# Introduction to Partial Differential Equations

Carlos Perez-Arancibia

c.a.perezarancibia@utwente.nl

Matthias Schlottbom

m.schlottbom@utwente.nl

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# Contents

# 1 Introduction

This introductory course consists of eight lectures on different topics in the broad field of partial differential equations (PDEs). The required pre-knowledge is basic analysis and linear algebra skills. The discussed topics are the following:

Chapter 2 This chapter derives the three classical PDEs from physical principles: Heat equation (parabolic) for diffusion processes; Poisson and Laplace equations (elliptic) for steady-state phenomena; wave equation (hyperbolic) for propagation dynamics. It also introduces well-posed problems and the classification of PDEs.

Chapter 3 We discuss the classical treatment of elliptic equations, focusing on: Maximum principle, which provides uniqueness and stability of solutions; Green's representation formula, used to express solutions in terms of boundary data. Lecture (This lecture is mainly based on [3, Secs. 3.1, 3.2 and 3.4] )

Chapter 4 We introduce weak formulations of elliptic PDEs. Key concepts include: Sobolev spaces, which generalize classical function spaces. Well-posedness and regularity of weak solutions, ensuring existence and stability. It is mainly based on [3, Sec. 3.5]. The interested student is referred to the master class PDE (201700033) for a more advanced treatment and detailed proofs.

Chapter 5 The Finite Element Method (FEM) explores the Ritz-Galerkin method for approximating solutions of elliptic PDEs, including: Error analysis and convergence of FEM. Construction of piecewise polynomial function spaces for practical computation. Lecture 4 is mainly based on [3, Chapter 5] and [**?**, Chapter 0]. We give the basic ideas of the finite element method; the interested student is referred to the master class 'Finite Element Methods' (202100097) for an advanced treatment.

Chapter 6 The Eigenvalue Problem for Elliptic PDEs Covers eigenvalue problems in PDEs, essential for understanding vibrations and stability. Topics include: Eigenfunction expansions for representing solutions. FEM for approximating eigenvalues and eigenfunctions.

Chapter 7 Parabolic PDEs – Variational and Numerical Treatment: Focuses on time-dependent PDEs, particularly the heat equation. Discusses: Lecture 7 is mainly based on [3, Sec. 8.3, Chapter 10]. Lecture 6 is mainly based on [3, Secs. 8.3 and 8.4] )

Chapter 8

After reading these notes, the reader should be able to read and understand most parts of the book [3]. Further, recommended further reading is contained in [].

# 2 Modeling using partial differential equations

## Lecture 1

In this lecture, we derive the three classical partial differential equations (PDEs) — the heat equation, Poisson's equation, and the wave equation — starting from fundamental physical principles, which will serve as main examples of three important types of linear second-order PDEs: parabolic, elliptic, and hyperbolic.

## 2.2 Heat Equation

Let us consider an open and bounded domain $V \subset \mathbb{R}^d$, $d = 2, 3$, with smooth boundary $S := \partial V$ (of class $C^2$), and let us denote by $\mathbf{n} : S \to \mathbb{R}^3$ the unit normal vector to $V$; $|\mathbf{n}| := \|\mathbf{n}\|_2 = \left(\sum_{j=1}^{d} n_j^2\right)^{1/2} = 1$.

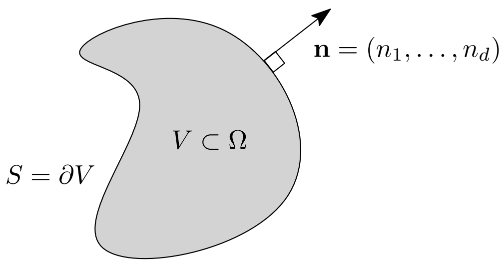


Figure 2.1: Illustration of the domain $V$ and its boundary $S$.

We seek to predict the temperature evolution and spatial variation of the temperature of a fluid enclosed in the open domain $\Omega \subseteq \mathbb{R}^d$ containing $V$.

Consider then the following quantity:

$$e := e(\mathbf{x}, t);\ \text{heat energy per unit mass at the point } \mathbf{x} \in \Omega \text{ at time } t \in \mathbb{R}_+ := (0, \infty).$$

The total thermal energy $E : \mathbb{R}_+ \to [0, \infty)$ enclosed within in a domain $V \subset \Omega$ can be computed from the energy density $e : \Omega \times \mathbb{R}_+ \to [0, \infty)$ via the volume integral:

$$E(t) = \int_V e(\mathbf{x}, t)\rho(\mathbf{x}, t)\,\mathrm{d}\mathbf{x}, \tag{2.1}$$

where

$$\rho := \rho(\mathbf{x}, t);\ \text{mass density at } \mathbf{x} \in \Omega \text{ and } t \in \mathbb{R}_+.$$

From (2.1), and assuming $e\rho$ is sufficiently regular, the time derivative of the total energy is

$$\frac{\mathrm{d}E}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_V e(\mathbf{x}, t)\rho(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = \int_V \frac{\partial}{\partial t}\big(e(\mathbf{x}, t)\rho(\mathbf{x}, t)\big)\,\mathrm{d}\mathbf{x}. \tag{2.2}$$

(See, e.g., [4, Thm. 11.9] for sufficient conditions under which exchanging the integral and the derivative is allowed.)

Since thermal energy can flow within $\Omega$, we introduce the vector quantity

$$\boldsymbol{q} := \boldsymbol{q}(\mathbf{x}, t); \text{ heat flux at } \mathbf{x} \in \Omega \text{ at } t \in \mathbb{R}_+.$$

In detail, heat flux refers to the rate at which heat energy flows through a given volume per unit time. It is a vector quantity, with the magnitude representing the amount of heat flowing through the volume, and the direction indicating the flow of heat.

The energy flowing **i**nto $D \subset \Omega$ is then given by:

$$\text{energy entering } V = -\int_S \boldsymbol{q}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) \, \mathrm{d}s.$$

Recall that $\mathbf{n}$ is the outward unit normal vector. Applying the divergence theorem (see, e.g., [4, Ch. 13.5]) the energy entering $V$ can be written as:

$$-\int_S q(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) \, \mathrm{d}s = -\int_V \operatorname{div} \boldsymbol{q}(\mathbf{x}, t) \, \mathrm{d}\mathbf{x}.$$

Combining this with (2.2) gives:

$$\frac{\mathrm{d}E}{\mathrm{d}t}(t) = \int_V \frac{\partial}{\partial t}\big(e(\mathbf{x}, t)\rho(\mathbf{x}, t)\big) \, \mathrm{d}\mathbf{x} = -\int_V \operatorname{div} \boldsymbol{q}(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} + \int_V F(\mathbf{x}, t)\rho(\mathbf{x}, t) \, \mathrm{d}\mathbf{x},$$

where $F(\mathbf{x}, t)$ is a heat source term.

Since this relation

$$\int_V \left\{ \frac{\partial}{\partial t}\big(e(\mathbf{x}, t)\rho(\mathbf{x}, t)\big) + \operatorname{div} \boldsymbol{q}(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} - F(\mathbf{x}, t)\rho(\mathbf{x}, t) \right\} \mathrm{d}\mathbf{x} = 0, \tag{2.3}$$

holds on any sufficiently regular domain $V$ immerse the fluid domain $\Omega$, we conclude that

$$\frac{\partial}{\partial t}\big(e(\mathbf{x}, t)\rho(\mathbf{x}, t)\big) = -\operatorname{div} \boldsymbol{q}(\mathbf{x}, t) + F(\mathbf{x}, t)\rho(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \ t \in \mathbb{R}_+, \tag{2.4}$$

assuming the quantity inside the curly brackets in (2.3) is continuous (as a function of $\mathbf{x}$). Why?

### Constitutive Relations

In (2.4), we have two unknown quantities, $e(\mathbf{x}, t)$ and $\boldsymbol{q}(\mathbf{x}, t)$. To reduce the equation to a single unknown, we apply the following physical laws that relate the heat flux and the energy density to the temperature:

a) Fourier's Law of heat conduction:

$$\boldsymbol{q}(\mathbf{x}, t) = -k(\mathbf{x})\nabla u(\mathbf{x}, t), \tag{2.5}$$

where $k(\mathbf{x}) > 0$ is the *thermal conductivity* and $u(\mathbf{x}, t)$ is the *temperature* at a the location $\mathbf{x}$ and time $t$.

b) Specific heat relation:

$$e(\mathbf{x}, t) = c(\mathbf{x})u(\mathbf{x}, t), \tag{2.6}$$

where $c(\mathbf{x})$ is the specific heat of the fluid at the location $\mathbf{x}$.

Substituting (2.5) and (2.6) into (2.4) yields:

$$\boxed{\frac{\partial}{\partial t}\big(c(\mathbf{x})\rho(\mathbf{x}, t)u(\mathbf{x}, t)\big) - \operatorname{div}\big(k(\mathbf{x})\nabla u(\mathbf{x}, t)\big) = F(\mathbf{x}, t)\rho(\mathbf{x}, t)} \tag{2.7}$$

for $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}_+$.

The unknown quantity of interest in the equation above is the temperature distribution $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$. Since $u$ and its partial derivatives appear in the equation, it is classified as a *Partial Differential Equation* (PDE). Specifically, this PDE is referred to as the *Heat Equation.*

**Remark 2.1.** The Laplace operator (or Laplacian) is defined as:

$$\Delta v(\mathbf{x}) = \operatorname{div} \nabla v(\mathbf{x}) = \sum_{j=1}^{d} \frac{\partial^2 v}{\partial x_j^2}(\mathbf{x}),$$

where $\nabla = (\partial_{x_1}, \dots, \partial_{x_d})$ denotes the gradient operator and div denotes the divergence operator, defined as:

$$\operatorname{div} \boldsymbol{f}(\mathbf{x}) = \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i}(\mathbf{x}),$$

for a vector field $\boldsymbol{f} = (f_1, f_2, \dots, f_d)$.

Note that in the case when $c$, $\rho$, and $k$ are constant positive scalars, the heat equation reduces to:

$$\boxed{\frac{\partial u}{\partial t}(\mathbf{x}, t) - \eta \Delta u(\mathbf{x}, t) = f(\mathbf{x}, t)} \tag{2.8}$$

for $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}_+$, where $\eta > 0$ and $f$ are commonly referred to as the thermal diffusivity and the source term, respectively.

### Initial and Boundary Conditions

Similar to Ordinary Differential Equations (ODEs), PDEs must be accompanied by additional conditions to ensure the existence and uniqueness of solutions. Unlike ODEs, time-evolution PDEs, such as the heat equation, require not only initial conditions but also information about the solution on the boundary of the domain where they are to be solved.

In order for (2.8) to be *solvable* in a fixed bounded domain $\Omega \subset \mathbb{R}^d$, the following conditions are required:

a) Initial Condition: The temperature distribution at $t = 0$ is prescribed as:

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

b) Boundary Conditions: For a bounded domain $\Omega$ with boundary $\Gamma := \partial\Omega$, we may have:

i) **D**irichlet Condition: Prescribed temperature at the boundary

$$u(\mathbf{x},t) = g(\mathbf{x},t), \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+.$$

ii) **N**eumann Condition: Prescribed energy flux $\boldsymbol{q} = -k\nabla u$ across the domain's boundary:

$$\frac{\partial u}{\partial n}(\mathbf{x},t) := \nabla u(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) = g(\mathbf{x},t), \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+.$$

(The *normal derivative* $\dfrac{\partial u}{\partial n}$ of $u$ at a boundary point is the directional derivative of $u$ in the direction normal to the boundary)

iii) **R**obin Condition: Newton's cooling law:

$$\frac{\partial u}{\partial n}(\mathbf{x},t) + \gamma(x)u(\mathbf{x},t) = \gamma(\mathbf{x})u_0(\mathbf{x},t), \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+,$$

where $\gamma(\mathbf{x}) > 0$ is quantity related to the heat transfer coefficient, and $u_0(\mathbf{x},t)$ is the known ambient temperature at the boundary.

iv) **M**ixed Conditions (Zaremba): On one part of $\Gamma$, Dirichlet conditions are applied, while on the complement, Neumann conditions are imposed:

$$u(\mathbf{x},t) = g_D(\mathbf{x},t), \quad x \in \Gamma_D, \quad \text{and} \quad \frac{\partial u}{\partial n}(\mathbf{x},t) = g_n(\mathbf{x},t) \quad \mathbf{x} \in \Gamma_N,\ t \in \mathbb{R}_+,$$

where $\Gamma = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

## 2.3 Poisson (and Laplace) Equation

Consider the heat equation of the previous section:

$$\frac{\partial}{\partial t}\big(c(\mathbf{x})\rho(\mathbf{x})u(\mathbf{x},t)\big) = \operatorname{div}\big(k(\mathbf{x})\nabla u(\mathbf{x},t)\big) + F(\mathbf{x})\rho(\mathbf{x}),\ \mathbf{x} \in \Omega,\ t \in \mathbb{R}_+, \tag{2.9}$$

with the following boundary and initial conditions:

$$u(\mathbf{x},0) = u_0(\mathbf{x}),\ \mathbf{x} \in \Omega, \quad u(\mathbf{x},t) = g(\mathbf{x}),\ \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+.$$

Given that neither the source $F$ nor the mass density $\rho$ nor the boundary data $g$ vary with time, we expect the temperature to reach a steady state, meaning that

$$\lim_{t\to\infty} \frac{\partial u(\mathbf{x},t)}{\partial t} = 0 \quad \text{and} \quad \lim_{t\to\infty} u(\mathbf{x},t) = v(\mathbf{x}) \text{ uniformly in } \mathbf{x} \in \Omega.$$

Therefore, assuming sufficient regularity of $u$ and $v$ (enough to exchange the time limit with the spatial partial derivatives), we obtain

$$\lim_{t\to\infty}\left(\frac{\partial}{\partial t}(c\rho u) - \operatorname{div}\big(k\nabla u\big) - F\rho\right) = -\operatorname{div}\big(k\nabla v\big) - F\rho = 0,$$

from where we obtain the following time-independent PDE for the stationary temperature distribution $v$:

$$\boxed{-\operatorname{div}\big(k(\mathbf{x})\nabla v(\mathbf{x})\big) = f(\mathbf{x})} \tag{2.10}$$

for $\mathbf{x} \in \Omega$ in this case.

The PDE (2.10) is known as the *Poisson Equation*, named after the early 19th-century French mathematician Siméon-Denis Poisson, who mentioned it in his works on electrostatic and gravitational potentials. The homogeneous version of the same equation (i.e., $-\operatorname{div}(k(\mathbf{x})\nabla v(\mathbf{x})) = 0$ for $\mathbf{x} \in \Omega$) is known as the *Laplace Equation*, which was studied earlier by another French mathematician, Pierre-Simon Laplace.

The boundary conditions for the Poisson/Laplace equation are the same as those for the heat equation: Dirichlet, Neumann, Robin, or mixed conditions.

## 2.4 Wave Equation

We now derive our third prototypical PDE by describing the propagation of sound or acoustic waves. These waves can be modeled as small-amplitude perturbations in the mass density ($\rho$) and pressure ($p$) of a compressible fluid, such as air, around a constant state characterized by the ambient mass density $\rho_0$ and pressure $p_0$. Specifically, we express the mass density and pressure as

$$\rho_T(\mathbf{x}, t) = \rho_0 + \rho(\mathbf{x}, t), \quad |\rho(\mathbf{x}, t)| \ll \rho_0, \tag{2.11}$$

$$p_T(\mathbf{x}, t) = p_0 + p(\mathbf{x}, t), \quad |p(\mathbf{x}, t)| \ll p_0. \tag{2.12}$$

To begin, we first discuss a PDE that describes the conservation of mass, also known as the advection equation. Consider a domain within a fluid, where the vector field $\boldsymbol{v} = \boldsymbol{v}(\mathbf{x}, t)$ represents the velocity of the fluid at a point $\mathbf{x}$ at time $t > 0$. Since $\rho_T(\mathbf{x}, t)$ denotes the mass density, the total mass enclosed in a domain $V$ at time $t > 0$ is given by

$$m(t) = \int_V \rho_T(\mathbf{x}, t)\,\mathrm{d}\mathbf{x}.$$

Assuming there are no sources or sinks of mass, the only mechanism by which the enclosed mass can change over time is due to the net mass flux through the boundary $S = \partial V$:

$$-\int_\Gamma \boldsymbol{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})\rho_T(\mathbf{x}, t)\,\mathrm{d}s.$$

Therefore, applying the divergence theorem, the time rate of change of the enclosed mass is given by

$$\frac{\mathrm{d}m}{\mathrm{d}t}(t) = \int_V \frac{\partial}{\partial t}\rho_T(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = -\int_S \boldsymbol{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})\rho_T(\mathbf{x}, t)\,\mathrm{d}s = -\int_V \operatorname{div}(\boldsymbol{v}(\mathbf{x}, t)\rho_T(\mathbf{x}, t))\,\mathrm{d}\mathbf{x},$$

which holds in an arbitrary domain $V \subset \Omega \subseteq \mathbb{R}^d$. Assuming continuity of the integrands above, we arrive at the *Advection Equation*:

$$\boxed{\frac{\partial \rho_T}{\partial t}(\mathbf{x}, t) + \operatorname{div}(\boldsymbol{v}(\mathbf{x}, t)\rho_T(\mathbf{x}, t)) = 0} \tag{2.13}$$

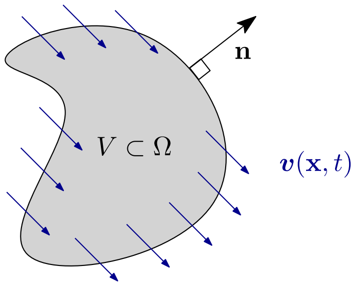


Figure 2.2: Fluid flow through a volume $V$.

for $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}_+$.

This PDE describes the transport of a quantity (in this case, mass) by a flow field, often represented by a velocity vector $\boldsymbol{v}$. It can also be applied to describe the transport of other physical quantities, such as momentum, for example.

Under the assumption $\rho(\mathbf{x}, t) \ll \rho_0$, we neglect $\rho(\mathbf{x}, t)$ in the mass advection equation to obtain

$$\frac{\partial \rho_T}{\partial t}(\mathbf{x}, t) + \rho_0 \operatorname{div} \boldsymbol{v}(\mathbf{x}, t) = 0. \tag{2.14}$$

Following a similar treatment, but now for enclosed momentum instead of mass, we apply Newton's second law of motion, which states that the rate of change of momentum is equal to the net force acting on the system. For a compressible fluid, the net force acting on the fluid element is related to the gradient of pressure $p$, resulting in the following simplified momentum conservation equation:

$$\frac{\partial \boldsymbol{v}}{\partial t}(\mathbf{x}, t) + \frac{1}{\rho_0} \nabla p(\mathbf{x}, t) = 0. \tag{2.15}$$

This equation represents the momentum balance, where the acceleration of the fluid is directly related to the pressure gradient.

To close the system, we apply the constitutive law:

$$p(\mathbf{x}, t) = c^2(\mathbf{x}) \rho(\mathbf{x}, t), \tag{2.16}$$

where $c^2 := \left(\frac{\partial P}{\partial \rho_0}\right)_s$ denotes the derivative of the pressure with respect to the mass density $\rho_0$, assuming constant entropy. The quantity $c(\mathbf{x}) > 0$ is called the *Speed of Sound*. The mathematical meaning of $c$ as "speed" will be discussed in more detail later in the course.

Now, taking divergence to both ides of (2.15) we get

$$\frac{\partial}{\partial t} \left(\operatorname{div} \boldsymbol{v}(\mathbf{x}, t)\right) + \frac{1}{\rho_0} \Delta p(\mathbf{x}, t) = 0.$$

On the other hand, from (2.14), we have:

$$\operatorname{div} \boldsymbol{v}(\mathbf{x}, t) = -\frac{1}{\rho_0} \frac{\partial \rho_T}{\partial t}(\mathbf{x}, t) = -\frac{1}{\rho_0} \frac{\partial \rho}{\partial t}(\mathbf{x}, t).$$

Therefore, combining these last two results, we obtain:

$$\frac{-1}{\rho_0} \frac{\partial^2 \rho}{\partial t^2}(\mathbf{x}, t) + \frac{1}{\rho_0} \Delta p(\mathbf{x}, t) = 0.$$

Finally, from (2.16), we know that $p(\mathbf{x},t) = c^2(\mathbf{x})\rho(\mathbf{x},t)$, so:

$$\boxed{\frac{\partial^2 p}{\partial t^2}(\mathbf{x},t) - c^2(\mathbf{x})\Delta p(\mathbf{x},t) = 0} \tag{2.17}$$

for $\mathbf{x} \in \Omega \subseteq \mathbb{R}^d$ and $t \in \mathbb{R}_+$, which is known as the *Wave Equation.*

Since the equations involve second-order time derivatives, we need two initial conditions. For the pressure, for instance, we need

$$p(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{and} \quad \frac{\partial p}{\partial t}(\mathbf{x},0) = v_0(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

where $\Omega$ is domain where (**??**) is to be solved and where $u_0$ and $v_0$ are given data for the problem.

To derive boundary conditions for acoustics, let us first consider what happens at the interface between two different fluids occupying disjoint domains $\Omega_1$ and $\Omega_2$ with a common boundary $\Gamma$. There, the following conditions need to be fulfilled at the interface.

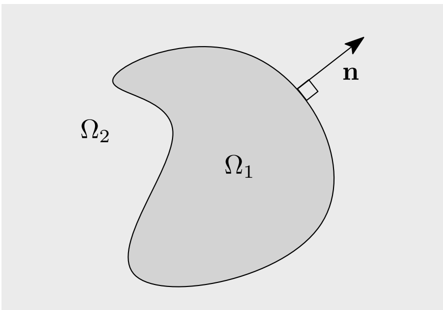


Figure 2.3: Acoustic Conditions at the Interface Between Two Compressible Fluids.

i) <u>Kinematic condition:</u> Continuity of the normal velocity:

$$\mathbf{n}(\mathbf{x}) \cdot \boldsymbol{v}_1(\mathbf{x},t) = \mathbf{n}(\mathbf{x}) \cdot \boldsymbol{v}_2(\mathbf{x},t), \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+.$$

Differentiating with respect to time, we obtain:

$$\mathbf{n}(\mathbf{x}) \cdot \frac{\partial \boldsymbol{v}_1}{\partial t}(\mathbf{x},t) = \mathbf{n}(\mathbf{x}) \cdot \frac{\partial \boldsymbol{v}_2}{\partial t}(\mathbf{x},t), \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+.$$

However, from (2.15), we know that:

$$\frac{\partial \boldsymbol{v}_j}{\partial t}(\mathbf{x},t) = -\frac{1}{\rho_{0j}} \nabla p_j(\mathbf{x},t), \quad \mathbf{x} \in \Omega_j, \quad j = 1,2,\ t \in \mathbb{R}_+.$$

Substituting this into the time derivative of the kinematic condition (assuming the relation above holds up to and including the boundary), we get:

$$\frac{1}{\rho_{01}} \frac{\partial p_1}{\partial n}(\mathbf{x},t) = \frac{1}{\rho_{02}} \frac{\partial p_2}{\partial n}(\mathbf{x},t) \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+. \tag{2.18}$$

ii) Dynamic condition: In the absence of external forces at the interface between the two fluids, we have that the pressure should remain continuous across the interface. Therefore, we obtain

$$p_1(\mathbf{x}, t) = p_2(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+. \tag{2.19}$$

Based on these condition, the following approximations are often used in practical applications:

1. Sound hard boundary condition; when $\rho_{02} \gg \rho_{01}$. For example, when $\Omega_1$ is air and $\Omega_2$ is water and we are interested in the sound wave propagating in the air.

   From the kinematic condition (2.18), using the approximation $\rho_{02}^{-1} \approx 0$ we get the *Neumann boundary condition*:

   $$\frac{\partial p_1}{\partial n}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+.$$

   The problem then reduces to find $p_1 : \Omega_1 \times \mathbb{R}_+ \to \mathbb{R}$ such that

   $$\begin{cases} \dfrac{\partial^2 p_1}{\partial t^2}(\mathbf{x}, t) - c_1^2(\mathbf{x})\Delta p_1(\mathbf{x}, t) &= f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega_1,\ t \in \mathbb{R}_+, \\ p_1(\mathbf{x}, 0) &= u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega_1 \\ \dfrac{\partial p_1}{\partial t}(\mathbf{x}, 0) &= v_0(\mathbf{x}), \quad \mathbf{x} \in \Omega_1, \\ \dfrac{\partial p_1}{\partial n}(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+, \end{cases}$$

   where we also allow for the possibility of a sound source, modeled by the function $f$.

2. Sound soft boundary condition; when $\rho_{01} \gg \rho_{02}$. For example, when $\Omega_1$ is water and $\Omega_2$ is air and we are interested in the sound wave propagating in the water.

   We consider the scaled quantities $u_j := p_j/\rho_{0j}$, $j = 1, 2$. From (2.19), we have $\rho_{01}u_1 = \rho_{02}u_2$ on $\Gamma$, or equivalently $u_1 = \frac{\rho_{02}}{\rho_{01}}u_2$. Therefore, using the approximation $\rho_{02} \approx 0$ we arrive at the following *Dirichlet boundary condition*:

   $$p_1(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+.$$

   The problem then reduces to find $p_1 : \Omega_1 \times \mathbb{R}_+ \to \mathbb{R}$ such that

   $$\begin{cases} \dfrac{\partial^2 p_1}{\partial t^2}(\mathbf{x}, t) - c_1^2(\mathbf{x})\Delta p_1(\mathbf{x}, t) &= f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega_1,\ t \in \mathbb{R}_+, \\ p_1(\mathbf{x}, 0) &= u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega_1 \\ \dfrac{\partial p_1}{\partial t}(\mathbf{x}, 0) &= v_0(\mathbf{x}), \quad \mathbf{x} \in \Omega_1, \\ p_1(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \Gamma,\ t \in \mathbb{R}_+. \end{cases}$$

## 2.5 Well-posed problems

We say that a problem is well-posed if it consists of a PDE in a domain, along with a set of initial and boundary conditions (or other auxiliary conditions) that lead to the following properties:

i) Existence: There exists at least one solution $u$ that satisfies all the conditions.

ii) Uniqueness: There is at most one solution.

iii) Stability: The unique solution $u$ depends stably on the data of the problem. This means that if the data of the problem changes slightly, the corresponding solution also changes slightly.

We will make each of these statements more precise when we introduce proper function spaces for PDE solutions. Meanwhile, let us consider two elementary examples:

**Example 2.2.** Consider the heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(\mathbf{x},t) - \eta\Delta u(\mathbf{x},t) &= f(\mathbf{x},t), \quad \mathbf{x}\in\Omega,\ t\in\mathbb{R}_+, \\ \dfrac{\partial u}{\partial n}(\mathbf{x},t) &= g(\mathbf{x},t), \quad \mathbf{x}\in\Gamma,\ t\in(0,T), \\ u(\mathbf{x},0) &= u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \end{cases}$$

where $\eta > 0$ and $\Omega$ is open and bounded. We want to study uniqueness of solutions of this problem.

Suppose for a contradiction that there are two distinct solutions, $u_1$ and $u_2$, i.e., $u_1 \neq u_2$. Then, by linearity, we have that $w = u_2 - u_1$ satisfies the following homogeneous problem:

$$\begin{cases} \dfrac{\partial w}{\partial t}(\mathbf{x},t) - \eta\Delta w(\mathbf{x},t) &= 0, \quad \mathbf{x}\in\Omega,\ t\in\mathbb{R}_+, \\ \dfrac{\partial w}{\partial n}(\mathbf{x},t) &= 0, \quad \mathbf{x}\in\Gamma,\ t\in\mathbb{R}_+, \\ w(\mathbf{x},0) &= 0, \quad \mathbf{x}\in\Omega. \end{cases}$$

Define then (the energy functional):

$$E(t) := \frac{1}{2}\int_\Omega w^2(\mathbf{x},t)\,\mathrm{d}\mathbf{x}, \quad t\in\mathbb{R}_+.$$

Clearly, $E(t) \geq 0$ for all $t\in\mathbb{R}_+$.

