

Introduction to Partial Differential Equations

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Exam

2024/2025, Q-2A

Give a suitable explanation of your answers.
The use of electronic devices is *not* allowed. A formula sheet is *not* handed out.

Question 1. Find the Green's function for the following boundary value problem

$$-u'' = f \quad \text{in } (0, 1), \quad \text{with } u(0) = 0, \quad u'(1) = 0. \quad \checkmark$$

Question 2. Let u be a classical solution of the boundary value problem

$$-u'' + \sin(u)u' + e^u = 0 \quad \text{in } (0, 1), \quad \text{with } u(0) = u(1) = 0. \quad \checkmark$$

Show that $u(x) < 0$ for all $0 < x < 1$.

Question 3. Let $f \in L^2(0, 1)$ and $g \in \mathbb{R}$. Consider the following boundary value problem

$$-u'' + u = f \quad \text{in } (0, 1), \quad \text{with } u(0) = 0 \quad \text{and} \quad u'(1) = g. \quad (1) \quad \checkmark$$

- (a) Derive a variational formulation for (1).
- (b) Prove existence and uniqueness of a weak solution for (1).

Hint: Consider the space $V = \{v \in H^1(0, 1) : v(0) = 0\}$.

Question 4. Consider the following eigenvalue problem: Find u and λ such that

$$-(au')' = \lambda u \quad \text{in } (0, 1), \quad \text{with } u(0) = 0 \quad \text{and} \quad u'(1) = 0. \quad (2) \quad \checkmark$$

- (a) Suppose $a(x) = 1$ for all $x \in [0, 1]$. Find all eigenfunctions and eigenvalues of (2).
- (b) Let $\alpha, \beta > 0$, and $a \in C^0([0, 1])$ satisfying $\alpha \leq a(x) \leq \beta$ for $x \in [0, 1]$. Denote λ_n the eigenvalues associated with (2) for this general a . Show that for all $n \in \mathbb{N}$ the following inequalities hold \checkmark

$$\left(\frac{2n+1}{2}\pi\right)^2 \alpha \leq \lambda_n \leq \left(\frac{2n+1}{2}\pi\right)^2 \beta.$$

Question 5. Let $S_h \subset H_0^1(0, 1)$ be the usual finite element space consisting of continuous, piecewise linear functions associated with the partition $x_j = jh$ with $h = 1/(J+1)$ and $j = 0, \dots, J+1$ of the interval $(0, 1)$. Denote $\Phi_j \in S_h$, $j = 1, \dots, J$, the associated basis functions satisfying $\Phi_j(x_i) = \delta_{i,j}$ for $i, j = 1, \dots, J$; recall that $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$. Define the matrix $C \in \mathbb{R}^{J \times J}$ by \checkmark

$$C_{i,j} = \int_0^1 \Phi_j'(x) \Phi_i(x) dx, \quad i, j = 1, \dots, J.$$

- (a) Show that C is skew-symmetric, $C^T = -C$, without calculating the matrix entries $C_{i,j}$ explicitly.
- (b) Calculate the matrix entries $C_{i,j}$ explicitly, i.e., give their numerical values.

Question 6. Let $S_h \subset H_0^1(0, 1)$ be the usual finite element space consisting of continuous, piecewise linear functions. Denote the L^2 projection operator $P_h : L^2(0, 1) \rightarrow S_h$ defined for $v \in L^2(0, 1)$ via

$$(P_h v, \chi) = (v, \chi) \quad \forall \chi \in S_h.$$

Show that

$$\|v - P_h v\| = \min_{\chi \in S_h} \|v - \chi\|.$$

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Question 7. Recall the finite element semi-discretization of the heat equation with homogeneous Dirichlet boundary conditions. For a finite element space $S_h \subset H_0^1(\Omega)$ find $u_h \in C^1([0, T]; S_h)$ such that

$$(u_h'(t), \chi) + (\nabla u_h(t), \nabla \chi) = (f(t), \chi) \quad \forall \chi \in S_h, \quad \text{with } u_h(0) = g_h.$$

Consider the discrete scheme of finding $u_h^n \in S_h$, $n \in \mathbb{N}_0$, such that

$$\frac{1}{k}(u_h^{n+1} - u_h^n, \chi) + \frac{1}{2}(\nabla(u_h^{n+1} + u_h^n), \nabla \chi) = (f(t_{n+1/2}), \chi) \quad \forall \chi \in S_h, \quad n \geq 0, \quad \text{with } u_h^0 = g_h,$$

where $t_n = nk$ and $t_{n+1/2} = (n + 1/2)k$ for step-size $k > 0$.

- (a) Show that u_h^{n+1} is well-defined.
- (b) Choose an appropriate test function χ in terms of u_h^n and u_h^{n+1} to show that

$$\|u_h^{n+1}\| \leq \|u_h^n\| + k\|f(t_{n+1/2})\| \quad \text{for all } n \geq 0.$$

Question 8. Consider the following first-order system

$$U_t(x, t) + \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} U_x(x, t) = 0 \quad \text{for } (x, t) \in \mathbb{R} \times (0, \infty),$$

$$U(x, 0) = \begin{pmatrix} \sin x \\ \cos x \end{pmatrix} \quad \text{for } x \in \mathbb{R},$$

where $U(x, t) = (U_1(x, t), U_2(x, t))^T \in \mathbb{R}^2$. Solve the system by the methods of characteristics.

Points:

Q1.	3	Q2.	3	Q3. (a)	3	Q4. (a)	4	Q5. (a)	2	Q6.	3	Q7. (a)	2	Q8.	5
				(b)	4	(b)	2	(b)	2			(b)	3		

Total: 36 + 4 = 40 points

Grade: (achieved points + 4)/4