hence not controllable



- Suppose R, L, C_1, C_2 are all greater than zero
- state (x_1, x_2, x_3) (one current, two voltages)
- RLC circuit is not controllable if ??

§ 3.4: Observability



- when controllable, we can force x to whatever
- ... using $u_*(t) = B^{\mathrm{T}}\, \mathrm{e}^{A^{\mathrm{T}}(t_1 - t)}\, P^{-1}(t_1)(x_1 - \mathrm{e}^{At_1}\, x_0)$
- this is not practical (it's like cycling blindfolded)
- for succesfull control we need to "look" around
- assume we "look" at y (the output, the part we can see/measure)
- ... can we then figure out the state?

VAGUE QUESTION:

$$\boxed{\text{Can we reconstruct / observe } x(t) \text{ on the basis of } u(t), y(t)\text{?}}$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Definition (Observability)

A system is observable if there exists a $t_1 > 0$ such that for every two triples (u_1, x_1, y_1) & (u_2, x_2, y_2) with the same external behavior,

$$u_1(t) = u_2(t), \quad y_1(t) = y_2(t) \qquad \forall t \in [0, t_1],$$

also the state is the same,

$$x_1(t) = x_2(t) \qquad \forall t \in [0, t_1].$$

Then x follows uniquely from u, y....

Example

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

has (among others) this constant solution

$$x(t) = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

but then output is zero for all time:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = 0.$$

Hence system is not observable

Suppose first $u(t) = 0, y(t) = 0$:

$$\dot{x} = Ax + B0$$
$$0 = Cx + D0$$

So

$$\dot{x} = Ax$$
$$0 = Cx$$

So

$$C\mathrm{e}^{At}x_0 = 0$$

Definition (Unobservable subspace)

$$\mathbb{X}^{\mathrm{uo}}(t_1) := \{x_0 \in \mathbb{R}^n \,|\, C\mathrm{e}^{At}x_0 = 0 \;\forall t \in [0, t_1]\}$$

$\mathbb{X}^{\mathrm{uo}}(t_1)$ is a subspace of $\mathbb{R}^n$

$$\mathbb{X}^{\mathrm{uo}}(t_1) := \{\eta \in \mathbb{R}^n \mid C\mathrm{e}^{At}\eta = 0 \ \forall t \in [0, t_1]\}$$

Lemma

Let $t_1 > 0$ and $\eta \in \mathbb{R}^n$. TFSAE:

1. $\eta \in \mathbb{X}^{\mathrm{uo}}(t_1)$, hence $C\mathrm{e}^{At}\eta = 0$ for all $t \in [0, t_1]$
2. $CA^k\eta = 0$ for all $k = 0, 1, 2, \ldots$
3. $CA^k\eta = 0$ for $k = 0, 1, \ldots, n-1$
4. $\eta \in \ker(\mathscr{O})$

$$\underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\mathscr{O}} \eta = 0$$

$\mathscr{O}$ is called observability matrix

- $\eta \in \mathbb{X}^{\mathrm{uo}}(t_1) \iff \mathscr{O}\eta = 0 \iff \eta \in \ker(\mathscr{O})$
- $\mathbb{X}^{\mathrm{ou}}(t_1) = \ker(\mathscr{O})$
- $\mathbb{X}^{\mathrm{ou}}(t_1)$ does not depend on $t_1 > 0$:

$$\{x_0 \in \mathbb{R}^n \,|\, C\,\mathrm{e}^{At}\,x_0 = 0 \forall\, t > 0\} = \ker(\mathscr{O})$$

- If $\ker(\mathscr{O})$ contains 2 or more entries, then not observable
- observability implies $\ker(\mathscr{O}) = \{0\}$
- In fact observability is equivalent to $\ker(\mathscr{O}) = \{0\}$ (see following slides):

Theorem (Observability)

TFAE:

1. system is observable (we say: "(A, C) is observable")
2. $\ker(\mathscr{O}) = \{0\}$
3. $\mathscr{O}$ has full column rank (rank n)

if $\mathscr{O}$ is square, then: observable $\iff$ $\mathscr{O}$ invertible

Proof.

$1 \implies 2$ old. $2 \implies 3$ is linear algebra. $3 \implies 1$:

$$y_1(t) = C\mathrm{e}^{At} x_1(0) + \int_0^t C\mathrm{e}^{A(t-\tau)} Bu_1(\tau)\,\mathrm{d}t + Du_1(t)$$

$$y_2(t) = C\mathrm{e}^{At} x_2(0) + \int_0^t C\mathrm{e}^{A(t-\tau)} Bu_2(\tau)\,\mathrm{d}t + Du_2(t)$$

$$0 = C\mathrm{e}^{At}[x_1(0) - x_2(0)] \qquad t \in [0, t_1]$$

So $[x_1(0) - x_2(0)] \in \ker(\mathscr{O}) = 0$. ∎

Example (Unobservable)

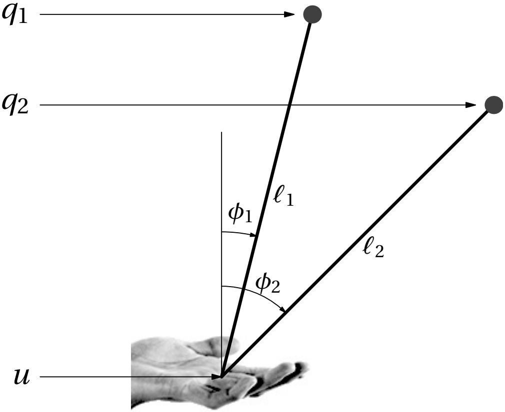
$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

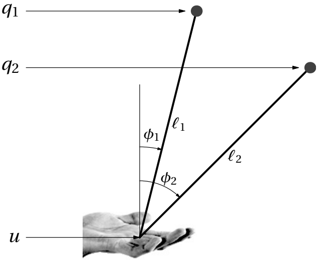
$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$\mathcal{O}$ singular so system not observable, and unobservable subspace is:

$$\ker \mathcal{O} = \begin{bmatrix} 0 \\ \mathbb{R} \end{bmatrix}$$



with just one output: $y := \phi_1 - \phi_2$.
When is it observable? (with state (q_1, q_2, v_1, v_2))



Example (Juggler (with $y = \phi_1 - \phi_2$))

$$\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot v_1 \\ \dot v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{g}{\ell_1} & 0 & 0 & 0 \\ 0 & \frac{g}{\ell_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{g}{\ell_1} \\ -\frac{g}{\ell_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{1}{\ell_1} & \frac{-1}{\ell_2} & 0 & 0 \end{bmatrix} x + \left(\frac{1}{\ell_2} - \frac{1}{\ell_1}\right) u$$

$$\mathcal{O} = \begin{bmatrix} \frac{1}{\ell_1} & \frac{-1}{\ell_2} & 0 & 0 \\ 0 & 0 & \frac{1}{\ell_1} & \frac{-1}{\ell_2} \\ \frac{g}{\ell_1^2} & \frac{-g}{\ell_2^2} & 0 & 0 \\ 0 & 0 & \frac{g}{\ell_1^2} & \frac{-g}{\ell_2^2} \end{bmatrix}$$

Observable iff $\ell_1 \neq \ell_2$!