Taking time derivative

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t}(t) &= \int_\Omega \frac{\partial w}{\partial t}(\mathbf{x},t) w(\mathbf{x},t)\,\mathrm{d}\mathbf{x} && (i) \\ &= \eta\int_\Omega w(\mathbf{x},t)\Delta w(\mathbf{x},t)\,\mathrm{d}\mathbf{x} && (ii) \\ &= \eta\left(\int_\Omega \mathrm{div}\left(w(\mathbf{x},t)\nabla w(\mathbf{x},t)\right)\,\mathrm{d}\mathbf{x} - \int_\Omega |\nabla w(\mathbf{x},t)|^2\,\mathrm{d}\mathbf{x}\right) && (iii) \\ &= \eta\left(\int_\Gamma \frac{\partial w}{\partial n}(\mathbf{x},t)\cdot w(\mathbf{x},t)\,\mathrm{d}s - \int_\Omega |\nabla w(\mathbf{x},t)|^2\,\mathrm{d}\mathbf{x}\right) && (iv) \\ &= -\eta\int_\Omega |\nabla w|^2\,\mathrm{d}\mathbf{x} \leq 0. && (v) \end{aligned}$$

Here, we used the fact that

$$\operatorname{div}(w\nabla w) = |\nabla w|^2 + w\Delta w$$

to move from the (*ii*) to (*iii*), and the divergence theorem to move from (*iii*) to (*iv*). Additionally, we used the fact that $\dfrac{\partial w}{\partial n} = 0$ on $\Gamma$ to move from (*iv*) to (*v*).

On the other hand, from the initial condition we have

$$E(0) = \frac{1}{2}\int_\Omega w^2(\mathbf{x}, 0)\,\mathrm{d}\mathbf{x} = 0.$$

In conclusion we have $\dfrac{\mathrm{d}E}{\mathrm{d}t}(t) \le 0$ and $E(t) \ge 0$ for $t \in \mathbb{R}_+$, and $E(0) = 0$, so $E(t) = 0$ for all $t \ge 0$, i.e.,

$$\int_\Omega w^2(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = 0, \quad t \in \mathbb{R}_+.$$

We conclude from here, assuming continuity of $w$, that $w = u_2 - u_1 = 0$, meaning $u_2 = u_1$, which is a contradiction. Therefore, the problem has at most one solution.

Does the same procedure work for the problem with Dirichlet boundary condition?

**Example 2.3.** Let us consider the heat equation in the "steady-state" (Poisson equation):

$$\begin{cases} \Delta u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \dfrac{\partial u}{\partial n}(\mathbf{x}) = 0, & \mathbf{x} \in \Gamma. \end{cases}$$

Integrating the source term, we obtain:

$$\int_\Omega f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Omega \Delta u(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Gamma \frac{\partial u}{\partial n}(\mathbf{x})\,\mathrm{d}s = 0.$$

Therefore, for a solution to exist, it is a necessary condition that:

$$\int_\Omega f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,$$

which is known as the compatibility condition.

On the other hand, if a solution $u$ exists for this problem, then any function of the form $u_\alpha = u + \alpha$, with $\alpha \in \mathbb{R}$, would also be a solution. Thus, the solution would not be unique.

## 2.6 Classification of second-order linear PDEs

In general, for PDEs, the question of what type of condition to use—initial conditions or boundary conditions—depends on the type of PDE.

A second-order PDE in $n$ variables $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in D \subset \mathbb{R}^n$ takes the form:

$$-\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{\xi}) \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}(\boldsymbol{\xi}) + \sum_{i=1}^{n} b_i(\boldsymbol{\xi}) \frac{\partial u}{\partial \xi_i}(\boldsymbol{\xi}) + c(\boldsymbol{\xi})u(\boldsymbol{\xi}) = f(\boldsymbol{\xi}), \tag{2.20}$$

with $u \in C^2(D)$. Since

$$\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 u}{\partial \xi_j \partial \xi_i},$$

we can always choose the coefficients $a_{ij}(\boldsymbol{\xi})$ such that $a_{ij}(\boldsymbol{\xi}) = a_{ji}(\boldsymbol{\xi})$. Therefore, the matrix $\mathbf{A}(\boldsymbol{\xi}) = [a_{ij}(\boldsymbol{\xi})] \in \mathbb{R}^{n\times n}$ is symmetric. Similarly, we let $\boldsymbol{b}(\boldsymbol{\xi}) = [b_i(\boldsymbol{\xi})] \in \mathbb{R}^n$.

**Remark 2.4.** If the functions $a_{ij}$, $b_i$, and $c$ do not depend on $\boldsymbol{\xi}$, the PDE is said to have constant coefficients.

**Definition 2.5** (Classification of Second-Order PDEs)**.** Let $\mathbf{A}(\boldsymbol{\xi}) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}(\boldsymbol{\xi}) \in \mathbb{R}^n$.

1) The equation (2.20) is called **elliptic** at the point $\boldsymbol{\xi}$ if $\mathbf{A}(\boldsymbol{\xi})$ is positive (negative) definite.
2) The equation (2.20) is called **hyperbolic** at the point $\boldsymbol{\xi}$ if $\mathbf{A}(\boldsymbol{\xi})$ has one negative (positive) eigenvalue and $n-1$ positive (negative) eigenvalues.
3) The equation (2.20) is called **parabolic** at the point $\boldsymbol{\xi}$ if $\mathbf{A}(\boldsymbol{\xi})$ is positive (negative) semidefinite but not positive (negative) definite, and if the rank of the matrix $[\mathbf{A}(\boldsymbol{\xi}), \boldsymbol{b}(\boldsymbol{\xi})]$ is $n$.

A PDE is said to be elliptic, hyperbolic, or parabolic in $D$ if the corresponding condition is satisfied for every $\boldsymbol{\xi} \in D$.

**Example 2.6.** Let $\mathbf{x} \in \mathbb{R}^3$, $\boldsymbol{\xi} = (t, \mathbf{x}) \in \mathbb{R}^4$, and define

$$\boldsymbol{A}(t,\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \end{bmatrix}, \quad \boldsymbol{b}(t,\mathbf{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c(t,\mathbf{x}) = 0$$

With these definitions (2.20) corresponds to the Heat Equation (2.8), which is a parabolic PDE provided $\eta > 0$.

**Example 2.7.** Let $\boldsymbol{\xi} = \mathbf{x} \in \mathbb{R}^d$ and define

$$\boldsymbol{A}(\mathbf{x}) = [k(\mathbf{x})\delta_{ij}] \in \mathbb{R}^{d\times d}, \quad \boldsymbol{b}(\mathbf{x}) = -\nabla k(\mathbf{x}) \in \mathbb{R}^d, \quad c(\mathbf{x}) = 0.$$

With these definitions (2.20) corresponds to the Poisson Equation (2.10), which is an elliptic PDE provided $k(\mathbf{x}) > 0$.

**Example 2.8.** Let $\mathbf{x} \in \mathbb{R}^3$, $\boldsymbol{\xi} = (t, \mathbf{x}) \in \mathbb{R}^4$, and define

$$\boldsymbol{A}(t,\mathbf{x}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & c^2(\mathbf{x}) & 0 & 0 \\ 0 & 0 & c^2(\mathbf{x}) & 0 \\ 0 & 0 & 0 & c^2(\mathbf{x}) \end{bmatrix}, \quad \boldsymbol{b}(t,\mathbf{x}) = \mathbf{0}, \quad c(t,\mathbf{x}) = 0.$$

With these definitions (2.20) corresponds to the Wave Equation (2.17), which is a hyperbolic PDE provided $c^2(\mathbf{x}) > 0$.

# 3 Elliptic PDEs - Classical Theory

## Lecture 2

We will focus on the following elliptic boundary value problem: Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\operatorname{div}(a\nabla u) + \mathbf{b} \cdot \nabla u + cu &= f \quad \text{in} \quad \Omega, \\ u &= g \quad \text{on} \quad \Gamma := \partial\Omega, \end{aligned} \tag{3.1}$$

where $\Omega$ is an open and bounded subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, with boundary $\partial\Omega =: \Gamma$ of class $C^1$ (this is a sufficient condition for the unit normal $\mathbf{n} : \Gamma \to \mathbb{R}^3$ to be well-defined and continuous).

For the remainder of this lecture, we consider the differential operator

$$\mathcal{A}u := \underbrace{-\operatorname{div}(a\nabla u)}_{\text{diffusion term}} + \underbrace{\mathbf{b} \cdot \nabla u}_{\text{convection term}} + \underbrace{cu}_{\text{reaction term}} .$$

A function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ that satisfies (3.1) is called a **classical solution**.

We work under the following assumptions on the given functions:

- $a = a(\mathbf{x})$, $\mathbf{b} = \mathbf{b}(\mathbf{x})$, and $c = c(\mathbf{x})$ are smooth functions, i.e., elements of $C^\infty(\mathbb{R}^d)$.
- There exists a constant $a_0 > 0$ such that $a(\mathbf{x}) \geq a_0$ for all $\mathbf{x} \in \overline{\Omega}$ (i.e., the PDE is of the elliptic type), and $c(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \overline{\Omega}$.

Note that by **l**inearity of the PDE and the (Dirichlet) boundary condition, $u$ solves (3.1) if and only if it can be decomposed as $u = v + w$ with $v$ and $w$ being solutions of

$$\begin{aligned} \mathcal{A}v &= 0 \quad \text{in} \quad \Omega, \quad v = g \quad \text{on} \quad \Gamma, \\ \mathcal{A}w &= f \quad \text{in} \quad \Omega, \quad w = 0 \quad \text{on} \quad \Gamma \end{aligned}$$

(We may consider other boundary conditions as well, such the Neumann boundary condition:

$$a\frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma,$$

or Robin boundary condition:

$$a\frac{\partial u}{\partial n} + hu = g \quad \text{on} \quad \Gamma \quad (h \geq 0)$$

which, for convenience, are written involving the coefficient $a$ in the definition of the partial differential operator $\mathcal{A}$.)

Our plan for this lecture is to investigate the well-posedness of the problem (3.1). In detail, we want to determine

- Existence of solutions (for the Poisson eqn. via the Green's function)

- Uniqueness of solutions (via the maximum principle)
- Stable dependence on the data $f$ and $g$ in our model problem (2.10) in the $C(\overline{\Omega})$-norm, i.e., in the norm $\|\cdot\| : C(\overline{}) \to [0,\infty)$ given by $\|u\|_{C(\overline{\Omega})} = \max_{\mathbf{x}\in\overline{\Omega}} |u(\mathbf{x})|$. (via the maximum principle)

The last two will follow as a consequence of the following important result:

## 3.1 The maximum principle and its consequences

**Theorem 3.1** (weak maximum (minimum) principle)**.** *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and suppose that*

$$\mathcal{A}u \le 0 \quad in \quad \Omega \qquad (resp. \quad \mathcal{A}u \ge 0 \quad in \quad \Omega).$$

*(i) If $c = 0$, then*

$$\max_{\mathbf{x}\in\overline{\Omega}} u(\mathbf{x}) = \max_{\mathbf{x}\in\Gamma} u(\mathbf{x}). \tag{3.2}$$

*(ii) If $c \ge 0$ in $\Omega$, then*

$$\max_{\mathbf{x}\in\overline{\Omega}} u(\mathbf{x}) \le \max\left\{0, \max_{\mathbf{x}\in\Gamma} u(\mathbf{x})\right\} \qquad \left(resp. \quad \min_{\mathbf{x}\in\overline{\Omega}} u(\mathbf{x}) \ge \min\left\{0, \min_{\mathbf{x}\in\Gamma} u(\mathbf{x})\right\}\right). \tag{3.3}$$

*Proof.* (i) Suppose that there exists $\mathbf{x}_0 \in \Omega$ such that both $u(\mathbf{x}) \le u(\mathbf{x}_0)$ for all $\mathbf{x} \in \overline{\Omega}$ and $u(\mathbf{x}) < u(\mathbf{x}_0)$ for all $\mathbf{x} \in \Gamma$. Then, $\mathbf{x}_0 \in \Omega$ is a local maximum of $u$. Necessary conditions for $\mathbf{x}_0 \in \Omega$ to be a (local) maximum are that $\nabla u(\mathbf{x}_0) = \mathbf{0}$ and that the Hessian matrix $D^2 u(\mathbf{x}_0)$ is negative semidefinite. Therefore,

$$\mathcal{A}u(\mathbf{x}_0) = -\underbrace{a(\mathbf{x}_0)}_{\ge 0} \underbrace{\Delta u(\mathbf{x}_0)}_{=\mathrm{trace}(D^2u(\mathbf{x}_0))\le 0} - \nabla a(\mathbf{x}_0)\cdot\underbrace{\nabla u(\mathbf{x}_0)}_{=0} + \mathbf{b}(\mathbf{x}_0)\cdot\underbrace{\nabla u(\mathbf{x}_0)}_{=0} \ge 0. \tag{3.4}$$

We now use the assumption that $\mathcal{A}u \le 0$ in $\Omega$ to reach a contradiction. We consider two cases:

(a) Suppose that $\mathcal{A}u(\mathbf{x}) < 0$ for all $\mathbf{x} \in \Omega$. Then, (3.4) readily yields a contradiction (i.e., $\mathbf{x}_0$ cannot exist) because $\mathcal{A}u \in C^0(\Omega)$. Therefore, (3.2) follows.

(b) Suppose that $\mathcal{A}u(\mathbf{x}) \le 0$ for all $\mathbf{x} \in \Omega$.

Idea: Additively perturb $u$ by a suitable function $\phi$ and use part (a).

Consider $\phi(\mathbf{x}) = \mathrm{e}^{\lambda x_1} > 0$ with $\lambda > 0$ to be chosen. Since $\nabla\phi = \lambda\phi\mathbf{e}_1$ and $\Delta\phi = \mathrm{div}(\nabla\phi) = \lambda^2\phi$, it follows that

$$\mathcal{A}\phi = \underbrace{\left(-a\lambda^2 + (b_1 - \frac{\partial a}{\partial x_1})\lambda\right)}_{\text{quadratic in }\lambda} \underbrace{\mathrm{e}^{\lambda x_1}}_{>0} = \Big(-\underbrace{a}_{\ge a_0 > 0}\lambda^2 + \underbrace{(b_1 - \frac{\partial a}{\partial x_1})}_{\text{bounded in }\overline{\Omega}}\lambda\Big)\mathrm{e}^{\lambda x_1}.$$

We can then select $\lambda > 0$ large enough such that

$$\mathcal{A}\phi = -a\lambda^2 + \left(b_1 - \frac{\partial a}{\partial x_1}\right)\lambda < 0 \quad \text{in } \overline{\Omega}. \tag{3.5}$$

Therefore, for all $\epsilon > 0$ it holds that

$$\mathcal{A}(u + \epsilon\phi) = \underbrace{\mathcal{A}u}_{\leq 0} + \underbrace{\epsilon\mathcal{A}\phi}_{<0} < 0 \tag{3.6}$$

Now, by part (a), since $\mathcal{A}\phi < 0$ in $\Omega$ (see (3.5)), we get

$$\max_{\mathbf{x}\in\Gamma} \phi(\mathbf{x}) = \max_{\mathbf{x}\in\overline{\Omega}} \phi(\mathbf{x}) =: M > 0.$$

Then, for all $\epsilon > 0$, it holds that

$$\max_{\mathbf{x}\in\Gamma}(u(\mathbf{x}) + \epsilon\phi(\mathbf{x})) \leq \max_{\mathbf{x}\in\Gamma} u(\mathbf{x}) + \epsilon \max_{\mathbf{x}\in\Gamma} \phi(\mathbf{x}) \leq \max_{\mathbf{x}\in\Gamma} u(\mathbf{x}) + \epsilon M.$$

Using the assumption $u(\mathbf{x}_0) > \max_{\mathbf{x}\in\Gamma} u(\mathbf{x})$ and selecting

$$\epsilon := \frac{u(\mathbf{x}_0) - \max_{\mathbf{x}\in\Gamma} u(\mathbf{x})}{M} > 0,$$

we finally arrive at

$$\max_{\mathbf{x}\in\Gamma}(u(\mathbf{x}) + \epsilon\phi(\mathbf{x})) \leq \max_{\mathbf{x}\in\Gamma} u(\mathbf{x}) + \epsilon M \leq u(\mathbf{x}_0) < u(\mathbf{x}_0) + \underbrace{\epsilon\phi(\mathbf{x}_0)}_{>0},$$

which contradicts the statement proved in (a) for the function $v = u + \epsilon\phi$ that satisfies $\mathcal{A}v < 0$ in $\Omega$ (see (3.6)). Therefore, (3.2) also holds in this case.

(ii) Suppose that there exists $\mathbf{x}_0 \in \Omega$ such that

$$\max_{\mathbf{x}\in\overline{\Omega}} u(\mathbf{x}) = u(\mathbf{x}_0) > 0.$$

Define the following domain:

$$\Omega_0 = \{\mathbf{x} \in \Omega : u(\mathbf{x}) > 0\},$$

which is open by virtue of the fact that $u \in C(\Omega)$. Note that $\Omega_0 \neq \emptyset$ since we are working under the assumption that $u(\mathbf{x}_0) > 0$ with $\mathbf{x}_0 \in \Omega$.

To use the result in part (i), consider the operator $\widetilde{\mathcal{A}}$ given by

$$\widetilde{\mathcal{A}}u := \underbrace{\underbrace{\mathcal{A}u}_{\leq 0} - \underbrace{c}_{\geq 0}\underbrace{u}_{>0}}_{=-\operatorname{div}(a\nabla u)+\mathbf{b}\cdot\nabla u} \leq 0 \quad \text{in} \quad \Omega_0.$$

By part (i), applied to the operator $\widetilde{\mathcal{A}}$ over $\Omega_0$, we obtain

$$\max_{\mathbf{x}\in\overline{\Omega}} u(\mathbf{x}) \overset{u(\mathbf{x})\leq 0\,\forall\mathbf{x}\in\overline{\Omega}\setminus\Omega_0}{=} \max_{\mathbf{x}\in\overline{\Omega}_0} u(\mathbf{x}) \overset{(i)}{=} \max_{\mathbf{x}\in\partial\Omega_0} u(\mathbf{x}) \overset{u(\mathbf{x})=0\,\forall\mathbf{x}\in\partial\Omega_0\cap\Omega}{\leq} \max\{0, \max_{\mathbf{x}\in\Gamma} u(\mathbf{x})\},$$

and hence (3.3) holds.

On the other hand, if $u(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega$, then (3.3) follows directly.

Finally, we point out that for the minimum, the statements follow by simply replacing $u$ by $(-u)$ in the proof above, using the fact that $\min u = -\max(-u)$. □

**Remark 3.2.** If $c \geq 0$, then the bound $\max\{0, \max_{\mathbf{x}\in\Gamma} u(\mathbf{x})\}$ in (3.3) cannot be replaced by $\max_{\mathbf{x}\in\Gamma} u(\mathbf{x})$. Consider, for example, the solution $u$ to the boundary value problem

$$-u'' + u = 0 \quad \text{in } (0,1), \quad u(0) = u(1) = -1.$$

Since the general solution of the ODE is given by $u(x) = \alpha\, \mathrm{e}^x + \beta\, \mathrm{e}^{-x}$ with

$$\alpha := \frac{-1}{1+\mathrm{e}}, \quad \beta := -\frac{\mathrm{e}}{1+\mathrm{e}},$$

it is easy to see that there exists $x_0 \in (0,1)$ such that $u(x_0) > \max_{x\in\{0,1\}} u(x) = -1$.

**Remark 3.3.** The condition $c \geq 0$ cannot be relaxed in general to allow for $c < 0$. Consider the boundary value problem

$$-u'' - \pi^2 u = 0 \quad \text{in } (0,1), \quad u(0) = u(1) = 0,$$

which has as a solution $u(x) = \sin(\pi x)$, and $\max_{x\in[0,1]} u(x) = 1$, yet $\max_{x\in\{0,1\}} u(x) = 0$. Moreover, interestingly, this boundary value problem has more than one solution. Indeed, $u(x) = \gamma \sin(\pi x)$ is a solution of the boundary value problem for any $\gamma \in \mathbb{R}$.

**Corollary 3.4** (Uniqueness)**.** *For $f \in C(\Omega)$, $g \in C(\Gamma)$, there is at most one solution in $C^2(\Omega) \cap C(\overline{\Omega})$ of* (3.1).

*Proof.* Suppose there exist two distinct solutions $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (3.1). Defining $v = u_1 - u_2$, we see that $v$ satisfies $\mathcal{A}v = 0$ in $\Omega$ and $v = 0$ on $\Gamma$. Applying the maximum and minimum principles, we obtain

$$\max_{\mathbf{x}\in\overline{\Omega}} v(\mathbf{x}) \leq \max\{0, \underbrace{\max_{\mathbf{x}\in\Gamma} v(\mathbf{x})}_{=0}\} = 0, \quad \text{and} \quad \min_{\mathbf{x}\in\overline{\Omega}} v(\mathbf{x}) \geq \min\{0, \underbrace{\min_{\mathbf{x}\in\Gamma} v(\mathbf{x})}_{=0}\} = 0.$$

Thus, $\|v\|_{C(\overline{\Omega})} = 0$ and hence $v = 0$ in $\overline{\Omega}$, contradicting the assumption that $u_1 \neq u_2$. □

**Theorem 3.5** (Stability)**.** *Let $u \in C^2(\overline{\Omega})$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{C(\overline{\Omega})} \leq \|u\|_{C(\Gamma)} + C\|\mathcal{A}u\|_{C(\overline{\Omega})}. \tag{3.7}$$

*Proof.* Let $\phi$ be such that $\phi \geq 0$ and $\mathcal{A}\phi \leq -1$ in $\Omega$ (we know from the proof of Theorem 3.1 that such a function exists). Define

$$v_+(\mathbf{x}) := u(\mathbf{x}) + \|\mathcal{A}u\|_{C(\overline{\Omega})}\phi(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

Applying the operator $\mathcal{A}$, we obtain

$$\mathcal{A}v_+(\mathbf{x}) = \mathcal{A}u(\mathbf{x}) + \|\mathcal{A}u\|_{C(\overline{\Omega})} \underbrace{\mathcal{A}\phi(\mathbf{x})}_{\leq -1} \leq \mathcal{A}u(\mathbf{x}) - \|\mathcal{A}u\|_{C(\overline{\Omega})} \leq 0, \quad \text{for all } \mathbf{x} \in \Omega,$$

where we used the fact that

$$-\|\mathcal{A}u\|_{C(\overline{\Omega})} \leq \mathcal{A}u(\mathbf{x}) \leq \|\mathcal{A}u\|_{C(\overline{\Omega})} \quad \text{for all } \mathbf{x} \in \overline{\Omega}.$$

Applying the maximum principle (Theorem 3.1), we obtain

$$v_+(\mathbf{x}) \leq \max\{0, \max_{\mathbf{x}\in\Gamma} v_+(\mathbf{x})\}.$$

Furthermore,

$$\max_{\mathbf{x}\in\Gamma} v_+(\mathbf{x}) \leq \max_{\mathbf{x}\in\Gamma} u(\mathbf{x}) + \|\mathcal{A}u\|_{C(\overline{\Omega})} \underbrace{\|\phi\|_{C(\Gamma)}}_{=:C}.$$

Since $u(\mathbf{x}) \leq v_+(\mathbf{x})$, we conclude that

$$u(\mathbf{x}) \leq \max_{\mathbf{x}\in\Gamma} v_+(\mathbf{x}) \leq \max_{\mathbf{x}\in\Gamma} u(\mathbf{x}) + C\|\mathcal{A}u\|_{C(\overline{\Omega})} \leq \|u\|_{C(\Gamma)} + C\|\mathcal{A}u\|_{C(\overline{\Omega})},$$

since $\max_{\mathbf{x}\in\Gamma} u(\mathbf{x}) \leq \max_{\mathbf{x}\in\Gamma} |u(\mathbf{x})| = \|u\|_{C(\Gamma)}$.

Similarly, considering the function

$$v_-(\mathbf{x}) := -u(\mathbf{x}) + \|\mathcal{A}u\|_{C(\overline{\Omega})}\phi(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

we obtain

$$\mathcal{A}v_-(\mathbf{x}) = -\mathcal{A}u(\mathbf{x}) + \|\mathcal{A}u\|_{C(\overline{\Omega})} \underbrace{\mathcal{A}\phi(\mathbf{x})}_{\leq -1} \leq -\mathcal{A}u(\mathbf{x}) - \|\mathcal{A}u\|_{C(\overline{\Omega})} \leq 0, \quad \text{for all } \mathbf{x} \in \Omega.$$

Since $-u(\mathbf{x}) \leq v_-(\mathbf{x})$ for all $\mathbf{x} \in \Omega$, applying the maximum principle again, we obtain

$$-u(\mathbf{x}) \leq \max_{\mathbf{x}\in\Gamma}(-u(\mathbf{x})) + C\|\mathcal{A}u\|_{C(\overline{\Omega})} \leq \|u\|_{C(\Gamma)} + C\|\mathcal{A}u\|_{C(\overline{\Omega})},$$

since $-u(\mathbf{x}) \leq |u(\mathbf{x})| \leq \max_{\mathbf{x}\in\Gamma} |u(\mathbf{x})| = \|u\|_{C(\Gamma)}$ for all $\mathbf{x} \in \Gamma$.

Thus,

$$|u(\mathbf{x})| \leq \|u\|_{C(\Gamma)} + C\|\mathcal{A}u\|_{C(\overline{\Omega})} \quad \text{for all} \quad \mathbf{x} \in \overline{\Omega}$$

and the statement follows by taking the maximum over $\overline{\Omega}$. □

**Remark 3.6.** Note that the inequality (3.7) implies continuous dependency of the solution of (3.1) on the data. Indeed, suppose that $\mathcal{A}u_j = f_j$ in $\Omega$, $u_j = g_j$ on $\Gamma$ for $j = 1, 2$ for sufficiently regular functions $u_j, f_j$ and $g_j$, $j = 1, 2$. Then, applying (3.7) to $u = u_1 - u_2$, we obtain

$$\|u_1 - u_2\|_{C(\overline{\Omega})} \leq \|g_1 - g_2\|_{C(\Gamma)} + C\|f_1 - f_2\|_{C(\overline{\Omega})},$$

or, in words, small changes in the problem data yield small changes in the solutions of (3.1).

