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Cursusnaam/Coursename Signals and Transforms		Datum/Date 28-03-2023	Bladnr./Page no.
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1.a) By Fourier Transform Convolution Theorem, we have that

$(f*g)(t) \overset{\mathcal{F}}{\longleftrightarrow} \hat{f}(\omega)\hat{g}(\omega)$

Thus we have in this case that $g_N(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}_{2N}(\omega)$

$a\,\text{sinc}(at/2) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\text{rect}_a(\omega)$

$2N\,\text{sinc}(\frac{2Nt}{2}) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\text{rect}_{2N}(\omega)$

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By linearity: $\frac{2N}{2\pi}\,\text{sinc}(\frac{2Nt}{2}) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}_{2N}(\omega)$

Thus we have that $g_N(t) = \frac{2N}{2\pi}\,\text{sinc}(\frac{2Nt}{2}) = \frac{N}{\pi}\,\text{sinc}(Nt)$ 1

1.b) $(f*g_N)(t) \overset{\mathcal{F}}{\longleftrightarrow} \hat{f}(\omega)\,\text{rect}_{2N}(\omega) = \begin{cases} \hat{f}(\omega) & \text{for } |\omega| < \frac{2N}{2} \\ 0 & \text{for } |\omega| > \frac{2N}{2} \\ \frac{1}{2}\hat{f}(\omega) & \text{for } |\omega| = \frac{2N}{2} \end{cases} = \begin{cases} \hat{f}(\omega) & \text{for } |\omega| < N \\ 0 & \text{for } |\omega| > N \\ \frac{1}{2}\hat{f}(\omega) & \text{for } |\omega| = N \end{cases}$

This implies that the Fourier transform of $(f*g_N)(t)$ equals $\hat{f}(\omega)$ but only on interval $(-N,N)$, thus only for all "small" frequencies. 1

1/1

Therefore $(f*g_N)(t)$ is an approximation of $f(t)$ obtained by removing all "large" frequencies from $f(t)$.

If we Fourier back we get $f(t)$ ~~but not for all values of t~~.

1.c) $(f*g_N)(t) = \int_{-\infty}^{\infty} g_N(\tau) f(t-\tau)\,d\tau = \int_{-\infty}^{\infty} g_N(\tau)(\mathbb{1}(t-\tau) - \mathbb{1}(t-\tau-T))\,d\tau$

$= \int_{-\infty}^{\infty} g_N(\tau)\mathbb{1}(t-\tau)\,d\tau - \int_{-\infty}^{\infty} g_N(\tau)\mathbb{1}(t-\tau-T)\,d\tau$

$= \int_{-\infty}^{t} g_N(\tau)\,d\tau - \int_{-\infty}^{\infty} g_N(\tau)\mathbb{1}(-\tau+t-T)\,d\tau$

$= \int_{-\infty}^{t} g_N(\tau)\,d\tau - \int_{-\infty}^{t-T} g_N(\tau)\,d\tau = \int_{t-T}^{t} g_N(\tau)\,d\tau \quad \text{for } T>0$ 1

$\mathbb{1}(-\tau+t-T) = \begin{cases} 1 & \text{if } -\tau+t-T>0 \\ 1/2 & \text{if } -\tau+t-T=0 \\ 0 & \text{if } -\tau+t-T<0 \end{cases} = \begin{cases} 1 & \text{if } \tau<t-T \\ 1/2 & \text{if } \tau=t-T \\ 0 & \text{if } \tau>t-T \end{cases}$

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1 d) $\max_{t\in\mathbb{R}} \int_{-\infty}^{t} g_N(\tau)\,d\tau = \max_{t\in\mathbb{R}} \int_{-\infty}^{t} \frac{N}{\pi}\,\text{sinc}(N\tau)\,d\tau = \max_{t\in\mathbb{R}} \frac{N}{\pi}\int_{-\infty}^{t} \text{sinc}(N\tau)\,d\tau$

Take $s = N\tau$, $\frac{ds}{d\tau} = N \Rightarrow d\tau = \frac{1}{N}ds$

$\Rightarrow \max_{t\in\mathbb{R}} \frac{N}{\pi}\int_{-\infty}^{t} \text{sinc}(N\tau)\,d\tau = \max_{t\in\mathbb{R}} \frac{N}{\pi}\int_{-\infty}^{Nt} \text{sinc}(s)\frac{1}{N}\,ds = \max_{t\in\mathbb{R}} \frac{1}{\pi}\int_{-\infty}^{Nt} \text{sinc}(s)\,ds.$