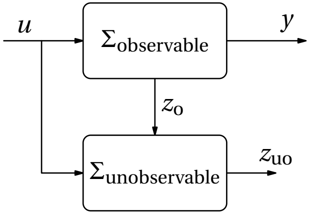
Lemma (Kalman Observability decomposition)

Suppose final q columns of some invertible T span $\ker(\mathscr{O}_x)$.
Then for $z := T^{-1}x$ the system becomes

$$\begin{bmatrix} \dot z_{\text{o}} \\ \dot z_{\text{uo}} \end{bmatrix} = \begin{bmatrix} A_{11} & 0_{(n-q)\times q} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_{\text{o}} \\ z_{\text{uo}} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & 0_{n_y \times q} \end{bmatrix} \begin{bmatrix} z_{\text{o}} \\ z_{\text{uo}} \end{bmatrix} + Du$$

with (C_1, A_{11}) observable. Moreover $\mathscr{O}_z = \mathscr{O}_x T$ and $\ker(\mathscr{O}_z) = \left[\begin{smallmatrix} 0 \\ \mathbb{R}^q \end{smallmatrix}\right]$.



Extra (💣): elegant proof of "Kalman"

Elegant (if you remember Lineaire Structures II):

- If $\mathcal{O}_x x = 0$ then $\mathcal{O}_x(Ax) = 0$.
- So $\ker(\mathcal{O}_x)$ is an A-invariant subspace
- Then $A|_{\ker(\mathcal{O}_x)}$ well defined
- we have $\ker(\mathcal{O}_z) = \{z | \mathcal{O}_z z = 0\} = \{z | \mathcal{O}_x T z = 0\} = \begin{bmatrix} 0 \\ \mathbb{R}^q \end{bmatrix}$
- So $A_z|_{\begin{bmatrix} 0 \\ \mathbb{R}^q \end{bmatrix}}$ well defined:

$$\begin{bmatrix} \dot{z}_o \\ \dot{z}_{uo} \end{bmatrix} = \begin{bmatrix} A_{11} & 0_{(n-q)\times q} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_o \\ z_{uo} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

- finally: $\mathcal{O}_z = \mathcal{O}_x T = \begin{bmatrix} * & 0 \end{bmatrix}$, so $C_z = \begin{bmatrix} * & 0 \end{bmatrix}$

Example (Construction of Kalman observability decomposition)

$$\dot{x} = \begin{bmatrix} 4 & -1 & -2 \\ 3 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} x, \qquad \mathscr{O}_x = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & -1 \\ 4 & -1 & -3 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} x$$

$\ker(\mathscr{O})$ has dimension 1, spanned by $(1, 1, 1)$:

$$\text{choose } T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad z = T^{-1}x, \ \ker(\mathscr{O}_z) = \begin{bmatrix} 0 \\ 0 \\ \mathbb{R} \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} z \qquad \mathscr{O}_z = \mathscr{O}_x T = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 4 & -1 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} z$$

It splits A-matrix (splits eigenvalues)

Hautustest (for observability)

Hautus controllability test we know:

Theorem (controllability)

$\dot{x} = Ax + Bu$ is controllable iff the $n \times (n + n_u)$ matrix

$$\begin{bmatrix} sI - A & B \end{bmatrix}$$

has full row rank $\forall s \in \mathbb{C}$

Likewise for observability:

Theorem (observability)

$\dot{x} = Ax + Bu, y = Cx + Du$ is observable iff $(n + n_y) \times n$ matrix

$$\begin{bmatrix} sI - A \\ C \end{bmatrix}$$

full column rank $\forall s \in \mathbb{C}$

proof of "(not observable) $\iff$ (Hautus matrix loses rank)".

Transformation $z = T^{-1}x$ does not change observability.
if not observable then "Hautusmatrix"

$$\begin{bmatrix} sI - A_{11} & 0 \\ -A_{21} & sI - A_{22} \\ C_1 & 0 \end{bmatrix}$$

loses rank voor all eigenvalues of A_{22}. Conversely, if Hautusmatrix $\left[\begin{smallmatrix} sI-A \\ C \end{smallmatrix}\right] = 0$ loses rank at some $s_0 \in \mathbb{C}$, then nonzero x_0 exists such that

$$\begin{bmatrix} s_0 I - A \\ C \end{bmatrix} x_0 = 0.$$

This x_0 is a eigenvector van A, so

$$\mathscr{O} x_0 = \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ s_0 C x_0 = 0 \\ \vdots \end{bmatrix} = 0,$$

so then not observable. ■

§ 3.5: Canonical Forms

Canonical forms are useful:

- $A \to D$ (diagonal)
- $A \to J$ (Jordan normal form)
- ....

Controllable & observable canonical forms:

State transformation

With state transformation $z = T^{-1}x$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \longrightarrow \quad \begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ y &= CTz + Du \end{aligned}$$

$$\begin{aligned} \mathscr{C}_z &= T^{-1}\mathscr{C}_x \\ \mathscr{O}_z &= \mathscr{O}_x T \\ \chi_{A_z}(\lambda) &= \chi_A(\lambda) \end{aligned}$$

we say:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{is isomorphic to} \quad \begin{bmatrix} T^{-1}AT & T^{-1}B \\ CT & D \end{bmatrix}$$

Lemma (A first form)

Suppose $n_u = 1$ *& define* $s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0 := \det(sI - A)$.
Every controllable $\dot{x} = Ax + Bu$ *is isomorphic to*

$$\dot{v} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u$$

and the transformation is unique (in fact $v = T^{-1}x$ *with* $T = \mathscr{C}_x^{-1}$*).*

Proof.