## 3.2 Green's representation formula for the Poisson equation

We want to investigate the existence of solutions to the boundary value problem (3.1). To do so, we focus on the solvability of the simpler boundary value problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma. \tag{3.8}$$

where $f \in C(\overline{\Omega})$. We begin with Green's second identity (see problem 5 in Tutorial 1):

$$\boxed{\int_\Omega (v(\mathbf{x})\Delta u(\mathbf{x}) - u(\mathbf{x})\Delta v(\mathbf{x}))\, \mathrm{d}\mathbf{x} = \int_\Gamma \left(\frac{\partial u}{\partial n}(\mathbf{x})v(\mathbf{x}) - u(\mathbf{x})\frac{\partial v}{\partial n}(\mathbf{x})\right) \mathrm{d}s} \tag{3.9}$$

**I**dea: Construct a function $v$ satisfying the following properties: $v = 0$ on $\Gamma$ and such that

$$-\int_\Omega \Delta u(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{y} = u(\mathbf{y}) \quad \text{for all } \mathbf{y} \in \Omega.$$

Then, Green's second identity would lead to the following formula for the solution of (3.8):

$$u(\mathbf{y}) = \int_\Omega f(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \mathbf{y} \in \Omega.$$

To construct such a function we look for harmonic functions, i.e., solutions of Laplace's equation:

$$-\Delta\Phi(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}.$$

To do so, we consider radial solutions $\Phi(\mathbf{x}) = \phi(|\mathbf{x}|) = \phi(r)$ where $r = |\mathbf{x}|$ (well look into this in more detail in Tutorial 2). Then, by the chain rule, it follows that

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_j}(\mathbf{x}) =& \phi'(|\mathbf{x}|)\frac{\partial |\mathbf{x}|}{\partial x_j} = \phi'(|\mathbf{x}|)\frac{x_j}{|\mathbf{x}|}, \\
\frac{\partial^2 \Phi}{\partial x_j^2}(\mathbf{x}) =& \phi''(|\mathbf{x}|)\frac{x_j^2}{|\mathbf{x}|^2} + \phi'(|\mathbf{x}|)\left(\frac{1}{|\mathbf{x}|} - \frac{x_j^2}{|\mathbf{x}|^3}\right), \\
\Delta\Phi(\mathbf{x}) =& \sum_{j=1}^d \left\{\phi''(|\mathbf{x}|)\frac{x_j^2}{|\mathbf{x}|^2} + \phi'(|\mathbf{x}|)\left(\frac{1}{|\mathbf{x}|} - \frac{x_j^2}{|\mathbf{x}|^3}\right)\right\} = \phi''(|\mathbf{x}|) + \frac{(d-1)}{|\mathbf{x}|}\phi'(|\mathbf{x}|).
\end{aligned}$$

From the last identity we get that the sought radial solution $\phi := (0,\infty) \to \mathbb{R}$ solves the following ODE:

$$\phi'' = \frac{(1-d)}{r}\phi',$$

where we have set $r = |\mathbf{x}|$. Integrating (assuming $\phi' \neq 0$) we obtain

$$\frac{\phi''}{\phi'} = \frac{1-d}{r} \implies \log|\phi'| = (1-d)\log r + c \implies |\phi'| = c' r^{1-d}.$$

So, integrating again, we get

$$\phi(r) = \begin{cases} \alpha \log r + \beta, & d = 2, \\ \alpha r^{2-d} + \beta, & d > 2 \end{cases}, \quad \alpha, \beta \in \mathbb{R}.$$

This derivation leads to the following definition:

**Definition 3.7.** The fundamental solution of Laplace's equation is defined as

$$\Phi(\mathbf{x}) := \begin{cases} -\dfrac{1}{\omega_d}\log|\mathbf{x}|, & d = 2, \\[2ex] \dfrac{1}{\omega_d |\mathbf{x}|} & d = 3, \end{cases} \tag{3.10}$$

for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, where

$$\omega_d := \begin{cases} 2\pi, & d = 2, \\ 4\pi, & d = 3, \end{cases}$$

denotes the "area" of the unit sphere $\partial B_1(\mathbf{0})$ in $\mathbb{R}^d$, $d = 2, 3$.

**Remark 3.8.** Note that both the fundamental solution and its gradient, which is given by

$$\nabla\Phi(\mathbf{x}) = -\frac{1}{\omega_d}\frac{\mathbf{x}}{|\mathbf{x}|^d}, \quad \mathbf{x} \neq \mathbf{0},$$

are locally integrable (in the improper sense). Indeed, let $\epsilon \leq 1$ and $B_\epsilon(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < \epsilon\}$. Then, we have

$$\int_{B_\epsilon(\mathbf{0})} |\Phi(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \omega_d \int_0^\epsilon |\phi(r)| r^{d-1} \, \mathrm{d}r = \begin{cases} -\int_0^\epsilon r \log r \, \mathrm{d}r = \frac{1}{4}\epsilon^2(1 - 2\log(\epsilon)), & d = 2, \\ \int_0^\epsilon r \, \mathrm{d}r = \frac{\epsilon^2}{2}, & d = 3, \end{cases} \tag{3.11}$$

and

$$\int_{B_\epsilon(\mathbf{0})} |\nabla\Phi(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_0^\epsilon 1 \, \mathrm{d}r = \epsilon.$$

However, higher order derivatives of the fundamental solution are not integrable on any ball $B_\epsilon(\mathbf{0})$.

We are now in a position to go back to the idea of finding a representation formula for the solution $u$ of (2.10).

Let $v(\mathbf{x}) = \Phi(\mathbf{x}-\mathbf{y})$ for $\mathbf{x} \neq \mathbf{y}$, where $\Phi$ is the fundamental solution of the Laplace equation defined in (3.10). Since $\Delta v$ is not integrable, we may not use (3.9) directly. However, let $\mathbf{y} \in \Omega$ and let $\epsilon > 0$ be sufficiently small such that $B_\epsilon(\mathbf{y}) \subset \Omega$ (see Figure 7.1).

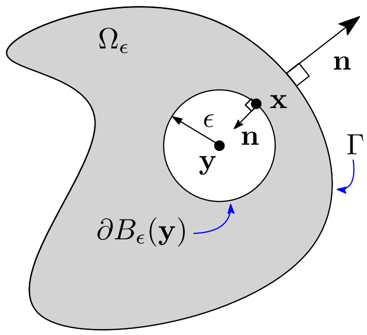


Figure 3.1: Illustration of the domain $\Omega_\epsilon$ and its boundary $\Gamma \cup \partial B_\epsilon(\mathbf{y})$.

Consider the open domain

$$\Omega_\epsilon := \Omega \setminus \overline{B_\epsilon(\mathbf{x})}.$$

Then, by Green's 2nd identity (3.9), we get

$$\begin{aligned}\int_{\Omega_\epsilon} \Delta u(\mathbf{x})\Phi(\mathbf{x}-\mathbf{y}) - \underbrace{\Delta\Phi(\mathbf{x}-\mathbf{y})}_{=0} u(\mathbf{x}) \, \mathrm{d}\mathbf{x} &= \int_{\partial\Omega_\epsilon} \frac{\partial u(\mathbf{x})}{\partial n}\Phi(\mathbf{x}-\mathbf{y}) - \frac{\partial\Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})} u(\mathbf{x}) \, \mathrm{d}s(\mathbf{x}) \\ &= \int_\Gamma \frac{\partial u(\mathbf{x})}{\partial n}\Phi(\mathbf{x}-\mathbf{y}) - \frac{\partial\Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})} u(\mathbf{x}) \, \mathrm{d}s(\mathbf{x}) \\ &\quad + \int_{\partial B_\epsilon(\mathbf{y})} \frac{\partial u(\mathbf{x})}{\partial n}\Phi(\mathbf{x}-\mathbf{y}) - \frac{\partial\Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})} u(\mathbf{x}) \, \mathrm{d}s(\mathbf{x})\end{aligned}$$

We want to take the limit as $\epsilon \downarrow 0$ in the above identity. Clearly, since $\Phi$ is integrable, from (3.11) we get

$$\begin{aligned}
\left|\int_\Omega \Delta u(\mathbf{x})\Phi(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{x} - \int_{\Omega_\epsilon} \Delta u(\mathbf{x})\Phi(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{x}\right| &= \left|\int_{B_\epsilon(\mathbf{y})} \Delta u(\mathbf{x})\Phi(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{x}\right| \\
&\le \|\Delta u\|_{C(\overline{\Omega})} \int_{B_\epsilon(\mathbf{y})} |\Phi(\mathbf{x}-\mathbf{y})|\,\mathrm{d}\mathbf{x} \qquad \text{(translation)} \\
&= \|\Delta u\|_{C(\overline{\Omega})} \int_{B_\epsilon(\mathbf{0})} |\Phi(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z} \\
&\le C\|\Delta u\|_{C(\overline{\Omega})}\epsilon^2(1+|\log\epsilon|) \to 0 \quad \text{as} \quad \epsilon \to 0+\,.
\end{aligned}$$

To compute the limit of the surface integrals over $\partial B_\epsilon(\mathbf{y})$, we note that for all $\mathbf{x} \in \partial B_\epsilon(\mathbf{y}) := \{\boldsymbol{z} \in \mathbb{R}^d : |\boldsymbol{z}-\mathbf{y}| = \epsilon\}$ it holds that

$$\Phi(\mathbf{x}-\mathbf{y}) = \frac{1}{\omega_d}\begin{cases} -\log\epsilon, & d=2, \\ \dfrac{1}{\epsilon}, & d=3, \end{cases}$$

$$\frac{\partial\Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})} = \nabla_{\mathbf{x}}\Phi(\mathbf{x}-\mathbf{y}) \cdot \underbrace{\mathbf{n}(\mathbf{x})}_{-(\mathbf{x}-\mathbf{y})/|\mathbf{x}-\mathbf{y}|} = \frac{1}{\omega_d}\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^d}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{\omega_d}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d-1}} = \frac{1}{\omega_d\epsilon^{d-1}}$$

Therefore,

$$\begin{aligned}
\left|\int_{\partial B_\epsilon(\mathbf{y})} \frac{\partial u}{\partial n}(\mathbf{x})\Phi(\mathbf{x}-\mathbf{y})\,\mathrm{d}s(\mathbf{x})\right| &\le \|\nabla u\|_{C(\overline{\Omega})} \int_{\partial B_\epsilon(\mathbf{y})} |\Phi(\mathbf{x}-\mathbf{y})|\,\mathrm{d}s(\mathbf{x}) \\
&= \begin{cases} \dfrac{\epsilon|\log\epsilon|}{2\pi}\displaystyle\int_0^{2\pi}\mathrm{d}t = \epsilon|\log\epsilon|, & d=2 \\ \dfrac{\epsilon}{4\pi}\displaystyle\int_0^{\pi}\int_0^{2\pi}\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta = \epsilon, & d=3 \end{cases} \\
&\le C\epsilon(1+|\log\epsilon|) \to 0 \quad \text{as} \quad \epsilon\downarrow 0.
\end{aligned}$$

Now, to compute the limit of the other surface integral, we note that

$$u(\mathbf{y}) = u(\mathbf{y})\underbrace{\frac{1}{\omega_d\epsilon^{d-1}}\int_{\partial B_\epsilon(\mathbf{y})}\mathrm{d}s(\mathbf{x})}_{=1} = \int_{\partial B_\epsilon(\mathbf{y})} u(\mathbf{y})\frac{\partial\Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})}\,\mathrm{d}s(\mathbf{x}). \tag{3.12}$$

Note also that since $u$ is uniformly continuous on $\overline{\Omega}$, i.e., for all $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$ satisfying $|\mathbf{x}-\mathbf{y}| < \delta_0$, it holds that $|u(\mathbf{x}) - u(\mathbf{y})| < \epsilon_0$. Then, for all $0 < \epsilon < \delta_0$, we have

$$\left|u(\mathbf{y}) - \int_{\partial B_\epsilon(\mathbf{y})} u(\mathbf{x}) \frac{\partial \Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})} \, \mathrm{d}s(\mathbf{x})\right| \overset{(3.12)}{=} \left|\int_{\partial B_\epsilon(\mathbf{y})} (u(\mathbf{y}) - u(\mathbf{x})) \underbrace{\frac{\partial \Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})}}_{=(\omega_d \epsilon^{d-1})^{-1}} \, \mathrm{d}s(\mathbf{x})\right|$$

$$\leq \frac{1}{\omega_d \epsilon^{d-1}} \underbrace{\int_{\partial B_\epsilon(\mathbf{y})}}_{|\mathbf{x}-\mathbf{y}|=\epsilon<\delta_0} \underbrace{|u(\mathbf{y}) - u(\mathbf{x})|}_{<\epsilon_0} \, \mathrm{d}s(\mathbf{x})$$

$$< \epsilon_0.$$

We conclude from here that

$$\int_{\partial B_\epsilon(\mathbf{y})} u(\mathbf{x}) \frac{\partial \Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})}, \, \mathrm{d}s(\mathbf{x}) \to u(\mathbf{y}) \quad \text{as} \quad \epsilon \downarrow 0,$$

and with that, we have proven the Green's representation formula (or Green's 3rd identity), which states the following:

**Theorem 3.9** (Green's representation formula)**.** *Let $u \in C^2(\overline{\Omega})$. Then for any $\mathbf{y} \in \Omega$:*

$$\boxed{u(\mathbf{y}) = -\int_\Omega \Delta u(\mathbf{x}) \Phi(\mathbf{x}-\mathbf{y}) \, \mathrm{d}\mathbf{x} + \int_\Gamma \frac{\partial u(\mathbf{x})}{\partial n} \Phi(\mathbf{x}-\mathbf{y}) - \frac{\partial \Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})} u(\mathbf{x}) \, \mathrm{d}s(\mathbf{x}).} \tag{3.13}$$

**Remark 3.10.** The terms in Green's representation formula go by the following names:

| | |
|---|---|
| $\int_\Omega f(\mathbf{x}) \Phi(\mathbf{x}-\mathbf{y}) \, \mathrm{d}\mathbf{x}$ | Newton potential of $f \in C(\overline{\Omega})$ |
| $\int_\Gamma \mu(\mathbf{x}) \Phi(\mathbf{x}-\mathbf{y}) \, \mathrm{d}\mathbf{x}$ | Single-layer potential of $\mu \in C(\Gamma)$ |
| $\int_\Gamma \lambda(\mathbf{x}) \frac{\partial \Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})} \, \mathrm{d}\mathbf{x}$ | Double-layer potential of $\lambda \in C(\Gamma)$ |

In absence of boundary conditions, we already have a representation formula. Indeed:

**Theorem 3.11.** *If $f \in C_0^2(\mathbb{R}^d)$, then*

$$u(\mathbf{y}) = \int_{\mathbb{R}^d} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad \mathbf{y} \in \mathbb{R}^d,$$

*is in $C^2(\mathbb{R}^d)$ and $-\Delta u = f$ in $\mathbb{R}^d$.*

*Proof.* Let $\Omega := \{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) \neq 0\}$ (open), so that the support of $f$ is given by $\overline{\Omega}$. Note that under the assumption $f \in C_0^2(\mathbb{R}^d)$, $f$ together with its first and second order derivatives, vanishes outside $\Omega$. Moreover, $f$ and its normal derivative $\partial f / \partial n$ vanish on $\partial \Omega$.

Applying the change of variables $\boldsymbol{z} = \mathbf{x} - \mathbf{y}$ if follows that

$$u(\mathbf{y}) = \int_{\mathbb{R}^d} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \underbrace{\Phi(\boldsymbol{z})}_{\substack{\text{locally} \\ \text{integrable}}} \underbrace{f(\boldsymbol{z}+\mathbf{y})}_{\substack{\text{compact support} \\ \text{and in } C^2}} \, \mathrm{d}\boldsymbol{z} \quad \in C^2(\mathbb{R}^d).$$

We denote by $\Omega + \mathbf{y} := \{\boldsymbol{z} \in \mathbb{R}^d : \boldsymbol{z} = \mathbf{x} + \mathbf{y}, \mathbf{x} \in \Omega\}$ the translation of $\Omega$ by $\mathbf{y} \in \mathbb{R}^d$. Then, using the fact that $f(\cdot + \mathbf{y})$ and therefore $\Delta f(\cdot + \mathbf{y})$ vanish outside $\Omega + \mathbf{y}$, we get

$$\begin{aligned}
\Delta u(\mathbf{y}) &= \int_{\mathbb{R}^d} \Phi(\boldsymbol{z}) \underbrace{\Delta_{\mathbf{y}} f(\boldsymbol{z}+\mathbf{y})}_{\substack{\text{Laplacian with} \\ \text{respect to } \mathbf{y}}} \,\mathrm{d}\boldsymbol{z} \\
&= \int_{\mathbb{R}^d} \Phi(\boldsymbol{z}) \underbrace{\Delta_{\boldsymbol{z}} f(\boldsymbol{z}+\mathbf{y})}_{\substack{\text{Laplacian with} \\ \text{respect to } \boldsymbol{z}}} \,\mathrm{d}\boldsymbol{z} \\
&= \int_{\Omega+\mathbf{y}} \Phi(\boldsymbol{z}) \Delta_{\boldsymbol{z}} f(\boldsymbol{z}+\mathbf{y}) \,\mathrm{d}\boldsymbol{z} \qquad \text{(integral over the support of } f(\cdot+\mathbf{y})\text{)} \\
&= \underbrace{\int_{\Omega} \Phi(\mathbf{x}-\mathbf{y}) \Delta f(\mathbf{x}) \,\mathrm{d}\mathbf{x}}_{\text{Newton potential of } f} \qquad \text{(reversing the change of variables)} \\
&= -f(\mathbf{y}) + \int_{\partial\Omega} \underbrace{\frac{\partial f(\mathbf{x})}{\partial n}}_{=0} \Phi(\mathbf{x}-\mathbf{y}) - \frac{\partial \Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{y})} \underbrace{f(\mathbf{x})}_{=0} \,\mathrm{d}s(\mathbf{x}) \qquad \text{(Green's rep. formula )} \\
&= -f(\mathbf{y}).
\end{aligned}$$

The proof is now complete. □

**Remark 3.12.** The assumed regularity $f \in C_0^2(\mathbb{R}^d)$ can be lowered; see, e.g. [1, Sec. 4.2].

To obtain an actual solution formula for (2.10) from Green's 3rd identity (3.13), we need to handle the term $\int_\Gamma \frac{\partial u}{\partial n} \Phi(\cdot - \mathbf{y}) \,\mathrm{d}s$. To do so, we consider for $\mathbf{y} \in \Omega$, the solution $\phi_{\mathbf{y}} : \overline{\Omega} \to \mathbb{R}$ to the boundary value problem

$$\begin{cases} -\Delta_{\mathbf{x}} \phi_{\mathbf{y}}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega \\ \phi_{\mathbf{y}}(\mathbf{x}) = \Phi(\mathbf{x}-\mathbf{y}), & \mathbf{x} \in \Gamma \end{cases} \tag{3.14}$$

Using this function in Green's 2nd indentity (3.9), we get

$$\int_\Omega \phi_{\mathbf{y}}(\mathbf{x}) \underbrace{\Delta u(\mathbf{x})}_{=-f(\mathbf{x})} - u(\mathbf{x}) \underbrace{\Delta_{\mathbf{x}} \phi_{\mathbf{y}}}_{=0} \,\mathrm{d}\mathbf{x} = \int_\Gamma \frac{\partial u(\mathbf{x})}{\partial n} \underbrace{\phi_{\mathbf{y}}(\mathbf{x})}_{=\Phi(\mathbf{x}-\mathbf{y})} - \underbrace{u(\mathbf{x})}_{=0} \frac{\partial \phi_{\mathbf{y}}(\mathbf{x})}{\partial n} \,\mathrm{d}s.$$

Therefore, subtracting this identity from Green's representation formula (3.13), we arrive at the following formula for the solution of (2.10):

$$\boxed{u(\mathbf{y}) = \int_\Omega f(\mathbf{x}) \underbrace{(\Phi(\mathbf{x}-\mathbf{y}) - \phi_{\mathbf{y}}(\mathbf{x}))}_{\text{Green's function of (2.10)}} \,\mathrm{d}\mathbf{x}, \quad \mathbf{y} \in \Omega.} \tag{3.15}$$

The Green's function

$$\boxed{G(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{x}, \mathbf{y}) - \phi_{\mathbf{y}}(\mathbf{x}), \quad \mathbf{x}, \mathbf{y} \in \overline{\Omega}, \ \mathbf{x} \neq \mathbf{y},}$$

derived above can also be used to find a solution formula for boundary value problems with inhomogeneous Dirichlet boundary conditions. Consider the boundary value problem

$$-\Delta v = f \quad \text{in} \quad \Omega, \qquad v = g \quad \text{on} \quad \Gamma, \tag{3.16}$$

with $f \in C(\overline{\Omega})$ and $g \in C(\Gamma)$. By Green's representation formula we have

$$v(\mathbf{y}) = \int_\Omega f(\mathbf{x})\Phi(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{x} + \int_\Gamma \frac{\partial v(\mathbf{x})}{\partial n}\Phi(\mathbf{x}-\mathbf{y}) - \frac{\partial \Phi(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{x})} g(\mathbf{x})\,\mathrm{d}s(\mathbf{x}), \quad \mathbf{y} \in \Omega.$$

On the other hand, from Green's 2nd identity applied to $v$ and $\phi_\mathbf{y}$, we get

$$-\int_\Omega f(\mathbf{x})\phi_\mathbf{y}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Gamma \frac{\partial v(\mathbf{x})}{\partial n}\Phi(\mathbf{x}-\mathbf{y}) - g(\mathbf{x})\frac{\partial \phi_\mathbf{y}}{\partial n}(\mathbf{x})\,\mathrm{d}s, \quad \mathbf{y} \in \Omega.$$

Subtracting these two identities, we thus obtain

$$\boxed{v(\mathbf{y}) = \int_\Omega f(\mathbf{x})G(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x} - \int_\Gamma \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})} g(\mathbf{x})\,\mathrm{d}s(\mathbf{x}), \quad \mathbf{y} \in \Omega.}$$

**Remark 3.13.** (i) If $\Omega$ is the unit disc in $\mathbb{R}^2$ then $\phi_\mathbf{y}$ can be obtained as in [3, Sec 3.3]. →read it yourself!

(ii) If $\Omega$ is a ball of radius $r_0 > 0$ in $\mathbb{R}^d$, see, e.g. [1, p. 19] for a generalization of (i). Perron's method [1, p. 23] can then be used for general domains $\Omega$ to find $\phi_\mathbf{y}$. Alternatively, the theory of boundary integral equations can be used to establish the existence of $\phi_\mathbf{y}$; see, e,g. [2].

(iii) We can obtain $\phi_\mathbf{y}$ via variational methods for $\Omega$ less regular → next lecture.

(iv) In view of the uniqueness result above, if (2.10) has a solution, it must be given by (3.15).

(v) It can be shown that $\phi_\mathbf{y}(\mathbf{x}) = \phi_\mathbf{x}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$.

(vi) For general $\mathcal{A}$ and $\Omega$, Green's functions are unknown → numerical methods can still be used to exploit the powerful Green's formula.

# 4 Variational formulation of elliptic PDEs of second order

## Lecture 3

Consider

$$-\operatorname{div}(a\nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \tag{4.1}$$

with $\Omega$ an open, bounded subset of $\mathbb{R}^d$ and boundary $\Gamma$. We assume that the coefficient functions $a, c : \overline{\Omega} \to \mathbb{R}$ and $b : \overline{\Omega} \to \mathbb{R}^d$ satisfy

$$a(x) \geq a_0 > 0, \qquad c(x) - \frac{1}{2}\operatorname{div}(b(x)) \geq 0 \quad \forall x \in \overline{\Omega}, \tag{4.2}$$

for some constant $a_0 > 0$. Furthermore, the function $f : \overline{\Omega} \to \mathbb{R}$, which models internal sources, is assumed to be given.

**Aim:** Verify the well-posedness of (4.1), i.e., the existence of solutions, their uniqueness and their stable dependence on the data.

**Challenges:**

- Since the coefficient functions may not be constant and the shape of $\Omega$ can be very complicated, we cannot find solution formulas or corresponding formulas for the Green's function in general.
- Moreover, discontinuous coefficient functions may not allow for continuously differentiable solutions; see Example 4.1. Hence, the classical solution concept, i.e., $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying (4.1) pointwise, is too limited.

**Example 4.1.** Consider (4.1) with $\Omega = (0, 2)$, $b = 0$, $c = 0$ and discontinuous functions

$$a(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 2, & 1 < x \leq 2, \end{cases}, \qquad f(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ 1, & 1 < x \leq 2, \end{cases} \quad x \in [0, 2]. \tag{4.3}$$

A solution $u$ to (4.1) must then satisfy the following two ordinary differential equations

$$\begin{aligned} -u'' &= 0 \qquad \text{on } (0,1), \\ -u'' &= \frac{1}{2} \qquad \text{on } (1,2). \end{aligned}$$

That is $u_L = u_{|(0,1)}$ is linear and $u_R = u_{|(1,2)}$ is quadratic, i.e.,

$$u_L(x) = c_1 x + c_0, \qquad u_R(x) = d_2 x^2 + d_1 x + d_0$$

for some constants $c_0, c_1, d_0, d_1, d_2 \in \mathbb{R}$. Using the boundary conditions $u_L(0) = 0$, $u_R(2) = 0$ and $u''(x) = -1/2$, we compute that

$$c_0 = 0, \quad d_2 = -\frac{1}{4}, \quad d_1 = \frac{1 - d_0}{2}.$$

How shall we determine the remaining two coefficients $c_1$ and $d_0$? Since (4.1) is used as a model for a stationary diffusion process (think of the heat equation), it is reasonable to require continuity of $u$ (of temperature), and $au'$ (of the heat flux), i.e.,

$$u_L(1) = u_R(1), \qquad \text{and} \qquad 1u_L'(1) = 2u_R'(1).$$

Using these two conditions, we determine $d_0 = -c_1$ and $c_1 = 1/6$, i.e.,

$$u(x) = \begin{cases} \frac{1}{6}x, & 0 \le x \le 1, \\ -\frac{1}{4}x^2 + \frac{7}{12}x - \frac{1}{6}, & 1 < x \le 2, \end{cases} \qquad x \in [0, 2].$$

We observe that $u \notin C^1(0, 2)$, because $u$ is not continuously differentiable in $x = 1$. However, $u$ and the flux $au'$ are continuous functions in $[0, 2]$ and $u$ solves (4.1) for the given coefficient functions in all points $x \neq 1$.

**Approach:** Find a solution concept for (4.1) that allows to characterize solutions in a sense that is weaker than the notion of classical derivative, such that well-posedness of (4.1) in general situations can be established.

**Remark 4.2.** One may compare our approach to some extend with the situation for initial value problems for ordinary differential equations $u'(t) = f(t, u(t))$, $t > 0$ and $u(0) = u_0$. In certain situations, e.g., if $f(t, u) = Au$ is a linear function, it is possible to find explicit solution formulas. In more general situations, we rely on abstract theorems that guarantee the existence of a solution, and sometimes its uniqueness; consult the theorems by Peano or Picard-Lindelöf.

## 4.1 Derivation of a weak formulation

Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies (4.1). Then we obtain for any $v \in C_0^\infty(\Omega)$ (infinitely often differentiable functions that vanish in a neighborhood of $\Gamma$) by integration over $\Omega$ that

$$\begin{aligned} \int_\Omega fv\,dx &= \int_\Omega \Big( -\operatorname{div}(a\nabla u) + b\cdot\nabla u + cu \Big) v\,dx \\ &= \int_\Omega a\nabla u\cdot\nabla v + b\cdot\nabla uv + cuv\,dx - \int_\Gamma (a\nabla uv)\cdot n\,ds \qquad \text{(divergence theorem)} \\ &= \int_\Omega a\nabla u\cdot\nabla v + b\cdot\nabla uv + cuv\,dx. \qquad (v = 0 \text{ on } \Gamma) \end{aligned}$$

Here, $n$ is the unit outer normal to $\Gamma$. In other words, we arrive at the necessary condition:

$$\int_\Omega a\nabla u\cdot\nabla v + b\cdot\nabla uv + cuv\,dx = \int_\Omega fv\,dx \quad \forall v \in C_0^\infty(\Omega). \tag{4.4}$$

**Observations:**

- In the derivation of (4.4), we used $v = 0$ on $\Gamma$ to treat the term $\int_\Gamma (a\nabla uv)\cdot n\,ds$, because (4.1) does not make any statement about $a\nabla u \cdot n = a\partial_n u$. If the Dirichlet condition $u = 0$ on $\Gamma$ would have been replaced by a Neumann ($a\partial_n u = 0$ on $\Gamma$) or Robin condition ($a\partial_n u + du = 0$ on $\Gamma$), then we could have taken $v \in C^\infty(\overline{\Omega})$, see exercises.
- The identity (4.4) involves only first order derivatives, while (4.1) involves second order derivatives.
- Since the identity (4.4) only involves integrals, the value of the integrands in single points does play no role. In particular, only integrability of the corresponding terms is important to make the integrals in (4.4) well-defined.

Since in many applications the coefficient functions $a, b, c$ in (4.1) ((4.4)) are bounded, we next consider conditions that make $uv$, $\nabla uv$ and $\nabla u \cdot \nabla v$ integrable. We start the discussion with a brief recap of the space $L^2(\Omega)$ that forms the basis for defining function spaces that are appropriate for our discussion.

