

Signals & Transforms (202001343) — TEST 2RE

Date: 19-6-2023
Place: NH-115
Time: 08:45–10:15 (till 10:40 for students with special rights)
Course coordinator: G. Meinsma
Allowed aids during test: None

The solutions of the problems should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.

1. Let $f(t) = |1 - t| \text{rect}_2(t)$.

- (a) Determine the Fourier transform of $f(t)$.
- (b) Determine $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$.

2. Prove the modulation property of Fourier transformation:

$$f(t) \cos(\omega_0 t) \xrightarrow{\mathcal{F}} \frac{1}{2} \hat{f}(\omega - \omega_0) + \frac{1}{2} \hat{f}(\omega + \omega_0).$$

(Of course you may NOT use the tables in this case.)

3. Suppose $f(t)$ is real, that $f(t) \geq 0$ and $\int_{-\infty}^{\infty} f(t) dt < \infty$.
Show that $\hat{f}(0) = \max_{\omega \in \mathbb{R}} |\hat{f}(\omega)|$.

4. Determine the convolution of $\cos(t) \mathbb{1}(t)$ and $g(t) = e^{-t} \mathbb{1}(t)$ using the Laplace transform.

5. Formulate the *initial value theorem* of the Laplace transform.

6. Given is the differential equation

$$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = u^{(2)}(t) + u(t). \quad (1)$$

- (a) Determine the frequency response of this differential equation.
- (b) Suppose that $u(t) = t$. Use Laplace transformation to determine the solution $y(t)$ for $t > 0$ of (1) for the case that $y(0^-) = 1$ and $y^{(1)}(0^-) = -2$.

problem:	1	2	3	4	5	6
points:	5+3	3	3	5	2	1+5

Test grade is $1 + 9p/p_{\max}$

Property	Time domain	Freq. domain	Condition
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega)$	
Duality	$\hat{f}(t)$	$2\pi f(-\omega)$	
Conjugation	$f^*(t)$	$\hat{f}^*(-\omega)$	
Time-scaling	$f(at)$	$\frac{1}{ a } \hat{f}\left(\frac{\omega}{a}\right)$	$a \in \mathbb{R}, a \neq 0$
Time-shift	$f(t - \tau)$	$\hat{f}(\omega) e^{-i\omega\tau}$	
Frequency-shift	$f(t) e^{i\omega_0 t}$	$\hat{f}(\omega - \omega_0)$	
Modulation Thm.	$f(t) \cos(\omega_0 t)$	$\frac{\hat{f}(\omega - \omega_0) + \hat{f}(\omega + \omega_0)}{2}$	
Differentiation (time)	$f^{(1)}(t)$	$(i\omega) \hat{f}(\omega)$	$\lim_{t \rightarrow \pm\infty} f(t) = 0$
Integration (time)	$\int_{-\infty}^t f(\tau) d\tau$	$\frac{\hat{f}(\omega)}{i\omega}$	$\hat{f}(0) = 0$
Differentiation (freq.)	$-itf(t)$	$\hat{f}'(\omega)$	

$f(t)$	$\hat{f}(\omega)$	Condition
$\text{rect}_a(t)$	$a \text{sinc}(a\omega/2)$	$a > 0$
$\text{trian}_a(t)$	$a \text{sinc}^2(a\omega/2)$	$a \in \mathbb{R}, a > 0$
$e^{-a t }$	$\frac{2a}{a^2 + \omega^2}$	$\text{Re}(a) > 0$
$\frac{t^n}{n!} e^{-at} \mathbb{1}(t)$	$\frac{1}{(a + i\omega)^{n+1}}$	$\text{Re}(a) > 0; n \in \mathbb{N}$
$-\frac{t^n}{n!} e^{-at} \mathbb{1}(-t)$	$\frac{1}{(a + i\omega)^{n+1}}$	$\text{Re}(a) < 0; n \in \mathbb{N}$
$e^{-(at)^2}$	$\frac{\sqrt{\pi}}{ a } e^{-(\omega/(2a))^2}$	$a \in \mathbb{R}, a \neq 0$
$a \text{sinc}(at/2)$	$2\pi \text{rect}_a(\omega)$	$a \in \mathbb{R}, a > 0$

$f(t)$	$\hat{f}(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t - b)$	$e^{-i\omega b}$
$e^{i\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
$\text{sgn}(t)$	$\frac{2}{i\omega}$
$\mathbb{1}(t)$	$\frac{1}{i\omega} + \pi\delta(\omega)$

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time-scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$ (if $a > 0$)
Time-shift	$f(t - t_0) \mathbb{1}(t - t_0^-)$	$F(s) e^{-st_0}$ (if $t_0 > 0$)
Shift in s -domain	$f(t) e^{s_0 t}$	$F(s - s_0)$
Differentiation (t)	$f^{(1)}(t)$	$sF(s) - f(0^-)$
	$f^{(2)}(t)$	$s^2 F(s) - sf(0^-) - f^{(1)}(0^-)$
Integration (t)	$\int_{0^-}^t f(\tau) d\tau$	$\frac{F(s)}{s}$
Differentiation (s)	$-tf(t)$	$F'(s)$

$f(t), (t > 0^-)$	$F(s)$
e^{at}	$\frac{1}{s - a}$
$\frac{t^n}{n!} (n \in \mathbb{N})$	$\frac{1}{s^{n+1}}$
$\frac{t^n}{n!} e^{at} (n \in \mathbb{N})$	$\frac{1}{(s - a)^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2 + b^2}$
$\sin(bt)$	$\frac{b}{s^2 + b^2}$
$e^{at} \cos(bt)$	$\frac{s - a}{(s - a)^2 + b^2}$
$e^{at} \sin(bt)$	$\frac{b}{(s - a)^2 + b^2}$
$\delta(t)$	1