

Kenmerk: EWI2021/TW/MOR/MU/Mod7/Exam1

Exam 1, Module 7, Codes 202001360 & 202001364

Discrete Structures & Efficient Algorithms

Monday, March 25, 2022, 8:45 - 11:45

Answers to questions 1-8 need to be motivated, arguments and proofs must be complete. You are allowed to use a handwritten cheat sheet (A4, both sides) per topic (ADS, DM) during the exam.

For information: This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	ca. 1h	(30 points)
Discrete Mathematics (DM)	ca. 2h	(60 points)

The total is $30+60=90$ points. Your grade is $1 + 0.1x$, x being the number of points, rounded to one digit. That means, you need 45 points to get a 5.5.

Please use a new sheet of paper for each part (ADS, DM), as the ADS and DM parts will be corrected separately!

Double students Discrete Mathematics & Algebra (202001364) only do the DM part. In that case, please write "Discrete Mathematics & Algebra" on top of your exam.

Algorithms & Data Structures

1. (10 points)

(a) Examine the following algorithm:

```
def func(n):  
    res=0  
    while n>0:  
        m=n  
        while m>0:  
            res=res+m  
            m=m-1  
        n=n-1  
    return res
```

Give the asymptotic time complexity of this algorithm expressed in the number of arithmetic operations.

(b) Suppose that the number of steps of an algorithm $T(n)$ with an input of n , has the recurrence relation

$$T(n) = 8T(n/2) + n^2 + 4n + 1/n$$

What is the asymptotic complexity class of this algorithm?

2. (10 points)

(a) Give an algorithm that deletes the smallest element in a minheap, returning a heap with the remaining elements. The complexity of this algorithm must be $O(\log n)$.

(b) Given a non-empty binary tree sorted post-order. Give an algorithm that yields the node with the biggest element and the node with the smallest element.

3. (10 points)

Given n positive integers a_1, \dots, a_n and positive integer G . We want to determine whether there is a subsequence of a_1, \dots, a_n with sum G . Example: for 7, 3, 2, 5, 8 and $G = 14$ the answer is "true" since $7 + 2 + 5 = 14$.

Define boolean $R(i, g)$ as:

$R(i, g)$ iff a_1, \dots, a_i has subsequence with sum g .

Note that

- $R(i, 0) = \text{true}$ for all $0 \leq i \leq n$

- $R(0, g) = \text{false}$ for all $g > 0$
- $R(i, g) = \text{false}$ for all $g < 0$

(a) Motivate which of the following recurrence relations holds (for all $1 \leq i \leq n$, $1 \leq g \leq G$):

- $R(i, g) = (R(i-1, g) \leq a_i + R(i, g))$
- $R(i, g) = R(i-1, g) \vee R(i-1, g - a_i)$
- $R(i, g) = \min\{R(i+1, g), R(i+1, g + a_i)\}$
- $R(i, g) = R(i-1, g) \vee R(i, g - a_i) \wedge R(i-1, g)$

(b) Give an algorithm to determine whether a_1, \dots, a_n has a subsequence with sum G . The complexity may be no worse than $\Theta(Gn)$.

Reminder: Use a new sheet of paper for each part (ADS, DM), as the ADS and the DM parts will be corrected separately!

Discrete Mathematics

4. (4+5 points)

(a) Consider $a \in \mathbb{Z}$, with $a > 1$ and its (unique) prime factorization

$$a = p_1^{a_1} \cdot p_2^{a_2} \cdot \dots \cdot p_k^{a_k}.$$

Show that a is a perfect square¹ if and only if for all $1 \leq i \leq k$, a_i is even.

(b) Consider $a, b \in \mathbb{Z}_{>0}$, with $\gcd(a, b) = 1$. Show that if ab is a perfect square then both a and b must be perfect squares.

[Hint: You may want to use the statement of (a).]

5. (8 points) Consider a bipartite graph $G = ((V_1, V_2), E)$ for which every vertex has the same positive degree. In other words, there exists an integer $\Delta \geq 1$ such that $\deg(v) = \Delta$ for all $v \in (V_1 \cup V_2)$. Prove that $|V_1| = |V_2|$.