## 4.2 The Sobolev space $H^1(\Omega)$

The space of square integrable (class of) functions

$$L^2(\Omega) = \{v : \Omega \to \mathbb{R} : \int_\Omega |v(x)|^2\,dx < \infty\}$$

is a Hilbert space when endowed with the inner product

$$(v, w) = \int_\Omega v(x)w(x)\,dx, \quad v, w \in L^2(\Omega).$$

The corresponding norm is given by

$$\|v\| = \sqrt{(v, v)} \quad v \in L^2(\Omega),$$

and the Cauchy-Schwarz inequality

$$(v, w) \le \|v\|\,\|w\|, \quad v, w \in L^2(\Omega)$$

holds. The space $L^2(\Omega)$ does not contain information about the derivative of a function. As can be seen for instance from (4.3), which displays functions $a, f \in L^2(\Omega)$ that are not differentiable, incorporating derivative information is an extra requirement. As one can verify, the solution $u$ found in Example 4.1 is not differentiable but satisfies (4.4) (homework!). Therefore, we arrive at the question how to properly generalize the notion of differentiability in our context. In order to transform (4.1) to (4.4), we used the divergence theorem,

$$\int_\Omega \mathrm{div}(F)\,dx = \int_\Gamma F \cdot n\,ds. \tag{4.5}$$

If $F = uve_i$, with $e_i$ denoting the $i$th unit vector and $u, v \in C^1(\overline{\Omega})$ with $v = 0$ on $\Gamma$, (4.5) yields that

$$\int_\Omega \partial_{x_i} uv\,dx = -\int_\Omega u\partial_{x_i} v\,dx.$$

The expression on the right-hand side makes sense for merely $u \in L^2(\Omega)$ (and even less, but we do not follow-up on this here). We next introduce a generalization of $\partial_{x_i} u$ on the left-hand side.

**Definition 4.3** (Weak derivative)**.** Let $u \in L^2(\Omega)$. We say that a function $w_i \in L^2(\Omega)$ is the $i$th weak partial derivative of $u$ if

$$\int_\Omega w_i v \, dx = -\int_\Omega u \partial_{x_i} v \, dx \quad \forall v \in C_0^\infty(\Omega).$$

In slight abuse of notation, we write $\partial_{x_i} u = w_i$.

One can show that the $i$th weak partial derivative of $u$ is unique in $L^2(\Omega)$. We next define the set of square integrable functions that have a weak derivative in the sense of Definition 4.3,

$$H^1(\Omega) = \{u \in L^2(\Omega) : \exists \partial_{x_i} u \in L^2(\Omega) \text{ for } 1 \le i \le d\}. \tag{4.6}$$

The linear space $H^1(\Omega)$ is called a *Sobolev space*. Without proof (see advanced classes on partial differential equations in the master program), we state the following:

**Properties of $H^1(\Omega)$:**

- $H^1(\Omega)$ is a Hilbert space when endowed with the inner product

$$(u, v)_1 = (u, v) + \sum_{i=1}^{d} (\partial_{x_i} u, \partial_{x_i} v) = (u, v) + (\nabla u, \nabla v).$$

  The corresponding norm is given by $\|u\|_1 = \sqrt{(u, u)_1}$.

- Functions $u \in H^1(\Omega)$ have boundary values (so-called traces) such that $u_{|\Gamma} \in L^2(\Gamma)$, and there exists a constant $C > 0$ that is independent of $u$ such that [3, Theorem A.4]

$$\|u\|_{L^2(\Gamma)} \le C \|u\|_1.$$

- The space $H_0^1(\Omega) = \{u \in H^1(\Omega) : u_{|\Gamma} = 0\}$ is a closed subspace of $H^1(\Omega)$, i.e., $H_0^1(\Omega)$ is a Hilbert space.

- Poincaré inequality: Let $\Omega$ be bounded. Then there is a constant $C_P > 0$ that does not depend on $u$ such that [3, Theorem A.6]

$$\|u\| \le C_P \|\nabla u\| \quad \forall u \in H_0^1(\Omega). \tag{4.7}$$

- $C_0^\infty(\Omega)$ is a dense subset of $H_0^1(\Omega)$, i.e., for any $u \in H_0^1(\Omega)$ there exists a sequence of functions $u_m \in C_0^\infty(\Omega)$ such that $\|u - u_m\|_1 \to 0$ as $m \to \infty$.

The space $H_0^1(\Omega)$ meets exactly our requirements to make (4.4) meaningful, i.e., $H_0^1(\Omega)$ contains square integrable functions that have square integrable derivatives and that vanish on $\Gamma$.

**Definition 4.4.** A function $u \in H_0^1(\Omega)$ is called a *weak solution* of (4.1) if (4.4) holds for all $v \in H_0^1(\Omega)$. The formulation (4.4) is called a weak formulation of (4.1).

**Observations:**

- Every classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ (if it exists), is a weak solution of (4.1).
- If a weak solution $u \in H_0^1(\Omega)$ satisfies $u \in C^2(\overline{\Omega})$ (and $a$ is continuously differentiable), then $u$ is a classical solution (undo integration-by-parts/divergence theorem).

**Example 4.5.** (i) Consider the function

$$u(x) = 1 - |x|, \quad |x| \leq 1.$$

Clearly, $u \in L^2(\Omega)$ with $\Omega = (-1, 1)$. We claim that $u \in H^1(\Omega)$. To that end, we verify that $u$ has a weak derivative. We compute for any $v \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\int_{-1}^1 u(x)v'(x)\,dx &= \int_{-1}^0 (1+x)v'(x)\,dx + \int_0^1 (1-x)v'(x)\,dx \\
&= [(1+x)v(x)]_{x=-1}^{x=0} - \int_{-1}^0 v(x)\,dx + [(1-x)v(x)]_{x=0}^{x=1} + \int_0^1 v(x)\,dx \\
&= -\int_{-1}^1 u'(x)v(x)\,dx,
\end{aligned}$$

where we used that $v(-1) = v(1) = 0$, and with weak derivative $u'$ defined by

$$u'(x) = \begin{cases} 1, & x < 0 \\ -1, & x > 0 \end{cases}.$$

(ii) Consider $\Omega = B_1(0) = \{x \in \mathbb{R}^2 : |x| < 1\}$ (unit disc), and $u(x) = |x|^\beta$ for some $\beta \in \mathbb{R}$. Using polar coordinates, we compute that

$$\int_B |u(x)|^2\,dx = 2\pi \int_0^1 r^{2\beta+1}\,dr,$$

which is finite if and only if $2\beta + 1 > -1$, i.e., $\beta > -1$. If $x \neq 0$, we have that $u$ is differentiable, and $\nabla u = \beta|x|^{\beta-2}x$. With the same argument as before, we obtain that $|\nabla u| = \beta|x|^{\beta-1} \in L^2(\Omega)$ if and only if $2(\beta-1)+1 > -1$, i.e., $\beta > 0$. Moreover, similar to Lecture 2 on Green's functions, one verifies that for $v \in C_0^\infty(\Omega)$ and $\beta > -1$

$$\begin{aligned}
\int_B u(x)\partial_{x_i}v(x)\,dx &= \lim_{\varepsilon \to 0} \int_{\{\varepsilon < |x| < 1\}} u(x)\partial_{x_i}v(x)\,dx \\
&= \lim_{\varepsilon \to 0} \left( -\int_{B_\varepsilon} \beta|x|^{\beta-2}x_i v(x)\,dx - \int_{\{|x|=\varepsilon\}} \varepsilon^{\beta-1} v(x) x_i\,ds \right) \\
&= -\int_B \partial_{x_i}u(x)v(x)\,dx,
\end{aligned}$$

where we applied the divergence theorem and that the outer unit normal is given by $-x/\varepsilon$ on $\{|x| = \varepsilon\}$. Summarizing, we have shown that $u \in H^1(\Omega)$ for $\beta > 0$. But notice, that $\nabla u$ is unbounded if $\beta < 1$.

## 4.3 Well-posedness of the weak formulation

The key tool, which we provide without proof, but see [3, Theorem A.3], is the following theorem, which will be proven in the master class on partial differential equations.

**Theorem 4.6** (Lax-Milgram)**.** *Let $V$ be a Hilbert space with inner product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$. Suppose that the bilinear form $a : V \times V \to \mathbb{R}$ satisfies*

- $\exists \beta > 0 : |a(v,w)| \le \beta \|v\|_V \|w\|_V$ *for all* $v, w \in V$. *(boundedness)*
- $\exists \alpha > 0 : a(v,v) \ge \alpha \|v\|_V^2$ *for all* $v \in V$. *(coercivity)*

*Furthermore, assume that $\ell : V \to \mathbb{R}$ is linear and satisfies*

- $\exists C_\ell > 0 : |\ell(v)| \le C_\ell \|v\|_V$ *for all* $v \in V$.

*Then, there exists a unique $u \in V$ such that*

$$a(u,v) = \ell(v) \quad \forall v \in V, \quad \text{and} \quad \|u\|_V \le \frac{C_\ell}{\alpha}.$$

The Lax-Milgram theorem is an abstract well-posedness results. Let us apply it in our context.

**Theorem 4.7.** *Suppose $f \in L^2(\Omega)$, $a, c$ are bounded functions and $b$ is differentiable and that* (4.2) *holds. Then* (4.1) *admits a unique weak solution $u \in H_0^1(\Omega)$ that satisfies the inequality*

$$\|u\|_1 \le \frac{1 + C_P^2}{a_0} \|f\|,$$

*where $C_P$ is the constant from* (4.7).

*Proof.* We want to apply the Lax-Milgram theorem. The space $V = H_0^1(\Omega)$ is a Hilbert space. Inspecting (4.4) leads us to defining the bilinear and linear forms

$$\begin{aligned} a(u,v) &= \int_\Omega a \nabla u \cdot \nabla v + b \cdot \nabla u v + c u v \, dx, \\ \ell(v) &= \int_\Omega f v \, dx. \end{aligned}$$

Next, we verify the assumptions of the Lax-Milgram theorem. We start with the boundedness of $\ell$. The Cauchy-Schwarz inequality implies that

$$|\ell(v)| = |(f,v)| \le \|f\|\|v\| \le \|f\|\|v\|_1,$$

i.e., $C_\ell = \|f\|$. Next, we verify boundedness of the bilinear form $a$, using again the Cauchy-Schwarz inequality,

$$\begin{aligned} |a(u,v)| &\le \int_\Omega |a|\,|\nabla u|\,|\nabla v| + |b|\,|\nabla u|\,|v| + |c|\,|u|\,|v| \, dx \\ &\le \|a\|_\infty \|\nabla u\|\|\nabla v\| + \|b\|_\infty \|\nabla u\|\|v\| + \|c\|_\infty \|u\|\|v\| \\ &\le \big(\|a\|_\infty + \|b\|_\infty + \|c\|_\infty\big) \|u\|_1 \|v\|_1, \end{aligned}$$

i.e., $\beta = \|a\|_\infty + \|b\|_\infty + \|c\|_\infty$. For the coercivity of $a$, we first observe that the Definition 4.3 with $v = b_i u \in H_0^1(\Omega)$ implies that

$$\int_\Omega b \cdot \nabla u u \, dx = \sum_{i=1}^d \int_\Omega b_i u_{x_i} u \, dx = -\sum_{i=1}^d \int_\Omega (b_i u)_{x_i} u \, dx = -\sum_{i=1}^d \int_\Omega \frac{\partial b_i}{\partial x_i} |u|^2 + b_i u_{x_i} u \, dx,$$

where we used the product rule for differentiation in the last step. Rearranging yields that

$$\int_\Omega b \cdot \nabla u u \, dx = -\frac{1}{2} \int_\Omega \operatorname{div}(b) |u|^2 \, dx.$$

Hence, the assumption (4.2) implies that

$$\begin{aligned} a(u,u) &= \int_\Omega a|\nabla u|^2 + b \cdot \nabla u u + c|u|^2 \, dx = \int_\Omega a|\nabla u|^2 + \left(c - \frac{1}{2}\operatorname{div}(b)\right)|u|^2 \, dx \\ &\geq \int_\Omega a_0 |\nabla u|^2 \, dx = a_0 \|\nabla u\|^2. \end{aligned}$$

The Poincaré inequality (4.7) shows that

$$\|u\|_1^2 = \|u\|^2 + \|\nabla u\|^2 \leq (C_P^2 + 1)\|\nabla u\|^2.$$

We conclude that

$$a(u,u) \geq \frac{a_0}{1 + C_P^2} \|u\|_1^2,$$

i.e., $\alpha = a_0/(1 + C_P^2)$. Hence, all assumptions of the Lax-Milgram theorem are satisfied. Therefore, there exists a unique $u \in H_0^1(\Omega)$ that satisfies

$$a(u,v) = \ell(v) \quad \forall v \in H_0^1(\Omega),$$

i.e., $u$ satisfies (4.4); or, in other words, $u$ is a weak solution of (4.1). Moreover

$$\|u\|_1 \leq \frac{C_\ell}{\alpha} = \frac{1 + C_P^2}{a_0} \|f\|.$$

□

## 4.4 Regularity of the weak solution.

The weak solution $u \in H_0^1(\Omega)$ of (4.1) provided by Theorem 4.7 has by construction only one weak derivative. A natural question to ask is whether $u$ enjoys extra regularity. For instance, one could ask if the weak derivatives $u_{x_i}$ do themselves have weak derivatives, and so on. Let us introduce the corresponding Sobolev spaces with index $k \in \mathbb{N}$ inductively by

$$H^k(\Omega) = \{v \in H^{k-1}(\Omega) : \partial_{x_i} v \in H^{k-1}(\Omega), 1 \leq i \leq d\}, \tag{4.8}$$

and norm $\|u\|_k^2 = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|^2$, with $\partial^\alpha u$ the weak derivative of order $\alpha$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ and $|\alpha| = \sum_{i=1}^d \alpha_i$. We have the following theorem, which we provide without proof. The interested student may follow the master course on partial differential equations.

**Theorem 4.8.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\Gamma$ and let $k \in \mathbb{N}_0$. Let $a \in C^{k+1}(\overline{\Omega})$, $b \in C^k(\overline{\Omega})^d$, $c \in C^k(\overline{\Omega})$ satisfy* (4.2)*. Furthermore suppose that $f \in H^k(\Omega)$. Then the weak solution $u \in H_0^1(\Omega)$ of* (4.1) *satisfies $u \in H^{k+2}(\Omega)$, and there exists a constant $C > 0$, which is independent of $u$ such that*

$$\|u\|_{k+2} \leq C\|f\|_k.$$

We mention that the statement of Theorem 4.8 holds also for $k = 0$ if $\Omega$ is a convex domain with polygonal boundary $\Gamma$. In view of [3, Theorem A.5], functions $f \in H^k(\Omega)$ are continuous, i.e., $f \in C^0(\overline{\Omega})$, if $\Gamma$ is smooth or polygonal and $k > d/2$. In view of the previous theorem, we then obtain, under these assumptions, that $u \in C^2(\overline{\Omega})$ is a classical solution. We emphasize that not only smoothness of $a, b, c, f$ is required for such a statement, but that also the regularity of the boundary plays an important role. If $\Gamma$ is a non-convex polygon, then $u \notin H^2(\Omega)$ in general (exercise).

# 5 The finite element method for elliptic PDEs of second order

## Lecture 4

### 5.1 Setting and recap of previous lecture

Let $\Omega = (0,1)$, $\Gamma = \{0,1\} = \partial\Omega$ and consider

$$-\operatorname{div}(a\nabla u) + cu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma,$$

We assume that the coefficient functions $a, c : \overline{\Omega} \to \mathbb{R}$ are bounded and satisfy

$$a(x) \geq a_0 > 0, \qquad c(x) \geq 0 \quad \forall x \in \overline{\Omega}. \tag{5.1}$$

Moreover, we assume that $a(x)$ is piecewise constant. Furthermore, the function $f \in L^2(\Omega)$, which models internal sources, is assumed to be given.

We recall the weak formulation (4.4) in our setting: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \ell(v) \qquad \forall v \in H_0^1(\Omega), \tag{5.2}$$

where

$$a(u,v) = \int_\Omega a(x)u'(x)v'(x) + c(x)u(x)v(x)\,dx, \quad \ell(v) = \int_\Omega f(x)v(x)\,dx. \tag{5.3}$$

We have seen that (5.2) has a unique solution. To proof the existence of a weak solution, we employed that $a$ is coercive and continuous, i.e., that there are constants $\alpha, \beta > 0$

$$\alpha\|v\|_1^2 \leq a(v,v) \leq \beta\|v\|_1^2 \quad \forall v \in H_0^1(\Omega). \tag{5.4}$$

**Remark 5.1.** Let $K \subset [0,1]$ be an interval on which $a(x)$ is constant. Then one can show that $u \in H^2(K)$. If $[0,1] = \cup_{j=1}^{J+1} K_j$ for intervals $K_j$ and $a(x)$ is constant within each $K_j$, we thus have $u \in H^2(K_j)$, and we write $u \in H^2_{pw}(\Omega)$ and for the piecewise $H^2$-semi norm

$$\|u''\|_{pw}^2 = \sum_{j=1}^{J+1} \|u''\|_{L^2(K_j)}^2. \tag{5.5}$$

In particular, there is a constant $C > 0$ such that $\|u''\|_{pw} \leq C\|f\|$. Recall that $H^2(\Omega)$-regularity of $u$ also follows if $a \in C^1(\overline{\Omega})$, but we do not consider this case in the following.

**Aim:** Construct and analyze a numerical method for the approximation of the weak solution.

## 5.2 The Ritz-Galerkin method

Our approach is to project the weak formulation onto a finite-dimensional subspace $S_h \subset H_0^1(\Omega)$ and consider (5.2) with $H_0^1(\Omega)$ replaced by $S_h$, i.e, find $u_h \in S_h$ such that

$$a(u_h, \chi) = \ell(\chi) \qquad \forall \chi \in S_h. \tag{5.6}$$

The following result follows in principle with the same arguments as Theorem 4.7 in the previous lecture. Here, we give a different proof, which introduces some useful notation.

**Theorem 5.2.** *Given $f \in L^2(\Omega)$, then (5.6) has a unique solution $u_h \in S_h$.*

*Proof.* Let $\{\Phi_j : 1 \le j \le J\}$ be a basis for $S_h$. We look for $u_h = \sum_{i=1}^J U_j \Phi_j$, i.e., need to determine $U_j \in \mathbb{R}$. The condition (5.6) is equivalent to (exercise)

$$a(u_h, \Phi_i) = \ell(\Phi_i), \qquad 1 \le i \le J.$$

Defining a matrix $A \in \mathbb{R}^{J\times J}$ via $A_{i,j} = a(\Phi_j, \Phi_i)$ and a vector $F \in \mathbb{R}^J$ by $F_i = \ell(\Phi_i)$ for $1 \le i, j \le J$, the coordinate vector $U \in \mathbb{R}^J$ with entries $U_i$ is characterized as the solution of the linear system

$$AU = F. \tag{5.7}$$

Since $A$ is square, the previous linear system is uniquely solvable if and only if $AV = 0$ implies $V = 0$ for $V \in \mathbb{R}^J$. Writing $v_h = \sum_{j=1}^J V_j \Phi_j$, the condition $AV = 0$ implies that

$$0 = V^T AV = \sum_{i,j=1}^J V_i A_{i,j} V_j = \sum_{i,j=1}^J V_i a(\Phi_j, \Phi_i) V_j = a(v_h, v_h) \ge \alpha \|v_h\|_1^2,$$

where we used the coercivity (5.4) and that $v_h \in S_h \subset H_0^1(\Omega)$. Hence $v_h = 0$ and $V = 0$. □

The proof of the previous result shows that we can obtain $u_h$, the solution to (5.6), numerically, by solving a linear system (5.7) as soon as we can construct the matrix $A$ and the right-hand side vector $F$. Before considering the practical implementation, let us perform an error analysis.

## 5.3 Error analysis for the Ritz-Galerkin method

The key identity is the following orthogonality relation (*Galerkin orthogonality*), which follows directly from subtracting (4.4) and (5.6) from each other,

$$a(u - u_h, \chi) = 0 \quad \forall \chi \in S_h. \tag{5.8}$$

Since, under the assumption that $b = 0$, the bilinear form $a$ is symmetric, we can introduce the *energy norm*

$$\|v\|_a = \sqrt{a(v, v)}, \quad \forall v \in H_0^1(\Omega). \tag{5.9}$$

We note that (5.4) shows that $a$ actually defines an inner product on $H_0^1(\Omega)$ that is equivalent to the standard inner product $(\cdot, \cdot)_1$. Let us state the basic abstract error estimate, compare to the projection theorem.

**Theorem 5.3.** $\|u - u_h\|_a = \min\{\|u - \chi\|_a : \chi \in S_h\}$.

*Proof.* Let $\chi \in S_h$ be arbitrary. We compute

$$\|u - u_h\|_a^2 = a(u - u_h, u - u_h) = a(u - u_h, u - \chi) + a(u - u_h, \chi - u_h) = a(u - u_h, u - \chi),$$

where we used $\chi - u_h \in S_h$ and (5.8). Using the Cauchy-Schwarz inequality, we further obtain

$$a(u - u_h, u - \chi) \leq \|u - u_h\|_a \|u - \chi\|_a.$$

Since $\chi \in S_h$ was arbitrary, the statement follows. □

**Remark 5.4.** Using (5.1), we also obtain an error bound in the usual $H^1$-norm, i.e.,

$$\|u - u_h\|_1 \leq \frac{\beta}{\alpha} \inf_{\chi \in S_h} \|u - \chi\|_1.$$

A similar bound can be obtained if $b \neq 0$ (exercise).

Theorem 5.3 says that the Ritz-Galerkin method yields a solution that is optimal in the energy norm. Sometimes it is useful to consider also error estimates using the $L^2(\Omega)$-norm. Since $\|v\| \leq \|v\|_1 \leq \|v\|_a / \sqrt{\alpha}$, for $v \in H_0^1(\Omega)$, it is clear that the previous energy error bound implies an error estimate in the $L^2(\Omega)$-norm. The next considerations show that we can actually do better, if the problem admits $H^2(\Omega)$-regular solutions.

## 5.4 $L^2$-error estimates

Let $e_h = u - u_h$ denote the error, and let $z \in H_0^1(\Omega)$ be a such that

$$a(v, z) = (e_h, v) \quad \forall v \in H_0^1(\Omega). \tag{5.10}$$

Since $a$ is symmetric, it is clear that $z$ exists and is unique. In view of Remark 5.1, we have that $z \in H_{pw}^2(\Omega)$ and

$$\|z\|_1 + \|z''\|_{pw} \leq C\|e_h\| \tag{5.11}$$

for some constant $C > 0$, compare Lecture 3, Theorem 4.8.

**Theorem 5.5.** *Assume the following approximation property for* $w \in H_{pw}^2(\Omega) \cap H_0^1(\Omega)$

$$\inf_{\chi \in S_h} \|w - \chi\|_a \leq h\|w''\|_{pw} \tag{5.12}$$

*for some* $h > 0$. *Then the following error bounds hold*

$$\|u - u_h\| \leq h\|u - u_h\|_a \leq Ch^2\|f\|. \tag{5.13}$$

*Proof.* Since $u - u_h \in H_0^1(\Omega)$, we can choose $v = u - u_h$ in (5.10) to obtain that

$$\|u - u_h\|^2 = (e_h, e_h) = a(e_h, z).$$

Let $\chi \in S_h$ be arbitrary. Galerkin orthogonality (5.8) then implies that

$$\|u - u_h\|^2 = a(e_h, z) = a(e_h, z - \chi) \leq \|e_h\|_a \|z - \chi\|_a \leq h\|e_h\|_a \|z''\|_{pw} \leq Ch\|e_h\|_a \|e_h\|,$$

where we used (5.12) for $w = z$, and (5.11). Division by $\|u - u_h\|$ shows the first inequality. Applying the approximation property (5.12) for $w = u$, which is possible by Remark 5.1, and using Theorem 5.3 the second bound follows. □

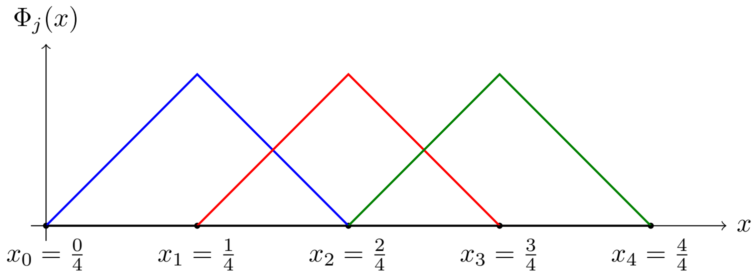


Figure 5.1: Illustration of the basis functions $\Phi_j$, $j = 1, 2, 3$ for a partition $x_j = j/4$ for $j = 0, 1, 2, 3, 4$.

**Remark 5.6.** The method of proof used in this section is sometimes referred to as Aubin-Nitsche trick or a duality argument, compare to (5.10), where we deliberately wrote $a(v, z)$ instead of $a(z, v)$, which makes a difference is $a$ is non-symmetric.

## 5.5 Piecewise polynomial spaces - the finite element method

In the previous paragraphs, the only requirement for the space $S_h$ was that it is finite-dimensional. The discussion did also not rely on the assumption that $\Omega = (0, 1)$ and $d = 1$, but was mainly related to coercivity and continuity properties of the bilinear form $a$ as well as regularity properties of the solution. In view of the proof of Theorem 5.2, it is important that we can effectively setup the matrix $A$ and the vector $F$ in (5.7) and that we can solve that linear system. In this paragraph, we consider exemplary one common choice for the space $S_h$.

Let $0 = x_0 < x_1 < \ldots < x_{J+1} = 1$ be a partition of $[0, 1]$. Also, assume throughout the rest of this lecture that the partition is such that the coefficient function $a$ is piecewise constant,

$$a_{|(x_{j-1}, x_j)} = a_j \quad \forall j = 1, \ldots, J + 1. \tag{5.14}$$

In this case the piecewise $H^2(\Omega)$-regularity of the dual solution $z$ and the solution $u$ is guaranteed, cf. Remark 5.1. Let $K_j = [x_{j-1}, x_j]$ and $h_j = x_j - x_{j-1}$ for $1 \leq j \leq J+1$, and denote $h = \max_j h_j$. Let $S_h$ be such that $\chi \in S_h$ if and only if

- $\chi \in C^0([0, 1])$,
- $\chi_{|K_j}$ is a linear polynomial for $j = 1, \ldots, J + 1$,
- $\chi(0) = \chi(1) = 0$.

As we have seen in the proof of Theorem 5.2, it is useful to have a basis for $S_h$. To that end, we consider the functions

$$\Phi_j \in S_h \quad \text{satisfying} \quad \Phi_j(x_i) = \delta_{i,j}, \quad j = 1, \ldots, J, \tag{5.15}$$

see Figure 5.1 for an illustration. Moreover, we introduce the *nodal interpolant*

$$I_h : C^0([0, 1]) \to S_h, \quad (I_h v)(x) = \sum_{j=1}^{J} v(x_j)\Phi_j(x). \tag{5.16}$$

**Lemma 5.7.** *The functions $\{\Phi_j : 1 \le j \le J\}$ form a basis of $S_h$.*

*Proof.* The functions $\Phi_j$, $1 \le j \le J$ are linearly independent: Let $\chi = \sum_{j=1}^{J} c_j \Phi_j = 0$. Then

$$0 = \chi(x_j) = c_j \quad \forall 1 \le j \le M. \tag{5.17}$$

To see that $\{\Phi_j\}$ span $S_h$, let $v \in S_h$. Since $v \in C^0([0,1])$, we can consider $\chi = v - I_h v \in S_h$. Clearly, $\chi(x_j) = 0$ for all $j = 0, \dots, J+1$ and $\chi$ is linear on each $K_j$. Hence, $\chi$ vanishes on each $K_j$, which shows that $v = I_h v \in \operatorname{span}\{\Phi_j : j = 1, \dots, J\}$. □

In view of Example 4.5 in Lecture 3, we see that $S_h \subset H_0^1(\Omega)$, and hence the Ritz-Galerkin projection $u_h \in S_h$ defined in (5.6) is well-defined. Next, we work out the abstract error estimates of the previous section in our context. We start with the following error estimates for the nodal interpolant, whose proof is left as an exercise.