Let $T = \mathscr{C}_x$. Then $\mathscr{C}_v = T^{-1}\mathscr{C}_x = I$ so has above structure.
And use companion form to determine p_i.
Conversely, since $I = \mathscr{C}_v$ it follows that T is unique. ∎

Theorem (Controller canonical form)

Suppose $n_u = 1$ and define $s^n + p_{n-1}s^{n-1} + \cdots + p_0 := \det(sI - A)$.
Every controllable $\dot{x} = Ax + Bu$ is isomorphic to

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

(and $T = \mathscr{C}_x \mathscr{C}_z^{-1}$)

Proof.

Above (A_z, B_z) is controllable (says Hautus).
Hence (A_z, B_z) is isomorphic to first form (A_v, B_v) [via $v = \mathscr{C}_z^{-1} z$].
Also (A, B) is isomorphic to first form (A_v, B_v) [via $v = \mathscr{C}_x^{-1} x$].
Hence $\mathscr{C}_z^{-1} z = \mathscr{C}_x^{-1} x$, that is, $x = \mathscr{C}_x \mathscr{C}_z^{-1} z$. ■

Tedious to determine the transformation

$$T := \mathscr{C}_x \mathscr{C}_z^{-1}$$

we can bypass $\mathscr{C}_z$:

$$T = \begin{bmatrix} \eta \\ \eta A \\ \vdots \\ \eta A^{n-1} \end{bmatrix}^{-1} \quad \text{in which} \quad \eta := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathscr{C}_x^{-1}.$$

Makes the manipulation easier (well a little bit easier)

Example

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Then (verify this yourself)

$$T^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

and then (after the dust settles):

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -8 & 3 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} z$$

Only the new $C_z := CT$ requires hard work

Lemma (Observer canonical form)

Suppose $n_u = n_y = 1$ & $s^n + p_{n-1}s^{n-1} + \cdots + p_0 := \det(sI - A)$.
Every observable $\dot{x} = Ax + Bu, y = Cx + 0u$ *is isomorphic to*

$$\dot{z} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{bmatrix} z + \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ \vdots \\ q_{n-1} \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} z$$

and then $T = \mathscr{O}_x^{-1}\mathscr{O}_z$. *This* T *can be determined via*

$$T = \begin{bmatrix} \eta & A\eta & \cdots & A^{n-1}\eta \end{bmatrix} \quad \textit{via} \quad \eta := \mathscr{O}_x^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

But that is the observer canonical form (of Chapter 2), so:

- observer canonical form of $P(\frac{\mathrm{d}}{\mathrm{d}t})y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u$ is, indeed, observable,
- Every observable $\dot{x} = Ax + Bu, y = Cx + Du$ is equivalent to a DE $P(\frac{\mathrm{d}}{\mathrm{d}t})y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u$

Part IV

Chapter 4

Overview

20 4.0 Open versus closed loop & James & Cornelis

21 4.1 Stabilizability

22 4.2 State feedback

23 4.3 Observers

24 4.4 Dynamic Output Feedback

25 Closing Remarks (fun with control)

- Recap (Kalman & canonical forms)
- 4.0: Open loop versus closed loop & history
- 4.1: stabilizability
- 4.2: Static State Feedback
- Theorem 4.2.4

Lemma (Kalman Controllability Decomposition)

Suppose first q columns of an invertible $T \in \mathbb{R}^{n\times n}$ span $\operatorname{im}(\mathscr{C}_x)$.
Then $z := T^{-1}x$ gives

$$\begin{bmatrix} \dot{z}_{\mathrm{c}} \\ \dot{z}_{\mathrm{uc}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_{\mathrm{c}} \\ z_{\mathrm{uc}} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u.$$

Then (A_{11}, B_1) is controllable, and $\mathscr{C}_z = T^{-1}\mathscr{C}_x$, and $\operatorname{im}(\mathscr{C}_z) = \left[\begin{smallmatrix} \mathbb{R}^q \\ 0 \end{smallmatrix}\right]$

Theorem (Controller canonical form)

Suppose $n_u = 1$ and define $s^n + p_{n-1}s^{n-1} + \cdots + p_0 := \det(sI - A)$.
Every controllable $\dot{x} = Ax + Bu$ is isomorphic to

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad T = \mathscr{C}_x \mathscr{C}_z^{-1}$$

§ 4.0: Open loop versus closed loop

Open loop:

$$\boxed{\mathcal{K}} \xrightarrow{\;u\;} \boxed{\mathcal{H}} \xrightarrow{\;y\;}$$

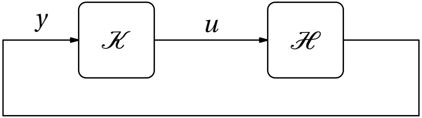
for example:

Example (blind juggler)

$$\boxed{\begin{matrix}\mathcal{K}\\ t_1, x_0, x_1\end{matrix}} \xrightarrow{\;u(t) = B^T \mathrm{e}^{A^T(t_1 - t)} P^{-1}(t_1)(x_1 - \mathrm{e}^{At_1} x_0)\;} \boxed{\mathcal{H}} \xrightarrow{\;y(t)\;}$$

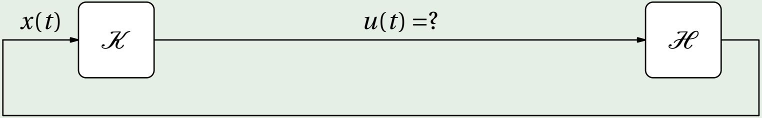
- It works in theory
- but not in practice

Closed loop:



This is natural (what a real juggler does):

Example (juggler (with eyes!))

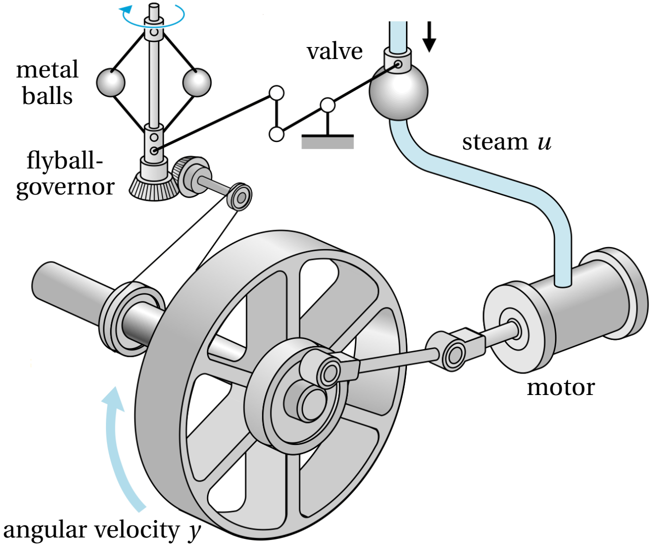


The juggler continuously looks at the pendulum, and uses it to determine u (the hand)

In this chapter: use u to stabilize the system.

