# Signals & Transforms (202001343) — TEST 2

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|---|---|
| Date: | 28-03-2023 |
| Place: | NH-115 |
| Time: | 13:45–15:15 (till 15:40 for students with special rights) |
| Course coordinator: | G. Meinsma |
| Allowed aids during test: | None |

**The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.**

1. *The Gibbs phenomenon of the Fourier transform.*

   (a) Let $N \gg 0$. Determine the function $g_N(t)$ such that

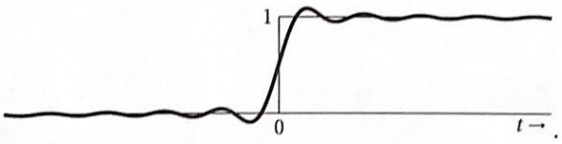
   $$(f * g_N)(t) \xleftrightarrow{\mathfrak{F}} \hat{f}(\omega)\,\mathrm{rect}_{2N}(\omega).$$

   (b) Interpretation: argue that $(f * g_N)(t)$ is an approximation of $f(t)$ obtained by removing all "large" frequencies from $f(t)$.

   (c) Let $T > 0$. Show that for $f(t) = \mathbb{1}(t) - \mathbb{1}(t-T)$ we have

   $$(f * g_N)(t) = \int_{t-T}^{t} g_N(\tau)\,\mathrm{d}\tau.$$

   (d) In the limit $T \to \infty$ the above $f(t)$ becomes the unit step, $f(t) = \mathbb{1}(t)$, and $(f * g_N)(t) = \int_{-\infty}^{t} g_N(\tau)\,\mathrm{d}\tau$. It can be shown that $\int_{-\infty}^{t} g_N(\tau)\,\mathrm{d}\tau$ looks like

   Let $N > 0$. Show that $\max_{t\in\mathbb{R}} \int_{-\infty}^{t} g_N(\tau)\,\mathrm{d}\tau$ does not depend on $N$.

2. Proof the conjugation property of Fourier transformation: $f^*(t) \xleftrightarrow{\mathfrak{F}} \hat{f}^*(-\omega)$. [Of course you may NOT use the tables in this case.]

3. Determine the convolution of $\sin(t)\,\mathbb{1}(t)$ and $g(t) = (1+\mathrm{e}^{-t})\,\mathbb{1}(t)$ using the Laplace transform.

4. Given is the differential equation

   $$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = u^{(2)}(t) - u^{(1)}(t). \tag{1}$$

   (a) Determine the frequency response of this differential equation.

   (b) Determine a solution $y(t)$ for the case that $u(t) = \mathrm{e}^{2t}\,\mathbb{1}(-t)$.

   (c) Suppose that $u(t) = \mathbb{1}(t)$. Use Laplace transformation to determine the solution $y(t)$ for $t > 0$ of (1) for the case that $y(0^-) = 1$ and $y^{(1)}(0^-) = -2$.

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| problem: | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| points: | 3+1+3+3 | 2 | 5 | 1+4+5 |

Test grade is $1 + 9p/p_{\max}$

| Property | Time domain | Freq. domain | Condition |
|---|---|---|---|
| Linearity | $a_1 f_1(t) + a_2 f_2(t)$ | $a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega)$ | |
| Duality | $\hat{f}(t)$ | $2\pi f(-\omega)$ | |
| Conjugation | $f^*(t)$ | $\hat{f}^*(-\omega)$ | |
| Time-scaling | $f(at)$ | $\frac{1}{\lvert a\rvert}\hat{f}\left(\frac{\omega}{a}\right)$ | $a \in \mathbb{R}, a \neq 0$ |
| Time-shift | $f(t-\tau)$ | $\hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}$ | |
| Frequency-shift | $f(t)\mathrm{e}^{\mathrm{i}\omega_0 t}$ | $\hat{f}(\omega-\omega_0)$ | |
| Modulation Thm. | $f(t)\cos(\omega_0 t)$ | $\frac{\hat{f}(\omega-\omega_0)+\hat{f}(\omega+\omega_0)}{2}$ | |
| Differentiation (time) | $f^{(1)}(t)$ | $(\mathrm{i}\omega)\hat{f}(\omega)$ | $\lim_{t\to\pm\infty} f(t) = 0$ |
| Integration (time) | $\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau$ | $\frac{\hat{f}(\omega)}{\mathrm{i}\omega}$ | $\hat{f}(0) = 0$ |
| Differentiation (freq.) | $-\mathrm{i}tf(t)$ | $\hat{f}'(\omega)$ | |