**Lemma 5.8.** *There is a constant $C > 0$ that does not depend on $h_j$ such that*

$$\|I_h v - v\|_{L^2(K_j)} \le C h_j^2 \|v''\|_{L^2(K_j)} \quad \forall v \in H^2(K_j), \tag{5.18}$$

$$\|(I_h v - v)'\|_{L^2(K_j)} \le C h_j \|v''\|_{L^2(K_j)} \quad \forall v \in H^2(K_j). \tag{5.19}$$

Using Lemma 5.8, and the previous error bounds, we arrive at the following result for the finite element method.

**Theorem 5.9.** *The following error bounds hold*

$$\|u - u_h\|_a \le C h \|u''\|_{pw}, \qquad \|u - u_h\| \le C h^2 \|u''\|_{pw}.$$

*Proof.* Setting $\chi = I_h u$ in Theorem 5.3 and using (5.1) yields that

$$\|u - u_h\|_a \le \|u - I_h u\|_a \le \sqrt{\beta} \|u - I_h u\|_1.$$

Furthermore, by summation, and applying Lemma 5.8, we obtain that

$$\begin{aligned}\|u - I_h u\|_1^2 &= \sum_{j=1}^{J+1} \left( \|u - I_h u\|_{L^2(K_j)}^2 + \|(u - I_h u)'\|_{L^2(K_j)}^2 \right) \le \sum_{j=1}^{J+1} C h_j^2 \|u''\|_{L^2(K_j)}^2 \\ &\le C h^2 \|u''\|_{pw}^2,\end{aligned}$$

which proves the first inequality. With a similar argument, it follows that (5.12) holds true for $h$ as defined here. The second claimed inequality then follows from (5.13) and the already proven estimate. □

## 5.6 The Ritz projection

The proven error estimates relied on the nodal interpolant $I_h$. For $d = 1$, we have that $H^1(\Omega) \subset C^0(\overline{\Omega})$. Hence, $I_h v$ makes sense for $v \in H^1(\Omega)$. This is not the case anymore for $d > 1$. For this and theoretical purposes, it is useful to consider also other interpolation operators. Here, we focus on the *Ritz projection* $R_h : H_0^1(\Omega) \to S_h$, which is, for $v \in H_0^1(\Omega)$, defined as the unique solution of the variational problem

$$a(R_h v, \chi) = a(v, \chi) \quad \chi \in S_h, \tag{5.20}$$

i.e., $R_h$ is just the orthogonal projection in the energy inner product, and $u_h = R_h u$.

**Theorem 5.10.** *The Ritz projection defined in* (5.20) *satisfies the following estimates*

$$h\|(R_h v - v)'\| + \|R_h v - v\| \leq Ch^2 \|v''\|_{pw} \qquad \forall v \in H_0^1(\Omega) \cap H_{pw}^2(\Omega), \tag{5.21}$$

$$\|R_h v - v\| \leq Ch\|v\|_1 \qquad \forall v \in H_0^1(\Omega). \tag{5.22}$$

*Proof.* The first inequality follows directly from Theorem 5.9. Using (5.1), we observe that

$$\alpha\|R_h v\|_1^2 \leq a(R_h v, R_h v) = a(v, R_h v) \leq \|v\|_a \|R_h v\|_a \leq \beta \|v\|_1 \|R_h v\|_1,$$

i.e., $\|R_h v\|_1 \leq C\|v\|_1$. Consider (5.10) with $e_h = R_h v - v$. We then obtain that

$$\|R_h v - v\|^2 = a(R_h v - v, z) = a(R_h v - v, z - \chi) \leq \|R_h v - v\|_a \|z - \chi\|_a \leq C\|v\|_1 \|z - \chi\|_a,$$

where $\chi \in S_h$ is arbitrary. The result then follows from the bound $\|z - I_h z\|_a \leq Ch\|R_h v - v\|$, which holds in view of (5.11). □

**Remark 5.11.** The quantitative error bounds, i.e., the ones which contain powers of $h$, in the statements above require regularity of the function that is to be approximated. For the solution $u$ and the dual solution $z$, this has been established by assumptions on the coefficients and the $L^2(\Omega)$-regularity of the corresponding source terms.

# 6 The eigenvalue problem for elliptic PDEs of second order

## Lecture 5

Eigenfunction expansions as considered here will be important in subsequent lectures for

- the analysis of partial differential equations,
- applications (think of stability of structures and resonances),
- time-dependent problems.

## 6.1 Motivation

We next outline how eigenfunctions can be helpful in solving partial differential equations. Note that we make a lot of assumptions and formal computations, which need to be justified. Consider Dirichlet's problem for the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma. \tag{6.1}$$

Suppose that we have at hand functions $\varphi_n$ ($\neq 0$), $n \in \mathbb{N}$, such that

$$-\Delta \varphi_n = \lambda_n \varphi_n \quad \text{in } \Omega, \qquad \varphi_n = 0 \quad \text{on } \Gamma, \tag{6.2}$$

and that the following expansion holds (compare to Fourier series)

$$f = \sum_{n=1}^{\infty} f_n \varphi_n. \tag{6.3}$$

Next, we consider the ansatz

$$u = \sum_{n=1}^{\infty} u_n \varphi_n \tag{6.4}$$

for the solution to (6.1) with unknown expansion coefficients $u_n \in \mathbb{R}$. A formal computation, which assumes that we can interchange summation and differentiation, yields that

$$-\Delta u = \sum_{n=1}^{\infty} u_n(-\Delta \varphi_n) = \sum_{n=1}^{\infty} u_n \lambda_n \varphi_n.$$

Combining this with (6.3) and (6.1) yields for $\lambda_n \neq 0$ that

$$u_n = \frac{f_n}{\lambda_n},$$

and $u$ given by (6.4) yields a solution to (6.1).

**Remark 6.1.** Generalizing the notion of eigenvector and eigenvalue of a matrix (a linear mapping between finite dimensional spaces), we call $\varphi_n$ an eigenfunction of the Dirichlet-Laplace operator. The suffix 'Dirichlet' indicates the type of boundary condition considered.

The following questions arise:

- Existence of eigenfunctions $\varphi_n$ with $\lambda_n$; (see Theorem 6.3).
- Positivity of eigenvalues: $\lambda_n > 0$? (see Theorem 6.2)
- Which $f$ can be written as (6.3)? (see Theorem 6.7)
- Does (6.4) converge? (see Theorem 6.7)

## 6.2 Variational approach for existence of eigenfunctions

To answer the previous questions, we will rely on a variational treatment of (6.2), as follows: Find $\varphi \in H_0^1(\Omega)$, $\varphi \neq 0$, and $\lambda \in \mathbb{R}$ such that

$$a(\varphi, v) = \lambda(\varphi, v) \qquad \forall v \in H_0^1(\Omega), \tag{6.5}$$

with $a(\varphi, v) = (\nabla\varphi, \nabla v)$; see Lecture 3 for the notation. Our first observation is that the eigenvalues $\lambda$ cannot be negative: Setting $v = \varphi$ in (6.5), we obtain that

$$\lambda\|\varphi\|^2 = a(\varphi, \varphi) = \|\nabla\varphi\|^2 > 0,$$

where we used the Poincaré inequality (4.7) for the last inequality. Since $\|\varphi\| > 0$, we have partly proven the following statement.

**Theorem 6.2.** *The eigenvalues $\lambda$ of (6.5) are positive. Two eigenfunctions $\varphi, \psi$ associated to different eigenvalues $\lambda, \mu$ are orthogonal in $L^2(\Omega)$ and in $H_0^1(\Omega)$.*

*Proof.* By (6.5) we have that

$$\begin{aligned} a(\varphi, v) &= \lambda(\varphi, v) \quad \forall v \in H_0^1(\Omega), \\ a(\psi, v) &= \mu(\psi, v) \quad \forall v \in H_0^1(\Omega). \end{aligned}$$

By setting $v = \psi$ and $v = \varphi$, respectively, and using the symmetry of $a$, we then obtain that

$$\mu(\psi, \varphi) = a(\psi, \varphi) = a(\varphi, \psi) = \lambda(\varphi, \psi), \quad \text{i.e., } (\mu - \lambda)(\varphi, \psi) = 0.$$

Since, by assumption $\mu \neq \lambda$, we obtain $(\varphi, \psi) = 0$ and then $a(\varphi, \psi) = 0$, which concludes the proof. □

In order to investigate the existence of eigenpairs $(\varphi, \lambda)$, we minimize the *Rayleigh quotient*

$$R_1 = \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{a(v, v)}{\|v\|^2} = \inf\{a(v, v) : v \in H_0^1(\Omega),\ \|v\| = 1\}. \tag{6.6}$$

**Theorem 6.3** (Existence of first eigenfunction)**.** *The infimum in (6.6) is attained by a function $\varphi_1 \in H_0^1(\Omega)$, which satisfies (6.5) with $\lambda_1 = R_1$.*

The proof of Theorem 6.3 relies on the following lemma, which will be proven in the master class partial differential equations.

**Lemma 6.4** (Rellich lemma)**.** *Let* $\{u_n\} \subset H_0^1(\Omega)$ *be a bounded sequence, i.e., there exists a constant* $C > 0$ *that does not depend on* $n$ *such that* $\|u_n\|_1 \le C$*. Then there exists a subsequence* $\{u_{n_j}\}$ *and* $u \in L^2(\Omega)$ *such that* $\lim_{j\to\infty} \|u_{n_j} - u\| = 0$*.*

Note that in Lemma 6.4, the convergence takes place in $L^2(\Omega)$ and *not* in $H_0^1(\Omega)$.

*Proof of Theorem 6.3.* **Step 1.** (*The direct method* in the calculus of variations.) Since $a(v,v)/\|v\|^2 \ge 0$, there exists a minimizing sequence $\{\tilde{u}_n\} \subset H_0^1(\Omega)$ such that

$$\lim_{n\to\infty} \frac{a(\tilde{u}_n, \tilde{u}_n)}{\|\tilde{u}_n\|^2} = R_1.$$

Setting $u_n = \tilde{u}_n/\|\tilde{u}_n\|$ shows that $\lim_{n\to\infty} a(u_n, u_n) = R_1$. Poincaré's inequality (4.7) gives the existence of a constant $C > 0$ such that

$$\|u_n\|_1 \le C \quad \forall n \in \mathbb{N}.$$

Rellich's lemma 6.4 yields $u \in L^2(\Omega)$ and a subsequence $\{u_{n_j}\}$ such that

$$\lim_{j\to\infty} \|u_{n_j} - u\| = 0, \qquad \|u\| = 1.$$

In the following, we relabel the subsequence and simply write $\{u_n\}$ instead of $\{u_{n_j}\}$. To show that actually $u \in H_0^1(\Omega)$, observe that, by (6.6)

$$\begin{aligned}\|\nabla u_n - \nabla u_m\|^2 &= 2\|\nabla u_n\|^2 + 2\|\nabla u_m\|^2 - \|\nabla(u_m + u_n)\|^2 \\ &\le 2\|\nabla u_n\|^2 + 2\|\nabla u_m\|^2 - R_1\|u_n + u_m\|^2.\end{aligned}$$

Since $\|\nabla u_n\|^2 \to R_1$, $\|\nabla u_m\|^2 \to R_1$ and $\|u_n + u_m\|^2 \to \|2u\|^2 = 4\|u\|^2 = 4$ for $m, n \to \infty$. We observe that $\{u_n\}$ is a Cauchy sequence in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is complete, there exists $w \in H_0^1(\Omega)$ such that $u_n \to w$ in $H_0^1(\Omega)$ for $n \to \infty$. Uniqueness of the limit implies that $u = w \in H_0^1(\Omega)$, and

$$\|\nabla u\|^2 = \lim_{n\to\infty} \|\nabla u_n\|^2 = R_1.$$

Setting $\varphi_1 = u$ proves the first part of the assertion.
**Step 2.** We claim that $\varphi_1$ is an eigenfunction. To prove this, let $\alpha \in \mathbb{R}$ with $\alpha < 0$ and $v \in H_0^1(\mathbb{R})$ be arbitrary. Since $w = \varphi + \alpha v \in H_0^1(\Omega)$, we get that

$$R_1\|w\|^2 \le a(w,w) = a(\varphi_1, \varphi 1) + 2\alpha a(\varphi_1, v) + \alpha^2 a(v,v). \tag{6.7}$$

Since $R_1 = a(\varphi, \varphi)$ by Step 1, and

$$\|w\|^2 = \|\varphi_1\|^2 + 2\alpha(\varphi_1, v) + \alpha^2\|v\|^2 = 1 + 2\alpha(\varphi_1, v) + \alpha^2\|v\|^2,$$

rearranging terms in (6.7) yields that

$$2\alpha\big(a(\varphi_1, v) - R_1(\varphi_1, v)\big) + \alpha^2\big(a(v,v) - R_1\|v\|^2\big) \ge 0.$$

Taking the limit $\alpha \to 0^-$ shows that $a(\varphi, v) = R_1(\varphi_1, v)$. Since $v$ was arbitrary in $H_0^1(\Omega)$, the theorem is proven. □

In order to obtain the next eigenfunctions, we minimize the Rayleigh quotient over functions that are orthogonal to $\varphi_1$,

$$R_2 = \inf\{a(v,v) : v \in H_0^1(\Omega),\ \|v\| = 1\,(v,\varphi_1) = 0\}. \tag{6.8}$$

**Observations.**

- $R_2 \geq R_1$.
- The infimum in (6.8) is attained (same proof as above). The minimizer, denoted by $\varphi_2$ satisfies

  $$a(\varphi_2,\varphi_2) = R_2, \quad \|\varphi_2\| = 1, \quad (\varphi_2,\varphi_1) = 0,$$

  and

  $$a(\varphi_2, v) = \lambda_2(\varphi_2, v) \qquad \forall v \in H_0^1(\Omega),$$

  i.e., (6.5) with $\lambda_2 = R_2$. To show the latter, write $v = \alpha\varphi_1 + w$ for some $w \in H_0^1(\Omega)$ with $(w,\varphi_1) = 0$. Then observe that

  $$a(\varphi_2, v) = \alpha a(\varphi_2,\varphi_1) + a(\varphi_2, w) = \alpha\lambda_1(\varphi_1,\varphi_2) + \lambda_2(\varphi_2, w) = \lambda_2(\varphi_2, v).$$

By iterating the above arguments, we construct a sequence $\{(\lambda_n,\varphi_n)\}$, $\varphi_n \in H_0^1(\Omega)$ with

$$0 < \lambda_n \leq \lambda_{n+1}, \quad \|\varphi_n\| = 1,$$

and

$$\lambda_n = \inf\{a(v,v) : v \in H_0^1(\Omega),\ \|v\| = 1,\ (v,\varphi_j) = 0 \forall 1 \leq j \leq n-1\}, \tag{6.9}$$

$$a(\varphi_n, v) = \lambda_n(\varphi_n, v) \quad \forall v \in H_0^1(\Omega). \tag{6.10}$$

## 6.3 The Min-Max Theorem

The next statement is another important characterization of the eigenvalues

**Theorem 6.5** (Min-Max Theorem)**.** *We have that*

$$\lambda_n = \min_{V_n} \max_{v \in V_n} \frac{a(v,v)}{\|v\|^2}, \tag{6.11}$$

*where $V_n$ varies over all subspaces of $H_0^1(\Omega)$ of dimension $n$. The minimum is attained for $V_n = span\{\varphi_1, \dots, \varphi_n\}$.*

*Proof.* Let $E_n = \text{span}\{\varphi_1, \dots, \varphi_n\}$. Then any $v \in E_n$ can be written as $v = \sum_{j=1}^n \alpha_j \varphi_j$ and

$$\frac{a(v,v)}{\|v\|^2} = \frac{\sum_{j=1}^n \alpha_j^2 \lambda_j}{\sum_{j=1}^n \alpha_j^2} \leq \lambda_n,$$

because $\lambda_j \le \lambda_n$ for $1 \le j \le n$. The value $\lambda_n$ is attained for $v = \varphi_n$. It thus remains to verify that

$$\max_{v \in V_n} \frac{a(v,v)}{\|v\|^2} \ge \lambda_n,$$

for any $V_n \subset H_0^1(\Omega)$ of dimension $n$. This is clear for $V_n = E_n$, if $V_n \ne E_n$, then there is $w \in V_n$ such that $\|w\| = 1$ and

$$(w, \varphi_j) = 0 \quad \forall 1 \le j \le n.$$

Using (6.9), we then deduce that $\lambda_n \le a(w,w)$, which proves the claim. □

Using Theorem 6.5, one can show that (exercise or [3, Example 6.1]).

**Theorem 6.6.** *There are positive constants $c_1, c_2 > 0$ that depend on $\Omega$ such that*

$$c_1 n^{2/d} \le \lambda_n \le c_2 n^{2/d}, \quad \forall n \in \mathbb{N}.$$

## 6.4 Eigenfunction expansions: completeness

**Theorem 6.7** (Completeness of eigenfunctions)**.** *The eigenfunctions $\{\varphi_n\}$ for (6.5) form an orthonormal basis of $L^2(\Omega)$. Furthermore, if $v \in H_0^1(\Omega)$, then*

$$a(v,v) = \sum_{n=1}^{\infty} \lambda_n (v, \varphi_n)^2 < \infty. \tag{6.12}$$

*Conversely, if $\sum_{n=1}^{\infty} \lambda_n |v_n|^2 < \infty$, then $v = \sum_{n=1}^{\infty} v_n \varphi_n \in H_0^1(\Omega)$ and (6.12) holds.*

Before proving the theorem, let us recall what it means that a system of functions $\{\phi_n\}$ is a basis of a (real) Hilbert space $H$. That is to say that for all $v \in H$ and any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and scalars $a_1, \dots, a_N \in \mathbb{R}$ such that $\|v - \sum_{j=1}^{N} a_j \phi_j\|_H < \varepsilon$. Furthermore, a set $M \subset H$ is called dense if for all $v \in H$ and $\varepsilon > 0$ there exists $\phi \in M$ such that $\|v - \phi\|_H < \varepsilon$. As an example, we may recall that $C_0^\infty(\Omega)$ is a dense subset of $L^2(\Omega)$.

*Proof.* **Step 1. $L^2$-basis.** Suppose first that $v \in H_0^1(\Omega)$, and consider the residual

$$r_N = v - \sum_{n=1}^{N} (v, \varphi_n) \varphi_n.$$

Since $N$ is finite, we can compute, using orthogonality of the eigenfunctions $(\varphi_n, \varphi_j) = \delta_{n,j}$,

$$(r_N, \varphi_j) = (v, \varphi_j) - \sum_{n=1}^{N} (v, \varphi_n)(\varphi_n, \varphi_j) = 0, \quad \forall 1 \le j \le N.$$

Using (6.9), we further have that

$$\|r_N\| \le \sqrt{\frac{a(r_N, r_N)}{\lambda_{N+1}}}.$$

We claim that the right-hand side vanishes for $N \to \infty$, by Theorem 6.6 and boundedness of $a(r_N, r_N)$. To show the latter, we compute

$$\begin{aligned} a(r_N, r_N) &= a(v,v) - 2\sum_{n=1}^{N}(v,\varphi_n)a(v,\varphi_n) + \sum_{n=1}^{N}(v,\varphi_n)^2 a(\varphi_n,\varphi_n) \\ &= a(v,v) - \sum_{n=1}^{N}\lambda_n(v,\varphi_n)^2 \le a(v,v), \end{aligned} \tag{6.13}$$

where we used that $a(v,\varphi_n) = \lambda(v,\varphi_n)$ and $a(\varphi_j,\varphi_n) = \lambda_n\delta_{j,n}$. This proves Step 1 for $v \in H_0^1(\Omega)$. If $v \in L^2(\Omega)$, let $\varepsilon > 0$ and choose $\psi \in C_0^\infty(\Omega)$ such that $\|\psi - v\| < \varepsilon/2$. Since $\psi \in H_0^1(\Omega)$, we find $N \in \mathbb{N}$ such that for $\psi_N = \sum_{n=1}^N (\psi,\varphi_n)\varphi_n$ we have $\|\psi - \psi_N\| < \varepsilon/2$. The triangle inequality then implies that

$$\|v - \psi_N\| \le \|v - \psi\| + \|\psi - \psi_N\| < \varepsilon.$$

**Step 2.** From (6.13), we observe that $\sum_{n=1}^\infty \lambda_n(v,\varphi_n)^2$ exists if $v \in H_0^1(\Omega)$. Moreover, for $v_N = \sum_{n=1}^N (v,\varphi_n)\varphi_n$, we have that

$$\|\nabla v_N\|^2 = a(v_N, v_N) = \sum_{n=1}^{N}\lambda_n(v,\varphi_n)^2.$$

Hence, $\{v_N\}$ is a Cauchy sequence in $H_0^1(\Omega)$, and converges to some limit in $H_0^1(\Omega)$. Since $v_N \to v$ in $L^2(\Omega)$ by Step 1, uniqueness of the limit implies that $v_N \to v$ in $H_0^1(\Omega)$. Thus,

$$a(v,v) = \lim_{N\to\infty}\|\nabla v_N\|^2 = \sum_{n=1}^{\infty}\lambda_n(v,\varphi_n)^2.$$

Conversely, if $\sum_{n=1}^\infty \lambda_n(v,\varphi_n)^2$ exists, then $\{v_N\}$ is Cauchy in $H_0^1(\Omega)$. Since $v_N \to v$ in $L^2(\Omega)$ by Step 1, we obtain that $v \in H_0^1(\Omega)$. □

**Remark 6.8.** In the situation of Step 1 in the previous proof and $v \in L^2(\Omega)$, note that the Cauchy-Schwarz inequality implies that $|(\psi - v, \varphi_n)| \le \|\psi - v\|$. Hence,

$$\begin{aligned} \|v - \sum_{n=1}^{N}(v,\varphi_n)\varphi_n\| &\le \varepsilon + \|\sum_{n=1}^{N}(\psi - v,\varphi_n)\varphi_n\| \le \varepsilon + \left(\sum_{n=1}^{N}(\psi - v,\varphi_n)^2\right)^{1/2}. \\ &\le \varepsilon(1+\sqrt{N}). \end{aligned}$$

Hence, by choosing $\varepsilon$ sufficiently small, $\lim_{N\to\infty}\sum_{n=1}^N (v,\varphi_n)\varphi_n = v$ in $L^2(\Omega)$.

To close this section, let us look back at the questions posed in the beginning of this chapter, i.e.: Which $f$ can be written as (6.3)? Does (6.4) converge? Since $\{\varphi_j\}_j$ forms a basis of $L^2(\Omega)$, it follows that every $f \in L^2(\Omega)$ can be written as (6.3) with

$$f_n = (f, \varphi_n),$$

where we recall that $\|\varphi_n\| = 1$. Furthermore, let us define $u_n = f_n/\lambda_n$. Since $\lambda_n \to \infty$ for $n \to \infty$, it follows that

$$\sum_{n=1}^{\infty} \lambda_n u_n^2 = \sum_{n=1}^{\infty} \frac{f_n^2}{\lambda_n} \leq C \sum_{n=1}^{\infty} f_n^2 = C\|f\|^2,$$

i.e., the series (6.4) converges in $H_0^1(\Omega)$, by Theorem 6.7. The same argument shows that

$$\sum_{n=1}^{\infty} \lambda_n^2 u_n^2 < \infty.$$

In view of the exercises, it then follows that $u \in H^2(\Omega) \cap H_0^1(\Omega)$, and

$$-\Delta u = \sum_{n=1}^{\infty} \lambda_n u_n \varphi_n = \sum_{n=1}^{\infty} f_n \varphi_n = f.$$

Therefore, we actually found the solution of (6.1) using eigenfunction expansions.

**Remark 6.9.** The theory developed in this class applies to more general elliptic equations of second order, for which the corresponding bilinear form $a$ is symmetric and coercive. We leave the details to the reader.

## 6.5 Finite element approximation of the eigenvalue problem

We consider $\Omega = (0, 1)$. Let $0 = x_0 < x_1 < \ldots < x_{J+1} = 1$ and

$$S_h = \{v \in C^0(\overline{\Omega}) : v \text{ is linear on } (x_{j-1}, x_j),\ v(0) = v(1) = 0\} \subset H_0^1(\Omega).$$

The discrete eigenvalue problem, which results from the projection of (6.5) onto $S_h$, is then: Find $\varphi_{n,h} \in S_h$, $\lambda_{n,h} \in \mathbb{R}$ such that

$$a(\varphi_{n,h}, \chi) = \lambda_{n,h}(\varphi_{n,h}, \chi) \quad \forall \chi \in S_h. \tag{6.14}$$

We recall the Ritz projector $R_h : H_0^1(\Omega) \to S_h$ defined by

$$a(R_h v, \chi) = a(v, \chi) \quad \forall \chi \in S_h,\ v \in H_0^1(\Omega). \tag{6.15}$$

In the previous lecture, we have shown the error estimates, cf. Theorem 5.10,

$$\|R_h v - v\| \leq Ch^2 \|v\|_s, \quad \|(R_h v - v)'\| \leq Ch^{s-1}\|v\|_s, \quad v \in H^s \cap H_0^1(\Omega), s \in \{1, 2\}. \tag{6.16}$$

The MinMax-Theorem 6.5 implies the following result. It states that we can approximate a fixed eigenvalue with second order accuracy in $h$. Note, however, that the required asymptotic error bound requires higher resolution (smaller $h$) for increasing $n$.

**Theorem 6.10.** *Let $\lambda_{n,h}$ and $\lambda_n$ be the nth eigenvalue of* (6.14) *and* (6.5)*, respectively. Then there exists a constants $C > 0$, $h_0 > 0$ (depending on $n$) such that for $h \leq h_0$:*

$$\lambda_n \leq \lambda_{n,h} \leq \lambda_n + Ch^2.$$

*Proof.* The MinMax-principle implies that

$$\lambda_n = \min_{V_n \subset H_0^1(\Omega)} \max_{v \in V_n} \frac{a(v,v)}{\|v\|^2},$$
$$\lambda_{n,h} = \min_{V_{n,h} \subset S_h} \max_{\chi \in V_{n,h}} \frac{a(\chi,\chi)}{\|\chi\|^2}. \tag{6.17}$$

Since $S_h \subset H_0^1(\Omega)$, it follows that $\lambda_n \leq \lambda_{n,h}$, which proves the first inequality. Introduce the spaces

$$E_n = \operatorname{span}\{\varphi_1, \dots, \varphi_n\},$$
$$E_{n,h} = \operatorname{span}\{R_h\varphi_1, \dots, R_h\varphi_n\}.$$

Then, (6.17) and (6.16) with $s = 1$ implies that

$$\lambda_{n,h} \leq \max_{\chi \in E_{n,h}} \frac{a(\chi,\chi)}{\|\chi\|^2} = \max_{v \in E_n} \frac{\|\nabla R_h v\|^2}{\|R_h v\|^2} \leq \max_{v \in E_n} \frac{\|\nabla v\|^2}{\|R_h v\|^2}.$$

To estimate the term $\|R_h v\|$ in the denominator, we use the second triangle inequality and (6.16) with $s = 2$ to obtain that

$$\|v\| - \|R_h v\| \leq \|v - R_h v\| \leq Ch^2\|v\|_2 \leq ch^2\|\Delta v\| \leq Ch^2\lambda_n\|v\|,$$

where we used the regularity bound from Theorem 4.8 in Lecture 3, and $v \in E_n$. We have thus $\|R_h v\| \geq (1 - Ch^2\lambda_n)\|v\|$. Then we can bound as follows, using $h \leq h_0$,

$$\max_{v \in E_n} \frac{\|\nabla v\|^2}{\|R_h v\|^2} \leq \max_{v \in E_n} \frac{\|\nabla v\|^2}{(1 - Ch^2\lambda_n)^2\|v\|^2} \leq \frac{\lambda_n}{(1 - Ch^2\lambda_n)^2} \leq \lambda_n + \tilde{C}h^2,$$

where we used $v = \varphi_n$ in the second inequality, compare the proof of Theorem 6.5. Note that, for the last step, we estimate, with constants $h_0, C' > 0$ that depend on $n$,

$$\frac{1}{1 - C\lambda_n h^2} = 1 + \frac{C\lambda_n h^2}{1 - C\lambda_n h^2} \leq 1 + \frac{C\lambda_n h^2}{1 - C\lambda_n h_0^2} = 1 + C'h^2.$$

This concludes the proof. □

We conclude this lecture with the following approximation result for the eigenfunctions, see [3, Theorem 6.8] for its proof.