Today: assume the entire state $x(t)$ is available for feedback

flyball governor of James Watt (1788?) (C. Huygens, 1658)



Cornelis Drebbel (1572–1633) — egg incubator (1609)

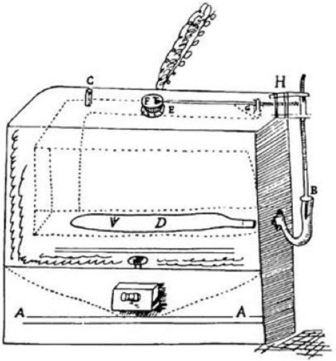


FIGURE 88.—Cornelis Drebbel's chicken incubator with temperature regulation, about 1620. Reprinted with permission of the Cambridge University Library from MS 2206, part 5, fol. 218.

He was known for his Perpetuum Mobile, built an incubator for eggs and a portable stove/oven with an optimal use of fuel, able to keep the heat on a constant temperature by means of a regulator/thermostat. *(one of the first recorded feedback-controlled devices)*

He designed a solar energy system for London (perpetual fire), demonstrated air-conditioning, made lightning and thunder "on command", and developed fountains and a fresh water supply for the city of Middelburg.

Discovered that stannous chloride makes the colour of carmine much brighter and more durable. His daughters and sons-in-laws set up a very successful dye works. The recipe was kept a family secret, and the new bright red colour was very popular in Europe.

"The idea of Drebbel as a universal wonderworker was as widespread in the seventeenth century as the idea of Einstein as a genius is today." [V. Keller, Princeton University]

Developed predecessors of the barometer and thermometer, and a harpsichords that played on solar energy.

Developed an automatic precision lens-grinding machine, build improved telescopes, constructs the first microscope ('lunette de Dreubells'), camera obscura, laterna magica.

Credited with the invention of the compound microscope. (In 1624 Galileo saw Drebbel's design for a microscope in Rome and created an improved version.)

... Drebbel went on to build two more submarines, each one bigger than the last. The final model had six oars and could carry 16 passengers. It was demonstrated to the king and thousands of Londoners on the Thames, and could stay submerged for three hours at a depth of 15 feet. How Drebbel maintained an air supply remains a mystery. (Might be an exaggeration.) [Wikipedia]

Example (Open loop versus closed loop)

Suppose

$$\dot{x} = x + u.$$

The two inputs

$$\textit{open-loop:}\;\; u_o(t) = -3\,e^{-2t}\,x(0),$$
$$\textit{closed-loop:}\;\; u_c(t) = -3x(t)$$

stabilize & are identical in that $u_o(t) \equiv u_c(t)$ and identical state:

$$x(t) = e^{-2t}\,x(0).$$

But they are very different if actual system is, say, $\dot{x} = 1.001x + u$:

$$\textit{open-loop:}\;\; x(t) = [\tfrac{3}{3.001}\,e^{-2t} + \tfrac{0.001}{3.001}\,e^{1.001t}]\,x(0),$$
$$\textit{closed-loop:}\;\; x(t) = e^{-1.999t}\,x(0).$$

Closed loop is much, much more robust against modeling errors (if given system unstable)

Definition (Stabilizability)

A system $\dot{x} = f(x,u)$ is *stabilizable* if for every

$$x(0) = x_0 \in \mathbb{R}^n,$$

there exists a $u : [0,\infty) \to \mathbb{R}^{n_u}$ such that

$$\lim_{t\to\infty} x(t) = 0.$$

It does not restrict u (open/closed loop, linear/nonlinear, ...)

Example (Three examples)

- $\dot{x} = +x + 1u$ is controllable "hence" stabilizable
- $\dot{x} = -x + 0u$ is not controllable, yet stabilizable
- $\dot{x} = +x + 0u$ is not controllable, not stabilizable

Stabilizability is weaker than controllability

§ 4.2: Static State feedback

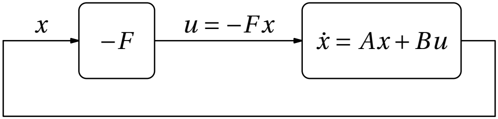
Assume $x(t)$ is available for feedback.
We focus on *(linear) static state feedback*:

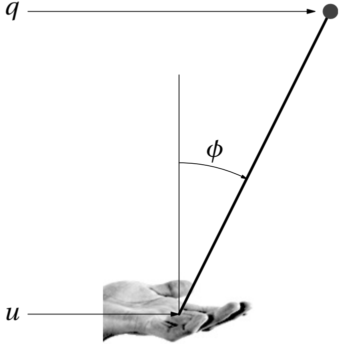
$$u(t) = -Fx(t).$$

Then

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= Ax - BFx \\ &= (A - BF)x.\end{aligned}$$

This u is stabilizing (for every $x(0)$) iff $A - BF$ is asymptotically stable.





$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} u$$

Example (Juggler — pole placement)

Let $u = -Fx = -\begin{bmatrix} f_1 & f_2 \end{bmatrix}\begin{bmatrix} q \\ v \end{bmatrix}$. Then

$$\begin{aligned}
\begin{bmatrix} \dot q \\ \dot v \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix}\begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} u \\
&= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix}\begin{bmatrix} q \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix}\begin{bmatrix} q \\ v \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell}(1 + f_1) & +\frac{g}{\ell} f_2 \end{bmatrix}\begin{bmatrix} q \\ v \end{bmatrix}.
\end{aligned}$$

$$\det(sI - (A - BF)) = s^2 - \tfrac{g}{\ell} f_2 s - \tfrac{g}{\ell}(1 + f_1)$$

This equals $(s+1)^2 = s^2 + 2s + 1$ iff

$$f_2 = -2\frac{\ell}{g}, \qquad f_1 = -1 - \frac{\ell}{g}.$$

Now the eigenvalues of $A - BF$ are -1 (twice).

