Appendix B

Answers to Some Exercises

Chapter 1

1.1 (a) $n = 1$
(b) nonlinear
(c) yes
(d) yes
(e) -

1.2 (a) $c_1 e^{-3t}$
(b) $c_1 e^{-3/2t}$
(c) $c_1 e^{-t} + c_2 t e^{-t}$
(d) $\hat{c}_1 e^{2it} + \hat{c}_2 e^{-2it} = c_1 \cos(2t) + c_2 \sin(2t)$
(e) $c_1 e^{2t} + c_2 e^{-2t}$
(f) $c_1 + c_2 t + c_3 t^2 + c_4 t^3$

1.3 (a) $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0$
(b) $\ddot{y}(t) + 4\dot{y}(t) + 4y(t) = 0$
(c) $\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 0$

1.4 (I,b), (II,b), (III,f), (IV,d), (V,a), (VI,e)

1.5 (a) $\hat{c}_1 e^{(-1-i)t} + \hat{c}_2 e^{(-1+i)t}$ or, equivalently, $e^{-t}(c_1 \sin(t) + c_2 \cos(t))$
(b) $c_1 e^{2t} + c_2 t e^{2t}$
(c) $c_1 e^{-3t} + c_2 e^{-4t}$
(d) $c_0 + c_1 t + c_2 t^2 + c_3 t^3$
(e) $c e^{-\beta t}$
(f) $c_1 + c_2 e^{(-1+i)t} + c_3 e^{(-1-i)t}$

1.6 (a) $c_1 + c_2 t + c_3 e^{-t}$
(b) $c_1 + c_2 \cos(t) + c_3 \sin(t)$
(c) $c_1 + c_2 e^{-10t} + c_3 e^{8t}$

1.7 (a) $\frac{1}{3} + c e^{-3t}$

(b) $\frac{1}{2}\mathrm{e}^{-t}+c\mathrm{e}^{-3t}$

(c) $5\times\frac{1}{3}+1\times\frac{1}{2}\mathrm{e}^{-t}+c\mathrm{e}^{-3t}$

(d) $\frac{1}{4}t^2-\frac{1}{8}+c_1\mathrm{e}^{2\mathrm{i}t}+c_2\mathrm{e}^{-2\mathrm{i}t}$

(e) $\frac{1}{4}t\mathrm{e}^{2t}+c_1\mathrm{e}^{2t}+c_2\mathrm{e}^{-2t}$

(f) $\frac{5}{4!}t^4+c_1+c_2t+c_3t^2+c_4t^3$

1.9 (a) $\dot{c}(t)=\mathrm{e}^{2t}$

(b) $c(t)=c_1+\frac{1}{2}\mathrm{e}^{2t}$

(c) $y(t)=c_1\mathrm{e}^{-3t}+\frac{1}{2}\mathrm{e}^{-t}$

1.10 (a), (c) and (e) iff $\beta>0$.

1.11 (a) yes

(b) yes

(c) yes

(d) no

(e) no

(f) no

1.12 Unstable. (Has two unstable zeros.)

Chapter 2

2.1 (a) No.

(b) Yes.

(c) yes

(d) yes

(e) yes [this should be removed from notes]

2.2 (a) $A^2=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$ and $A^k=0$ for all $k>2$. So $\mathrm{e}^{At}=\begin{bmatrix}1&t&t^2/2\\0&1&t\\0&0&1\end{bmatrix}$

(b) $x_3(t)=c_3$ so $x_2(t)=c_2+c_3t$ so $x_1(t)=c_1+c_2t+c_3t^2/2$. Which is $x(t)=\mathrm{e}^{At}c$ for $c=(c_1,c_2,c_3)$

2.3 (a) $\dfrac{1}{3}\begin{bmatrix}2e^{-t}+\mathrm{e}^{2t} & -2\mathrm{e}^{-t}+2\mathrm{e}^{2t}\\ -\mathrm{e}^{-t}+e^{2t} & \mathrm{e}^{-t}+2\mathrm{e}^{2t}\end{bmatrix}$

(b) $\dfrac{1}{2}\begin{bmatrix}\mathrm{e}^{-t}+\mathrm{e}^{3t} & -\mathrm{e}^{-t}+\mathrm{e}^{3t}\\ -\mathrm{e}^{-t}+\mathrm{e}^{3t} & \mathrm{e}^{-t}+\mathrm{e}^{3t}\end{bmatrix}$