**Theorem 6.11.** *Let $\varphi_{1,h}$ and $\varphi_1$ be the normalized first eigenfunctions of the eigenvalue problems* (6.14) *and* (6.5)*, respectively. Then there exists a constant $C > 0$*

$$\|\varphi_{1,h} - \varphi_1\| \leq Ch^2, \quad \|\nabla\varphi_{1,h} - \nabla\varphi_1\| \leq Ch \qquad \forall h > 0.$$

# 7 Parabolic PDEs of second order: classical treatment

## Lecture 6

We consider the following initial-boundary value problem: Find $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \Omega \times \mathbb{R}_+, \\ u &= 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \\ u(\cdot, 0) &= v \quad \text{in } \Omega, \end{aligned} \tag{7.1}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded domain with smooth boundary $\Gamma := \partial\Omega$, and $v \in L^2(\Omega)$.

We first seek a candidate solution by referring to Theorem 6.4 in [3]. Assuming $u(\cdot, t) \in H_0^1(\Omega)$, we expand it as

$$u(x,t) = \sum_{j=1}^{\infty} \hat{u}_j(t)\varphi_j(x), \quad x \in \Omega,\ t > 0, \tag{7.2}$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$ given by the solutions of the eigenvalue problem:

$$-\Delta\varphi_j = \lambda_j\varphi_j \quad \text{in } \Omega, \qquad \varphi_j = 0 \quad \text{on } \Gamma, \tag{7.3}$$

with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ satisfying $\lambda_j \to \infty$ as $j \to \infty$.

We recall that by "orthonormal" we mean:

$$(\varphi_i, \varphi_j) = \int_\Omega \varphi_i\varphi_j \,\mathrm{d}x = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{7.4}$$

To find an expression for the time-dependent coefficients $\{\hat{u}_j(t)\}_{j=1}^{\infty}$, we use the PDE. Applying the operator $\partial_t - \Delta$ to (7.2) and assuming we can push it inside the sum, we obtain

$$\sum_{j=1}^{\infty} (\hat{u}_j'(t) + \lambda_j\hat{u}_j(t))\varphi_j(x) = 0.$$

Since $\{\varphi_j\}_{j=1}^{\infty}$ is a basis of $L^2(\Omega)$, the identity above holds if and only if

$$\hat{u}_j'(t) + \lambda_j\hat{u}_j(t) = 0 \quad t > 0, \quad j \in \mathbb{N}.$$

Moreover, since at $t = 0$ we must satisfy the initial condition $u(\cdot, 0) = v$ in $\Omega$, we require

$$\hat{u}_j(0) = \hat{v}_j, \qquad \text{where} \quad \hat{v}_j := (v, \varphi_j) = \int_\Omega v\varphi_j \,\mathrm{d}x, \quad j \in \mathbb{N}.$$

Thus, the coefficients are given by

$$\hat{u}_j(t) = \hat{v}_j \,\mathrm{e}^{-\lambda_j t}, \quad t > 0, \quad j \in \mathbb{N},$$

and our candidate solution for (7.1) is formally given by

$$\boxed{u(x,t) = \sum_{j=1}^{\infty} \hat{v}_j \,\mathrm{e}^{-\lambda_j t} \,\varphi_j(x), \quad x \in \Omega, \quad t > 0.} \tag{7.5}$$

The first question we should ask ourselves is whether the sum in (7.5) defines a function and in what space. It is straightforward to see that the sum defines an element of $L^2(\Omega)$. Indeed, by Parseval's identity, we have

$$\|u(\cdot,t)\|^2 = \sum_{j=1}^{\infty} (\hat{v}_j e^{-\lambda_j t})^2 \le e^{-2\lambda_1 t} \underbrace{\sum_{j=1}^{\infty} \hat{v}_j^2}_{=\|v\|^2} = \underbrace{\mathrm{e}^{-2\lambda_1 t}}_{\le 1} \|v\|^2 \le \|v\|^2, \quad t \ge 0, \tag{7.6}$$

where we used the fact that $0 < \lambda_1 \le \lambda_2 \le \cdots$, ensuring that $1 \ge \mathrm{e}^{-\lambda_1 t} \ge \mathrm{e}^{-\lambda_j t} > 0$ for all $j \in \mathbb{N}$ and $t \ge 0$.

Furthermore, since $\lambda_1 > 0$, we immediately get that

$$|u(\cdot,t)| \to 0 \quad \text{as} \quad t \to \infty.$$

Now, the question of whether (7.5) is indeed a solution of (7.1), and in what sense, is answered by the following theorem:

**Theorem 7.1.** *For any $v \in L^2(\Omega)$, the function $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ given by* (7.5) *is a classical solution of the initial-boundary value problem* (7.1)*. Moreover,* (7.5) *satisfies the estimate*

$$\|D_t^m u(\cdot,t)\|_s \le C t^{-m-s/2} \|v\|, \quad t > 0, \quad s, m \in \mathbb{N}_0.$$

*Proof.* We will repeatedly use the fact that for all $k \ge 0$, there exists a constant $C_k > 0$ such that

$$s^k \,\mathrm{e}^{-s} \le C_k, \quad \text{for all } s \ge 0.$$

We begin by showing that $u$ in (7.5) is an element of $H_0^1(\Omega)$. Since we already know that $u(\cdot,t) \in L^2(\Omega)$ for all $t > 0$, it remains to prove that

$$\int_\Omega |\nabla u(\cdot,t)|^2 \,\mathrm{d}t = \|\nabla u(\cdot,t)\|^2 = |u(\cdot,t)|_1^2 < \infty.$$

To do so, we apply Theorem 6.4, which states that

$$\sum_{j=1}^{\infty} \lambda_j (v, \varphi_j)^2 < \infty \iff v \in H_0^1(\Omega),$$

and that

$$\|\nabla v\|^2 = \sum_{j=1}^{\infty} \lambda_j (v, \varphi_j)^2.$$

Then

$$|u(\cdot,t)|_1^2 = \sum_{j=1}^{\infty} \lambda_j \hat{v}_j^2 \,\mathrm{e}^{-2\lambda_j t} = t^{-1} \sum_{j=1}^{\infty} \hat{v}_j^2 \underbrace{\mathrm{e}^{-\lambda_j t}}_{\leq 1} \underbrace{(\lambda_j t \,\mathrm{e}^{-\lambda_j t})}_{\leq C_1} \leq t^{-1} C_1 \|v\|^2, \quad t > 0.$$

This shows that $u(\cdot,t) \in H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$ for all $t > 0$. Therefore, we can then evaluate the trace of $u(\cdot,t)$ to get that

$$\boxed{u(\cdot,t) = 0 \quad \text{on} \quad \Gamma, \quad t > 0,}$$

which shows that (7.5) satisfies the boundary condition of the problem.

(Note that the expansion of $u$ in terms of eigenfunctions and the fact that $u(\cdot,t) \in L^2(\Omega)$ are not sufficient to conclude that $u(\cdot,t)$ vanishes on the boundary.)

To show that (7.5) is a classical solution, our strategy is to establish that $\|D_t^m u(\cdot,t)\|_s$ is finite for $t > 0$ and any $s, m \in \mathbb{N}_0$. Then, by applying the Sobolev inequality [3, eq. A28], we will be able to conclude that $u$ has the required smoothness to be a classical solution of (7.1).

With that in mind, we apply $(-\Delta)^k$ to our solution candidate (7.5) to formally get

$$(-\Delta)^k u(x,t) = \sum_{j=1}^{\infty} \hat{v}_j \lambda_j^k \,\mathrm{e}^{-\lambda_j t} \varphi_j(x), \quad t > 0.$$

To verify that this operation is well-defined, we compute its $L^2$-norm using Parseval's identity:

$$\|\Delta^k u(\cdot,t)\|^2 = \sum_{j=1}^{\infty} (\hat{v}_j \lambda_j^k \,\mathrm{e}^{-\lambda_j t})^2 = t^{-2k} \sum_{j=1}^{\infty} (\hat{v}_j \underbrace{(t\lambda_j)^k \,\mathrm{e}^{-\lambda_j t}}_{\leq C_k})^2 \leq t^{-2k} C_k^2 \|v\|^2 < \infty, \quad t > 0.$$

Then, it follows that $\Delta^k u(\cdot,t) \in L^2(\Omega)$ for all $t > 0$.

Similarly, for the semi-norm we have

$$|\Delta^k u(\cdot,t)|_1^2 = \sum_{j=1}^{\infty} \lambda_j (\lambda_j^k \,\mathrm{e}^{-2\lambda_j t} \,\hat{v}_j)^2 \leq t^{-2k-1} C_{k+1/2}^2 \|v\|^2 < \infty, \quad t > 0.$$

These two bounds imply that $\Delta^k u \in H_0^1(\Omega)$. Thus, by the trace theorem [3, Theorem A.4], we conclude that $\Delta^k u(\cdot,t) = 0$ on $\Gamma$ for all $t > 0$ and $k \in \mathbb{N}_0$.

Similarly, applying the time derivative, since $D_t^m \,\mathrm{e}^{-\lambda_j t} = (-\lambda_j)^m \,\mathrm{e}^{-\lambda_j t}$, we get

$$\|D_t^m \Delta^k u(\cdot,t)\| = \|\Delta^k D_t^m u(\cdot,t)\| \leq t^{-k-m} C_{k+m} \|v\| < \infty, \quad t > 0,$$

and

$$|D_t^m \Delta^k u(\cdot,t)|_1 = |\Delta^k D_t^m u(\cdot,t)|_1 \leq t^{-k-m-1/2} C_{k+m+1/2} \|v\| < \infty, \quad t > 0.$$

We therefore conclude from here that $D_t^m \Delta^k u = \Delta^k D_t^m u \in H_0^1(\Omega)$ for all $t > 0$ and $m, k \in \mathbb{N}_0$.

To obtain a bound for the Sobolev norm $\|D_t^m u(\cdot,t)\|_s$ for arbitrary $s \in \mathbb{N}_0$, we need to convert the Laplacians into Sobolev norms. This can be achieved by applying the elliptic regularity estimate [3, eq. 3.37]:

$$\|w\|_s \leq C \|\Delta w\|_{s-2},$$

which holds for all $w \in H^s(\Omega) \cap H^1_0(\Omega)$, with $s \geq 2$. Indeed, assuming that $w \in H^{2k+1}(\Omega)$ and that $\Delta^j w = 0$ on $\Gamma$ for all $j < k$, applying this estimate repeatedly, we obtain

$$\|w\|_{2k} \leq C\|\Delta w\|_{2k-2} \leq C^2\|\Delta^2 w\|_{2k-4} \leq \cdots \leq C^k\|\Delta^k w\|_0$$

and

$$\|w\|_{2k+1} \leq C\|\Delta w\|_{2k-1} \leq C^2\|\Delta^2 w\|_{2k-3} \leq \cdots \leq C^k\|\Delta^k w\|_1.$$

Therefore, applying these bounds to $\Delta^k D_t^m u(\cdot, t)$ we get that

$$\|D_t^m u(\cdot, t)\|_{2k} \leq C^k\|\Delta^k D_t^m u(\cdot, t)\|_0 \leq C^k C_{k+m} t^{-k-m}\|v\|, \quad t > 0,$$

and

$$\|D_t^m u(\cdot, t)\|_{2k+1} \leq C^k\|\Delta^k D_t^m u(\cdot, t)\|_1 \leq C^k(|\Delta^k D_t^m u(\cdot, t)|_0^2 + |\Delta^k D_t^m u(\cdot, t)|_1^2)^{1/2}$$
$$\leq C^k \max\{C_{k+m}, C_{k+m+1/2}\} t^{-k-m-1/2}\|v\|, \quad t > 0.$$

Therefore, in conclusion, we have that there exists a constant $\tilde{C} > 0$ such that

$$\|D_t^m u(\cdot, t)\|_s \leq \tilde{C} t^{-m-s/2}\|v\|, \quad t > 0, \quad s, m \in \mathbb{N}_0,$$

and by the Sobolev inequality [3, A.28] we obtain that $D_t^m u(\cdot, t) \in C^p(\Omega)$ for all $p \in \mathbb{N}_0$. This shows that $u$ is smooth and it is therefore a classical solution of the boundary value problem. □

The bound blows up as $t \to 0$, which raises the question of what happens to the solution as $t \downarrow 0$. Indeed, it is interesting to note that we have just shown that the solution is smooth for any $t > 0$, regardless of how small. However, the initial condition is only in $L^2(\Omega)$. It is then natural to ask whether

$$\|u(\cdot, t) - v\|^2 = \sum_{j=1}^{\infty} (\mathrm{e}^{-\lambda_j t} - 1)^2 \hat{v}_i^2 \to 0 \quad \text{as} \quad t \to 0.$$

To examine whether this limit holds, let $\epsilon > 0$. Since the series $\sum_{j=1}^{\infty} \hat{v}_j^2$ converges, there exists an $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty} \hat{v}_j^2 < \epsilon.$$

Then

$$\|u(\cdot, t) - v\|^2 = \sum_{j=1}^{\infty} (\mathrm{e}^{-\lambda_j t} - 1)^2 \hat{v}_i^2 = \sum_{j=1}^{N} (\mathrm{e}^{-\lambda_j t} - 1)^2 \hat{v}_i^2 + \sum_{j=N+1}^{\infty} \underbrace{(\mathrm{e}^{-\lambda_j t} - 1)^2}_{\leq 4} \hat{v}_i^2$$

$$\leq \sum_{j=1}^{N} (\mathrm{e}^{-\lambda_j t} - 1)^2 \hat{v}_i^2 + 4\epsilon.$$

Taking now the limit as $t \to 0$ we arrive at

$$\lim_{t \to 0} \|u(\cdot, t) - v\|^2 \leq 4\epsilon,$$

but since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{t\to 0} \|u(\cdot,t) - v\|^2 = 0$.

Although this shows convergence to the initial condition, we would like to obtain a bound for the solution in terms of a norm of $v$ that is valid up to and including the initial time. To achieve this, we may require additional regularity of the initial condition. Suppose that $v \in H_0^1(\Omega)$. Then, from Theorem 6.4, we can obtain

$$|u(\cdot,t)|_1^2 = \int_\Omega |\nabla u(\cdot,t)|^2 \,\mathrm{d}x = \sum_{j=1}^\infty \lambda_j \hat{v}_j^2 \underbrace{\mathrm{e}^{-2\lambda_j t}}_{\le 1} \le \sum_{j=1}^\infty \lambda_j \hat{v}_j^2 = |v|_1^2, \quad t \ge 0,$$

from where it follows that

$$\|u(\cdot,t)\|_1 = (\|u(\cdot,t)\|^2 + |u(\cdot,t)|_1^2)^{1/2} \le (\|v\|^2 + |v|_1^2)^{1/2} = \|v\|_1, \quad t \ge 0.$$

## 7.1 An inhomogeneous problem

Consider now the following inhomogeneous initial-boundary value problem:

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \Omega \times \mathbb{R}_+, \\ u &= 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \\ u(\cdot,0) &= v \quad \text{in } \Omega, \end{aligned} \tag{7.7}$$

where $v \in L^2(\Omega)$ as before, and $f(\cdot,t) \in L^2(\Omega)$ for $t > 0$. Additionally, we require that $\|D_t f(\cdot,t)\|_{L^2} < \infty$.

As above, we can construct a candidate solution

$$u(x,t) = \sum_{j=1}^\infty \hat{u}_j(t)\varphi_j(x), \quad x \in \Omega, \quad t > 0,$$

expanding in terms of the eigenfunctions $\{\varphi_j\}_{j=1}^\infty$. The assumption $f(\cdot,t) \in L^2(\Omega)$ implies the existence of the coefficients

$$\hat{f}_j(t) := (f(\cdot,t), \varphi_j) = \int_\Omega f(x,t)\varphi_j(x)\,\mathrm{d}x, \quad t > 0, \quad j \in \mathbb{N}.$$

As before, applying the operator $\partial_t - \Delta$ to our formal solution we arrive at

$$\sum_{j=1}^\infty (\hat{u}_j'(t) + \lambda_j \hat{u}_j(t) - \hat{f}_j(t))\varphi_j(x) = 0, \quad x \in \Omega, \quad t > 0,$$

from where we obtain, since $\{\varphi_j\}_{j=1}^\infty$ is a basis of $L^2(\Omega)$, that

$$\hat{u}_j'(t) + \lambda_j \hat{u}_j(t) = \hat{f}_j(t), \quad t > 0, \quad \hat{u}_j(0) = \hat{v}_j, \quad j \in \mathbb{N}.$$

Solving the ODE above we get

$$\hat{u}_j(t) = \hat{v}_j\,\mathrm{e}^{-\lambda_j t} + \int_0^t \hat{f}_j(s)\,\mathrm{e}^{-\lambda_j(t-s)}\,\mathrm{d}s,$$

so that the natural candidate for a solution to (7.7) is given by

$$\boxed{u(x,t) = \sum_{j=1}^{\infty} \hat{v}_j \, \mathrm{e}^{-\lambda_j t} \, \varphi_j(x) + \int_0^t \left( \sum_{j=1}^{\infty} \hat{f}_j(s) \, \mathrm{e}^{-\lambda_j (t-s)} \, \varphi_j(x) \right) \mathrm{d}s, \quad x \in \Omega, \quad t > 0.} \tag{7.8}$$

Letting

$$E(t)v = \sum_{j=1}^{\infty} \hat{v}_j e^{-\lambda_j t} \varphi_j(x),$$

denote the solution (evolution) operator for the heat equation (where we omit the dependence on $x$ for brevity), we recognize that (7.8) can be expressed as

$$\boxed{u(x,t) = E(t)v + \int_0^t E(t-s) f(s) \, \mathrm{d}s.} \tag{7.9}$$

Using the already proved fact that

$$\|D_t^m E(t)v\|_s \leq C t^{-m-s/2} \|v\|, \quad \text{for all } t > 0, \quad s, m \in \mathbb{N}_0,$$

for some constant $C > 0$, and the assumptions on $f$, it can then be shown that

$$F(t) = \int_0^t E(t-s) f(s) \, \mathrm{d}s$$

satisfies

$$F_t - \Delta F = f \quad \text{in } \Omega \times \mathbb{R}_+, \quad F = 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \quad \text{and} \quad F(0) = 0.$$

Thus, (7.9) is indeed a classical solution of (7.7).

Moreover, since we have shown that $\|E(t)v\| \leq \|v\|$ for all $v \in L^2(\Omega)$ and $t \geq 0$, we obtain from (7.9) that

$$\|u(\cdot, t)\| \leq \|v\| + \int_0^t \|f(\cdot, s)\| \, \mathrm{d}s, \quad t \geq 0. \tag{7.10}$$

**Uniqueness:** Clearly, (7.10) can be used to prove that(7.7) admits at most one solution $u(\cdot, t) \in L^2(\Omega)$ for all $t \geq 0$. Suppose it does not, then there are two different solutions $u_1$ and $u_2$, with the difference $w = u_2 - u_1$ satisfying (7.7) with $f = 0$ and $v = 0$. From the bound above, we then conclude that $\|w\| = 0$, and from the fact that $\| \cdot \|$ is a norm (the $L^2$ norm), we have that $w(\cdot, t) = 0$

**Stability:** (7.10) can also be used to establish stability. Indeed, suppose $u_j$, $j = 1, 2$ solve (7.7) with sources $f_j$ and initial conditions $v_j$, respectively. Then the difference $w = u_2 - u_1$ satisfies (7.7) with source $f_2 - f_1$ and initial condition $v_2 - v_1$. From (7.10), it follows that

$$\|u_2(\cdot, t) - u_1(\cdot, t)\| \leq \|v_2 - v_1\| + \int_0^t \|f_2(\cdot, s) - f_1(\cdot, s)\| \, \mathrm{d}s, \quad t \geq 0.$$

In words, small changes in the source and small changes in the initial condition lead to small changes in the solution of (7.7).

## 7.2 A maximum principle

As in the case of the elliptic PDE, we can now investigate whether the solution of the heat equation satisfies some kind of maximum principle. To do so, let us consider the problem of finding $u$ on $\Omega \times I$ such that

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \Omega \times I, \\ u &= g \quad \text{on } \Gamma \times I, \\ u(\cdot, 0) &= v \quad \text{in } \Omega, \end{aligned} \tag{7.11}$$

where, as before, $\Omega \subset \mathbb{R}^d$ is open and bounded, and $I = (0, T)$, where $T > 0$. We assume that $f$, $g$, and $v$ are smooth on $\overline{\Omega} \times [0, T]$, $\Gamma \times [0, T]$, and $\overline{\Omega}$, respectively.

To study this, we need the concept of a parabolic boundary, which is defined as follows (see Figure 7.1):

$$\begin{aligned} \Gamma_p &:= (\Gamma \times \overline{I}) \cup (\Omega \times \{t = 0\}) \\ &= \partial(\Omega \times I) \setminus (\Omega \times \{t = T\}) \end{aligned}$$

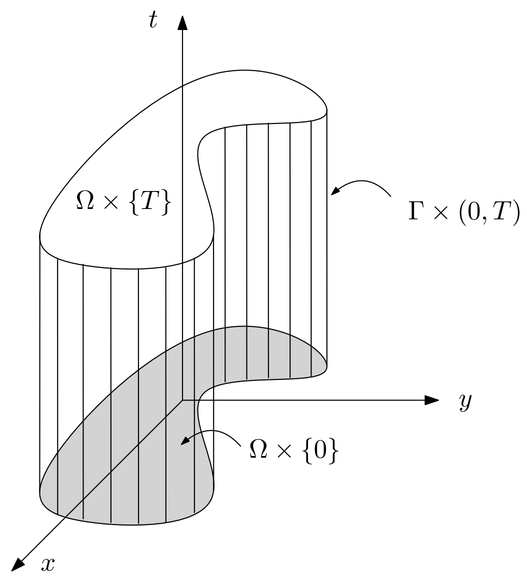


Figure 7.1: Illustration of the domains involve in the maximum principle for the heat equation.

We then have the following theorem:

**Theorem 7.2.** *Let $u$ be smooth, and assume that $u_t - \Delta u \le 0$ in $\Omega \times I$. Then, $u$ attains its maximum on the parabolic boundary $\Gamma_p$.*

*Proof.* We proceed by contradiction. Suppose the maximum is attained at a point $(\bar{x}, \bar{t}) \in \Omega \times (0, T]$, so that

$$u(\bar{x}, \bar{t}) = \max_{\overline{\Omega} \times [0,T]} u = M \ge m = \max_{\Gamma_p} u.$$

Consider the function

$$w(x, t) = u(x, t) + \epsilon |x|^2$$

for $\epsilon > 0$ to be specified. Note that

$$\max_{\Gamma_p} w \le m + \epsilon \max_{\Gamma} |x|^2.$$

We then choose $\epsilon$ small enough such that

$$\max_{\Gamma_p} w \le m + \epsilon \max_{\Gamma} |x|^2 < M = \max_{\overline{\Omega}\times[0,T]} u \le \max_{\overline{\Omega}\times[0,T]} w, \tag{7.12}$$

which is possible in view of the fact that $M > m$.

Now, for the function $w$, we have

$$w_t - \Delta w = u_t - \Delta u - \underbrace{2d\epsilon}_{\Delta(\epsilon|x|^2)} < u_t - \Delta u \le 0,$$

so then

$$w_t - \Delta w < 0 \quad \text{in} \quad \Omega \times I. \tag{7.13}$$

Suppose now that $w$ attains its maximum at a point $(\tilde{x}, \tilde{t}) \in \Omega \times (0, T]$ not on the parabolic boundary, such that

$$w(\tilde{x}, \tilde{t}) = \max_{\overline{\Omega}\times(0,T]} w.$$

Since $\tilde{x} \in \Omega$ and $\Omega$ is open, a necessary second-order condition for $(\tilde{x}, \tilde{t})$ to be a local maximum is that the Hessian matrix $A = D_x^2 w(\tilde{x}, \tilde{t})$ must be negative semi-definite. Therefore,

$$e_j^T A e_j = w_{x_j,x_j}(\tilde{x}, \tilde{t}) \le 0, \quad \forall j,$$

which implies that

$$-\Delta w(\tilde{x}, \tilde{t}) \ge 0.$$

On the other hand, the first-order condition implies that either $w_t(\tilde{x}, \tilde{t}) = 0$ or $w_t(\tilde{x}, \tilde{t}) \ge 0$. Combining these conditions, we obtain

$$w_t(\tilde{x}, \tilde{t}) - \Delta w(\tilde{x}, \tilde{t}) \ge 0,$$

which contradicts (7.13). Therefore, $w$ must attain its maximum on the parabolic boundary $\Gamma_p$. This, in turn, contradicts (7.12). Thus, $u$ must also attain its maximum on $\Gamma_p$.

The proof is now complete.

□

Finally, back to the problem (7.11), suppose that $f = 0$. Then, from the maximum principle we have that $u$ satisfies the inequality

$$u(x,t) \le \max_{\overline{\Omega}\times[0,T]} u \le \max\left\{ \max_{\Gamma\times[0,T]} g, \max_{\overline{\Omega}} v \right\}, \quad (x,t) \in \overline{\Omega} \times [0,T].$$

Similarly, $w = -u$ satisfies $w_t - \Delta w \le 0$ in $\Omega \times I$, so again by the maximum principle it follows that

$$w(x,t) \le \max_{\overline{\Omega}\times[0,T]} w \le \max\left\{ \max_{\Gamma\times[0,T]} (-g), \max_{\overline{\Omega}} (-v) \right\}, \quad (x,t) \in \overline{\Omega} \times [0,T].$$

However, since:

$$\max\left\{ \max_{\Gamma\times[0,T]} (-g), \max_{\overline{\Omega}} (-v) \right\} = -\min\left\{ \min_{\Gamma\times[0,T]} g, \min_{\overline{\Omega}} v \right\},$$

and

$$-\|v\|_{C(\overline{\Omega})} \leq \min_{\overline{\Omega}} v \leq v(x) \leq \max_{\overline{\Omega}} v \leq \|v\|_{C(\overline{\Omega})}, \quad x \in \overline{\Omega}$$

$$-\|g\|_{C(\Gamma\times[0,T])} \leq \min_{\Gamma\times[0,T]} g \leq g(x,t) \leq \max_{\Gamma\times[0,T]} g \leq \|g\|_{C(\Gamma\times[0,T])}, \quad (x,t) \in \Gamma \times [0,T]$$

we conclude

$$-\max\{\|g\|_{C(\Gamma\times[0,T])}, \|v\|_{C(\overline{\Omega})}\} \leq u(x,t) \leq \max\{\|g\|_{C(\Gamma\times[0,T])}, \|v\|_{C(\overline{\Omega})}\}, \quad (x,t) \in \overline{\Omega} \times [0,T],$$

which leads to

$$\|u\|_{C(\overline{\Omega}\times[0,T])} \leq \max\{\|g\|_{C(\Gamma\times[0,T])}, \|v\|_{C(\overline{\Omega})}\}.$$

# 8 Parabolic PDEs of second order: Variational and numerical treatment

## Lecture 7

In this lecture we consider the following model problem

$$
\begin{aligned}
u_t - \Delta u &= f \quad \text{in } \Omega \times (0, T), && (8.1)\\
u &= 0 \quad \text{on } \Gamma \times (0, T), && (8.2)\\
u(\cdot, 0) &= g \quad \text{on } \Omega \times \{0\}. && (8.3)
\end{aligned}
$$

Here, $\Omega$ is an open, bounded subset of $\mathbb{R}^d$ with smooth boundary $\Gamma$. The aim is to develop a well-posedness theory using variational techniques, similar to the elliptic case. Based on a corresponding weak formulation, we investigate the numerical approximation of the model problem. We proceed in two steps: (i) we approximate in space ($\Omega$) using finite elements while keeping time continuous; (ii) the spatially semidiscrete problem from (i) is a system of ordinary differential equations, and we use time-stepping methods for these to obtain a fully discrete scheme.

