Give a suitable explanation of your answers.
The use of electronic devices is *not* allowed. A formula sheet is *not* handed out.

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**Question 1.** Find the Green's function for the following boundary value problem

$$-u'' = f \quad \text{in } (0,1), \quad \text{with } u(0) = 0, \quad u'(1) = 0.$$

✓

**Question 2.** Let $u$ be a classical solution of the boundary value problem

$$-u'' + \sin(u)u' + e^u = 0 \quad \text{in } (0,1), \quad \text{with } u(0) = u(1) = 0.$$

✓

Show that $u(x) < 0$ for all $0 < x < 1$.

**Question 3.** Let $f \in L^2(0,1)$ and $g \in \mathbb{R}$. Consider the following boundary value problem

$$-u'' + u = f \quad \text{in } (0,1), \text{ with } u(0) = 0 \quad \text{and} \quad u'(1) = g. \tag{1}$$

✓

(a) Derive a variational formulation for (1).
(b) Prove existence and uniqueness of a weak solution for (1).
**Hint:** Consider the space $V = \{v \in H^1(0,1) : v(0) = 0\}$.

**Question 4.** Consider the following eigenvalue problem: Find $u$ and $\lambda$ such that

$$-(au')' = \lambda u \quad \text{in } (0,1), \text{ with } u(0) = 0 \quad \text{and} \quad u'(1) = 0. \tag{2}$$

(a) Suppose $a(x) = 1$ for all $x \in [0,1]$. Find all eigenfunctions and eigenvalues of (2). ✓
(b) Let $\alpha, \beta > 0$, and $a \in C^0([0,1])$ satisfying $\alpha \le a(x) \le \beta$ for $x \in [0,1]$. Denote $\lambda_n$ the eigenvalues associated with (2) for this general $a$. Show that for all $n \in \mathbb{N}$ the following inequalities hold ✓

$$\left(\frac{2n+1}{2}\pi\right)^2 \alpha \le \lambda_n \le \left(\frac{2n+1}{2}\pi\right)^2 \beta.$$

**Question 5.** Let $S_h \subset H_0^1(0,1)$ be the usual finite element space consisting of continuous, piecewise linear functions associated with the partition $x_j = jh$ with $h = 1/(J+1)$ and $j = 0, \dots, J+1$ of the interval $(0,1)$. Denote $\Phi_j \in S_h$, $j = 1, \dots, J$, the associated basis functions satisfying $\Phi_j(x_i) = \delta_{i,j}$ for $i, j = 1, \dots, J$; recall that $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$. Define the matrix $C \in \mathbb{R}^{J \times J}$ by ✓

$$C_{i,j} = \int_0^1 \Phi_j'(x)\, \Phi_i(x)\, dx, \qquad i, j = 1, \dots, J.$$

(a) Show that $C$ is skew-symmetric, $C^T = -C$, without calculating the matrix entries $C_{i,j}$ explicitly.
(b) Calculate the matrix entries $C_{i,j}$ explicitly, i.e., give their numerical values.

**Question 6.** Let $S_h \subset H_0^1(0,1)$ be the usual finite element space consisting of continuous, piecewise linear functions. Denote the $L^2$ projection operator $P_h : L^2(0,1) \to S_h$ defined for $v \in L^2(0,1)$ via

$$(P_h v, \chi) = (v, \chi) \quad \forall \chi \in S_h.$$

Show that

$$\|v - P_h v\| = \min_{\chi \in S_h} \|v - \chi\|.$$

**Question 7.** Recall the finite element semi-discretization of the heat equation with homogeneous Dirichlet boundary conditions. For a finite element space $S_h \subset H_0^1(\Omega)$ find $u_h \in C^1([0,T]; S_h)$ such that

$$(u_h'(t), \chi) + (\nabla u_h(t), \nabla \chi) = (f(t), \chi) \quad \forall \chi \in S_h, \quad \text{with} \quad u_h(0) = g_h.$$

Consider the discrete scheme of finding $u_h^n \in S_h$, $n \in \mathbb{N}_0$, such that

$$\frac{1}{k}(u_h^{n+1} - u_h^n, \chi) + \frac{1}{2}(\nabla(u_h^{n+1} + u_h^n), \nabla \chi) = (f(t_{n+1/2}), \chi) \quad \forall \chi \in S_h,\ n \geq 0, \quad \text{with} \quad u_h^0 = g_h,$$

where $t_n = nk$ and $t_{n+1/2} = (n + 1/2)k$ for step-size $k > 0$.

(a) Show that $u_h^{n+1}$ is well-defined.

(b) Choose an appropriate test function $\chi$ in terms of $u_h^n$ and $u_h^{n+1}$ to show that

$$\|u_h^{n+1}\| \leq \|u_h^n\| + k\|f(t_{n+1/2})\| \qquad \text{for all } n \geq 0.$$

**Question 8.** Consider the following first-order system

$$U_t(x,t) + \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} U_x(x,t) = 0 \qquad \text{for } (x,t) \in \mathbb{R} \times (0,\infty),$$

$$U(x,0) = \begin{pmatrix} \sin x \\ \cos x \end{pmatrix} \qquad \text{for } x \in \mathbb{R},$$

where $U(x,t) = (U_1(x,t), U_2(x,t))^T \in \mathbb{R}^2$. Solve the system by the methods of characteristics.

**Points:**

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Q1.** | 3 | **Q2.** | 3 | **Q3.** | (a) | 3 | **Q4.** | (a) | 4 | **Q5.** | (a) | 2 | **Q6.** | 3 | **Q7.** | (a) | 2 | **Q8.** | 5 |
| | | | | | (b) | 4 | | (b) | 2 | | (b) | 2 | | | | (b) | 3 | | |

**Total:** 36 + 4 = 40 points

**Grade:** (achieved points + 4)/4