Exam: Linear Structures 1.
Applied Mathematics, 2023-1A: Structures and Models
November 6 2023; 8:45 - 10:15

This exam consists of 11 problems which are divided into two parts:
Grasple (digital): 9 problems
Open Questions (written): 2 problems.

Grasple
Enter your answers in Grasple in the required form. Follow the instructions precisely.
For the statements, you choose one of three options: true (T), false (F), or no answer (N).
For each correct T or F you will receive (partial) points.
One incorrect T or F results in zero points for that entire question.
Total score for Grasple: 50 points.
Required score: 25 points.

Open questions
Write the solutions following the four steps.
Step 1. State the important information and summarize the problem.
Step 2. Devise a plan.
Step 3. Execute the plan.
Step 4. Analyze your solution and the answer.
Think, for example, of the following questions:
– Can you interpret the solution and/or the result intuitively?
– Does the solution/result make sense in special cases?
– Which role each condition played in the solution?
– Could you relax some assumptions of the problem?
– Is the problem related to other problems or results?
Total score for open questions part: 40 points.
Steps 1+2: 40%, Step 3: 40%, Step 4: 20%.
Required score: 20 points.

A (graphical) calculator is not needed and is **not allowed** at the exam.

PART 1: Multiple choice and final answer questions

1. [5pt] Let $\mathbb{P}^2(\mathbb{R})$ be a vector space over $\mathbb{R}$ with the standard operations of addition and scalar multiplication.

For each of the following subsets, indicate whether the set is a subspace of $\mathbb{P}^2(\mathbb{R})$ over $\mathbb{R}$ under the standard operations (T), whether it is no subspace (F), or give no answer (N).

(1) $W_1 = \{ax^2 + bx + c : b = c = 0\}$.

(2) $W_2 = \{ax^2 + bx + c : b = 2c\}$.

(3) $W_3 = \{ax^2 + bx + c : a + b + c = 2\}$

2. [5pt] Let A be an $n \times n$ matrix. Suppose that for some **b**, the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

For each of the following statements, indicate whether the statement **must be** true (T), can be false (F) or give no answer (N).

(1) For some $\mathbf{c} \in \mathbb{R}^n$, the linear system $A\mathbf{x} = \mathbf{c}$ has more than one solution.

(2) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

(3) There is an $n \times n$ matrix B with $AB = I_n$.

(4) The linear system $A\mathbf{x} = \mathbf{0}$ only has a trivial solution.

3. [5pt] Let S be a subset of a vector space V with $\dim(V) = n$ and let W be the span of S.

For each of the following statements, indicate whether the statement **must be** true (T), can be false (F) or give no answer (N).

(1) $\dim(W) \leq \dim(V)$.

(2) If there exists a linearly dependent subset of S, then S is also a linearly dependent set.

(3) There are n linearly independent vectors in the span of V.

(4) If $S = \{v_1, v_2, \ldots, v_m\}$ with $m > n$, then $V = \text{span}(S)$.

4. [8pt] Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ and α and β are two bases for $\mathbb{R}^3$. Given is

$$[I]_\beta^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix},$$

and

$$[T]_\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$$

What is $[T]_\beta$?

5. [7pt] Let $\mathbb{P}^3(\mathbb{R})$ denote the vector space of polynomials of degrees less than or equal to 3. Given is the linear transformation $T : \mathbb{P}^3(\mathbb{R}) \to \mathbb{P}^3(\mathbb{R})$ defined as

$$T(p(x)) = x^2 p''(x),$$

where $p''(x)$ is the second derivative of the function $p(x)$. The set $\alpha = \{1, 1+x, 1-x^2, 1+x^3\}$ is a basis for $\mathbb{P}^3(\mathbb{R})$.

What is $[T]_\alpha^\alpha$?

6. [5pt] Let $L_A : F^3 \to F^3$ be the left-multiplication transformation, where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 3 & -1 & -4 \end{pmatrix}.$$

Give a basis for $N(L_A)$.

7. [5pt] Given are the following vectors:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}.$$

For which values of $t \in \mathbb{R}$ is the span($\{v_1, v_2, v_3\}$) equal to $\mathbb{R}^3$?

A. Only if $t = 1$.

B. For all $t \neq 1$.

C. Only if $t = 1/3$.

D. For all $t \neq 1/3$.

E. There is no t.

8. [5pt] Let

$$A = \begin{pmatrix} 2 & 1 & 3 & 0 & 1 \\ 0 & 3 & 0 & -6 & 9 \\ 0 & 1 & 0 & -2 & 3 \end{pmatrix}.$$

Let S be the set of all vectors $y \in \mathbb{R}^3$ for which the system of linear equations, denoted by $Ax = y$, has a solution. Give a basis for S.

9. [5pt] Find the volume of the parallelepiped with one vertex at the origin, and adjacent vertices at $(1, 3, 9)$, $(1, 2, 3)$ and $(2, 1, 1)$.