So the integral does not depend on N, since N is now just a scalar for t and $t\in\mathbb{R} \Rightarrow$ does not depend on N. (take new $z\in\mathbb{R}$, where $z = Nt$)

✓ ✓ take. t/a

3/3

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2. $f^*(t) \overset{?}{\overset{\mathcal{F}}{\longleftrightarrow}} \hat{f}^*(-\omega)$ $\quad\quad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$ ✓

$\hat{f}(\omega) = \int_{-\infty}^{\infty} f^*(t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} \left(f(t)e^{i\omega t}\right)^* dt$ ✗

$= \left(\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt\right)^* = \left(\int_{-\infty}^{\infty} f(t)e^{-i(-\omega)t}\,dt\right)^*$ ✓

1.5/2

$= \left(\hat{f}(-\omega)\right)^* = \hat{f}^*(-\omega)$

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3. $f(t) = \sin(t)\mathbb{1}(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s) = \frac{1}{s^2+1}$

$g(t) = (1+e^{-t})\mathbb{1}(t) \overset{\mathcal{L}}{\longleftrightarrow} G(s) = \frac{1}{s} + \frac{1}{s+1}$

By convolution theorem for Laplace we have $(f*g)(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s)G(s)$

$F(s)G(s) = \frac{1}{s^2+1}\left(\frac{1}{s} + \frac{1}{s+1}\right) = \frac{1}{(s^2+1)s} + \frac{1}{(s^2+1)(s+1)} = \frac{s+1}{(s^2+1)(s+1)s} + \frac{s}{(s^2+1)(s+1)s}$ ✓

$= \frac{2s+1}{(s^2+1)(s+1)s} = \frac{A}{s+1} + \frac{B}{s} + \frac{Cs+D}{s^2+1} = \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{s} + \frac{1}{2}\cdot\frac{-3s+1}{s^2+1}$ ✓

Hand-opleg-methode (H.O.M)

$A = \frac{2s+1}{(s^2+1)s}\Big|_{s=-1} = \frac{2\cdot -1+1}{((-1)^2+1)\cdot -1} = \frac{-1}{2\cdot -1} = \frac{1}{2}$

$B = \frac{2s+1}{(s^2+1)(s+1)}\Big|_{s=0} = 1$

$\frac{A}{s+1} + \frac{B}{s} + \frac{Cs+D}{s^2+1} = \frac{As(s^2+1) + B(s+1)(s^2+1) + Cs^2(s+1) + Ds(s+1)}{(s+1)(s^2+1)s} = \frac{As^3 + As + B(s^3+s^2+s+1) + Cs^3 + Cs^2 + Ds^2 + Ds}{(s+1)(s^2+1)s}$

$= \frac{(A+B+C)s^3 + (B+C+D)s^2 + (A+B+D)s + B}{(s+1)(s^2+1)s}$ $\Rightarrow A+B+C = 0 \Rightarrow \frac{1}{2}+1+C = 0 \Rightarrow C = -1\frac{1}{2}$

$\Rightarrow A+B+D = 2 \Rightarrow \frac{1}{2}+1+D = 2 \Rightarrow D = \frac{1}{2}$ ✓

$$F(s)G(s) = \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{s} + \frac{1}{2}\cdot\frac{-3s+1}{s^2+1} = \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{s} - \frac{3}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$\overset{\mathcal{L}}{\longleftrightarrow} (f*g)(t) = \left(\tfrac{1}{2}e^{-t} + 1 - \tfrac{3}{2}\cos(t) + \tfrac{1}{2}\sin(t)\right)\mathbb{1}(t) = \tfrac{1}{2}\mathbb{1}(t)\left(e^{-t} - 3\cos(t) + \sin(t) + 2\right)$$