Example

Suppose system is in *controller canonical form*:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u.$$

Then $u = \begin{bmatrix} +p_0 - r_0 & \cdots & +p_{n-1} - r_{n-1} \end{bmatrix} z$ gives

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & \cdots & -r_{n-1} \end{bmatrix} z$$

This has characteristic polynomial $R(s)$

Theorem (Pole placement)

Consider $\dot{x} = Ax + Bu$. For every polynomial

$$R(s) := s^n + r_{n-1}s^{n-1} + \cdots + r_0, \qquad r_k \in \mathbb{R},$$

there exists an $F \in \mathbb{R}^{n_u \times n}$ such that

$$\det(sI - (A - BF)) = R(s)$$

if and only if the system is controllable.

This implies:

Corollary

Every controllable system is stabilizable through $u(t) = -Fx(t)$

We may apply transformation $z = T^{-1}x$:

$$\begin{aligned} \dot{x} &= Ax + Bu & \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ u &= -Fx & u &= -FTz \\ \chi_A(s) &= \det(sI - A) & \chi_{A_z}(s) &= \chi_A(s) \\ \chi_{A-BF}(s) &= \det(sI - (A - BF)) & \chi_{A_z - B_z F_z}(s) &= \chi_{A-BF}(s) \end{aligned}$$

Proof 1/2.

If not controllable then

$$(A, B) \quad \to \quad \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right)$$

which after state feedback $u = -Fx = -\tilde{F}z$ gives

$$\begin{aligned} \det(sI - (A - BF)) &= \det\left(s \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \end{bmatrix}\right)\right) \\ &= \det \begin{bmatrix} sI - (A_{11} - B_1 \tilde{F}_1) & -(A_{12} - B_1 \tilde{F}_2) \\ 0 & sI - A_{22} \end{bmatrix} \end{aligned}$$

Hence the eigenalvalues of A_{22} are fixed (can not be moved).
So then not "pole place-able" ■

Proof continued.

If controllable (and $n_u = 1$) then controllable canonical form exists:

$$A_z - B_z F_z = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} F_z$$

for $F_z := \begin{bmatrix} -p_0 + r_0 & \cdots & -p_{n-1} + r_{n-1} \end{bmatrix}$ this becomes

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & \cdots & -r_{n-1} \end{bmatrix}$$

This has char.pol $R(s)$. So then "pole place-able" ■

Useful, but derivation not fun:

Lemma (Ackermann)

If $n_u = 1$ then

$$F = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathscr{C}_x^{-1} R(A)$$

MATLAB:

```
A=[1 0; 0 -1];
B=[1; 1];
F=acker(A,B,[-2 -3])    % set closed loop poles at -2 & -3
eig(A-B*F)              % check
```

Can also use `place` although that comand cannot handle closed loop poles of higher multiplicity (weird!)

Example

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u.$$

If we want closed loop "poles" at $-1, -4$ then $R(s) = (s+1)(s+4)$:

$$\begin{aligned} F &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathscr{C}_x^{-1} R(A) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}^{-1} (A+I)(A+4I) \\ &= \underbrace{\underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} 1/2 & 1/2 \end{bmatrix}} \begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix}}_{\begin{bmatrix} 2 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 5 \end{bmatrix} \end{aligned}$$

QUESTION: are there stabilizable systems $\dot{x} = Ax + Bu$ that are not stabilizable through $u(t) = -Fx(t)$?

ANSWER: No (that's good news):

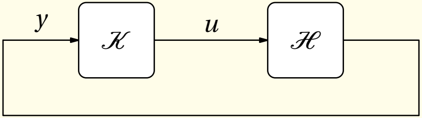
Theorem (4.2.4 — Stabilizability)

Consider $\dot{x} = Ax + Bu$. TFSAE:

1. $\exists F$ such that $A - BF$ is asymptotically stable.
   (So system is stabilizable through static state feedback $u = -Fx$.)
2. The system is stabilizable.
3. In the Kalman controllability decomposition of $\dot{x} = Ax + Bu$, the eigenvalues of A_{22} have negative real part.
4. $\begin{bmatrix} sI - A & B \end{bmatrix}$ has full row rank for all $s \in \mathbb{C}$ with $\mathrm{re}(s) \geq 0$.
5. $\begin{bmatrix} sI - A & B \end{bmatrix}$ has full row rank for all eigenvalues $s \in \mathbb{C}$ of A with $\mathrm{re}(s) \geq 0$.

Hence the choice $u(t) = -Fx(t)$ is not restrictive.

- Recap: state feedback
- § 4.3 Observers
- § 4.4 Dynamic Output feedback



Theorem (Pole placement (via static state feedback $u = -Fx$))

Consider $\dot{x} = Ax + Bu$. For every polynomial

$$R(s) := s^n + r_{n-1}s^{n-1} + \cdots + r_0, \qquad r_k \in \mathbb{R},$$

there exists an $F \in \mathbb{R}^{n_u \times n}$ such that

$$\det(sI - (A - BF)) = R(s)$$

if and only if the system is controllable.

Example (Inverted pendulum — juggler — $u = -Fx$)

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} u.$$

Then

$$u = -\begin{bmatrix} -2\ell/g & -1-\ell/g \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix}$$

places closed loop poles (eigenvalues) at -1 (double)

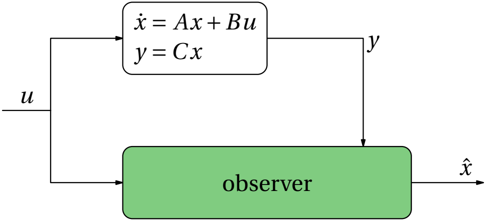
Theorem (4.2.4 — Stabilizability)

Consider $\dot{x} = Ax + Bu$. TFSAE:

1. $\exists F$ such that $A - BF$ is asymptotically stable.
   (So system is stabilizable through static state feedback $u = -Fx$.)
2. The system is stabilizable.
3. In the Kalman CD of $\dot{x} = Ax + Bu$, the A_{22} is as.stable.
4. $\begin{bmatrix} sI - A & B \end{bmatrix}$ has full row rank for all $s \in \mathbb{C}$ with $\operatorname{re}(s) \geq 0$.
5. $\begin{bmatrix} sI - A & B \end{bmatrix}$ has full row rank for all eigenvalues $s \in \mathbb{C}$ of A with $\operatorname{re}(s) \geq 0$.

If only we would have access to the entire state....

§ 4.3: Observers



observer = mapping from signals (u, y) to signal $\hat{x}$.

