Course : **Model Reduction**

Date : April 14, 2025
Time : 13:45-15:45

Motivate all answers and calculations.
The use of electronic devices and/or lecture notes is not permitted.

[2p] **1.** Consider the standard state space equation

$$\dot{x}(t) = Ax(t).$$

We assume that $A = JH$, with $J^* = -J$, $H^* = H$, and H is a (strictly) positive matrix, i.e. $x^*Hx > 0$ whenever $x \neq 0$.

Show that all eigenvalues of A lie on the imaginary axis.

Hint: Consider x^*HAx.

[1.5p] **2.** Consider the standard state space system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{1}$$

for which we assume that the input and output take values in $\mathbb{C}$. Its transfer function is denoted by $G(s)$. Suppose that along solutions of (1) the following holds

$$\|x(t)\|^2 \leq \text{Re}(u(t)y(t)).$$

Show that $\text{Re}(G(s)) \geq 0$ whenever $\text{Re}(s) > 0$.

[1.5p] **3.** Given a state space system with observability and controllability gramiians given by

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 8 & 1 \\ 1 & 4 \end{bmatrix}$$

Argue whether or not you would reduce this model.

4. Consider the Burgers equation with periodic boundary condition

$$\begin{cases} \mathcal{B}(u) \equiv u_t + (\frac{1}{2}u^2)_x - \nu u_{xx} = 0 & \text{for } x \in]0,\pi[\,, \; t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0 \\ u(x,0) = \sin(2kx) & \text{for } x \in]0,\pi[\end{cases}$$

where $\mathcal{B}$ denotes the Burgers operator, $\nu > 0$ is a constant viscosity, $k \in \mathbb{Z}$ the wavenumber ($k \neq 0$) and (x,t) the spatial and temporal variables respectively. The top-hat filter L is a convolution filter, defined as

$$\overline{u}(x,t) \equiv L(u)(x,t) \equiv \int_{x-\Delta/2}^{x+\Delta/2} \frac{u(\xi,t)}{\Delta}\, d\xi,$$

where $\Delta > 0$ denotes the filter-width.

[1p] a) Apply the top-hat filter to $f(x) = \sin(kx)$ and express the result as $\overline{f}(x) = H(k\Delta)f(x)$ for suitable H. Discuss the effect of filtering for small and large $k\Delta$.

[2p] b) Apply the top-hat filter to the Burgers equation and derive the unclosed equation governing the evolution of $\overline{u}$ in the form $\mathcal{B}(\overline{u}) = (\tau(u,\overline{u}))_x$. Discuss the closure term $(\tau(u,\overline{u}))_x$ for small and large scales. Motivate the steps in your derivation.

[1p] c) A closure model $m(\overline{u}) = C\overline{u}_x$, with constant C, is adopted to approximate $\tau(u,\overline{u})$. Should C be positive or negative to enhance the decay rate of the resolved kinetic energy $E(t) = \int_0^1 (\overline{u}^2(\xi,t)/2)d\xi$?

Total: 1+5+4 points