4a) frequency response: $\hat{h}(\omega)$

$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = u^{(2)}(t) - u^{(1)}(t)$, taking Fourier of both sides, we get

$$\overset{\mathcal{F}}{\longleftrightarrow} (i\omega)^2\hat{y}(\omega) + 3i\omega\hat{y}(\omega) + 2\hat{y}(\omega) = (i\omega)^2\hat{u}(\omega) - i\omega\hat{u}(\omega)$$

$$\Rightarrow \hat{y}(\omega)(-\omega^2 + 3i\omega + 2) = \hat{u}(\omega)(-\omega^2 - i\omega)$$

$$\Rightarrow \hat{y}(\omega) = \hat{u}(\omega)\cdot\frac{-\omega^2 - i\omega}{-\omega^2+3i\omega+2} \Rightarrow \hat{h}(\omega) = \frac{-\omega^2 - i\omega}{-\omega^2+3i\omega+2}$$

4b) $u(t) = e^{2t}\mathbb{1}(-t) \overset{\mathcal{F}}{\longleftrightarrow} \hat{u}(\omega) = \frac{-1}{-2+i\omega}$

from a) we get $\hat{y}(\omega) = \hat{u}(\omega)\cdot\frac{-\omega^2-i\omega}{-\omega^2+3i\omega+2} = \frac{-1}{-2+i\omega}\cdot\frac{-\omega^2-i\omega}{-\omega^2+3i\omega+2} = \frac{\omega^2+i\omega}{(-2+i\omega)(1+i\omega)(2+i\omega)}$

$$= \frac{A}{-2+i\omega} + \frac{B}{1+i\omega} + \frac{C}{2+i\omega} = -\frac{1}{6}\cdot\frac{1}{-2+i\omega} - \frac{2}{3}\cdot\frac{1}{1+i\omega} + \frac{3}{2}\cdot\frac{1}{2+i\omega}$$

$$\overset{\mathcal{F}}{\longleftrightarrow} y(t) = \frac{1}{6}e^{2t}\mathbb{1}(-t) - \frac{2}{3}e^{-t}\mathbb{1}(t) + \frac{3}{2}e^{-2t}\mathbb{1}(t)$$

H.O.M. $A = \frac{\omega^2+i\omega}{(1+i\omega)(2+i\omega)}\Big|_{\omega=-2i} = \frac{-4+2}{3\cdot 4} = -\frac{1}{6}$

$B = \frac{\omega^2+i\omega}{(-2+i\omega)(2+i\omega)}\Big|_{\omega=i} = \frac{\omega^2+i\omega}{-4-\omega^2}\Big|_{\omega=i} = \frac{-1-1}{-4+1} = \frac{-2}{3}$

$C = \frac{\omega^2+i\omega}{(1+i\omega)(-2+i\omega)}\Big|_{\omega=2i} = \frac{-4-2}{-1\cdot -4} = \frac{3}{2}$

4c) $u(t) = \mathbb{1}(t) \overset{\mathcal{L}}{\longleftrightarrow} U(s) = \frac{1}{s}$, $u(0^-) = 0$, $u^{(1)}(0^-) = 0$

$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = u^{(2)}(t) - u^{(1)}(t)$

$\overset{\mathcal{L}}{\longleftrightarrow} \mathcal{L}\{y^{(2)}(t) + 3y^{(1)}(t) + 2y(t)\} = \mathcal{L}\{u^{(2)}(t) - u^{(1)}(t)\}$

By linearity and differentiation: $s^2Y(s) - sy(0^-) - y^{(1)}(0^-) + 3(sY(s) - y(0^-)) + 2Y(s) = s^2U(s) - sU(0^-) - u^{(1)}(0^-) - sU(s) + u(0^-)$

$$\Rightarrow Y(s)(s^2+3s+2) - s + 2 - 3 = U(s)(s^2 - s)$$

$$\Rightarrow Y(s)(s^2+3s+2) = U(s)(s^2-s) + s + 1 = \frac{s^2-s}{s} + s + 1 = s - 1 + s + 1 = 2s$$

$$\Rightarrow Y(s) = \frac{2s}{s^2+3s+2} = \frac{2s}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} = 4\cdot\frac{1}{s+2} - 2\cdot\frac{1}{s+1}$$

$$\overset{\mathcal{L}}{\longleftrightarrow} y(t) = (4e^{-2t} - 2e^{-t})\mathbb{1}(t)$$

H.O.M. $A = \frac{2s}{s+1}\Big|_{s=-2} = \frac{-4}{-1} = 4$

$B = \frac{2s}{s+2}\Big|_{s=-1} = \frac{-2}{1} = -2$