(c) $\begin{bmatrix}6\mathrm{e}^{2t}-5\mathrm{e}^{4t} & -4\mathrm{e}^{2t}+4\mathrm{e}^{4t}\\ 7.5\mathrm{e}^{2t}-7.5\mathrm{e}^{4t} & -5\mathrm{e}^{2t}+6\mathrm{e}^{4t}\end{bmatrix}$

(d) $\dfrac{1}{4}\begin{bmatrix}2\mathrm{e}^{-t}+2\mathrm{e}^{3t} & -\mathrm{e}^{-t}+\mathrm{e}^{3t}\\ -4\mathrm{e}^{-t}+4\mathrm{e}^{3t} & 2\mathrm{e}^{-t}+2\mathrm{e}^{3t}\end{bmatrix}$

(e) $\begin{bmatrix}\mathrm{e}^{A_1t} & 0\\ 0 & \mathrm{e}^{4t}\end{bmatrix}$ with e^{A_1t} the answer from (a)

(f) $e^t \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}$

(g) $e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

2.7 (a) By induction in n

(b) -

2.10 (a) $d_1 c_2 \neq c_1 d_2$

(b) $\det(X(t)) = d_1 c_2 - c_1 d_2$

(c) $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

2.12 (a) not asymptotically stable

(b) not asymptotically stable

(c) asymptotically stable

(d) not asymptotically stable

(e) not asymptotically stable

(f) asymptotically stable iff both A_{11} and A_{22} are

2.13 $y_\infty = (-CA^{-1}B + D)u_\infty$ (note: A^{-1} exits because it is asymptotically stable)

2.16 (a) $\dot{x}_1 = -4x_1 + 2u, y = x_1$

(b) $\dot{x}_1 = -4x_1 - 2u, y = x_1 + u$

(c)

(d)

(e)

(f) $\dot{x}_1 = 0, \dot{x}_2 = x_1, \dot{x}_3 = x_2, y = x_3 + u$.
(Interesting: $\dot{x}_1 = 0$ means x_1 is an arbitrary constant. Then $\dot{x}_2 = x_1$ means x_2 is an arbitrary degree ≤ 1 polynomial. Then $\dot{x}_3 = x_2$ means that x_3 is an arbitrary degree ≤ 2 polynomial. Then $y = x_3 + u$ means that $y = u$ upto a polynomial of degree ≤ 2. Indeed that is what $\dddot{y} = \dddot{u}$ says.)

Chapter 3

3.1 (a) yes

(b) yes

(c) observable for no $n > 0$.

(d) no (always observable for $C = I$).

(e) yes (because $\left[\begin{smallmatrix} C \\ s_0 I - A \end{smallmatrix}\right]$ has rank at most $n-1$)

(f) no

3.5 (a) not controllable

(b) not controllable

(c) controllable if and only if $b_3 \neq 0$

3.7 -

3.8 $\dot{x} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] x + \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] u.$

3.10 (a) several possibilities: for $T = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, it is $A_z = \left[\begin{smallmatrix} -1 & 1 \\ 0 & -3 \end{smallmatrix}\right]$ and $B_z = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$

(b) several possibilities: for $T = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$, it is $A_z = \left[\begin{smallmatrix} 5 & 2 \\ 0 & -1 \end{smallmatrix}\right]$, $B_z = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$

(c) for $T = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{smallmatrix}\right]$, it becomes $A_z = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right]$, $B_z = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$

3.11 No.

3.17 (a)

$$\begin{bmatrix} sE - A & B \end{bmatrix} = \begin{bmatrix} sL + R & sRC_1 + 1 & 0 & \boxed{+1} \\ 0 & sRC_1 + 1 & -sRC_2 - 1 & 0 \\ \boxed{-1} & sC_1 & sC_2 & 0 \end{bmatrix}$$

Controllable iff $C_1 \neq C_2$.

(b) If $C_1 = C_2$ then the two capacitors behave the same (assuming they are the same at time zero) no matter what you do with u. Hence then $x_2(t) = x_3(t)$ for all time and they can not be controlled to, say, $x_2(t_1) = +2$ and $x_3(t_1) = -2$.

3.19 (a) yes

(b) yes because $\mathscr{O}$ is invertible:

$$\mathscr{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{M}{m}g & 0 \\ 0 & 0 & 0 & -\frac{M}{m}g \end{bmatrix}$$

(c) no because for $y = x_m$ it is already observable

3.20 (a) $\mathscr{O} = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, not observable because not invertible.

(b) $\mathscr{O} = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 5 \end{smallmatrix}\right]$, observable because invertible.

