

Final Exam: Analysis-I (202200143), MOD-01-AM: Structures and Models

Instructors: Pranab Mandal and Carlos Pérez Arancibia

Date/Time: 15-December-2022, 08:45 – 11:45

- Closed book exam! Use of own text-material or an electronic calculator is not allowed.
- All answers must be motivated, including the answers of Section C.
- Answers for Section A *must* use the four steps (practised during Tutor Sessions).
 - (i.) Get Started: describe what the problem is about and your initial thoughts
 - (ii.) Devise Plan: provide an outline how you plan to solve (or have solved) the problem
 - (iii.) Execute: execute your plan (and try) to reach your solution
 - (iv.) Evaluate: reflect on your solution and/or approach

Points are distributed (roughly) as: steps (i.)+(ii.) 35%, step (iii.) 50% and step (iv.) 15%.

- Section Grade: $\frac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- Course Grade: $0.7 \times \text{Grade_Section_A} + 0.3 \times \text{Grade_Section_C}$ (rounded off according to EER)
- Good Luck!

Section C:

Total Points : 15

1. (a) Let $z \in \mathbb{C}$ with $|z| = 1$ and $\text{Re}(z^2) = -1/2$. Compute $|z^2 + z^4|$. [4]

(b) Solve for all the roots of the following equation: [3]

$$z^4 = 8\sqrt{2}(1+i)e^{3i\pi/4} \quad (z \in \mathbb{C}).$$

Express them in the form $a + ib$, where $a, b \in \mathbb{R}$.

2. Let the function $f : \mathbb{R} \setminus \{-2, 2\} \rightarrow \mathbb{R}$ be given by

$$f(x) = \frac{x-1}{x^2-4}, \quad x \in \mathbb{R} \setminus \{-2, 2\}.$$

Show, by mathematical induction, that the n th-order derivative of f is given by [8]

$$f^{(n)}(x) = \frac{n!(-1)^n}{4} \left(\frac{3}{(x+2)^{n+1}} + \frac{1}{(x-2)^{n+1}} \right), \quad n \in \mathbb{N}, \quad x \in \mathbb{R} \setminus \{-2, 2\}.$$

[Hint: You may want to start by showing that $f(x) = \frac{3}{4} \cdot \frac{1}{x+2} + \frac{1}{4} \cdot \frac{1}{x-2}$.]

[Section A is on page 2.]

Section A: [Follow the four-step procedure.]

Total Points : 35

3. Answer the following. [You may give one (combined) "Evaluation".] [4+6+2]

(a) Let the sequence $\{x_n\}$ be given by

$$x_n = \frac{2n^2 + 2n + 1}{3n^2 + 1}, \quad n \in \mathbb{N}.$$

Prove, using the definition, that the sequence converges.

(b) Using different laws of limit, compute the following:

$$\lim_{n \rightarrow \infty} \left[\sqrt{4n^2 + n} - 2n + \frac{e^{-n}}{n} \right].$$

[Hint: Splitting into different suitable sequences may help.]

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Prove that if $f(a) < M$ for some $M \in \mathbb{R}$, then there is an open interval I containing a such that $f(x) < M$ for all $x \in I$. [4+5+1]

5. (a) Let $a < b$ be two real numbers. Suppose that $f : (a, b) \rightarrow \mathbb{R}$ is a differentiable function such that $f'(x) < 0$ for all $x \in (a, b)$.

Prove that f is strictly decreasing on (a, b) . [2+2+1]

(b) Show that [3+4+1]

$$x^2 e^{-x} < 0.8 - e^{-x} \quad \text{for all } x > 1.$$

[Recall that $e \approx 2.7183$.]