6. (10 points) Consider a simple, capacitated network $G = (V, E, c)$, where V is the set of vertices, $s, t \in V$, E is the set of directed edges, and $c : E \rightarrow \mathbb{Z}_{\geq 0}$ gives the edge capacities. Let $|V| = n$ and $|E| = m$. Suppose you are given a maximum (s, t) -flow f . Now suppose that the capacity of a specific edge $e \in E$ is increased by one unit, and let G' be the resulting capacitated network. Suggest how to compute the maximum (s, t) -flow in f' in G' , in $O(n+m)$ time. Briefly explain (i) why your suggested algorithm

¹An $x \in \mathbb{Z}$ is a perfect square if and only if there exists $y \in \mathbb{Z}$ such that $x = y^2$.

is correct (ii) why it achieves the desired running time.

[Hint: You may want to use the (s, t) -flow f as well as the residual graph for G' with respect to f .]

7. (4+3 points)

(a) Compute the solution to the recurrence relation

$$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n \quad (n \geq 2) \quad \text{with} \quad a_0 = 1 \quad \text{and} \quad a_1 = 2.$$

(b) Let a_n with $n \geq 0$ be the number of bit strings (i.e., strings in $\{0, 1\}$) of length n that contain three consecutive zeros. Find a recurrence relation for computing a_n . (You do not need to solve this recurrence relation.)

8. (8 points) Assume that Alice has published modulus $n = 55$, and exponent $e = 7$. Bob sends ciphertext $C = 5$ to Alice. You are eavesdropper Eve and you are interested in Bob's secret message M . Compute Bob's secret message M from ciphertext C . Write down all of the computational steps that you need to perform in order to obtain Bob's secret message M .

9. (3 points each) For each of the following six claims, decide if true or false or if you would rather not give an answer. A correct answer gives 3, an incorrect answer -3 and not giving an answer 0 points (minimum total number of points for Question 9 is 0 points). **Instead of guessing, it may be better not giving an answer.**

(a) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$, $e \in E$. Consider a cycle C in G , and let e be an edge of the cycle C such that $w_e \leq w_{e'}$ for any edge e' along C . Then e must belong to a minimum spanning tree of G .

True

False

I prefer to not give an answer

(b) Consider a capacitated network $G = (V, A, c)$, where V is the set of vertices, A is the set of directed arcs, and $c : A \rightarrow \mathbb{Z}_{\geq 0}$, are the arc capacities. Let f be some (s, t) -flow in G respecting the flow conservation and capacity constraints, and let $val(f)$ be its value. Then there must exist an (s, t) -cut $[S, T]$ with capacity $cap([S, T]) = val(f)$.

True

False

I prefer to not give an answer

(c) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w : E \rightarrow \mathbb{R}$, and let the minimum edge weight be -10 (i.e., $\min_e \{w(e)\} = -10$). The following algorithm can be used to compute the shortest path between $u, v \in V$ in G : (i) Construct graph $G' = (V, E)$ with edge weights $w' : E \rightarrow \mathbb{R}_{\geq 0}$ defined as $w'(e) = w(e) + 10$, for all $e \in E$. (ii) Run Dijkstra's algorithm to compute the shortest path $P(u, v)$ between u, v in G' . (Note that by construction G' only has non-negative weight edges). (iii) Output $P(u, v)$ as the shortest path between u, v in G .

- True
- False
- I prefer to not give an answer

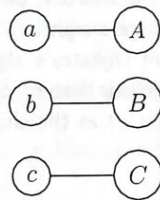
(d) Consider an undirected, connected, and simple graph $G = (V, E)$ with non-negative edge weights. Also consider a minimum spanning tree T of G . Then for every pair of vertices $u, w \in V$, the shortest (u, w) -path in G is contained in T .

- True
- False
- I prefer to not give an answer

(e) Consider a connected graph $G = (V, E)$ with 13 vertices (i.e., $|V| = 13$) such that G has an Euler path but no Euler tour. Then G must have exactly two vertices of odd degree and eleven vertices of even degree. *is simple*

- True
- False
- I prefer to not give an answer

(f) Consider the depicted matching M and corresponding preference lists. Then M is a stable matching.



- $B >_a A >_a C$
- $C >_b B >_b A$
- $A >_c C >_c B$
- $b >_A a >_A c$
- $c >_B b >_B a$
- $a >_C c >_C b$

[Reminder: $x >_z y$ indicates that z prefers to be matched with x over y .]

- True
- False
- I prefer to not give an answer