(c) $\mathscr{O} = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{smallmatrix}\right]$, not observable because first and last column the same.

(d) observable if $n = 1$ or $n = 2$ but not for $n > 2$.

3.21 (a) $\mathscr{O} = \left[\begin{smallmatrix} 2 & 2 \\ 4 & 4 \end{smallmatrix}\right]$. One possible choice of T is $T = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$. That one gives $A_z = T^{-1}AT = \left[\begin{smallmatrix} 2 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $C_z = CT = \left[\begin{smallmatrix} 2 & 0 \end{smallmatrix}\right]$. (You might have choosen another T, but whichever correct T you choose, the diagonal entries of A_z must equal 2 and 0.)

(b) -

3.24 (a) -

(b)

$$\mathscr{O} = \begin{bmatrix} C_1 & C_2 \\ C_1 A_1 & C_2 A_2 \\ \vdots & \vdots \end{bmatrix}$$

So if $C_1 = C_2$ and $A_1 = A_2$ then.....

3.27 (a) $\mathscr{C} = \left[\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right], \mathscr{C}^{-1} = \left[\begin{smallmatrix} 4 & -3 \\ -1 & 1 \end{smallmatrix}\right], T^{-1} = \left[\begin{smallmatrix} -1 & 1 \\ 2 & -1 \end{smallmatrix}\right], T = \left[\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right]$: $A_z = \left[\begin{smallmatrix} 0 & 1 \\ 5 & 2 \end{smallmatrix}\right], B_z = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], C_z = \left[\begin{smallmatrix} 3 & 1 \end{smallmatrix}\right]$

(b) $\mathscr{O} = \left[\begin{smallmatrix} -1 & 2 \\ 5 & 0 \end{smallmatrix}\right], \mathscr{O}^{-1} = \frac{1}{10}\left[\begin{smallmatrix} 0 & 2 \\ 5 & 1 \end{smallmatrix}\right]$, $T = \frac{1}{10}\left[\begin{smallmatrix} 2 & 4 \\ 1 & 7 \end{smallmatrix}\right], T^{-1} = \left[\begin{smallmatrix} 7 & -4 \\ -1 & 2 \end{smallmatrix}\right]$: $A_z = \left[\begin{smallmatrix} 0 & 5 \\ 1 & 2 \end{smallmatrix}\right], B_z = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right], C_z = \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]$.

Chapter 4

4.1 (a) No.
(b) Yes.
(c) Yes.
(d) No.
(e) Yes.
(f) Yes.
(g) Yes.

4.2 (a) $\beta \neq \pm 1$
(b) Let $z = -\beta x_1 + x_2$. For $\beta = 1$ it says that $\dot{z} = -z$ so if $z(0) = 0$ then it stays $z(t) = 0$ for all t. Hence $x_1(t) = x_2(t)$ irrespective of our choice of u: not controllable.
For $\beta = -1$ it says that $\dot{z} = z$ so if $z(0) = 0$ then, again, it stays $z(t) = 0$ for all t. Hence then $x_1(t) = -x_2(t)$ so again not controllable.
(c) yes
(d) $F = \begin{bmatrix} 1 & 0 \end{bmatrix}$
(e) $L = \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$ so $\dot{\hat{x}} = \left[\begin{smallmatrix} -2 & 1 \\ -1 & 0 \end{smallmatrix}\right]\hat{x} + \left[\begin{smallmatrix} 1 \\ \beta \end{smallmatrix}\right]u + \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]y$.

4.3 (a) $\alpha \neq 0$
(b) $\alpha \neq 0$
(c) $\det(sI - A) = (s-1)^2$
(d) -
(e) -
(f) Then $F = \begin{bmatrix} 4.5 & 0.75 \end{bmatrix}$
(g) -

4.4 (a) yes and yes
(b) no
(c) no: $A - BHC = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -H & 0 & 0 \end{smallmatrix}\right]$ so....
(d) $(s+1+2\mathrm{i}))(s+1-2\mathrm{i})(s+2) = (s^2+2s+5)(s+2) = s^3+4s^2+9s+10$, so $F = \begin{bmatrix} 10 & 9 & 4 \end{bmatrix}$

4.6 $F = \begin{bmatrix} -p_0 & -p_1 & \cdots & -p_{n-1} \end{bmatrix}$. Then $A - BF$ has characteristic polynomial s^n. According to Cayley-Hamilton this implies $(A - BF)^n = 0$ (the zero matrix).