Definition (Detectability)

A system is *detectable* if there exists an observer such that

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| = 0$$

for all initial conditions $x(0)$ and all inputs u.

Important: $x(t)$ and $\hat{x}(t)$ need not converge as $t \to \infty$!

given system: $\dot{x} = Ax + Bu, \quad y = Cx$

$$\begin{aligned} \text{observer: } \quad \dot{\hat{x}} &= P\hat{x} + Qu + Ly \\ &= P\hat{x} + Qu + LCx \end{aligned}$$

Define estimation error $e := x - \hat{x}$:

$$\begin{aligned} \dot{e} &= (A - LC)x - P\hat{x} + (B - Q)u \\ &= (A - LC)(x - \hat{x}) + (A - LC - P)\hat{x} + (B - Q)u \end{aligned}$$

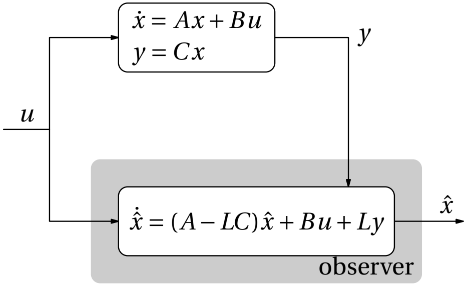
Choose $P := A - LC$ and $Q := B$:

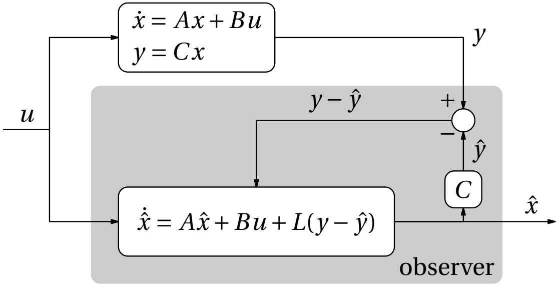
$$\dot{e} = (A - LC)e$$

$$\begin{aligned} \text{observer: } \quad \dot{\hat{x}} &= (A - LC)\hat{x} + Bu + Ly \\ &= A\hat{x} + Bu + L(y - C\hat{x}) \end{aligned}$$

Lemma

$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ is an observer if $A - LC$ as.stable


$\dot{x} = Ax + Bu$
$y = Cx$
y
u
$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$
$\hat{x}$
observer


$\dot{x} = Ax + Bu$
$y = Cx$
y
u
$y - \hat{y}$
+
−
$\hat{y}$
C
$\hat{x}$
$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$
observer

Example (Juggler)

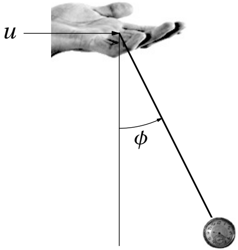
given system: $$\begin{bmatrix}\dot q\\ \dot v\end{bmatrix}=\begin{bmatrix}0 & 1\\ \frac{g}{\ell} & 0\end{bmatrix}\begin{bmatrix}q\\ v\end{bmatrix}+\begin{bmatrix}0\\ -\frac{g}{\ell}\end{bmatrix}u, \qquad y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}q\\ v\end{bmatrix}$$

observer: $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat q\\ \hat v\end{bmatrix}=\begin{bmatrix}0 & 1\\ \frac{g}{\ell} & 0\end{bmatrix}\begin{bmatrix}\hat q\\ \hat v\end{bmatrix}+\begin{bmatrix}0\\ -\frac{g}{\ell}\end{bmatrix}u+\begin{bmatrix}l_1\\ l_2\end{bmatrix}(y-\hat y)$$

$$\det(sI-(A-LC))=\det\begin{bmatrix}s+l_1 & -1\\ -\frac{g}{\ell}+l_2 & s\end{bmatrix}=s^2+l_1 s-\frac{g}{\ell}+l_2$$

Let's put observer poles at -2, so above char.pol. is $(s+2)^2=s^2+4s+4$. We must therefore choose $l_1=4$ and $l_2=4+\frac{g}{\ell}$. Ackermann the same:

$$L=(A+2I)^2\mathscr{O}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}4+g/\ell & 4\\ 4g/\ell & 4+g/\ell\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}4\\ 4+g/\ell\end{bmatrix}$$



Example (Hypnotist)

Take $m = 0.1, \ell = 0.4, g = 10$ (and friction coef 0.05), then:

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2.5 & -0.5 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix}$$

It has eigenvalues $-0.25 \pm \mathrm{i}1.56$

Do you know/remember "time constants"?

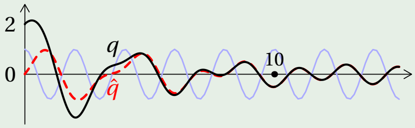
Example (Hypnotist — continued)

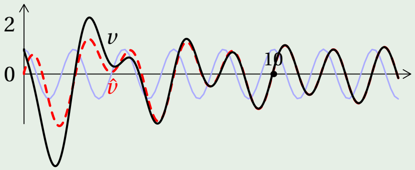
A possible "non-agressive" $L = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ gives

$$\begin{bmatrix} \dot{\hat{q}} \\ \dot{\hat{v}} \end{bmatrix} = \begin{bmatrix} -0.5 & 1.0 \\ -3.0 & -0.5 \end{bmatrix} \begin{bmatrix} \hat{q} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} u + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} y$$

having eigenvalues (aka "observer poles") $-0.5 \pm i1.73$.

Simulation for $u(t) = \cos(\pi t)$, $\begin{bmatrix} q(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} \hat{q}(0) \\ \hat{v}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:





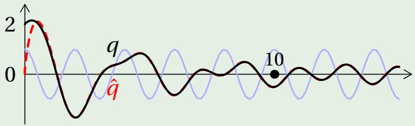
Example (Hypnotist — continued)

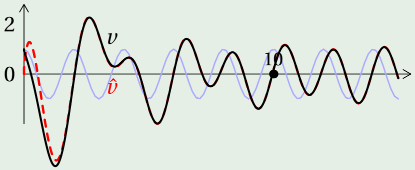
A possible "agressive" $L = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ gives

$$\begin{bmatrix} \dot{\hat{q}} \\ \dot{\hat{v}} \end{bmatrix} = \begin{bmatrix} -5.0 & 1.0 \\ -7.5 & -0.5 \end{bmatrix} \begin{bmatrix} \hat{q} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} u + \begin{bmatrix} 5.0 \\ 5.0 \end{bmatrix} y.$$

having eigenvalues (aka "observer poles") $-2.75 \pm i1.56$.

