Final Exam: Analysis-1 (202200143), MOD-01-AM: Structures and Models

Instructors: Pranab Mandal and Carlos Perez Arancibia

Date/Time: 13-December-2023, 13:45 – 16:45

- Closed book exam! May use one single-sided handwritten A4-paper.
- All answers must be motivated, including the answers of Section C.
- Answers for Section A *must* use the four steps (practised during Tutor Sessions).
  - (i.) Get Started: describe what the problem is about and your initial thoughts
  - (ii.) Devise Plan: provide an outline how you plan to solve (or have solved) the problem
  - (iii.) Execute: execute your plan (and try) to reach your solution
  - (iv.) Evaluate: reflect on your solution and/or approach

  Points are distributed (roughly) as: steps (i.)+(ii.) 35%, step (iii.) 50% and step (iv.) 15%.
- Section Grade: $\dfrac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- Course Grade: According to the assessment scheme (published elsewhere)
- Good Luck!

Section C:

Total Points : 15

1. Define the function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) := \begin{cases} x, & x \in \mathbb{Q}, \\ -x, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   (a) Find $\lim_{x \to 0} f(x)$. Justify your answer. [3]

   (b) Find the set, $E \subset \mathbb{R}$, of all points where f is continuous. Justify your answer. [4]

2. Let I be in an open interval containing 0 and $f, g : I \to \mathbb{R}$ are $\mathcal{C}^2(I)$-functions. Furthermore, the following relations hold:

$$f(0) = 2/g(0), \qquad f'(0) = 2g'(0) = 4g(0), \qquad g''(0) = \frac{2}{\pi} f''(0) = 6f(0) = 3.$$

   (a) Let $h_a : I \to \mathbb{R}$ be given by $h_a(x) := f(x)/g(x),\ x \in I$. Compute $h_a'(0)$. [1]

   (b) Let $h_b : I \to \mathbb{R}$ be given by $h_b(x) := f(x)g(x)\sin(x),\ x \in I$. Compute $h_b'(0)$. [1]

   (c) Let $h_c : I \to \mathbb{R}$ be given by $h_c(x) := f\left((g(x) - 4)^2\right),\ x \in I$. Compute $h_c'(0)$. [2]

   (d) Compute $\lim\limits_{x \to 0} \sin\left(\dfrac{f(x) - 2f(0) + f(-x)}{g(x) - 2g(0) + g(-x)}\right)$. [4]

Section A: [Follow the four-step proecdure]

Total Points : 35

3. Consider the function $f : (1, 2] \to \mathbb{R}$ defined as

$$f(x) := \frac{1}{x^2 - 1}, \quad x \in (1, 2].$$

Prove the following two properties of f. [5+5+1]

[You may provide one combined evaluation/reflection for the whole question, for example, by providing the limit of the function at infinity and the location of the fixed point w.r.t. 1.5.]

(a) Use the definition of limits (of a function) to prove that $\lim_{x \to 1+} f(x) = \infty$.

(b) Prove that there is a fixed point $c \in (1, 2)$ such that $f(c) = c$.

4. Consider the expression $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}$.

It is not difficult to understand what the expression means, at least, intuitively. However, to comment on whether it represents a real number, one needs to work a bit more.

One way to formalize the expression is by identifying it as the limit of the sequence $\{a_n\}$ defined by

$$a_n := \sqrt{2 + a_{n-1}}, \quad n \in \mathbb{N}, \quad \text{where } a_0 := \sqrt{2}.$$

Argue that the expression is indeed a number through the follwing steps. [4+7+1]

(a) Prove using the method of induction that the sequence $\{a_n\}$ is increasing.

(b) Prove that the sequence $\{a_n\}$ converges.

5. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if for all $x, y \in \mathbb{R}$ it holds that

$$f\big(tx + (1 - t)y\big) \leq tf(x) + (1 - t)f(y), \quad \forall t \in [0, 1].$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that f is convex if and only if [5+6+1]

$$f(y) \geq f(x) + (y - x)f'(x), \quad \forall x, y \in \mathbb{R}.$$

[Hint: The limit-characterization of derivatives can be useful, together with the relationships/(in)equalities among $f(z) - f(x)$, $f(y) - f(x)$, $(z - x)$ and $(y - x)$ for arbitrary x, y and any in-between point z. Explore, also, the relationships among $f(x)$, $f(y)$, $f(z)$ and $f'(z)$.]