4.9 -

4.12 The matrix $\left[\begin{smallmatrix} C \\ sI-A \end{smallmatrix}\right]$ for Kalman's observer decomposition takes the form

$$\begin{bmatrix} C_1 & 0 \\ sI - A_{11} & 0 \\ -A_{21} & sI - A_{22} \end{bmatrix} \ldots..$$

4.15 (a) $F = \left[\begin{smallmatrix} -1 & 3 \end{smallmatrix}\right]$
(b) $L = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$, so $\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly = \left[\begin{smallmatrix} -3 & 1 \\ 0 & -2 \end{smallmatrix}\right]\hat{x} + \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]u + \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]y$

(c) $A - BF - LC = \left[\begin{smallmatrix} -2 & -2 \\ 0 & -2 \end{smallmatrix}\right]$, so the controller is $\dot{\hat{x}} = \left[\begin{smallmatrix} -2 & -2 \\ 0 & -2 \end{smallmatrix}\right]\hat{x} + \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]y$ with $u = \left[\begin{smallmatrix} 1 & -3 \end{smallmatrix}\right]\hat{x}$

4.16 (a) iff $\alpha = 0$.

(b) $\frac{4}{\alpha}\left[\begin{smallmatrix} \alpha-1 \\ 1 \end{smallmatrix}\right]$

(c) The formula for L_α is not defined at $\alpha = 0$ because of division by zero. Makes perfect sense because by part (a) of this exercise we know that such L can not exist at $\alpha = \alpha_{\text{nondetec}} = 0$.

Chapter 5

5.1 $P = 1/2, F = 2, A - BB^{\mathrm{T}}P = -5$.

5.2 $P = F = \sqrt{2}. A - BB^{\mathrm{T}}P = 1 - \sqrt{2}$.

5.3 (a) No, so existence of P is not guaranteed (by that theorem) but it might exist

(b) $P = 1.5, F = 3, A - BB^{\mathrm{T}}P = -3$

(c) $u = F = P = 0$...

5.4 -

5.5 (a) yes and yes

(b) -

(c) See Exercise 5.6(e): $P_{1,1} = 2 > 0, \det(P) = 1 > 0$ so yes

(d) Yes says Thm. 5.4.2. Also follows directly: $A - BB^{\mathrm{T}}P = \left[\begin{smallmatrix} 0 & 1 \\ -1 & -2 \end{smallmatrix}\right]$ which has eigenvalues -1 (double).

(e) $F := B^{\mathrm{T}}P = \left[\begin{smallmatrix} 1 & 3 \end{smallmatrix}\right]$ does the job.

(f) it must be -1 (double) and $+1$ (double)....

(g) No: $H + I$ has rank 3, not 2. (Comment: to compute P from the Hamiltonian you might want to compute $(H + I)^2$ and determine its null space (kernel).)

5.8 If P would have been singular, then every $x_0 \in \ker(P)$ makes the optimal cost equal to zero. So $y(t) = 0$ but by observability that would mean $x_0 = 0$.

5.13 (a)

(b)

(c)

(d)

(e) $-\delta$ (double)

(f) $\left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right]$ spans null space of $(H + \delta I)^2$.

$$(H + \delta I)^2 = \begin{bmatrix} \delta^2 & 2\delta & 0 & -1 \\ 0 & 3\delta^2 & 1 & -2\delta \\ -2\delta^5 & -\delta^4 & \delta^2 & 0 \\ \delta^4 & -4\delta^3 & -2\delta & 3\delta^2 \end{bmatrix}$$

Apply row operations:

$$
\begin{aligned}
&\sim \begin{bmatrix} \delta^2 & 2\delta & 0 & -1 \\ -2\delta^3 & -\delta^2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 2\delta^3 & \delta^2 & -1 & 0 \\ \delta^2 & 2\delta & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\implies \quad & P = \begin{bmatrix} 2\delta^3 & \delta^2 \\ \delta^2 & 2\delta \end{bmatrix}.
\end{aligned}
$$

(g) (If $\delta = 0$ then $P = 0$ so positive semi-definite). If $\delta > 0$ then $P_{1,1} = 2\delta^3 > 0$ and $\det(P) = 3\delta^4$.

(h) Then cost function converges to $\int_0^\infty u^2(t)\,\mathrm{d}t$. Optimal would be to take $u = 0$ but that one destabilizes. However a very small $u = -\begin{bmatrix} \delta^2 & 2\delta \end{bmatrix} x$ stabilizes........