Final Exam: Analysis-I (202200143), MOD-01-AM: Structures and Models

Instructors: Pranab Mandal and Carlos Perez Arancibia

Date/Time: 09-November-2022, 13:45 – 16:45

- Closed book exam! Use of own text-material or an electronic calculator is not allowed.
- All answers must be motivated, including the answers of Section C.
- Answers for Section A *must* use the four steps (practised during Tutor Sessions).
  - (i.) Get Started: describe what the problem is about and your initial thoughts
  - (ii.) Devise Plan: provide an outline how you plan to solve (or have solved) the problem
  - (iii.) Execute: execute your plan (and try) to reach your solution
  - (iv.) Evaluate: reflect on your solution and/or approach

  Points are distributed (roughly) as: steps (i.)+(ii.) 35%, step (iii.) 50% and step (iv.) 15%.
- Section Grade: $\dfrac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- Course Grade: $0.7 \times \text{Grade\_Section\_A} + 0.3 \times \text{Grade\_Section\_C}$ (rounded off according to EER)
- Good Luck!

Section C: Total Points : 15

1. Write the following quantities in the form $a + ib$, where $a, b \in \mathbb{R}$. [3+4]

   (a) $\left| z + \dfrac{1}{z} \right|$, where $|z| = 2$ and $\mathrm{Re}(z^2) = 1$.

   (b) $\left(\dfrac{1 + i\sqrt{3}}{1 - i\sqrt{3}} \right)^{10}$. [Recall that $\cos(\pi/3) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$.]

2. Compute $\lim_{x \to 1} \left| (1 + \ln x)^{1/\ln x} - \pi \right|$ [4]

3. Given a number $a > 0$, find the minimum value of $(a + x)/\sqrt{ax}$ where $x > 0$. Subsequently, use this result to prove the arithmetic-geometric mean inequality: [4]

$$\frac{a+b}{2} \geqslant \sqrt{ab} \quad \text{for every} \quad a > 0, \quad b > 0.$$

Section A: [Follow the four-step proecdure] Total Points : 35

4. (a) Let $\{x_n\}$ be a sequence of real numbers. Suppose that there exists an $\epsilon > 0$ and $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $m, n \geq N$.

   Prove that the sequence $\{x_n\}$ is bounded. [2+3+1]

   (b) Let $A \subseteq \mathbb{R}$ be a set. Express the statement "A is not bounded" using the quantifier notations $\exists \ldots (\cdots)$ and $\forall \ldots (\cdots)$. [2]

5. (a) Prove the following (part of the sequential characterization of continuity). [3+4+1]

Let E be a nondegenerate interval in $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous on E. Suppose that $\{x_n\}$ is a sequence in E, i.e., $x_n \in E$ for all $n \in \mathbb{N}$, and $x_n \to a \in E$ as $n \to \infty$. Then $f(x_n) \to f(a)$ as $n \to \infty$.

(b) Determine the following limit (of a sequence): $\lim_{n\to\infty} n^2 \sin^2\left(\frac{\pi}{n}\right)$. [2+4+1]

[You may use part (a) in answering part (b). Also, you may assume that the function $f(x) = \frac{\sin(x)}{x}$, if $x \neq 0$, and $= 1$, if $x = 0$ is continuous on $\mathbb{R}$.]

6. In the following, you may use part (a) in answering part (b). Also, you may give one (combined) "Evaluation". [5+6+1]

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ with $f'(x) \leq 2$ for all $x \in \mathbb{R}$. Prove that there is at most one $a > 1$ such that $f(a) = a^2$. [Hint: Mean Value Theorem may be helpful.]

(b) Show that there is a unique $a > 1$ satisfying $e^{-a} + 2a - a^2 = 0$.