## 8.1 Derivation of a weak formulation

Suppose there is a sufficiently smooth function $u$ that satisfies (8.1)–(8.3). As for elliptic equations in Lecture 4, we take a test function $\varphi \in H_0^1(\Omega)$, multiply (8.1) by $\varphi$, integrate the result over $\Omega$, perform integration by parts to rewrite the integral of $\Delta u \varphi$, and arrive at

$$
\int_\Omega u_t(t)\varphi \, dx + \int_\Omega \nabla u(t) \cdot \nabla \varphi \, dx = \int_\Omega f(t)\varphi \, dx, \quad \forall \varphi \in H_0^1(\Omega), t \in (0, T). \tag{8.4}
$$

**Notation and observations:**

- We write $u(t)$ for the function $u(t)(x) = u(x, t)$.
- Inspecting (8.4), we require $\nabla u(t) \in L^2(\Omega)^d$.
- Inspecting (8.2), we require $u(t) = 0$ on $\Gamma$.

In view of the previous observations $u(t) \in H_0^1(\Omega)$ is a sensible requirement. Rewriting (8.4), and using the Cauchy-Schwarz inequality, we then observe that

$$
(u_t(t), \varphi) = \int_\Omega f(t)\varphi \, dx - \int_\Omega \nabla u(t) \cdot \nabla \varphi \, dx \leq \|f\|\|\varphi\| + \|\nabla u\|\|\nabla \varphi\| \leq \big(\|f\| + \|u\|_1\big)\|\varphi\|_1,
$$

i.e., $u_t(t)$ defines a bounded linear functional via $\varphi \mapsto (u_t(t), \varphi)$ for $\varphi \in H_0^1(\Omega)$. We denote the space of all linear bounded functionals on $H_0^1(\Omega)$ by

$$H^{-1}(\Omega) = \{\ell : H_0^1(\Omega) \to \mathbb{R} : \ell \text{ is linear and } \|\ell\|_{-1} = \sup_{0 \neq v \in H_0^1(\Omega)} \frac{\ell(v)}{\|v\|_1} < \infty\}.$$

(Compare the definition of the norm $\|\ell\|_{-1}$ with the definition of a matrix norm that is induced by a vector norm.) We hence require that $u_t(t) \in H^{-1}(\Omega)$. In the following, we use the notation

$$\langle \ell, \varphi \rangle = \ell(\varphi), \quad \varphi \in H_0^1(\Omega), \ \ell \in H^{-1}(\Omega),$$

which is called duality pairing, and note that $\langle w, v \rangle = (w, v)$ as soon as $w \in L^2(\Omega)$. These observations lead us to define a weak solution concept for (8.1)–(8.3) as follows.

**Definition 8.1.** We say that $u \in L^2(0, T; H_0^1(\Omega))$, with $u' \in L^2(0, T; H^{-1}(\Omega))$ is a weak solution of (8.1)–(8.3), provided

- $\langle u'(t), \varphi \rangle + (\nabla u(t), \nabla \varphi) = (f(t), \varphi)$ for all $\varphi \in H_0^1(\Omega)$, $t \in (0, T)$;
- $u(0) = g$.

**Remark 8.2.** (i) The condition $v \in L^2(0, T; H_0^1(\Omega))$ ($v' \in L^2(0, T; H^{-1}(\Omega))$) means that the following integral exists,

$$\int_0^T \|v(t)\|_1^2 \, dt, \qquad \left( \int_0^T \|u'(t)\|_{-1}^2 \, dt \right).$$

It can be shown that functions $v \in L^2(0, T; H_0^1(\Omega))$ with $v' \in L^2(0, T; H^{-1}(\Omega))$ satisfy $v \in C^0([0, T]; L^2(\Omega))$, i.e.,

$$\lim_{t \to t_0} \|v(t) - v(t_0)\| = 0 \quad \forall t_0 \in [0, T].$$

Hence, prescribing initial conditions $u(0) = g$ makes sense for $g \in L^2(\Omega)$.
(ii) $u'$ denotes the weak time derivative of $u$, i.e.,

$$\int_0^T u(t) \phi_t \, dt = -\int_0^T u'(t) \phi(t) \, dt \quad \forall \phi \in C_0^\infty(0, T).$$

(iii) For $v, w \in L^2(0, T; H_0^1(\Omega))$ with $v', w' \in L^2(0, T; H^{-1}(\Omega))$, the following integration by parts formula is valid for $0 \leq t_1 \leq t_2 \leq T$,

$$(v(t_2), w(t_2)) - (v(t_1), w(t_1)) = \int_{t_1}^{t_2} \langle v'(t), w(t) \rangle + \langle w'(t), v(t) \rangle dt. \tag{8.5}$$

Using $v = w$, one can then deduce that

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 = \langle v'(t), v(t) \rangle \qquad \text{a.e. } t \in [0, T]. \tag{8.6}$$

(iv) The in-depth study of abstract space $L^p(0, T; X)$ with a Banach space $X$ is deferred to the master classes.

**Aims:**

- Show stable dependence of the weak solution on the data $f, g$. see Theorem 8.3
- Approximation of the weak solution using finite elements and time-stepping schemes; see Theorem 8.7, Theorem 8.8, Theorem 8.11.

**Theorem 8.3.** *Let $u$ be a weak solution of* (8.1)–(8.3)*. Then*

$$\|u(t)\|^2 + \int_0^t \|\nabla u(t)\|^2 \, dt \le \|g\|^2 + C_P^2 \int_0^t \|f(s)\|^2 \, ds, \quad t \ge 0, \tag{8.7}$$

*with $C_P$ the Poincaré constant from* (4.7)*. If in addition $u' \in L^2(0,T; H_0^1(\Omega))$ and $g \in H_0^1(\Omega)$, then we have that*

$$\|\nabla u(t)\|^2 + \int_0^t \|u'(s)\|^2 \, ds \le \|\nabla g\|^2 + \int_0^t \|f(s)\|^2 \, ds.$$

*Proof.* Set $\varphi = u(t) \in H_0^1(\Omega)$ in Definition 8.1 and use (8.6) with $v = u$ to obtain the identity

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 = (f(t), u(t)).$$

The right-hand side can be estimated from above using the Cauchy-Schwarz inequality, the Poincaré inequality (4.7) and the basic inequality $ab \le a^2/2 + b^2/2$ for $a, b \in \mathbb{R}$, as follows

$$(f(t), u(t)) \le C_P \|f(t)\| \|\nabla u(t)\| \le \frac{C_P^2}{2}\|f(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2.$$

Therefore, we have that

$$\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 \le C_P^2 \|f(t)\|^2.$$

Integration in $t$ then yields that

$$\|u(t)\|^2 - \|u(0)\|^2 + \int_0^t \|\nabla u(s)\|^2 \, ds \le C_P^2 \int_0^t \|f(s)\|^2 \, ds,$$

which proves the first inequality, because $u(0) = g$. For the second inequality we use $\varphi = u'(t)$ in Definition 8.1 and obtain that

$$\langle u'(t), u'(t) \rangle + (\nabla u(t), \nabla u'(t)) = (f, u'(t)).$$

Since $\langle u'(t), u'(t) \rangle = \|u'(t)\|^2$ and $(\nabla u(t), \nabla u'(t)) = \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2$, integration over $(0,t)$ yields

$$\frac{1}{2}\|\nabla u(t)\|^2 + \int_0^t \|u'(s)\|^2 \, dx = \frac{1}{2}\|\nabla g\|^2 + \int_0^t (f(s), u'(s)) \, ds.$$

Using the Cauchy-Schwarz inequality, we estimate further

$$\begin{aligned}\int_0^t (f(s), u'(s)) \, ds &\le \left(\int_0^t \|f(s)\|^2 \, ds\right)^{1/2} \left(\int_0^t \|u'(s)\|^2 \, ds\right)^{1/2} \\ &\le \frac{1}{2}\int_0^t \|f(s)\|^2 \, ds + \frac{1}{2}\int_0^t \|u'(s)\|^2 \, ds,\end{aligned}$$

from which the claim follows. □

**Questions**

- Why does the statement of Theorem 8.3 imply stability?
- What can you say about uniqueness of weak solutions?

**Remark 8.4.** In the previous proof, we used that for $u \in L^2(0,T;H_0^1(\Omega))$ with $u' \in L^2(0,T;H_0^1(\Omega))$. Therefore, $u \in C^0([0,T];H_0^1(\Omega))$. In particular, $u(0) \in H_0^1(\Omega)$. Since $u(0) = g$, we therefore must have $g \in H_0^1(\Omega)$, too, for consistency reasons.

## 8.2 Spatially semidiscrete problem using finite elements

We consider $\Omega = (0,1)$ with a uniform partition $\{x_j = jh\}$ for $j = 0,\ldots,J+1$ and $h = 1/(J+1)$. We recall the space of continuous piecewise linear functions

$$S_h = \{v \in C^0(\overline{\Omega}) : v \text{ is linear on } (x_{j-1}, x_j),\ v(0) = v(1) = 0\} \subset H_0^1(\Omega).$$

We look for $u_h \in C^1([0,T];S_h)$ as the solution to

$$(u_h'(t), \chi) + (\nabla u_h(t), \nabla \chi) = (f(t), \chi) \quad \forall \chi \in S_h, \tag{8.8}$$

$$u_h(0) = g_h, \tag{8.9}$$

where $g_h \in S_h$ is an approximation to $g$. In comparison to (8.4), we have replaced $H_0^1(\Omega)$ by the subspace $S_h$ in (8.8); compare to the formulation of the Ritz-Galerkin (finite element) method as discussed in Lecture 5.

**Theorem 8.5.** *For each $f \in C^0([0,T];L^2(\Omega))$, there exists a unique solution $u_h \in C^1([0,T];S_h)$ of* (8.8)–(8.9).

*Proof.* Let $S_h = \operatorname{span}\{\Phi_j : 1 \leq j \leq J\}$, see Section 5.5. Then $u_h$ has the representation

$$u_h(t) = \sum_{j=1}^{J} U_j(t)\Phi_j,$$

with unknown functions $U_j \in C^1([0,T])$. Similar to the proof of Theorem 8.3, we set $\chi = \Phi_k$ for $k = 1,\ldots,J$ in (8.8), to obtain the following linear system of ordinary differential equations

$$BU'(t) + AU(t) = F(t), \quad t > 0, \quad U(0) = G, \tag{8.10}$$

where $U(t), F(t), G \in \mathbb{R}^J$ with $F_j(t) = (f, \Phi_j)$, and $U(t)$ and $G$ vectors containing the coefficients of $u_h(t)$ and $g_h$. The stiffness and mass matrices $A, B \in \mathbb{R}^{J\times J}$ are given by

$$B_{k,j} = (\Phi_j, \Phi_k), \quad A_{k,j} = (\nabla\Phi_j, \nabla\Phi_k).$$

Since the matrix $B$ is invertible, see Lemma 8.6 below, (8.10) is equivalent to

$$U'(t) + B^{-1}AU(t) = B^{-1}F(t), \quad t > 0, \quad U(0) = G. \tag{8.11}$$

This inhomogeneous system of linear ordinary differential equations has a unique solution (Picard-Lindelöf Theorem) $U \in C^1([0,T],\mathbb{R}^J)$ if $F \in C^0([0,T],\mathbb{R}^J)$. □

**Lemma 8.6.** *The matrix $B$ is invertible.*

*Proof.* Let $V \in \mathbb{R}^J$ with $V \neq 0$, and $v_h = \sum_{j=1}^J V_j \Phi_j \in S_h$. Then

$$V^T B V = \sum_{j,k=1}^J V_k B_{k,j} V_j = \sum_{j,k=1}^J V_k (\Phi_j, \Phi_k) V_j = (v_h, v_h) > 0.$$

□

Similar to Theorem 8.3, we obtain the following estimates for the solution $u_h$ of (8.8)–(8.9)

$$\|u_h(t)\|^2 + \int_0^t \|\nabla u_h(s)\|^2 \, ds \le \|g_h\|^2 + C \int_0^t \|f(s)\|^2 \, ds, \quad t > 0, \tag{8.12}$$

where we may choose $C$ as in Theorem 8.3. The stability estimate (8.12) combined with the properties of the Ritz projection $R_h : H_0^1(\Omega) \to S_h$, defined by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \chi), \quad \forall \chi \in S_h, \tag{8.13}$$

allows us to quantify the error between $u$ and $u_h$. To that end, let us recall the error estimates shown in Theorem 5.10

$$\|R_h v - v\| + h^{s-1} \|\nabla R_h v - v\| \le c h^s \|v\|_s \quad \text{for } s = 1, 2, \quad v \in H_0^1(\Omega) \cap H^s(\Omega). \tag{8.14}$$

**Theorem 8.7.** *Let $u_h$ be the solution to* (8.8)–(8.9) *and let $u$ be a weak solution of* (8.1)–(8.3) *with the additional regularity requirements $u' \in L^2(0, T; H^2(\Omega))$ and $g \in H^2(\Omega)$. Then there is a constant $C > 0$ that does not depend on $h, t, u, u_h$ such that*

$$\|u_h(t) - u(t)\| \le \|g - g_h\| + Ch^2 \left( \|g\|_2 + \int_0^t \|u'(s)\|_2 \, ds \right).$$

*Proof.* The proof relies on splitting the error

$$u_h - u = \big(u_h - R_h(u)\big) + \big(R_h(u) - u\big) = \theta + \zeta,$$

into a discrete error $\theta$ and an approximation error $\zeta$.

**Step 1: Bounding $\zeta$.** We use the error estimates for the Ritz projection (8.14)

$$\|\zeta(t)\| \le Ch^2 \|u(t)\|_2 \le Ch^2 \left( \|g\|_2 + \int_0^t \|u'(s)\|_2 \, ds \right),$$

where we used that $u(t) = u(0) + \int_0^t u'(s) \, ds$.

**Step 2: Bounding $\theta$.** We observe that $\theta = u_h - R_h u$ satisfies for $\chi \in S_h$ and $t > 0$

$$(\theta', \chi) + (\nabla \theta, \nabla \chi) = (f, \chi) - ((R_h u)', \chi) - (\nabla R_h u, \nabla \chi) = (u' - (R_h u)', \chi) = (u' - R_h u', \chi),$$

where we used that $(\nabla R_h u, \chi) = (\nabla u, \chi) = (f, \chi) - (u', \chi)$ in the second last step, and that time differentiation commutes with the Ritz projection. Moreover, $\theta(0) = g_h - R_h g = g_h - g + g - R_h g$. Hence, (8.12) applied to $\theta$ implies that

$$\|\theta(t)\| \le \|\theta(0)\| + C \int_0^t \|u'(s) - R_h u'(s)\| \, ds \le \|g_h - g\| + Ch^2 \left( \|g\|_2 + \int_0^t \|u'(s)\|_2 \, ds \right),$$

where used the estimates $\|g - R_h g\| \le Ch^2 \|g\|_2$ and $\|u'(s) - R_h u'(s)\| \le Ch^2 \|u'\|_2$.

**Step 3:** Combining the bounds derived in steps 1 and 2, we arrive at the claimed inequality.

□

Similar to the previous considerations, error bounds for the gradient can be derived. Testing (8.8) with $\chi = u_h'$, one shows as in the proof of Theorem 8.3 that

$$\|\nabla u_h(t)\|^2 + \int_0^t \|u_h'(s)\|^2\, ds \le \|\nabla g_h\|^2 + \int_0^t \|f(s)\|^2\, ds. \tag{8.15}$$

Proceeding as in the proof of Theorem 8.7, one then shows the following result.

**Theorem 8.8.** *Under the hypothesis of Theorem 8.7, we have for $t > 0$ the estimate*

$$\|\nabla u_h(t) - \nabla u(t)\| \le \|\nabla g_h - \nabla g\| + Ch\left(\|g\|_2 + \|u(t)\|_2 + \Big(\int_0^t \|u'(s)\|_1^2\, ds\Big)^{1/2}\right).$$

**Remark 8.9.** (i) If $g_h = I_h g$ (nodal interpolation) or $g_h = R_h g$, then $\|\nabla g_h - \nabla g\| \le Ch\|g\|_2$. Thus $\|u_h(t) - u(t)\|_1$ is $O(h)$ as $h \to 0$.
(ii) Superconvergence: If $g_h = R_h g$, then $\theta(0) = 0$ and (8.15) implies that

$$\|\nabla\theta(t)\|^2 \le C\int_0^t \|u'(s) - R_h u'(t)\|^2\, ds \le Ch^4 \int_0^t \|u'(s)\|_2^2\, ds,$$

where we used that $\|u'(s) - R_h u'(t)\| \le Ch^2\|u'\|_2$ (assuming the required regularity!). Hence, $\|\nabla\theta(t)\|$ (gradient of discrete error), converges faster, i.e., $O(h^2)$ for $h \to 0$, than $\|\nabla u_h(t) - \nabla u(t)\|$.

**Remark 8.10.** So far, we did not discuss the existence of weak solutions of (8.1)–(8.3). The estimates (8.12) and (8.15) can be used to show that there is $\tilde{u} \in L^2(0,T;H_0^1(\Omega))$ such that $\lim_{h\to 0} u_h = \tilde{u}$ in $L^2(0,T;L^2(\Omega))$ and that $\tilde{u}$ is a weak solution of (8.1)–(8.3). This is done in the master class on PDEs.

## 8.3 Fully discrete schemes

As can be seen from (8.11), the semidiscrete solution $u_h$ can be obtained as a solution of a system of ordinary differential equations. Therefore, a fully discrete numerical solution $U^n$ with $U^n \approx U(t_n)$ can be obtained by using a numerical scheme for ordinary differential equations; see Numerical Mathematics course (202200240). As we have seen in the Numerical Mathematics course, many methods for solving the ODE system numerically will converge eventually (explicit Euler, implicit Euler). 'Eventually' here means for sufficiently small time-step and fixed finite element discretization. In the following, we discuss the interplay between the finite element discretization and time discretization for the implicit Euler scheme; for other schemes see the exercises. Let $k > 0$ be a time-step size, $t_n = kn$ denote discrete times, and define $U^n$ (approximation to $U(t_n)$) by

$$(B + kA)U^n = BU^{n-1} + kF(t_n), \quad n \ge 1,\ U^0 = G. \tag{8.16}$$

Since $B + kA$ is symmetric and positive definite, the scheme defined in (8.16) is well-defined. Moreover, the coordinate vectors $U^n$ correspond to finite element functions

$$u_h^n = \sum_{j=1}^{J} U_j^n \Phi_j \in S_h. \tag{8.17}$$

It is easy to see that (8.16) is equivalent to

$$(\overline{\partial_t} u_h^n, \chi) + (\nabla u_h^n, \nabla \chi) = (f^n, \chi) \quad \forall \chi \in S_h, \tag{8.18}$$

where $\overline{\partial_t} u_h^n = (u_h^n - u_h^{n-1})/k$ and $f^n = f(t_n)$. Testing with $\chi = u_h^n$, we obtain the estimate

$$\|u_h^n\|^2 + k\|\nabla u_h^n\|^2 = (u_h^{n-1} + kf^n, u_h^n) \leq \|u_h^{n-1} + kf^n\| \, \|u_h^n\|.$$

Thus, after division by $\|u_h^n\|$ and neglecting the gradient-term, we obtain the following stability estimate for the implicit Euler scheme

$$\|u_h^n\| \leq \|g_h\| + k \sum_{m=1}^{n} \|f^n\|. \tag{8.19}$$

The previous inequality shows that the implicit Euler scheme is *unconditionally stable*, i.e., inequality (8.19) holds for any choice of $k > 0$. The interested reader may compare (8.19) to (7.10). Certain schemes, such as the explicit Euler scheme (see exercises), are only *conditionally stable*, i.e., require a smallness condition on $k$ in terms of the mesh-size parameter $h$ to allow for a stability estimate. The approximation properties of the finite element space $S_h$, in combination with the stability estimate (8.19) then yield the error estimate.

**Theorem 8.11.** *Let $u_h^n$ be defined by* (8.18)*, and let $u$ be a weak solution of* (8.1)–(8.3) *such that $u' \in L^2(0, T; H^2(\Omega))$, $u'' \in L^2(0, T; L^2(\Omega))$ and $\|g - g_h\| \leq Ch^2\|g\|_2$. Then the following estimate holds for some $C > 0$ that does not depend on $h$ and $k$,*

$$\|u_h^n - u(t_n)\| \leq Ch^2 \left( \|g\|_2 + \int_0^{t_n} \|u'(s)\| \, ds \right) + k \int_0^{t_n} \|u''(s)\| \, ds.$$

*Proof.* The idea of the proof is similar to the one employed in proving Theorem 8.7. We split the error

$$u_h^n - u(t_n) = \big(u_h^n - R_h u(t_n)\big) + \big(R_h u(t_n) - u(t_n)\big) = \theta^n + \zeta^n.$$

**Step 1.** Using the error estimates for the Ritz projection, (8.14), we have for the approximation error $\zeta^n$ that

$$\|\zeta^n\| \leq Ch^2\|u(t_n)\|_2 \leq Ch^2 \left( \|g\|_2 + \int_0^t \|u'(s)\| \, ds \right).$$

**Step 2.** The discrete error $\theta^n$ satisfies the implicit Euler scheme with right-hand side $\omega^n = u'(t_n) - R_h(\overline{\partial_t} u_h^n)$ instead of $f^n$, cf. Step 2. in the proof of Theorem 8.7, that is

$$(\overline{\partial_t} \theta^n, \chi) + (\nabla \theta^n, \nabla \chi) = (\omega^n, \chi).$$

The stability estimate (8.19) thus implies that

$$\|\theta^n\| \leq \|\theta^0\| + k \sum_{m=1}^{n} \big(\|\omega_1^m\| + \|\omega_2^m\|\big),$$

where $\omega^n = \omega_1^n + \omega_2^n$ with $\omega_1^n = u'(t_n) - \overline{\partial_t} u(t_n)$ and $\omega_2^n = \overline{\partial_t} u(t_n) - R_h \overline{\partial_t} u(t_n)$ . It remains to bound the right-hand side in the previous inequality. We start with the initial data

$$\|\theta^0\| = \|g_h - R_h g\| \leq \|g_h - g\| + \|g - R_h g\| \leq Ch^2\|g\|_2,$$

where we used (8.14). To proceed, we observe that

$$\overline{\partial_t} u(t_m) = \frac{u(t_m) - u(t_{m-1})}{k} = \frac{1}{k} \int_{t_{m-1}}^{t_m} u'(s)\, ds.$$

Therefore, using that $R_h$ commutes with integration in $s$, the bound (8.14) implies that

$$k \sum_{m=1}^{M} \|\omega_2^m\| \le \int_0^{t_n} \|R_h u'(s) - u'(s)\|\, ds \le Ch^2 \int_0^{t_n} \|u'(s)\|_2\, ds. \tag{8.20}$$

To estimate the time-differencing error, we observe that

$$\begin{aligned}
\omega_1^m &= u'(t_m) - \frac{u(t_m) - u(t_{m-1})}{k} = -\frac{1}{k} \int_{t_{m-1}}^{t_m} u'(s) - u'(t_m)\, ds \\
&= \frac{1}{k} \int_{t_{m-1}}^{t_m} \int_s^{t_m} u''(r)\, dr\, ds = \frac{1}{k} \int_{t_{m-1}}^{t_m} \int_{t_{m-1}}^{r} u''(r)\, ds\, dr \qquad \text{(Fubini)} \\
&= \frac{1}{k} \int_{t_{m-1}}^{t_m} (r - t_{m-1}) u''(r)\, dr.
\end{aligned}$$

When applying Fubini, we used that

$$\{(r, s) : t_{m-1} \le s \le t_m,\ s \le r \le t_m\} = \{(r, s) : t_{m-1} \le r \le t_m,\ t_{m-1} \le s \le r\}.$$

Since $(r - t_{m-1}) \le k$, we hence obtain that

$$k \sum_{m=1}^{M} \|\omega_2^m\| \le \sum_{m=1}^{n} \| \int_{t_{m-1}}^{t_m} (r - t_{m-1}) u''(r)\, dr \| \le k \int_0^{t_n} \|u''(r)\|\, dr,$$

which concludes the proof. □

# 9 Hyperbolic PDEs

## Lecture 8

### 9.0.1 The wave equation

In this section we first consider the initial-boundary value problem for the wave equation:

$$\begin{aligned} &u_{tt} - \Delta u = 0, && \text{in } \Omega \times \mathbb{R}_+ \\ &u = 0, && \text{on } \Gamma \times \mathbb{R}_+ \\ &u(\cdot,0) = v, \quad u_t(\cdot,0) = w, && \text{in } \Omega \end{aligned} \tag{9.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^d$ with smooth boundary $\Gamma$, and $v$ and $w$ are given functions of $x$ in $\Omega$. We defer stating the required regularity of the initial data $v$ and $w$ until after we derive a formal solution using separation of variables.

We proceed as in the case of the heat equation. Let $\{\varphi_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ denote eigenfunctions and eigenvalues of the Laplacian $-\Delta$ with homogeneous Dirichlet boundary condition, i.e., $-\Delta\varphi_j = \lambda_j\varphi_j$ in $\Omega$, $\varphi_j = 0$ on $\Gamma$. Since $\{\varphi_j\}_{j=1}^{\infty}$ is a basis of $L^2(\Omega)$, we look for a solution of the form

$$u(x,t) = \sum_{j=1}^{\infty} \hat{u}_j(t)\varphi_j(x), \quad x \in \Omega,\ t \in \mathbb{R}_+.$$

Replacing this series into the differential equation, we find

$$\sum_{j=1}^{\infty} \left(\hat{u}_j''(t) + \lambda_j \hat{u}_j(t)\right)\varphi_j(x) = 0, \quad x \in \Omega,\ t \in \mathbb{R}_+.$$

We also expand the initial conditions:

$$\sum_{j=1}^{\infty} \hat{u}_j(0)\varphi_j(x) = v(x), \quad \sum_{j=1}^{\infty} \hat{u}_j'(0)\varphi_j(x) = w(x).$$

Since the $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$ we then have, for $j \geq 1$, the following second-order ODEs:

$$\begin{aligned} &\hat{u}_j'' + \lambda_j \hat{u}_j = 0, \quad \text{for } t > 0 \\ &\hat{u}_j(0) = \hat{v}_j = (v,\varphi_j)\,, \quad \hat{u}_j'(0) = \hat{w}_j = (w,\varphi_j) \end{aligned}$$

Solving this initial-value problem, we arrive at

$$\hat{u}_j(t) = \hat{v}_j \cos\left(\sqrt{\lambda_j}t\right) + \hat{w}_j \frac{1}{\sqrt{\lambda_j}} \sin\left(\sqrt{\lambda_j}t\right), \quad \text{for } j \geq 1, \quad t \geq 0,$$

and thus, formally, the solution of the initial-boundary value problem (9.1) is given by

$$\boxed{u(x,t) = \sum_{j=1}^{\infty} \left(\hat{v}_j \cos\left(\sqrt{\lambda_j}t\right) + \hat{w}_j \lambda_j^{-1/2} \sin\left(\sqrt{\lambda_j}t\right)\right)\varphi_j(x), \quad x \in \Omega, \quad t \in \mathbb{R}_+.} \tag{9.2}$$

**Theorem 9.1.** *Assume that $v \in H^2(\Omega) \cap H_0^1(\Omega)$ and $w \in H_0^1(\Omega)$. Then the series* (9.2) *is a solution of* (9.1) *in the $L^2$-sense, i.e., $\|u_{tt}(\cdot, t) - \Delta u(\cdot, t)\| = 0$ for all $t > 0$, $\|u(\cdot, 0) - v\| = 0$ and $\|u_t(\cdot, 0) - w\| = 0$.*

*Proof.* To prove that (9.2) satisfies the PDE, it suffices to verify that if $v$ and $w$ are sufficiently regular to ensure the convergence of the series (9.2) and its derivatives, then (9.2) represents a solution of (9.1). To verify that the initial conditions are satisfies, see Problem 11.4 in the book. □

We now focus on stability, for which we can derive the following energy estimate—without relying on the eigenfunction expansion.