| $f(t)$ | $\hat{f}(\omega)$ | Condition |
|---|---|---|
| $\mathrm{rect}_a(t)$ | $a\,\mathrm{sinc}(a\omega/2)$ | $a > 0$ |
| $\mathrm{trian}_a(t)$ | $a\,\mathrm{sinc}^2(a\omega/2)$ | $a \in \mathbb{R},\ a > 0$ |
| $\mathrm{e}^{-a\lvert t\rvert}$ | $\frac{2a}{a^2+\omega^2}$ | $\mathrm{Re}(a) > 0$ |
| $\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(t)$ | $\frac{1}{(a+\mathrm{i}\omega)^{n+1}}$ | $\mathrm{Re}(a) > 0;\ n \in \mathbb{N}$ |
| $-\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(-t)$ | $\frac{1}{(a+\mathrm{i}\omega)^{n+1}}$ | $\mathrm{Re}(a) < 0;\ n \in \mathbb{N}$ |
| $\mathrm{e}^{-(at)^2}$ | $\frac{\sqrt{\pi}}{\lvert a\rvert}\mathrm{e}^{-(\omega/(2a))^2}$ | $a \in \mathbb{R},\ a \neq 0$ |
| $a\,\mathrm{sinc}(at/2)$ | $2\pi\,\mathrm{rect}_a(\omega)$ | $a \in \mathbb{R},\ a > 0$ |

| $f(t)$ | $\hat{f}(\omega)$ |
|---|---|
| $\delta(t)$ | $1$ |
| $1$ | $2\pi\delta(\omega)$ |
| $\delta(t-b)$ | $\mathrm{e}^{-\mathrm{i}\omega b}$ |
| $\mathrm{e}^{\mathrm{i}\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ |
| $\cos(\omega_0 t)$ | $\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$ |
| $\mathrm{sgn}(t)$ | $\frac{2}{\mathrm{i}\omega}$ |
| $\mathbb{1}(t)$ | $\frac{1}{\mathrm{i}\omega}+\pi\delta(\omega)$ |

| Property | $f(t)$ | $F(s)$ |
|---|---|---|
| Linearity | $a_1 f_1(t) + a_2 f_2(t)$ | $a_1 F_1(s) + a_2 F_2(s)$ |
| Time-scaling | $f(at)$ | $\frac{1}{a}F\left(\frac{s}{a}\right)$ (if $a > 0$) |
| Time-shift | $f(t-t_0)\,\mathbb{1}(t-t_0^-)$ | $F(s)\mathrm{e}^{-st_0}$ (if $t_0 > 0$) |
| Shift in $s$-domain | $f(t)\mathrm{e}^{s_0 t}$ | $F(s-s_0)$ |
| Differentiation ($t$) | $f^{(1)}(t)$ | $sF(s) - f(0^-)$ |
| | $f^{(2)}(t)$ | $s^2F(s) - sf(0^-) - f^{(1)}(0^-)$ |
| Integration ($t$) | $\int_{0^-}^{t} f(\tau)\,\mathrm{d}\tau$ | $\frac{F(s)}{s}$ |
| Differentiation ($s$) | $-tf(t)$ | $F'(s)$ |

| $f(t),\ (t > 0^-)$ | $F(s)$ |
|---|---|
| $\mathrm{e}^{at}$ | $\frac{1}{s-a}$ |
| $\frac{t^n}{n!}$ $(n \in \mathbb{N})$ | $\frac{1}{s^{n+1}}$ |
| $\frac{t^n}{n!}\mathrm{e}^{at}$ $(n \in \mathbb{N})$ | $\frac{1}{(s-a)^{n+1}}$ |
| $\cos(bt)$ | $\frac{s}{s^2+b^2}$ |
| $\sin(bt)$ | $\frac{b}{s^2+b^2}$ |
| $\mathrm{e}^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| $\mathrm{e}^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ |
| $\delta(t)$ | $1$ |