Sample Exam: Analysis-2 (202200237), MOD-02-AM: Structures and Systems

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Date/Time: 30-January-2023, 13:45 – 16:45

- Closed book/calculator exam! May use one single-sided handwritten A4-paper.
- All answers must be motivated, including the answers for Section C.
- Answers for Section A *must* use the four steps (practised during Tutor Sessions).
  - (i.) Get Started: describe what the problem is about and your initial thoughts
  - (ii.) Devise Plan: provide an outline how you plan to solve (or have solved) the problem
  - (iii.) Execute: execute your plan (and try) to reach your solution
  - (iv.) Evaluate: reflect on your solution and/or approach [with something new not yet mentioned].

  Points are distributed (roughly) as: steps (i.)+(ii.) 35%, step (iii.) 50% and step (iv.) 15%.
- Section Grade: $\dfrac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- Course Grade: $0.6 \times \text{Grade\_Section\_A} + 0.4 \times \text{Grade\_Section\_C}$ (see Assessment Policy for details)
- Good Luck!

Section C: Total Points : 30

1. (a) Use a suitable Riemann sum to evaluate the following limit: [5]

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right].$$

(b) Let the function f be defined as $f(x) := \dfrac{x^2}{(1+3x)^2}$, for $x \in \mathbb{R}\setminus\{-\frac{1}{3}\}$.

√ Show that the Maclaurin series of f is given by $\displaystyle\sum_{k=0}^{\infty}(-1)^k\,(k+1)\,3^k\,x^{k+2}$ and find the radius and the interval of convergence of the series. [5]

[Hint: Instead of obtaining the derivatives of f, using some known series may be wiser.]

2. (a) Let $I := \displaystyle\int_0^{\pi/6} \tan(x)e^{\sin(x)}\,dx$. Express the following integrals in terms of I. [5]

√

$$\int_0^{1/2} \frac{xe^x}{1-x^2}\,dx \qquad \text{and} \qquad \int_0^{1/2} \ln(1-x^2)e^x\,dx.$$

(b) The well-known gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ is defined as the improper integral

√

$$\Gamma(x) := \int_0^{\infty} t^{x-1}e^{-t}\,dt, \quad \text{for } x > 0.$$

Show that, for $0 < x \le 1$, $\Gamma(x)$ indeed exists and $\Gamma(x+1) = x\Gamma(x)$. [3+2]

[Hint: Splitting the integral over $(0,1]$ and $[1,\infty)$ will help.]

3. (a) Suppose that $g : \mathbb{R} \to \mathbb{R}$ has a continuous derivative and the bivariate function f is defined as $f(x,y) := g\left(\frac{x+y}{x-y}\right)$ for $x \neq y$.

Find the numerical value of $\quad x\,\dfrac{\partial f(x,y)}{\partial x} + y\,\dfrac{\partial f(x,y)}{\partial y} \quad$ (for $x \neq y$). [5]

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x,y) := g(x)g(y)$.

Suppose that $\displaystyle\iint_D f(x,y)\,\mathrm{d}A = 4$, where D is the square $[a,b] \times [a,b]$ $(a, b \in \mathbb{R})$.

√

Find $\quad \displaystyle\int_a^b g(x)\,dx \quad$ and $\quad \displaystyle\int_a^b \int_y^b f(x,y)\,dx\,dy$. [2+3]

Section A: [Follow the four-step procedure]

Total Points : 20

4. Prove that the following series [2+2+2]

$$\sum_{k=1}^{\infty} (-1)^k \frac{2k+3}{(k+1)(k+2)}$$

converges. Determine, also, whether the series converges absolutely.

[In the evaluation-step, you must comment on the value of the infinite sum.
For this, split $\frac{2k+3}{(k+1)(k+2)}$ in terms of simple/partial fractions $\frac{\text{constant}}{\text{simple expression in } k}$.]

5. Suppose that $\{a_k\}_{k\in\mathbb{N}}$ is a real-valued sequence and $f(x)$ is formally defined as the series of functions

$$f(x) := \sum_{k=1}^{\infty} a_k \frac{1}{k^x}, \qquad x \in \mathbb{R}.$$

Prove that if the series converges at some $x_0 \in \mathbb{R}$ i.e., $f(x_0)$ exists, then the series converges absolutely on the interval $(x_0 + 1, \infty)$. [2+3+1]

[Hint: Argue and use the boundedness of $\left\{\frac{a_k}{k^{x_0}}\right\}_{k\in\mathbb{N}}$ and $x = x_0 + x - x_0$.
In the evaluation-step, comment on the existence of $f(x_0+1)$.]

6. Prove that [3+4+1]

$$\lim_{n\to\infty} \int_1^2 e^{-nx^2}\,dx = 0.$$