**Theorem 9.2.** *Let $u = u(x, t)$ be a sufficiently smooth/classical solution of* (9.1)*. Then the total energy $\mathcal{E}(t)$ of $u$ is constant in time, i.e.,*

$$\mathcal{E}(t) := \frac{1}{2} \int_\Omega \left(u_t^2 + |\nabla u|^2\right) \mathrm{d}x = \mathcal{E}(0).$$

*Proof.* Multiplying the $u_{tt} - \Delta u = 0$ by $u_t$ and integrating with respect to $x$ over $\Omega$, using also Green's formula, we get

$$0 = \int_\Omega u_t(u_{tt} - \Delta u) \, \mathrm{d}x = \int_\Omega u_{tt} u_t \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla u_t \, \mathrm{d}x$$

Then, using the facts that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_t^2 \, \mathrm{d}x = \int_\Omega u_t u_{tt} \, \mathrm{d}x \qquad \text{and} \qquad \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla u|^2 \, \mathrm{d}x = \int_\Omega \nabla u_t \cdot \nabla u \, \mathrm{d}x$$

we arrive at

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u_t\|^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u\|^2 = 0,$$

or, equivalently,

$$\frac{d}{dt} \mathcal{E}(t) = 0, \qquad \text{for } t > 0.$$

Therefore $\mathcal{E}$ is constant and equal to $\mathcal{E}(0)$. □

It follows from Poincaré's inequality and the result above that

$$\|u_t\|^2 + c_P^{-2} \|u\|^2 \leq \|u_t\|^2 + \|\nabla u\|^2 = 2\mathcal{E}(t) = 2\mathcal{E}(0) = \|w\|^2 + \|\nabla v\|^2 \quad \text{for all } t \geq 0,$$

from which we conclude stability.

An interesting aspect of wave phenomena, which is particularly useful in applications such as radio communications, is that waves can travel long distances. This motivates the study of wave propagation in unbounded domains, where waves can extend infinitely. The simplest unbounded domain to consider is $\mathbb{R}^d$. Let us then consider the following pure initial-value problem for the wave equation:

$$\begin{aligned} &u_{tt} - \Delta u = 0, && \text{in } \mathbb{R}^d \times \mathbb{R}_+, \\ &u(\cdot, 0) = v, \quad u_t(\cdot, 0) = w, && \text{in } \mathbb{R}^d. \end{aligned} \tag{9.3}$$

The following results introduces the concept of the *light cone* in the context of the wave equation, which represents the region within which disturbances propagate at a finite speed, defining the domain of dependence and influence of the solution.

**Theorem 9.3.** *Let $u$ be a solution of the wave equation in (9.3). For $(\bar{x},\bar{t})$ a given point in $\mathbb{R}^d \times \mathbb{R}_+$, let $K$ denote the circular cone,*

$$K = \Big\{(x,t) \in \mathbb{R}^d \times \mathbb{R}_+ : |x-\bar{x}| \leq \bar{t} - t, t \leq \bar{t}\Big\} \tag{9.4}$$

*and set*

$$\mathcal{E}_K(t) = \frac{1}{2}\int_{B_t} \left(u_t(x,t)^2 + |\nabla u(x,t)|^2\right) \mathrm{d}x$$

*where $B_t = \big\{x \in \mathbb{R}^d : (x,t) \in K\big\}$. Then*

$$\mathcal{E}_K(t) \leq \mathcal{E}_K(0), \qquad \textit{for } 0 \leq t \leq \bar{t}.$$

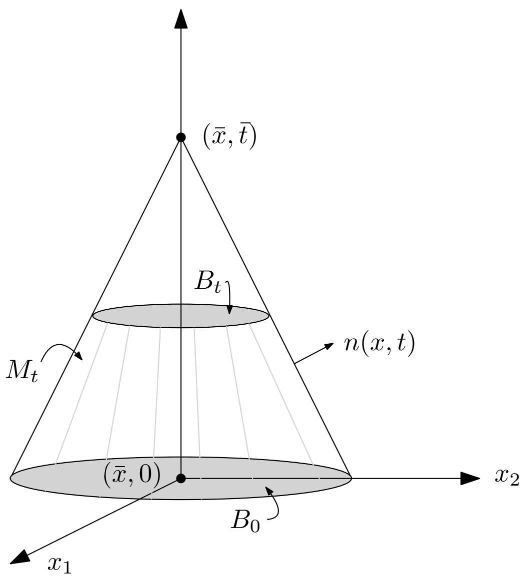


Figure 9.1: Light cone $K$ defined in (9.4) and related domains involved in the proof of Theorem 9.3.

**Remark 9.4.** Note that if $v = w = 0$ in $B_0 = \{x \in \mathbb{R}^d : |x - \bar{x}| \leq \bar{t}\}$, then $u = 0$ in all of $K$ and in particular at $(\bar{x},\bar{t})$. This shows that $u(\bar{x},\bar{t})$ depends only on the values of $v$ and $w$ in the ball $B_0$ defined by the circular cone $K$ with vertex $(\bar{x},\bar{t})$ , and not on the values of $v$ and $w$ outside this ball.

*Proof.* Multiplying $u_{tt} - \Delta u = 0$ by $2u_t$ we obtain

$$\begin{aligned} 0 &= 2\left(u_{tt} - \operatorname{div} \nabla u\right) u_t = 2u_{tt}u_t + 2\nabla u \cdot \nabla u_t - 2\operatorname{div}\left(\nabla u u_t\right) \\ &= \partial_t \left(u_t^2 + |\nabla u|^2\right) - 2\operatorname{div}\left(\nabla u u_t\right) = \operatorname{div}_{x,t}(-2u_t \nabla u, , u_t^2 + |\nabla u|^2) \end{aligned}$$

where $\operatorname{div}_{x,t}$ denotes the divergence operator in space-time domain. Integrating over the domain

$$K_t = \{(x,\tau) \in K : 0 \leq \tau \leq t\}$$

and using the divergence theorem

$$
\begin{aligned}
0 =& \int_{\partial K_t} \left(n_t \left(u_t^2 + |\nabla u|^2\right) - 2n_x \cdot \nabla u u_t\right) \mathrm{d}s \\
=& \int_{B_t} \left(u_t^2 + |\nabla u|^2\right) \mathrm{d}x - \int_{B_0} \left(u_t^2 + |\nabla u|^2\right) \mathrm{d}x + \int_{M_t} \left(n_t \left(u_t^2 + |\nabla u|^2\right) - 2u_t n_x \cdot \nabla u\right) \mathrm{d}s \\
=& \mathcal{E}_K(t) - \mathcal{E}_K(0) + \int_{M_t} \left(n_t \left(u_t^2 + |\nabla u|^2\right) - 2u_t n_x \cdot \nabla u\right) \mathrm{d}s
\end{aligned}
$$

where $n = (n_x, n_t) = (n_{x_1}, \ldots, n_{x_d}, n_t) \in \mathbb{R}^{d+1}$ denotes the exterior unit normal to $K_t$ and and where

$$M_t = \{(x, \tau) \in M : \tau \leq t\}$$

with

$$M = \{(x, t) : |x - \bar{x}| = \bar{t} - t\},$$

To complete the proof it suffices to show that the last integral above is nonnegative. To do so, we note that on $M$ the unit normal is given by

$$n = (\underbrace{\frac{\sqrt{2}}{2} \frac{x - \bar{x}}{|x - \bar{x}|}}_{=n_x}, \underbrace{\frac{\sqrt{2}}{2}}_{=n_t}).$$

In particular, we have $n_t^2 = |n_x|^2 = \frac{1}{2}$ on $M$, so

$$|n_x \cdot \nabla u| \leq |n_x| \, |\nabla u| = n_t |\nabla u|$$

Finally, from the inequality $2ab \leq a^2 + b^2$, we obtain

$$2 \, |u_t n_x \cdot \nabla u| = 2 \, |u_t| \, |n_x \cdot \nabla u| \leq 2n_t \, |u_t| \, |\nabla u| \leq n_t \left(u_t^2 + |\nabla u|^2\right)$$

which completes the proof. □

### 9.0.2 First Order Scalar Equations and the Method of the Characteristics

A powerful method for solving certain types of hyperbolic PDEs is the method of characteristics, which we now explain—first for scalar PDEs and then for hyperbolic symmetric systems—allowing certain second-order hyperbolic PDEs, such as the wave equation, to be recast in a more tractable form.

Consider the the following first-order scalar PDE:

$$\sum_{j=1}^{d} a_j(\zeta) \frac{\partial u}{\partial \zeta_j} + a_0(\zeta) u = f(\zeta), \quad \zeta \in \Omega, \tag{9.5}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with boundary $\Gamma$, the vector field $a : \Omega \to \mathbb{R}^d$,

$$a(\zeta) = (a_1(\zeta), \ldots, a_d(\zeta)) \,,$$

is smooth and nowhere vanishing (i.e., $|a(\zeta)| \neq 0$ for all $\zeta \in \Omega$), and $a_0$ and $f$ are given smooth functions.

We say that

$$\zeta = \zeta(s) := (\zeta_1(s), \ldots, \zeta_d(s)), \quad s \in \mathbb{R},$$

is a characteristic curve, or simply a characteristic, for (9.5) if

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s) = a(\zeta(s))$$

or, equivalently,

$$\frac{\mathrm{d}\zeta_j}{\mathrm{d}s} = a_j(\zeta), \quad \text{for } j = 1, \ldots, d,$$

i.e., if the curve in $\mathbb{R}^d$ defined by $\zeta = \zeta(s)$ has the vector $a(\zeta)$ as a tangent at each of its points.

From the local existence and uniqueness theorems for (autonomous) ODE systems (i.e., the Picard–Lindelöf theorem also known as the Cauchy–Lipschitz theorem), since $a$ does not vanish, it follows that that for each $x_0 \in \Omega$ there exists a unique such curve in some neighborhood of $\zeta_0$ such that $\zeta(0) = \zeta_0$.

We distinguish three parts of the boundary $\Gamma = \partial\Omega$. Let $n(\zeta)$ denote the exterior normal to $\Gamma$ at $\zeta$. We then define:

- **The inflow boundary**:

$$\Gamma_- = \{\zeta \in \Gamma : n(\zeta) \cdot a(\zeta) < 0\}. \tag{9.6}$$

  Through each point of $\Gamma_-$, there is a unique characteristic that enters $\Omega$. We prescribe the solution of (9.5) on $\Gamma_-$ via the boundary condition

$$u = v, \quad \text{on } \Gamma_-, \tag{9.7}$$

  where $v$ is a given smooth function on $\Gamma_-$.

- **The outflow boundary**:

$$\Gamma_+ = \{\zeta \in \Gamma : n(\zeta) \cdot a(\zeta) > 0\}. \tag{9.8}$$

- **The characteristic boundary**:

$$\Gamma_0 = \{\zeta \in \Gamma : n(\zeta) \cdot a(\zeta) = 0\}. \tag{9.9}$$

Consider a solution $u$ of the first-order PDE with the boundary condition (9.6). Let us now examine the solution $u$ restricted along a characteristic curve $\zeta = \zeta(s)$. Define the function

$$w(s) = u(\zeta(s)).$$

By applying the chain rule, we obtain:

$$\frac{\mathrm{d}w}{\mathrm{d}s} = \nabla u \cdot \frac{\mathrm{d}\zeta}{\mathrm{d}s} = a(\zeta) \cdot \nabla u.$$

Thus, from the PDE, it follows that $w$ satisfies the equation

$$\frac{\mathrm{d}w}{\mathrm{d}s} + a_0(\zeta(s))w = f(\zeta(s)), \quad \text{for } s > 0, \quad w(0) = v(\zeta_0), \quad \text{with } \zeta(0) = \zeta_0 \in \Gamma_-. \tag{9.10}$$

This is an initial value problem for a linear ODE, which can be solved to determine the value of $w$ at points along the characteristic. To find the solution of the first-order PDE (9.5) with boundary condition (9.6) at a point $\bar{\zeta} \in \Omega$, we proceed as follows:

1. Determine the characteristic passing through $\bar{\zeta} \in \Omega$.
2. Find its intersection $\zeta_0$ with $\Gamma_-$. For that it is useful to parametrize $\Gamma_-$ using a parameter, say $\xi$, so that $\zeta_0 := \zeta_0(\xi) \in \Gamma_-$.
3. Solve the equation (9.10) with the initial condition $\zeta(0) = \zeta(0; \xi) = \zeta_0(\xi)$.

Clearly, the solution at $\bar{\zeta}$ depends only on $v(\zeta_0)$ and the values of $f$ along the characteristic $\zeta$.

In the special case that $a_0 = f = 0$ in $\Omega$, the equation (11.14) reduces to

$$\frac{\mathrm{d}w}{\mathrm{d}s} = 0, \quad \text{for } s > 0, \quad \text{with } w(0) = v\left(\zeta_0\right), \zeta(0) = \zeta_0 \in \Gamma_-$$

Thus in this case $u(\zeta(s))$ is constant along the characteristic and the value of the solution at $\bar{\zeta}$ is the same as at $\zeta(0)$, i.e., $u(\zeta(s)) = u(\zeta(0)) = v(\zeta(0))$.

This procedure is often referred to as the method of characteristics.

**Example 9.5.** Consider the problem

$$\begin{aligned} u_t + \lambda u_x &= 0, \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u(\cdot, 0) &= v, \qquad \text{in } \mathbb{R}. \end{aligned}$$

The characteristic variables in this case are given by $\zeta = (x, t)$, with domain $\Omega = \mathbb{R} \times \mathbb{R}_+$, boundary $\Gamma_- = \mathbb{R} \times \{0\}$, and characteristic velocity

$$a(x, t) = (\lambda, 1).$$

The characteristic equations are:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lambda, \qquad \frac{\mathrm{d}t}{\mathrm{d}s} = 1.$$

Additionally, along the characteristic curves, the function $w(s) = u(x(s), t(s))$ satisfies:

$$\frac{\mathrm{d}w}{\mathrm{d}s} = 0.$$

Integrating these equations, we obtain:

$$x(s) = \lambda s + x(0), \qquad t(s) = s + t(0), \qquad w(s) = w(0).$$

The boundary condition is given by

$$u(\xi, 0) = v(\xi), \quad \xi \in \mathbb{R}.$$

Since we parametrize the boundary $\Gamma_-$ as $(\xi, 0)$ for $\xi \in \mathbb{R}$, we set:

$$x(0) = \xi, \quad t(0) = 0.$$

Thus, we obtain:

$$w(s) = w(0; \xi) = u(\underbrace{x(0)}_{=\xi}, \underbrace{t(0)}_{=0}) = v(\xi).$$

Using $x = \lambda s + \xi$ and $t = s$, we solve for $\xi$:

$$\xi = x - \lambda t.$$

Substituting back, we conclude:

$$\boxed{u(x, t) = v(x - \lambda t).}$$

**Example 9.6.** Consider now the problem:

$$xu_x + (x+y)u_y = 1, \quad \text{with } u(1,y) = y, \quad 0 < y < 1.$$

The characteristic equations are:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = x, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = x + y, \quad \frac{\mathrm{d}w}{\mathrm{d}s} = 1,$$

where $w(s) = u(x(s), y(s))$.

We parametrize the boundary curve where the initial condition is given:

$$x_0(\xi) = 1, \quad y_0(\xi) = \xi, \quad w_0(\xi) = \xi, \quad \xi \in (0,1).$$

Solving for $x(s)$:

$$x(s) = x(0)e^s = e^s.$$

For $y(s)$, solving the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}s} - y = e^s,$$

we rewrite it as:

$$\left(ye^{-s}\right)' = 1.$$

Integrating, we obtain:

$$y = e^s(s + \xi).$$

Since $\frac{\mathrm{d}w}{\mathrm{d}s} = 1$, we integrate:

$$w(s) = s + w(0) = s + \xi.$$

To express $u = u(x,y)$, we need to solve for $s$ and $\xi$ in terms of $x$ and $y$. We solve the system:

$$x = e^s, \quad y = e^s(s + \xi).$$

Dividing the second equation by $x = e^s$ gives:

$$\frac{y}{x} = s + \xi.$$

Using $s = \ln x$ (since $x > 0$), we obtain:

$$\xi = \frac{y}{x} - \ln x.$$

Substituting into $w(s;\xi) = u(x,y)$, we conclude:

$$\boxed{u(x,y) = \frac{y}{x}.}$$

### 9.0.3 Symmetric hyperbolic systems

Consider now the system

$$\begin{aligned}\frac{\partial u}{\partial t} + A(x,t)\frac{\partial u}{\partial x} + B(x,t)u = & \; f(x,t), \quad \text{for } x \in \mathbb{R}, t > 0 \\ u(x,0) = & \; v(x), \quad \text{for } x \in \mathbb{R}\end{aligned} \tag{9.11}$$

with the following assumptions:

- $u = u(x,t)$ and $f = f(x,t)$ are $\mathbb{R}^N$-values valued functions
- $A = A(x,t)$ and $B = B(x,t)$ are smooth $N \times N$ matrices,
- $A$ is symmetric (therefore, $A$ then has real eigenvalues $\{\lambda_j\}_{j=1}^N$, with $\lambda_j = \lambda_j(x,t)$);
- The eigenvalue $\{\lambda_j\}_{j=1}^N$ of $A$ are distinct.

The system (9.11) satisying these assumption is called hyperbolic.

Under these assumptions, we can find a smooth orthogonal matrix $P = P(x,t)$ that diagonalizes $A$, so that

$$P^\top AP = \Lambda = \text{diag}(\lambda_j)_{j=1}^N.$$

Introducing a new dependent variable $w$ by setting $u = Pw$, we obtain

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + Bu = P\frac{\partial w}{\partial t} + AP\frac{\partial w}{\partial x} + \left(\frac{\partial P}{\partial t} + A\frac{\partial P}{\partial x} + BP\right)w = f.$$

Rearranging, we obtain

$$\frac{\partial w}{\partial t} + \Lambda\frac{\partial w}{\partial x} + \tilde{B}w = P^\top f,$$

where

$$\tilde{B} = P^\top\left(\frac{\partial P}{\partial t} + A\frac{\partial P}{\partial x} + BP\right).$$

Thus, we have transformed the system (9.11) into a system with $A$ in diagonal form.

Now, without loss of generality, we assume that $A$ in (9.11) is already diagonal, with eigenvalues arranged in increasing order:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N.$$

In the case when $B = 0$, the system consists of $N$ uncoupled equations:

$$\boxed{\frac{\partial u_j}{\partial t} + \lambda_j(x,t)\frac{\partial u_j}{\partial x} = f_j(x,t), \quad \text{with } u_j(x,0) = v_j(x), \quad \text{for } j = 1,\dots,N.}$$

Each equation represents a scalar problem of the type analyzed earlier, with $\zeta = (x,t)$. For each index $j$, the characteristic curve passing through $(\bar{x},\bar{t})$ is determined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_j(x,t), \quad \text{with } x(\bar{t}) = \bar{x}.$$

Let $x_j(t)$ denote the solution of this initial value problem, so that the characteristic through $(\bar{x}, \bar{t})$ is given by $x = x_j(t)$. The solution to the PDE along this characteristic is

$$\boxed{u_j(\bar{x}, \bar{t}) = v_j\left(x_j(0)\right) + \int_0^{\bar{t}} f_j\left(x_j(s), s\right) \mathrm{d}s.}$$

Thus, $u_j(\bar{x}, \bar{t})$ depends only on $v_j$ at a single point and on $f_j$ along the characteristic curve passing through $(\bar{x}, \bar{t})$.

**Example 9.7.** Consider the initial value problem for the wave equation:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, && \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u(\cdot, 0) = v, \quad & \frac{\partial u}{\partial t}(\cdot, 0) = w, && \text{in } \mathbb{R}. \end{aligned} \tag{9.12}$$

We introduce the new variables

$$U_1 = \frac{\partial u}{\partial t}, \quad U_2 = \frac{\partial u}{\partial x},$$

and define $U = (U_1, U_2)^\top$, which satisfies the system:

$$\begin{aligned} \frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial x} &= 0, && \text{in } \mathbb{R} \times \mathbb{R}_+, \\ \frac{\partial U_2}{\partial t} - \frac{\partial U_1}{\partial x} &= 0, \\ U_1(\cdot, 0) = w, \quad & U_2(\cdot, 0) = v', && \text{in } \mathbb{R}. \end{aligned}$$

Rewriting in matrix form, we obtain

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0, \quad \text{with } U(x, 0) = \begin{bmatrix} w(x) \\ v'(x) \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of $A$ are $\lambda_1 = -1$, $\lambda_2 = 1$. Setting

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad U = PV,$$

we transform the system into

$$\frac{\partial V}{\partial t} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial V}{\partial x} = 0, \quad \text{in } \mathbb{R} \times \mathbb{R}_+.$$

This decouples into two transport equations:

$$\begin{aligned} \frac{\partial V_1}{\partial t} - \frac{\partial V_1}{\partial x} &= 0, \\ \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x} &= 0. \end{aligned}$$

Thus, their solutions are

$$V_1(x,t) = V_1(x+t,0), \quad V_2(x,t) = V_2(x-t,0).$$

Since $V(x,t) = PU(x,t)$, we compute

$$V(x,0) = PU(x,0) = P\begin{bmatrix} w(x) \\ v'(x) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} w(x)+v'(x) \\ w(x)-v'(x) \end{bmatrix}.$$

Thus,

$$V_1(x,t) = \frac{1}{\sqrt{2}}(w(x+t)+v'(x+t)), \quad V_2(x,t) = \frac{1}{\sqrt{2}}(w(x-t)-v'(x-t)).$$

Transforming back,

$$U(x,t) = PV(x,t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}}(w(x+t)+v'(x+t)) \\ \frac{1}{\sqrt{2}}(w(x-t)-v'(x-t)) \end{bmatrix}.$$

Simplifying,

$$U(x,t) = \frac{1}{2}\begin{bmatrix} w(x+t)+w(x-t)+v'(x+t)-v'(x-t) \\ w(x+t)-w(x-t)+v'(x+t)+v'(x-t) \end{bmatrix}.$$

To retrieve $u$, we use $U_1 = u_t$:

$$u(x,t) = \int_0^t U_1(x,s)\,\mathrm{d}s = \frac{1}{2}(v(x+t)+v(x-t)) + \frac{1}{2}\int_0^t (w(x+s)+w(x-s))\,\mathrm{d}s.$$

Since

$$\int_0^t (w(x+s)+w(x-s))\,\mathrm{d}s = \int_{x-t}^{x+t} w(s)\,\mathrm{d}s,$$

we obtain the well-known d'Alembert's formula:

$$\boxed{u(x,t) = \frac{1}{2}(v(x+t)+v(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} w(s)\,\mathrm{d}s.}$$

Consider now the generalization to $d$ space dimensions:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \sum_{j=1}^{d} A_j \frac{\partial u}{\partial x_j} + Bu = f, && \text{in } \mathbb{R}^d \times \mathbb{R}_+, \\ &u(\cdot,0) = v, && \text{in } \mathbb{R}^d, \end{aligned} \tag{9.13}$$

where $u = u(x,t)$ is an $N$-vector-valued function, $A_j = A_j(x,t)$ are symmetric $N \times N$ matrices, $B = B(x,t)$ is an $N \times N$ matrix, and $f = f(x,t)$ and $v = v(x)$ are $N$-vectors. All of these functions depend smoothly and remain bounded on their respective domains. We also assume that solutions decay sufficiently fast for large $|x|$ to ensure the validity of the following analysis.

In general, we cannot proceed as above because it is not possible to simultaneously diagonalize all the matrices $A_j$. We therefore restrict ourselves to apply the energy method to derive a stability estimate for this problem with respect to the norm $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^d)}$. For that, we need the following famous lemma:

**Lemma 9.8** (Gronwall's Lemma)**.** *Suppose that $\varphi$ is a nonnegative continuous function satisfying*

$$\varphi(t) \le a + b\int_0^t \varphi(s)\,\mathrm{d}s, \quad \textit{for } t > 0,$$

*where $a$ and $b$ are nonnegative constants. Then,*

$$\varphi(t) \le a\mathrm{e}^{bt}, \quad \textit{for } t > 0.$$

*Proof.* Define

$$v(\tau) = b\mathrm{e}^{-b\tau}\int_0^\tau \varphi(s)\,\mathrm{d}s, \quad \tau > 0.$$

Differentiating, we obtain

$$v'(\tau) = \left(\varphi(\tau) - \int_0^\tau \varphi(s)\,\mathrm{d}s\right) b\mathrm{e}^{-b\tau} \le ab\mathrm{e}^{-b\tau}, \quad \tau > 0.$$

Since $v(0) = 0$, integration from 0 to $t$ gives

$$v(t) \le ab\int_0^t \mathrm{e}^{-b\tau}\,\mathrm{d}\tau = a(1 - \mathrm{e}^{-bt}), \quad t > 0.$$

This leads to

$$v(t)\mathrm{e}^{bt} = b\int_0^t \varphi(s)\,\mathrm{d}s \le a\mathrm{e}^{bt} - a.$$

Thus,

$$\varphi(t) \le a + v(t)\mathrm{e}^{bt} = a + b\int_0^t \varphi(s)\,\mathrm{d}s \le a\mathrm{e}^{bt}, \quad t > 0.$$

□

**Theorem 9.9.** *For the solution of* (9.13)*, there exists a constant $C = C(T)$ such that*

$$\|u(t)\| \le C\left(\|v\| + \left(\int_0^T \|f\|^2\,\mathrm{d}s\right)^{1/2}\right), \quad \textit{for } 0 \le t \le T.$$

*Proof.* Multiplying the equation by $u$ and integrating over $\mathbb{R}^d$ gives

$$\left(\frac{\partial u}{\partial t}, u\right) + \sum_{j=1}^d \left(A_j\frac{\partial u}{\partial x_j}, u\right) + (Bu, u) = (f, u).$$

We compute

$$\left(\frac{\partial u}{\partial t}, u\right) = \int_{\mathbb{R}^d} \frac{\partial u}{\partial t}\cdot u\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2.$$

For the term involving $A_j$,

$$\left(A_j\frac{\partial u}{\partial x_j}, u\right) = \int_{\mathbb{R}^d}\left(A_j\frac{\partial u}{\partial x_j}\right)\cdot u\,\mathrm{d}x.$$

Using integration by parts and assuming $u$ decays at infinity, we obtain

$$\left(A_j \frac{\partial u}{\partial x_j}, u\right) = -\frac{1}{2}\left(\frac{\partial A_j}{\partial x_j} u, u\right).$$

Thus,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + (\tilde{B}u, u) \leq \|f\|\|u\|,$$

where

$$\widetilde{B} = B - \frac{1}{2}\sum_{j=1}^{d} \frac{\partial A_j}{\partial x_j}.$$

Using the inequality $2\|f\|\|u\| \leq \|f\|^2 + \|u\|^2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \leq 2\|\widetilde{B}\|_{C(\mathbb{R}^d\times\mathbb{R}_+)}\|u\|^2 + 2\|f\|\|u\| \leq C_0\|u\|^2 + \|f\|^2,$$

where $C_0 = 2\|\widetilde{B}\|_{C(\mathbb{R}^d\times\mathbb{R}_+)} + 1$. Integrating, we obtain

$$\|u(t)\|^2 \leq \|v\|^2 + \int_0^T \|f\|^2 \,\mathrm{d}s + C_0 \int_0^t \|u\|^2 \,\mathrm{d}s, \quad \text{for } 0 \leq t \leq T.$$

Applying Gronwall's lemma,

$$\|u(t)\|^2 \leq \mathrm{e}^{C_0 T}\left(\|v\|^2 + \int_0^T \|f\|^2 \,\mathrm{d}s\right), \quad \text{for } 0 \leq t \leq T.$$

□

As usual, this inequality implies uniqueness and stability for problem (9.13).

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