

## Test Linear Optimization

Wednesday, January 31, 2024, 8.45-11.45

Give motivations for all your answers. You are not allowed to use a calculator.

1. We have a set of foods  $F$  and a set of vitamins  $V$ .  
Any unit of food  $i \in F$  contains  $a_{ij}$  units of vitamin  $j \in V$  and  $c_i$  calories, and costs  $d_i$  dollars.  
To stay healthy, for each vitamin  $j$ , we need at least  $u_j$  units per day.  
We have a budget of  $b$  dollars per day.

- a) Construct a linear program that finds the diet with the smallest amount of calories among all healthy diets that fit the budget.
- b) Suppose that, instead of finding the healthy diet with the smallest amount of calories, we are interested in a healthy diet that contains an amount of calories  $c$  as close to some target  $c^*$  of calories per day as possible. That is, we want to minimize  $|c - c^*|$ , where  $c$  denotes the amount of calories in our diet. Construct a linear program that finds such a diet.

2. Consider a polyhedron  $P \subseteq \mathbb{R}^n$ , vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ , and a set  $S = \{\mathbf{x} \mid \mathbf{x} = \mathbf{a} + \lambda \mathbf{b}, \lambda \in \mathbb{R}\}$ . Prove that  $S$  contains at most two vertices of  $P$ .

3. Consider the following linear program:

$$\begin{aligned} \max \quad & x \\ \text{s.t.} \quad & x + y \geq 4 \\ & x - 2y \geq -6 \\ & x - y \leq 6 \end{aligned}$$

- a) Use Fourier-Motzkin to determine all objective values that correspond to feasible solutions.
- b) Use Fourier-Motzkin to determine an optimal solution.

You do **not** get points for using any method other than Fourier Motzkin.

4. Solve the following linear program using the simplex method.

$$\begin{aligned} \min \quad & -3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 - 6x_2 \leq 1 \\ & 4x_1 - 9x_2 \leq 2 \\ & -3x_1 + 8x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Turn the page for exercises 5, 6, and 7!

5. Consider the following tableau.

$$\begin{array}{c|cccc} -3 & 1 & a & 0 & 0 \\ \hline b & 1 & c & 1 & 0 \\ 2 & d & 4 & 0 & 1 \end{array}$$

- During first step objective value does not change.*
- a) Find all values for  $a, b, c, d$  for which the simplex method takes at least one more step, but in the next step, the objective value does not change.
- b) Find all values for  $a, b, c, d$  for which this can be the initial tableau in phase 1 of the 2-phase simplex method.

6. Consider the following linear program.

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 1 \\ & 5x_1 - x_2 \leq 0 \\ & x_1 \geq 0, x_2 \text{ free} \end{array}$$

- a) Construct its dual program, using the definition of dual program.
- b) Determine a linear program that is equivalent to the primal program, and for which all variables have a non-negativity constraint.
- c) Construct the dual program of the LP found in b), using the definition of dual program.
- d) Give a short reason that the duals found in parts a) and c) are equivalent (without referring to the result that equivalent primals always yield equivalent duals).

7. Let  $A, B \in \mathbb{R}^{m \times n}$ . Let  $\mathbf{b}, \mathbf{d} \in \mathbb{R}^m$ . Consider the following two statements.

S1) There exists some vector  $\mathbf{x}$  for which  $A\mathbf{x} = \mathbf{b}$  and  $B\mathbf{x} \leq \mathbf{d}$ .

S2) There exist some vectors  $\mathbf{u}, \mathbf{v}$  for which:  
 $\mathbf{v} \geq \mathbf{0}, \mathbf{u}'A + \mathbf{v}'B = \mathbf{0}'$ , and  $\mathbf{u}'\mathbf{b} + \mathbf{v}'\mathbf{d} < \mathbf{0}$ .

- a) Prove that at most one of these statements is true.
- b) Prove that at least one of these statements is true by constructing the dual of the following LP:

$$\begin{array}{ll} \max & 0 \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & B\mathbf{x} \leq \mathbf{d} \end{array}$$

exercise	1	2	3	4	5	6	7
points	6	6	6	6	6	6	6