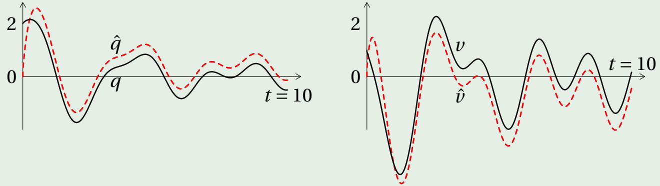
Simulation for $u(t) = \cos(\pi t)$, $\begin{bmatrix} q(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} \hat{q}(0) \\ \hat{v}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:



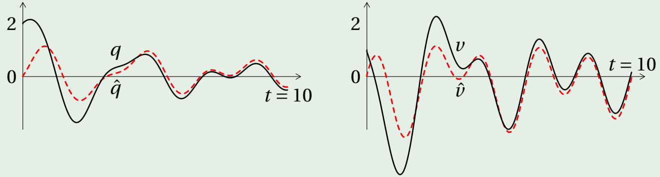


Example (Hypnotist — continued (final))

If there are (measurement) errors, say, $y_{\text{measured}} = y_{\text{real}} + 0.5$, then fast (agressive) observer not so good



Slow (non-agressive) is better:



Theorem (Observer pole placement)

Consider system $\dot{x} = Ax + Bu$, $y = Cx$. For every real polynomial $R(s) = s^n + r_{n-1}s^{n-1} + \cdots + r_0$, there exists an $L \in \mathbb{R}^{n \times n_y}$ such that

$$\det(sI - (A - LC)) = R(s)$$

iff (A, C) is observable.
If $n_y = 1$, then L can be determined using Ackermann's formula

$$L = R(A)\mathcal{O}_x^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Corollary

Every observable system has an observer of the special form $\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$ (for some L).

Is this special form restrictive? No:

Lemma (Detectability)

Consider $\dot{x} = Ax + Bu, y = Cx$. TFSAE:

1. There exists an L such that $A - LC$ is as.stable.
   (hence an observer exists of the form $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$)
2. The system is detectable
3. In Kalman Observability Decomp, the A_{22} is as.stable
4. $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ has full column rank for all $\mathrm{re}(s) \geq 0$
5. $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ has full column rank for all eigenvalues $s \in \mathbb{C}$ of A for which $\mathrm{re}(s) \geq 0$

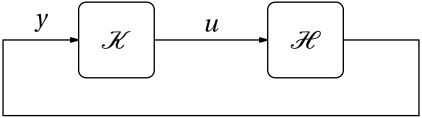
Hence the form $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ is not a restriction!

§ 4.4: Dynamic Output Feedback

What we did so far:

$$\begin{aligned} \text{dynamic observer:} \quad & \dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly \\ \text{static state feedback:} \quad & u = -Fx \end{aligned}$$

Now it is time to design a stabilizing controller $u = \mathscr{K}(y)$:



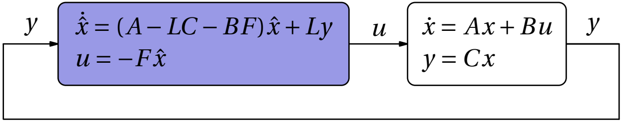
Bold idea: why not try this controller $u = \mathcal{K}(y)$:

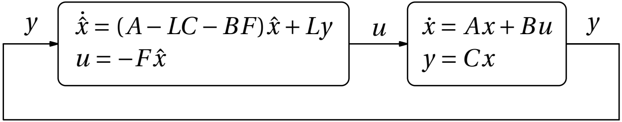
$$\text{dynamic observer:} \quad \dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$
$$\text{static state feedback with a twist:} \quad u = -F\hat{x}$$

Eliminate Bu:

$$\dot{\hat{x}} = (A - LC - BF)\hat{x} + Ly$$
$$u = -F\hat{x}$$

This a dynamic system (controller) with input y and output u:





Closed-loop is described by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BF \\ LC & A - BF - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \qquad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & -F \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Example (Closed loop state transformation)

$$\begin{bmatrix} x \\ e \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}}_{T^{-1}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \qquad \underbrace{\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}}_{T} \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}.$$

Similarity transformation:

$$\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & -BF \\ LC & A - BF - LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - LC \end{bmatrix}$$

So we proved:

Theorem (Stabilizing controller — separation principle)

If a system is stabilizable & detectable, then matrices F en L exist such that $A-BF$ & $A-LC$ asymptotically stable.

In that case the controller

$$\begin{aligned}\dot{\hat{x}} &= (A-LC-BF)\hat{x}+Ly\\ u &= -F\hat{x}\end{aligned}$$

stabilizes the given system (aka the "plant"), in the sense that

$$\begin{aligned}\lim_{t\to\infty} x(t) &= 0\\ \lim_{t\to\infty} \hat{x}(t) &= 0\end{aligned}$$

for every initial condition $x(0)=x_0$ and $\hat{x}(0)=\hat{x}_0$.

By the way: the controller itself need not be stable!

Example (Juggler with $\ell = g/2$)

$$\begin{bmatrix} \dot q \\ \dot v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u,$$
$$y = q.$$

- determined earlier: $F = \begin{bmatrix} -3/2 & -1 \end{bmatrix}$, then $\chi_{A-BF}(s) = (s+1)^2$
- determined earlier: $L = \left[\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right]$, then $\chi_{A-LC}(s) = (s+2)^2$

Gives this controller

$$\dot{\hat x} = \underbrace{\begin{bmatrix} -4 & +1 \\ -7 & -2 \end{bmatrix}}_{A-LC-BF} \hat x + \underbrace{\begin{bmatrix} 4 \\ 6 \end{bmatrix}}_{L} y$$

$$u = \underbrace{\begin{bmatrix} 3/2 & 1 \end{bmatrix}}_{-F} \hat x$$

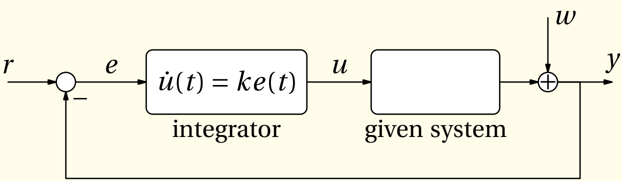
MATLAB code `simujuggler.m` on CANVAS:

```
n=2;                          % number of states of system %%
g=10;                         % gravitational acceleration %%
l=0.5;                        % whatever length of pendulum %

A=[0 1; g/l 0];               % A matrix for x=[q;v] %%%%%%%
B=[0; -g/l];                  % B matrix for u=position hand%
C=[1 0];                      % y=q %%%%%%%%%%%%%%%%%%%%%%%%
F=[-1-l/g -2*l/g];            % so A-BF has eigenv -1 (twice)
L=[4; 4+g/l];                 % so A-LC has eigenv -2 (twice)

eigABF=eig(A-B*F)             % check the stability of A-BF %
eigALC=eig(A-L*C)             % and of A-LC %%%%%%%%%%%%%%%%%

f =@(x,u) [x(2); +g/l*sin(x(1))]+[0; -g/l*u]; % nonlinear %
fl=@(x,u) [A*x+B*u];          % linearized %%%%%%%%%%%%%%%%%

FWX=@(t,z) [f(z(1:n),-F*z(n+(1:n)));      % $f(x,u)=f(x,-F\hat{x})$
    L*C*z(1:n)+(A-L*C-B*F)*z(n+(1:n))]; % $LCx+(A-LC+BF)\hat{x}$
FLW=@(t,z) [fl(z(1:n),-F*z(n+(1:n)));     % $Ax+Bu=Ax+B(-F\hat{x})$
    L*C*z(1:n)+(A-L*C-B*F)*z(n+(1:n))]; % $LCx+(A-LC+BF)\hat{x}$
```

```
%z0=[0;0.2;0;0];             % okay initial z(0) %%%%%%%%%%
%z0=[.3;.3;0;0];             % too big (for nonlinear case)
z0=[0;0.3;0;0];              % scary   (for nonlinear case)
tspan =[0 20];
[tt,z]=ode45(FWX,tspan,z0);  % simulate nonlinear %%%%%%%%%
%[tt,z]=ode45(FLW,tspan,z0); % simulate linear %%%%%%%%%%%%

% Plot all four state components against time: %%%%%%%%%%%%
figure(1),   plot(tt,z'),    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
legend('q','v','qh','vh'),   grid,  set(gca,'Fontsize',14);
% Simplistic movie: %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure(2);
for k=1:(length(tt)-1);
  u=-F*z(k,1:n)';
  q=z(k,1);
  plot([u,q],[0,sqrt(l^2-(q-u)^2)],[-1 1],[0 0],[0 0],[-0.1 .6]);
  axis off;
  drawnow;
  pause(tt(k+1)-tt(k));
end
```

Closing Remarks (fun with control)

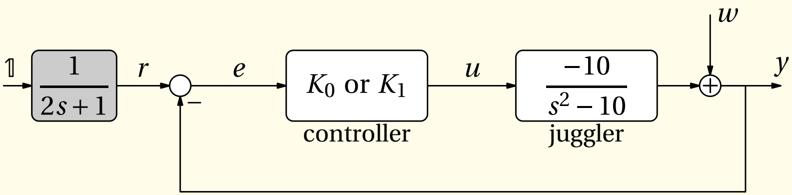
- Here we stabilized (everything $\to 0$ as $t \to \infty$)
- Can also steer to other “desired” value
- Classic control: put integrator in loop:



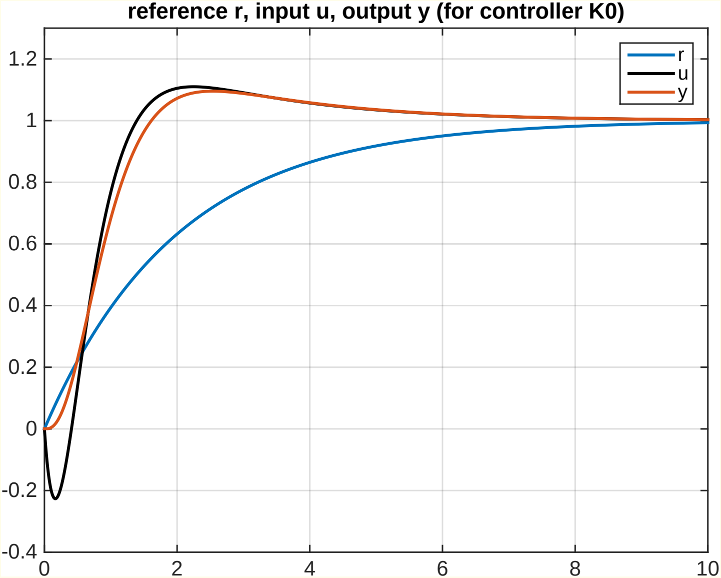
If closed loop as.stable then $\lim_{t\to\infty} e(t) = 0$ if $r(t)$ “constant”

- So if $r(t), w(t)$ constant then $\lim_{t\to\infty} y(t) = r(t)$!!!!!

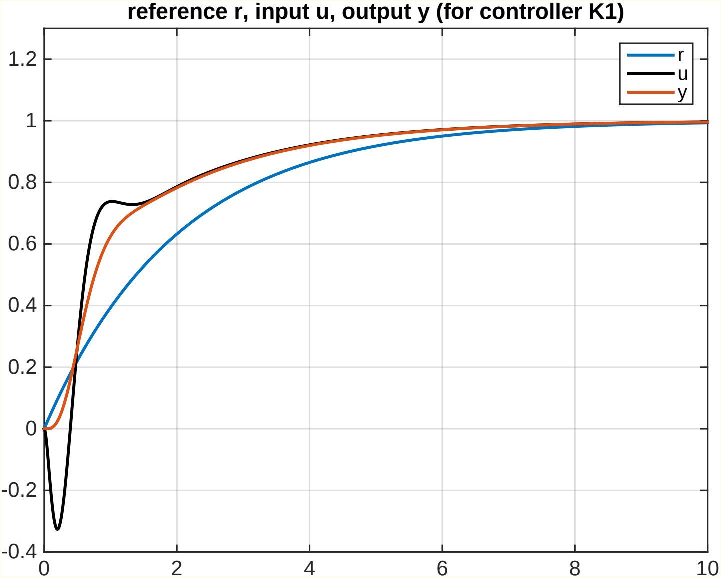
Walking with an inverted pendulum of length 1:



Simulation on next two slides:



reference r, input u, output y (for controller K0)
r
u
y
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
2
4
6
8
10


reference r, input u, output y (for controller K1)
r
u
y
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
2
4
6
8
10